On cyclic Higgs bundles

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Set-up

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- $\pi_1(S)$ be the fundamental group of S.
- Σ denote a Riemann surface structure (S, J), where J is a complex structure.
- K denote the canonical line bundle $T^{*1,0}\Sigma$ of Σ.

Non-abelian Hodge theory

The non-abelian Hodge theory relates the following moduli spaces.

- **Betti moduli space** $\mathcal{M}_{Betti}(G)$: $Hom^+(\pi_1(S), G)/G$, consists of conjugacy classes of semisimple representations.

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- **(Intermediate)** the deformation space of equivariant harmonic maps from $\widetilde{\Sigma}$ into G/H (H is the maximal compact subgroup of G).
- **Dolbeault moduli space** $\mathcal{M}_{Dolbeault}(G)$: $\mathcal{M}_{Higgs}(G)$, gauge equivalence classes of polystable G-Higgs bundles over Σ.

Non-abelian Hodge theory

The moduli spaces are homeomorphic to each other.

Topology		Diff Geom		Riem Geom		Complex Geom
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$\mathcal{M}_{Betti}(G)$	$\cong$	$\mathcal{M}_{DeRham}(G)$	$\cong$	"Intermediate"	$\cong$	$\mathcal{M}_{Dolbeault}(G)$
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- The first homeomorphism is the generalized de Rham theorem.
- The second and third homeomorphisms are the non-abelian Hodge theory: relating the de Rham moduli space and the Dolbeault moduli space through harmonic maps.

Higgs bundles

Definition

A $SL(n, \mathbb{C})$-Higgs bundle over Σ is a pair (E, ϕ) where

- $E \to \Sigma$ is a rank n holomorphic vector bundle satisfying $\det E = \mathcal{O}$;
- ϕ is a holomorphic bundle map: $E \to E \otimes K$ satisfying $\operatorname{tr} \phi = 0$. (Higgs field)

Similar extra conditions for G-Higgs bundles.

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Theorem (Hitchin, Simpson)

Let (E,ϕ) be a stable $SL(n,\mathbb{C})$-Higgs bundle, then there exists a unique metric h on E, called the harmonic metric, solving the Hitchin equation

$$F_{\nabla^h} + [\phi, \phi^{*_h}] = 0$$

where

- F_{∇^h} — the curvature of the Chern connection ∇^h,
- ϕ^{*_h} — the hermitian adjoint of ϕ.

Non-abelian Hodge theory

By the work of Hitchin and Simpson, given a stable Higgs bundle (E, ϕ),

- $\rightsquigarrow$ a flat $SL(n, \mathbb{C})$-connection $D = \nabla^h + \phi + \phi^{*_h}$;
  (D being flat is equivalent to h satisfying the Hitchin equation.)

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Conversely, by the work of Donaldson and Corlette, given a semisimple representation $\rho : \pi_1(\Sigma) \to SL(n, \mathbb{C})$, there exists a ρ-equivariant harmonic map $f : \tilde{\Sigma} \to SL(n, \mathbb{C})/SU(n)$.

Remark

The nonabelian Hodge correspondence is not explicit since it is through solving a highly nontrivial second-order elliptic PDE. For example, if we have a natural action on one side, it is hard to see how the action affects on the other side.

Example: Teichmüller space $\mathit{Teich}(S)$

$$\begin{aligned} \mathit{Teich}(S) &:= \{\text{complex structures } \Sigma \text{ on } S \text{ up to isotopy}\} \\ &\cong \{\text{hyperbolic metrics } g \text{ on } S \text{ up to isotopy}\}. \end{aligned}$$

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Fix a base Riemann surface Σ_0,

- any point $\Sigma = (S, g)$ in $Teich(S)$ corresponds to a $SL(2, \mathbb{R})$-Higgs bundle over Σ_0:

$$E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \quad \phi = \begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix} : E \to E \otimes K,$$

$$(1 : K^{\frac{1}{2}} \to K^{-\frac{1}{2}} \otimes K, \quad q_2 : K^{-\frac{1}{2}} \to K^{\frac{1}{2}} \otimes K)$$

where $q_2 \in H^0(K^2)$ is the Hopf differential of the harmonic diffeomorphism $f : \Sigma_0 \to (S, g)$, i.e. the $(2, 0)$-part of the pullback metric.

