Hyperbolic p-barycenters, circumcenters, and Moebius maps

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Kolkata.

Moebius group in n dimensions

Moeb($\mathbb{R}^n \cup \{\infty\}$)

$=$ < reflections in hyperplanes, inversions in spheres >

$=$ group of homeomorphisms of $\mathbb{R}^n \cup \{\infty\}$ preserving cross-ratios

where cross-ratio of a quadruple of distinct points given by

$$[\xi, \xi', \eta, \eta'] = \frac{||\xi - \eta|| ||\xi' - \eta'||}{||\xi - \eta'|| ||\xi' - \eta||}$$

Conjugating by stereographic projection gives

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Isometries of real hyperbolic space

$\mathbb{H}^n = (\mathbb{R}^{n-1} \times \mathbb{R}^+, \frac{ds^2}{x_n^2})$ upper half-space model of n-dim real hyperbolic space.

$\partial\mathbb{H}^n = \mathbb{R}^{n-1} \cup \{\infty\}$ boundary.

$\mathrm{Isom}(\mathbb{H}^n) = <$ reflections/inversions in planes/spheres perpendicular to $\partial\mathbb{H}^n >$

$$\mathrm{Isom}(\mathbb{H}^n) \to \mathrm{Moeb}(\mathbb{R}^{n-1} \cup \{\infty\})$$
$$f \mapsto f_{|\partial\mathbb{H}^n}$$

$\mathrm{Isom}(\mathbb{H}^n) \simeq \mathrm{Moeb}(\mathbb{R}^{n-1} \cup \{\infty\})$

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Theorem

(Mostow) For $n \geq 3$, any isomorphism $\phi : \pi_1(M) \to \pi_1(N)$ between fundamental groups of closed hyperbolic n-manifolds M, N is induced by an isometry $f : M \to N$.

Step 1. Choosing basepoints $x_0, y_0 \in \mathbb{H}^n$, ϕ induces equivariant quasi-isometry

$$F_0 : \pi_1(M) \cdot x_0 \to \pi_1(N) \cdot y_0$$
$$g \cdot x_0 \mapsto \phi(g) \cdot y_0$$

Step 2. F_0 extends to an equivariant homeomorphism $f : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$.

Step 3. f equivariant and quasi-conformal implies f Moebius.

Step 4. f Moebius implies f extends to an equivariant isometry $F : \mathbb{H}^n \to \mathbb{H}^n$.

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Marked length spectrum rigidity

X closed negatively curved n-manifold

Each free homotopy class of closed curves contains a unique closed geodesic

Length function $l_X : \pi_1(X) \to \mathbb{R}^+$

Question: Given X, Y closed negatively curved n-manifolds, and $\phi : \pi_1(X) \to \pi_1(Y)$ an isomorphism such that $l_Y \circ \phi = l_X$, is X isometric to Y?

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(Otal) Yes, if $n = 2$.

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(Hamenstadt) X, Y have same marked length spectrum if and only if geodesic flows of X, Y are topologically conjugate.

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CAT(-1) spaces

(X, d) metric space is CAT(-1) if:

(1) X is a length space: For all $p, q \in X$, exists isometric embedding $\gamma : [0, T = d(p, q)] \to X$ with $\gamma(0) = p, \gamma(T) = q$.

(2) X satisfies CAT(-1) inequality: Geodesic triangles thinner than in $\mathbb{H}^2$, $d(s, t) \leq d_{\mathbb{H}^2}(\overline{s}, \overline{t})$.

Facts:

Unique geodesic joining any two points.

Contractible.

Examples:

X complete simply connected manifold, $K \leq -1$.

X metric tree.

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Boundary at infinity

$\partial X := \{[\gamma] : \gamma : [0,\infty) \to X \quad \text{geodesic ray}\}/\sim$

where $\gamma_1 \sim \gamma_2$ if $\{d(\gamma_1(t), \gamma_2(t)) : t \geq 0\}$ bounded.

$\gamma(\infty) := [\gamma]$.

