Continuity of Lyapunov exponents for non-uniformly fiber-bunched linear cocycles

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Preliminaries

Let

$$M \text{ compact metric space,}$$

$$f\colon M \to M \text{ continue,}$$

$$\mu \text{ ergodic } f \text{ -invariant,}$$

$$A\colon M \to SL(2,\mathbb{R}) \text{ continue.}$$

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We define linear cocycle by

$$F_A : M \times \mathbb{R}^2 \to M \times \mathbb{R}^2$$

$$F_A(x, v) = (f(x), A(x)v).$$

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$$A^n(x) = A(f^{n-1}(x)) \dots A(x).$$

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By Furstenberg-Kesten, the limits

$$\lambda_+(A) = \lim_{n \to \infty} \frac{1}{n} \log \|A^n(x)\|$$

and

$$\lambda_-(A) = \lim_{n \to \infty} \frac{1}{n} \log \|(A^n(x))^{-1}\|^{-1}$$

are well defined μ-almost every point.

History of the problem

Theorem - Mañé-Bochi [2002]

If f is an homeomorphism and μ is aperiodic. Then A is a continuity point in the C^0 topology if and only if

a) A is uniformly hyperbolic,

b) $\lambda_+(A) = 0 = \lambda_-(A)$.

History of the problem

Recently, Viana and Yang (2016) prove that when the base is non-invertible Mañé-Bochi is not true: there exists a C^0 continuity point with $\lambda_+(A) > 0$ and non-uniformly hyperbolic.

Let $f\colon M \to M$ be a sub-shift of finite type with expansion rate σ and μ an ergodic f- invariant measure with local product structure and $\text{supp}(\mu) = M$.

Definition

An *uniform stable holonomy* for A over f are isomorphisms $H^s_{x,y}\colon \mathbb{R}^2 \to \mathbb{R}^2$

- $H^s_{y,z} \circ H^s_{x,y} = H^s_{x,z}$ e $H^s_{x,x} = Id$;
- $A(y) \circ H^s_{x,y} = H^s_{f(x),f(y)} \circ A(x)$;
- $(x, y, \xi) \mapsto H^s_{x,y}(\xi)$ is continuous.

History of the problem

Theorem (Backes, Brown e Butler, 2018)

$$\text{If } (A_k, H^{s,k}, H^{u,k}) \xrightarrow{C^0} (A, H^s, H^u), \text{ then } \lambda_+(A_k) \to \lambda_+(A).$$

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Conjecture (M.Viana)

Se $(A_k, H^{s,k}) \xrightarrow{C^0} (A, H^s)$, then $\lambda_+(A_k) \to \lambda_+(A)$.

Results

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- $H^s_{y,z} \circ H^s_{x,y} = H^s_{x,z}$ and $H^s_{x,x} = Id$;
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- $(x, y, \xi) \mapsto H^s_{x,y}(\xi)$ is measurable.
  There exist $\mathcal{D}_l \subset \mathcal{D}_{l+1}$ with $\mu(\mathcal{D}_l) < 1 - \frac{1}{l}$. If $x \in \mathcal{D}_l$

$$\left\| H^s_{y,z} - id \right\| \leq C_l d(y, z) \text{ with } y, z \in W^s_{loc}(x).$$

Case $\lambda_+(A) < \frac{\log \sigma}{2}$ or non-uniformly fiber-bunched.

Theorem A (.-Marin)

If

$$A_k \xrightarrow{Lip} A \text{ and } H^{s,k} \xrightarrow{C^0} H^s$$

then

$$\lambda_+(A_k) \to \lambda_+(A)$$

Results

Case $\lambda_+(A) < \frac{\alpha \log \sigma}{2}$ or non-uniformly fiber-bunched.

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$$A_k \xrightarrow{\alpha} A \text{ and } H^{s,k} \xrightarrow{C^0} H^s$$

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Results

A locally constant cocycle A is *irreducible* if there is no proper subspace of $\mathbb{R}^2$ invariant by $A(x)$ for μ-a.e.p.

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Theorem B (.-Marin)

Let A be irreducible locally constant and non-uniformly fiber-bunched. If $A_k \xrightarrow{Lip} A$, then $\lambda_+(A_k) \to \lambda_+(A)$

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We consider $\mathbb{P}(F_A)$ as a differential cocycle, thus

$$\lambda(\mathbb{P}(F_A), m^u) = \lambda_-(A) - \lambda_+(A) = 2\lambda_-(A) < 0 < \log \sigma,$$

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and

$$\lambda(\mathbb{P}(F_A), m^s) = -\lambda_-(A) + \lambda_+(A) = 2\lambda_+(A) \geq \log \sigma,$$

Muchas gracias!
Muito obrigada!
Thanks you!