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- In this case, the existence of solutions to Hitchin equation gives a proof of the uniformization theorem for Riemann surfaces, called the base Fuchsian case.

Higgs bundles

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- The Hitchin fibration and Hitchin section.
- The $\mathbb{C}^*$-action.

We'll also introduce a family of Higgs bundles: cyclic Higgs bundles.

Main goals

- Understanding the $\mathbb{C}^*$-action;
- Find out special feature of Hitchin section;

for cyclic Higgs bundles.

Hitchin fibration and Hitchin section

- The Hitchin fibration is a map from the moduli space of $SL(n,\mathbb{C})$-Higgs bundles over Σ to the direct sum of holomorphic differentials.

$$\begin{aligned} h : \mathcal{M}_{Higgs}(SL(n,\mathbb{C})) &\longrightarrow \bigoplus_{j=2}^{n} H^0(\Sigma, K^j) \ni (q_2, q_3, \cdots, q_n) \\ (E,\phi) &\longmapsto (tr(\phi^2), tr(\phi^3), \cdots, tr(\phi^n)). \end{aligned}$$

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$$\text{Bundle: } E = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}$$

$$\text{Higgs field: } \quad \phi = \begin{pmatrix} 0 & q_2 & & \cdots & q_{n-1} & q_n \\ 1 & 0 & q_2 & \cdots & q_{n-2} & q_{n-1} \\ & \ddots & & \ddots & & \\ & & & & q_2 & \\ & & & 1 & 0 & q_2 \\ & & & & 1 & 0 \end{pmatrix}$$

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The holonomy of the corresponding representation lies in $SL(n,\mathbb{R})$.

Hitchin fibration and Hitchin section (continue)

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- The representations arising from Hitchin section form a connected component of $Hom(\pi_1(S), PSL(n, \mathbb{R}))$, called the Hitchin component. It is a natural generalization of $Teich(S)$ and indeed shares lots of similar properties.

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- The representations arising from Hitchin section form a connected component of $Hom(\pi_1(S), PSL(n, \mathbb{R}))$, called the Hitchin component. It is a natural generalization of $Teich(S)$ and indeed shares lots of similar properties.
- The base n-Fuchsian point is $s(0, 0, \cdots, 0) = (E, \phi)$ with

$$E = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{\frac{1-n}{2}}, \quad \phi = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

  The corresponding representation is the base Fuchsian representation composing with the unique irreducible representation

$$\tau : SL(2, \mathbb{R}) \to SL(n, \mathbb{R}).$$

Hitchin fibration and Hitchin section (continue)

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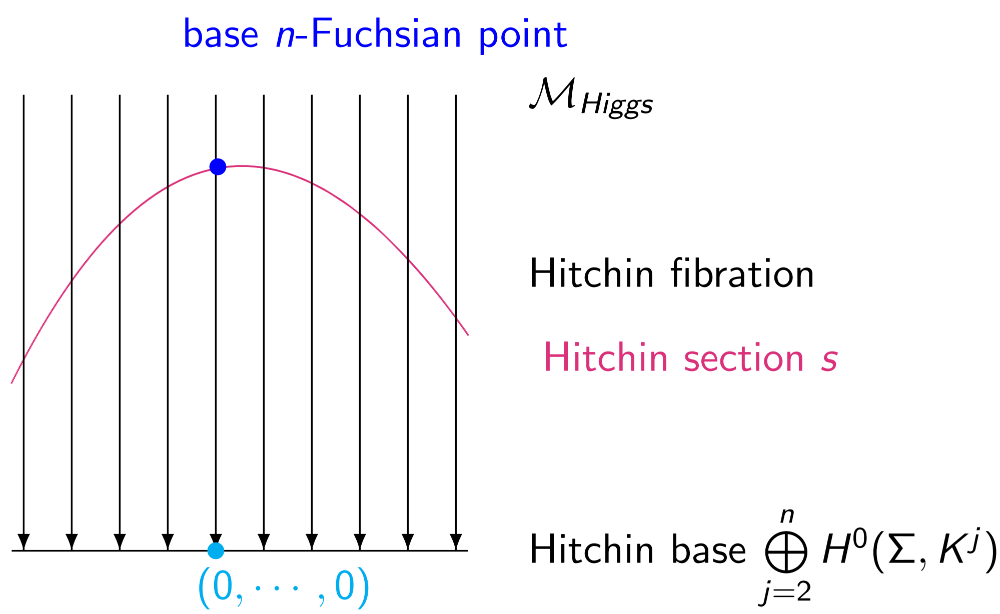