For $x \in X, \xi \in \partial X$, unique geodesic ray γ with $\gamma(0) = x, \gamma(\infty) = \xi$.

For $\xi, \eta \in \partial X$ distinct points, unique bi-infinite geodesic $\gamma : \mathbb{R} \to X$ with $\gamma(-\infty) = \xi, \gamma(\infty) = \eta$.

Cone topology on $\overline{X} = X \cup \partial X$:

Neighbourhood of $\xi = \gamma(\infty) \in \partial X$: Points $y \in \overline{X}$ outside ball of radius R such that geodesic joining $\gamma(0)$ to y stays close to γ up to time R.

$\overline{X}$ compact iff X proper.

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Neighbourhood of $\xi = \gamma(\infty) \in \partial X$: Points $y \in \overline{X}$ outside ball of radius R such that geodesic joining $\gamma(0)$ to y stays close to γ up to time R.

$\overline{X}$ compact iff X proper.

Boundary at infinity

$\partial X := \{[\gamma] : \gamma : [0, \infty) \to X \quad \text{geodesic ray}\}/\sim$

where $\gamma_1 \sim \gamma_2$ if $\{d(\gamma_1(t), \gamma_2(t)) : t \geq 0\}$ bounded.

$\gamma(\infty) := [\gamma]$.

For $x \in X, \xi \in \partial X$, unique geodesic ray γ with $\gamma(0) = x, \gamma(\infty) = \xi$.

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Examples:

X simply connected complete manifold, $K \leq -1$, then the map

$$\begin{aligned} T_x^1 X &\to \partial X \\ v &\mapsto \gamma_v(\infty) \end{aligned}$$

(where γ_v unique geodesic ray with $\gamma_v'(0) = v$) is a homeomorphism, $X \cup \partial X \simeq \mathbb{B}^n \cup \partial \mathbb{B}^n$.

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Visual metrics

For $x, y, z \in X$, $\angle^{(-1)} yxz$ comparison angle between $[x, y), [x, z)$ at x.

For $x \in X$, $\xi, \eta \in \partial X$, limiting comparison angle

$$\angle^{(-1)} \xi x \eta := \lim_{y \to \xi, z \to \eta} \angle^{(-1)} yxz$$

exists.

Visual metric on ∂X based at $x \in X$:

$$\rho_x(\xi, \eta) := \sin\left(\frac{\angle^{(-1)} \xi x \eta}{2}\right)$$

Example. $X = \mathbb{H}^n$ (ball model), $\rho_0(\xi, \eta) = \frac{1}{2}||\xi - \eta||$.

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Gromov inner product between $y, z \in X$ at $x \in X$:

$$(y|z)_x := \frac{1}{2}(d(x,y) + d(x,z) - d(y,z))$$

For X metric tree, $(y|z)_x =$ length of common segment of $[x,y], [x,z]$.

For $\xi, \eta \in \partial X$, the limit

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Moebius and conformal maps between metric spaces

(Z, ρ) metric space, cross-ratio defined by

$$[\xi, \xi', \eta, \eta']_\rho := \frac{\rho(\xi, \eta)\rho(\xi', \eta')}{\rho(\xi, \eta')\rho(\xi', \eta)},$$

Injective map $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ between metric spaces is **Moebius** if it preserves cross-ratios.

f is **conformal** if

$$df_{\rho_1, \rho_2}(\xi) := \lim_{\eta \to \xi} \frac{\rho_2(f(\xi), f(\eta))}{\rho_1(\xi, \eta)}$$

exists for all $\xi \in Z_1$.

$df_{\rho_1, \rho_2} : Z_1 \to \mathbb{R}$ is called the derivative of f with respect to the metrics ρ_1, ρ_2.

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Any Moebius map is C^1 conformal.

Any Moebius map satisfies the **Geometric Mean-Value Theorem**:

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Spaces of Moebius metrics

For two metrics ρ_1, ρ_2 on the same space Z, if $[\ \ ]_{\rho_1} \equiv [\ \ ]_{\rho_2}$, define the derivative of ρ_2 with respect to ρ_1 to be

$$\frac{d\rho_2}{d\rho_1} := \text{derivative of the Moebius map } id : (Z, \rho_1) \to (Z, \rho_2)$$

Let (Z, ρ_0) be a compact metric space with $\text{diam}(Z, \rho_0) = 1$.

Assume ρ_0 is **antipodal**, i.e. for any $\xi \in Z$ there exists $\eta \in Z$ such that $\rho_0(\xi, \eta) = 1$.

Space of Moebius metrics:

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Cross-ratio on ∂X

X CAT(-1) space, $x \in X$, then

$$[\xi, \xi', \eta, \eta']_{\rho_x} = \exp\left(-(\xi|\eta)_x - (\xi'|\eta')_x + (\xi|\eta')_x + (\xi'|\eta)_x\right)$$

does not depend on x, so ∂X has a canonical cross-ratio.

Any isometry $F : X \to Y$ between CAT(-1) spaces extends to a Moebius map $f : \partial X \to \partial Y$.

For $x, y \in X$,

$$\frac{d\rho_y}{d\rho_x}(\xi) = \exp(B(x, y, \xi))$$

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$$B(x, y, \xi) := \lim_{z \to \xi} (d(x, z) - d(y, z))$$

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Let $\mathcal{M}(\partial X) := \mathcal{M}(\partial X, \rho_x)$ (independent of choice of $x \in X$).

The map

$$i_X : X \to \mathcal{M}(\partial X)$$
$$x \mapsto \rho_x$$

is an isometric embedding.

The image of the embedding is $\frac{1}{2}\log 2$-dense in $\mathcal{M}(\partial X)$.

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Geodesic flow of a CAT(-1) space

X CAT(-1) space. Define the geodesic flow space of X by

$$\mathcal{G}X := \{\gamma : \mathbb{R} \to X \text{ bi-infinite geodesic } \}$$

(endowed with the topology of uniform convergence on compacts).

(For X a simply connected manifold with $K \leq -1$, the map $\gamma \mapsto \gamma'(0)$ identifies $\mathcal{G}X$ with T^1X)

The geodesic flow of X is the 1-parameter group of homeomorphisms $(\phi_t : \mathcal{G}X \to \mathcal{G}X)_{t \in \mathbb{R}}$ defined by

$$(\phi_t \gamma)(s) := \gamma(s + t)$$

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Given X, Y CAT(-1) spaces and $f : \partial X \to \partial Y$ a conformal map, f induces a homeomorphism $\phi : \mathcal{G}X \to \mathcal{G}Y$ conjugating the geodesic flows, which is defined as follows:

Given $\gamma \in \mathcal{G}X$, let $x = \gamma(0), \xi = \gamma(-\infty), \eta = \gamma(+\infty)$.

There is a unique $y \in (f(\xi), f(\eta))$ such that

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Conformal maps and geodesic conjugacies

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If f is Moebius, then the geodesic conjugacy ϕ is **flip-equivariant**:

By the Geometric Mean-Value Theorem,

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Integrated Schwarzian of a conformal map

$f : \partial X \to \partial Y$ conformal map.

The **integrated Schwarzian** of f is the function $S(f) : \partial^2 X \to \mathbb{R}$ defined by

$$S(f)(\xi, \eta) := -\log(df_{\rho_X,\rho_Y}(\xi) \cdot df_{\rho_X,\rho_Y}(\eta))$$

Measures deviation of ϕ from being flip-equivariant:

$$\phi(-\gamma) = -\phi_{-t}(\phi(\gamma)) \quad \text{where } t = S(f)(\xi, \eta)$$

$S(f) = 0$ if f is Moebius.