$\mathbb{C}^*$-action and Morse funciton

- A key attribute of Higgs bundles: there is a natural $\mathbb{C}^*$-action on $\mathcal{M}_{Higgs}(G)$ (the focus of Simpson's Ph.D. thesis) as follows:

$$\begin{array}{rcl} \mathbb{C}^* \times \mathcal{M}_{Higgs}(G) & \to & \mathcal{M}_{Higgs}(G) \\ t \cdot (E, \phi) & \to & (E, t\phi) \end{array}$$

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$$f(E, \phi) = \int_\Sigma \mathrm{tr}(\phi\phi^*)$$

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- The $\mathbb{C}^*$-action then acts on the space of representations in an unexplicit way.

Cyclic Higgs bundles

- The cyclic Higgs bundles are of the following form

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_n, \quad \phi = \begin{pmatrix} 0 & & & \gamma_n \\ \gamma_1 & 0 & & \\ & \ddots & \ddots & \\ & & \gamma_{n-1} & 0 \end{pmatrix}$$

where L_i is a holomorphic line bundle, $\gamma_j : L_j \to L_{j+1} \otimes K$ is nonzero for all $j < n$.

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Main goals (recall)

For cyclic Higgs bundles,

Goal I

Understanding the $\mathbb{C}^*$-action.

Goal II

Find out special feature of Hitchin section.

Goal I: Monotonicity of the pullback metric ($\mathbb{C}^*$-flow)

Motivation

Along the $\mathbb{C}^*$-flow of the moduli space of Higgs bundles, we want to understand how the associated harmonic (minimal) maps vary through solving the Hitchin equation. (via the pullback metric)

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The Riemannian metric on $SL(n, \mathbb{C})/SU(n)$ is induced by the Killing form on $sl(n, \mathbb{C})$. If the harmonic map $f : \tilde{\Sigma} \to SL(n, \mathbb{C})/SU(n)$ is conformal, the pullback metric of f is given by

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Theorem (Dai-L)

Let (E,ϕ) be a cyclic Higgs bundle with $\gamma_n \neq 0$. Then along the $\mathbb{C}^*$-orbit of (E,ϕ), as $|t|$ increases, the pullback metric g^t of its corresponding branched minimal immersion strictly increases away from branch points.

Goal I: Monotonicity of the pullback metric ($\mathbb{C}^*$-flow)

Integrating the metric over the surface,

Corollary (Dai-L)

Let (E, ϕ) be a cyclic Higgs bundle. Then along the $\mathbb{C}^*$-orbit of (E, ϕ), as $|t|$ increases, the Morse funciton $f(E, t\phi)$ strictly increases.

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- The monotonicity for Morse function is not new. In fact, Hitchin showed that with respect to the Kähler metric on $\mathcal{M}_{Higgs}$, the gradient flow of the Morse function is exactly the $\mathbb{R}^*$-part of $\mathbb{C}^*$-action.
- So we improve the result for cyclic Higgs bundles from integral monotonicity to pointwise monotonicity.

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We expect this to be true in general.