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$$S(f \circ g) = S(f) \circ (g, g) + S(g)$$

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Integrated Schwarzian and cross-ratio distortion

Theorem

(B.) X complete, simply connected manifold with $-b^2 \leq K \leq -1$, Y proper, geodesically complete CAT(-1) space. For $f : U \subset \partial X \to \partial Y$ a C^1 conformal map,

$$\log \frac{[f(\xi), f(\xi'), f(\eta), f(\eta')]}{[\xi, \xi', \eta, \eta']}$$

$$= \frac{1}{2} \left(S(f)(\xi, \eta) + S(f)(\xi', \eta') - S(f)(\xi, \eta') - S(f)(\xi', \eta) \right)$$

for all quadruples in $U \subset \partial X$.

In particular f is Moebius if and only if $S(f) = 0$.

Marked length spectrum and Moebius maps

Given closed negatively curved manifolds X, Y and an isomorphism $\phi : \pi_1(X) \to \pi_1(Y)$.

ϕ induces an equivariant homeomorphism $f : \partial\tilde{X} \to \partial\tilde{Y}$.

Theorem

(Otal) X, Y have same marked length spectrum (i.e. $l_Y \circ \phi = l_X$) if and only if $f : \partial\tilde{X} \to \partial\tilde{Y}$ is Moebius.

Question: X, Y CAT(-1) spaces. Does a Moebius map $f : \partial X \to \partial Y$ extend to an isometry $F : X \to Y$?

Theorem

(Bourdon) For X a rank one symmetric space (normalized so that $K_{max} = -1$), Y a CAT(-1) space, any Moebius embedding $F : \partial X \to \partial Y$ extends to an isometric embedding $f : X \to Y$.

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X proper CAT(-1) space.

Circumcenter of a bounded set $A \subset X$ is the unique $x = c(A) \in X$ minimizing the function

$$p \in X \mapsto \sup_{q \in A} d(p, q)$$

Let $K \subset \mathcal{G}X$ be a compact set such that $p(K) \subset \partial X$ is not a singleton.

Let $A_t = (\pi \circ \phi_t)(K) \subset X$. Then the limit

$$c_\infty(K) := \lim_{t \to \infty} c(A_t)$$

exists. We call $c_\infty(K)$ the **asymptotic circumcenter** of K.

Asymptotic circumcenters in CAT(-1) spaces

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$\mathcal{F}(-1)$-convex functions

A function $u : X \to \mathbb{R}$ is $\mathcal{F}(-1)$-convex if it is continuous and its restriction to any geodesic satisfies $f'' - f \geq 0$ in the barrier sense, i.e. if $g'' - g = 0$ and g agrees with f at the endpoints of a geodesic segment then $f \leq g$.

A proper, positive, $\mathcal{F}(-1)$-convex function on X has a unique minimum.

The asymptotic circumcenter $c_\infty(K)$ is the unique minimum of the proper, positive, $\mathcal{F}(-1)$-convex function

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Circumcenter extension of Moebius maps

Let $f : \partial X \to \partial Y$ be a Moebius homeomorphism between proper, geodesically complete CAT(-1) spaces.

Let $\phi : \mathcal{G}X \to \mathcal{G}Y$ be the associated geodesic conjugacy.

Pushing forward metrics by f gives an isometry $f_* : \mathcal{M}(\partial X) \to \mathcal{M}(\partial Y)$.

For $x \in X$, let

$$F(x) = c_\infty(\phi(T^1_x X)) \in Y$$

where $T^1_x X = \pi^{-1}(x) \subset \mathcal{G}X$.

The point $F(x)$ is the unique minimum of the function

$$y \in Y \mapsto d_{\mathcal{M}}(f_* \rho_x, \rho_y)$$

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Results

Theorem

(B.) For X, Y proper, geodesically complete CAT(-1) spaces and $f : \partial X \to \partial Y$ a Moebius homeomorphism, the circumcenter extension $F : X \to Y$ is a $(1, \log 2)$-quasi-isometry with image $\frac{1}{2}\log 2$-dense in Y. Moreover, F is locally $1/2$-Holder continuous:

$$d(F(x), F(y)) \leq d(x, y)^{1/2}$$

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(B.) If X, Y are metric trees, then the circumcenter extension F is a surjective isometry.

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(B.) Let X, Y be complete, simply connected manifolds with $-b^2 \leq K \leq -1$, and let $f : \partial X \to \partial Y$ be a Moebius homeomorphism with inverse $g : \partial Y \to \partial X$. Then:

(1) The circumcenter extensions $F : X \to Y$ and $G : Y \to X$ of f and g are inverses of each other.

(2) The maps F, G are $\sqrt{b}$-bi-Lipschitz.

(3) The maps F, G are $(1, (1 - 1/b)\log 2)$-quasi-isometries.

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Compactly supported deformations of metrics

(X, g_0) complete, simply connected, $K(g_0) \leq -1$.

Let g be a metric on X such that $K(g) \leq -1$, and $g = g_0$ outside a compact.

Then $id : (X, g_0) \to (X, g)$ induces a boundary map $\hat{id} : \partial_{g_0} X \to \partial_g X$ which is locally Moebius, hence conformal.

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Infinitesimal and local rigidity

Theorem

(B.) Given $(g_t)_{0 \leq t \leq 1}$ smooth 1-parameter family of compactly supported deformations of (X, g_0). If $\hat{id}_t : \partial_{g_0} X \to \partial_{g_t} X$ is Moebius for all t, then $\hat{id}_t$ extends to an isometry $F_t : (X, g_0) \to (X, g_t)$ for all t.

Theorem

(B.) Assume $-b^2 \leq K(g_0) \leq -1$. Given a compact $K \subset X$, there exists $\epsilon = \epsilon(K)$ such that the following holds:

Let g be such that $g = g_0$ outside K, and such that $Vol_g(K) = Vol_{g_0}(K)$.

If $\hat{id} : \partial_{g_0} X \to \partial_g X$ is Moebius, and $||g - g_0||_{C^2} < \epsilon$, then $\hat{id}$ extends to an isometry $F : (X, g_0) \to (X, g)$.

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Infinitesimal and local rigidity

Theorem

(B.) Given $(g_t)_{0 \leq t \leq 1}$ smooth 1-parameter family of compactly supported deformations of (X, g_0). If $\hat{id}_t : \partial_{g_0} X \to \partial_{g_t} X$ is Moebius for all t, then $\hat{id}_t$ extends to an isometry $F_t : (X, g_0) \to (X, g_t)$ for all t.

Theorem

(B.) Assume $-b^2 \leq K(g_0) \leq -1$. Given a compact $K \subset X$, there exists $\epsilon = \epsilon(K)$ such that the following holds:

Let g be such that $g = g_0$ outside K, and such that $Vol_g(K) = Vol_{g_0}(K)$.

If $\hat{id} : \partial_{g_0} X \to \partial_g X$ is Moebius, and $||g - g_0||_{C^2} < \epsilon$, then $\hat{id}$ extends to an isometry $F : (X, g_0) \to (X, g)$.

Both proofs use the integrated Schwarzian $S(\hat{id})$ of the conformal map $\hat{id} : \partial_{g_0} X \to \partial_g X$.

Hyperbolic p-barycenters

X CAT(-1) space.

μ probability measure on X with compact support which is not a singleton.

Let $1 \leq p < \infty$. The hyperbolic p-barycenter of μ is the unique point $x = c^p(\mu) \in X$ which minimizes the positive, proper, $\mathcal{F}(-1)$-convex function

$$z \in X \mapsto \int_X \cosh^p(d(z,y))d\mu(y)$$

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Asymptotic hyperbolic p-barycenters

ν probability measure on $\mathcal{G}X$ with compact support K such that $p(K) \subset \partial X$ is not a singleton.

Let $\mu_t = (\pi \circ \phi_t)_* \nu$, probability measure on X with compact support.

For $1 \leq p < \infty$, the limit

$$c^p_\infty(\nu) := \lim_{t \to \infty} c^p(\mu_t)$$

exists. We call $c^p_\infty(\nu)$ the asymptotic hyperbolic p-barycenter of the measure ν.