Conjecture

Along the $\mathbb{C}^*$-orbit of a Higgs bundle (E, ϕ), as $|t|$ increases, the pullback metric g^t of its corresponding branched minimal immersion strictly increases.

Goal II: Curvature of cyclic Higgs bundles in Hitchin section

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Find out special feature of Hitchin section.

Recall the cyclic Higgs bundles in Hitchin sections are

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Motivation

We want to investigate that, as an immersed submanifold, how the image $f(\widetilde{\Sigma})$ sits inside the symmetric space X. (via the extrinsic curvature)

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Theorem (Dai-L)

Let (E, ϕ) be a cyclic Higgs bundle in Hitchin section and f be the associated equivariant harmonic map into the symmetric space X. Let σ be the tangent plane of the image of f, then the curvature K_σ in X satisfies

$$-\frac{1}{n(n-1)^2} \leq K_\sigma < 0.$$

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Goal II: Curvature of cyclic Higgs bundles in Hitchin section

Theorem (Dai-L)

Let (E, ϕ) be a cyclic Higgs bundle in Hitchin section and f be the associated equivariant harmonic map into the symmetric space X. Let σ be the tangent plane of the image of f, then the curvature K_σ in X satisfies

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- The sectional curvature of the symmetric space X satisfies $-\frac{1}{n} \leq K \leq 0$. So we give a nontrivial upper and lower bound on the curvature K_σ.
- The upper bound means that the minimal immersion is never tangential to any flat. Moreover, it is sharp in asymptotic sense.
- In the base n-Fuchsian case, the sectional curvature K_σ is $-\frac{6}{n^2(n^2-1)}$.
- However, in case $n > 3$, the curvature $-\frac{6}{n^2(n^2-1)}$ for the base n-Fuchsian case cannot serve as a lower bound for K_σ. This is because we show that at the zeros p of q_n, the sectional curvature K_σ satisfies $K_\sigma < -\frac{6}{n^2(n^2-1)}$.

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For example, in the case of cyclic Higgs bundles parametrized by $(\gamma_1, \gamma_2, \cdots, \gamma_n)$, if $n-1$ terms of γ_i's have a common zero point, then the curvature of the tangent plane σ at that point achieves the most negative, i.e., $K_\sigma = -\frac{1}{n}$. (Note that the sectional curvature K in $SL(n, \mathbb{C})/SU(n)$ satisfies $-\frac{1}{n} \le K \le 0$.)

Goal II: Comparison inside real Hitchin fiber

- Note that cyclic Higgs bundles (E, ϕ) lie in the Hitchin fiber at $(0, \cdots, 0, n \cdot q_n)$, where $q_n = (-1)^{n-1} \det(\phi)$.
- There is one special point in each Hitchin fiber at $(0, \cdots, 0, n \cdot q_n)$: the cyclic Higgs bundle in the Hitchin component parametrized by q_n.

Theorem (Dai-L)

Let $(\tilde{E}, \tilde{\phi})$ be a cyclic Higgs bundle in the Hitchin component parameterized by q_n and (E, ϕ) be a distinct cyclic $SL(n, \mathbb{R})$-Higgs bundle such that $\det \phi = \det \tilde{\phi} = (-1)^{n-1} q_n$. For $n = 2, 3, 4$, the pullback metrics $g, \tilde{g}$ of the corresponding harmonic maps satisfy

$$g < \tilde{g}.$$

Case $n = 2$ is already shown by Deroin-Tholozan.

Goal II: Comparison inside real Hitchin fiber (continue)

We expect this to be true in general case.

Conjecture

Let $(\tilde{E}, \tilde{\phi})$ be a Higgs bundle in the Hitchin component and (E, ϕ) be a distinct $SL(n, \mathbb{R})$-Higgs bundle in the same Hitchin fiber at $(q_2, q_3, \cdots, q_n)$. Then the pullback metrics $g, \tilde{g}$ of corresponding harmonic maps satisfy $g < \tilde{g}$. As a result, the Morse function satisfies $f(E, \phi) < f(\tilde{E}, \tilde{\phi})$.