The asymptotic hyperbolic p-barycenter $c^p_\infty(\nu)$ is the unique minimizer of the proper, positive, $\mathcal{F}(-1)$-convex function

$$z \in X \mapsto \int_{\mathcal{G}X} \exp(pB(z, \pi(\gamma), p(\gamma))) d\nu(\gamma)$$

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Hyperbolic p-barycenter extension of Moebius maps

$f : \partial X \to \partial Y$ Moebius map, $\phi : \mathcal{G}X \to \mathcal{G}Y$ associated geodesic conjugacy.

$\mathcal{M} = \{\mu_x : x \in X\}$ family of probability measures on ∂X such that $\text{supp}(\mu_x) = \partial X$.

For $x \in X, \xi \in \partial X$ let $\overrightarrow{x\xi} = \gamma'(0) \in T^1_x X$, where $\gamma(0) = x, \gamma(\infty) = \xi$.

Let $q_x : \xi \in \partial X \mapsto \overrightarrow{x\xi} \in T^1_x X$.

The hyperbolic p-barycenter extension of f with respect to the family $\mathcal{M}$ is defined by

$$F_{p,\mathcal{M}}(x) := c^p_\infty((\phi \circ q_x)_*(\mu_x))$$

Let F be the circumcenter extension of f, then

$$F_{p,\mathcal{M}} \to F$$

pointwise on X as $p \to \infty$.

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Application of hyperbolic p-barycenter extension

X, Y complete, simply connected manifolds with $-b^2 \leq K \leq -1$.

Fix a probability measure μ on ∂X with full support.

Fix constant family $\mathcal{M} := \{\mu_x := \mu | x \in X\}$.

Let $F_p : X \to Y$ be the hyperbolic p-barycenter extension of f with respect to this family $\mathcal{M}$.

The point $y = F_p(x) \in Y$ is the unique minimizer of the strictly convex function

$$z \in Y \mapsto \int_{\partial X} \exp(pB(z, \pi(\phi(\overrightarrow{x\xi})), f(\xi)))d\mu(\xi)$$

The map F_p is C^1, and $F_p \to F$ pointwise as $p \to \infty$, where F is the circumcenter extension.

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Balanced measures

For $x \in X$, the probability measure μ_p^x on ∂X defined by

$$d\mu_p^x(\xi) = c_{x,p} \cdot \exp(pB(F_p(x), \pi(\phi(\overrightarrow{x\xi})), f(\xi)))d\mu(\xi)$$

satisfies

$$\int_{\partial X} < w, \overrightarrow{F_p(x)f(\xi)} > d\mu_p^x(\xi) = 0$$

for all $w \in T_{F_p(x)}Y$, or

$$\int_{\partial Y} < w, \overrightarrow{F_p(x)\eta} > df_*\mu_p^x(\eta) = 0$$

We say the measure $f_*\mu_p^x$ on ∂Y is *balanced* at $F_p(x) \in Y$.

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Letting $p \to \infty$, gives a probability measure μ^x_∞ on ∂X such that the measure $f_* \mu^x_\infty$ on ∂Y is balanced at $F(x) \in Y$.

Moreover μ^x_∞ has support contained in the set $K \subset \partial X$ where the function

$$\xi \in \partial X \mapsto \frac{df_* \rho_x}{d\rho_{F(x)}}(f(\xi))$$

attains its maximum.

Existence of a measure μ^x_∞ satisfying the above two properties characterizes the point $y = F(x)$.

$x = G(y)$ is shown by using this criterion for the circumcenter map: it is proved that μ^x_∞ is balanced at x, by showing

$$\left(\int_{\partial X} < v, \overrightarrow{x\xi} > d\mu^x_p(\xi) \right)^2 \leq \frac{C}{p}$$

and letting $p \to \infty$.

Letting $p \to \infty$, gives a probability measure μ_∞^x on ∂X such that the measure $f_* \mu_\infty^x$ on ∂Y is balanced at $F(x) \in Y$.

Moreover μ_∞^x has support contained in the set $K \subset \partial X$ where the function

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