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- Even the Morse function level is unknown.
- The conjecture in the $SL(2, \mathbb{C})$ case is already shown by Deroin-Tholozan.

Applications to maximal $Sp(4,\mathbb{R})$-representations

- For each reductive representation ρ into a non-compact Hermitian Lie group G, we can define the Toledo invariant $\tau(\rho)$ satisfying the Milnor-Wood inequality $|\tau(\rho)| \leq \mathrm{rank}(G)(g-1)$. The representation ρ with $|\tau(\rho)| = \mathrm{rank}(G)(g-1)$ is called maximal.

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- In the case for $Sp(4,\mathbb{R})$, Gothen showed that there are $3 \cdot 2^{2g} + 2g - 4$ connected components of maximal representations containing 2^{2g} isomorphic components of Hitchin representations and $2g-3$ exceptional components called Gothen components.

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- Combine with the work of Labourie and Collier, for any maximal $Sp(4, \mathbb{R})$ representation ρ in the Hitchin components and Gothen components, there exists a unique ρ-equivariant minimal surface of $\tilde{S}$ in the symmetric space $Sp(4, \mathbb{R})/U(2)$.

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We can then state the monotonicity and the curvature bounds for the equivariant minimal surfaces for maximal $Sp(4, \mathbb{R})$ representations in the Hitchin components and Gothen components, mostly as corollaries of previous theorems.

Applications to maximal $Sp(4, \mathbb{R})$-representations

We can then state the monotonicity and the curvature bounds for the equivariant minimal surfaces for maximal $Sp(4, \mathbb{R})$ representations in the Hitchin components and Gothen components, mostly as corollaries of previous theorems.
Lastly, as a corollary of the comparison inside real Hitchin fiber, we can compare these two kinds of maximal representations.

Corollary

For any maximal representation $Sp(4, \mathbb{R})$-representation ρ in the Gothen components, there exists a Hitchin representation j in $Sp(4, \mathbb{R})$ such that the pullback metric of the unique j-equivariant minimal surface strictly dominates the one for ρ.

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Let (Σ, g) be a closed Riemannian manifold. For each $1 \leq i \leq n$, let u_i be a C^2 function on $\Sigma \setminus P_i$, where P_i is an isolated subset of Σ (P_i can be empty). Suppose u_i approaches to $+\infty$ around P_i. Let $P = \bigcup_{i=1}^n P_i$.

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(a) cooperative: $c_{ij} \geq 0,\ i \neq j$,
(b) column diagonally dominant: $\sum_{i=1}^n c_{ij} \leq 0,\ 1 \leq j \leq n$,
(c) fully coupled: the index set $\{1, \cdots, n\}$ cannot be split up in two disjoint nonempty sets α, β such that $c_{ij} \equiv 0$ for $i \in \alpha, j \in \beta$.

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Why introduce the pole set P_i? For our use, they arise as the set of zeros of γ_i's.

Main Tool: Maximum principles for system (continue)

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Suppose the above assumptions hold. Let f_i be non-positive continuous functions on $\Sigma \setminus P$, $1 \leq i \leq n$.

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(3) $\sum_{i=1}^n u_i \geq 0$.
Then either condition (1) or (2) imply $u_i > 0$, $1 \leq i \leq n$. And condition (3) implies either $u_i > 0$, $1 \leq i \leq n$ or $u_i \equiv 0$, $1 \leq i \leq n$.

Remarks on the proof

- The monotonicity and the comparison inside Hitchin fiber can be proved by deriving the right equation system (needs trick!) to apply the maximum principle for system by applying different conditions (1)(2)(3).

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- The monotonicity and the comparison inside Hitchin fiber can be proved by deriving the right equation system (needs trick!) to apply the maximum principle for system by applying different conditions (1)(2)(3).
- However, for the lower bound of extrinsic curvature of the harmonic maps, it needs much more work other than applying the maximum principle.

Thank You!