Convex real projective Dehn fillings

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Joint work with Suhyoung Choi (KAIST)
& Ludovic Marquis (Université de Rennes)

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Motivation

Thuston's Dehn filling theorem

If the interior of a compact three-manifold M with toral boundaries admits a complete hyperbolic structure of finite volume, then almost all Dehn fillings of M admit a hyperbolic structure.

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Anderson and Bamler proved that many features of Dehn filling theory for hyperbolic 3-manifolds can be generalized to Einstein metric in any dimension.

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$$\mathbb{S}^n = (\mathbb{R}^{n+1} \backslash \{0\}) / \mathbb{R}_+$$

$$\mathrm{SL}^{\pm}_{n+1}(\mathbb{R}) = \{A \in \mathrm{GL}_{n+1}(\mathbb{R}) \mid \det(A) = \pm 1\}$$

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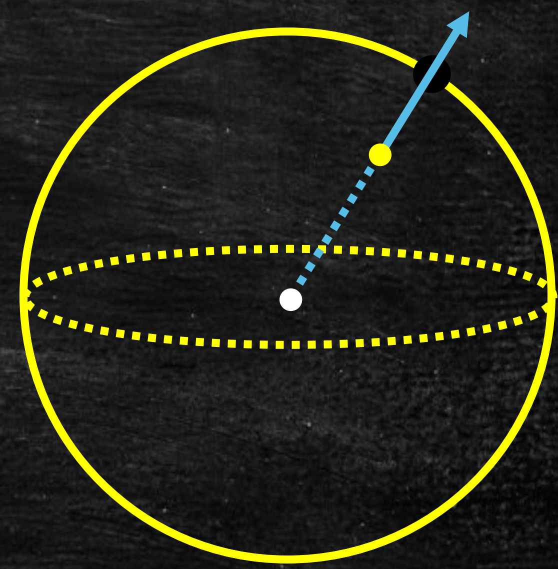

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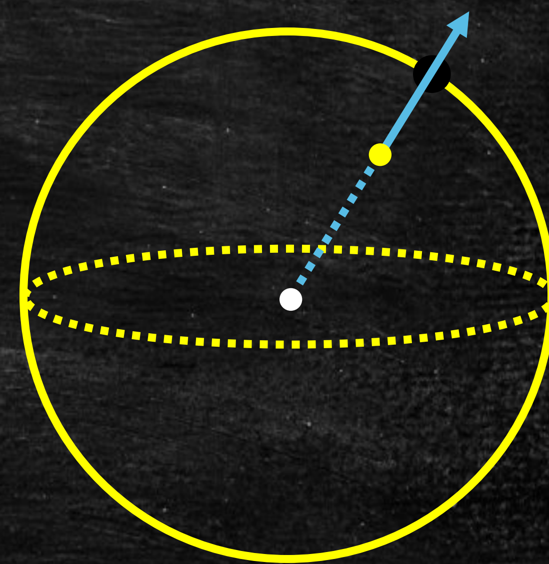

$$\mathrm{Isom}(S^n) = \mathrm{O}_{n+1}(\mathbb{R}) = \{ A \in \mathrm{SL}^{\pm}_{n+1}(\mathbb{R}) \mid (Ax) \cdot (Ay) = x \cdot y \}$$

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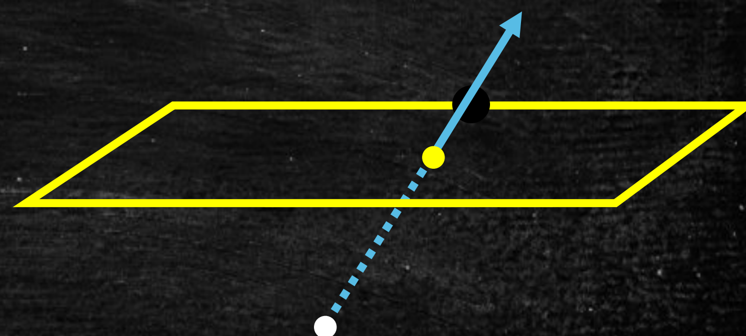

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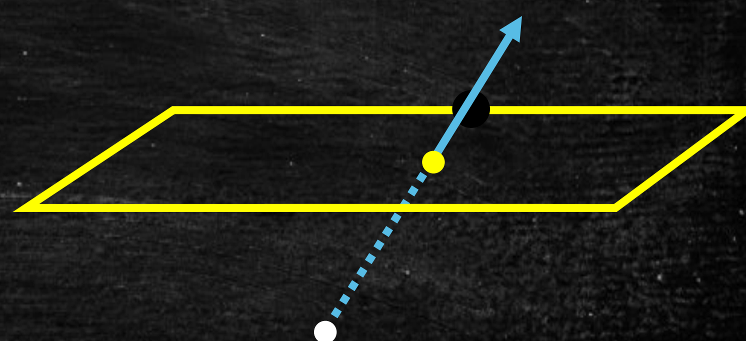

$$\mathrm{Isom}(E^n) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}^{\pm}_{n+1}(\mathbb{R}) \mid A \in \mathrm{O}_n(\mathbb{R}), v \in \mathbb{R}^n \right\}$$

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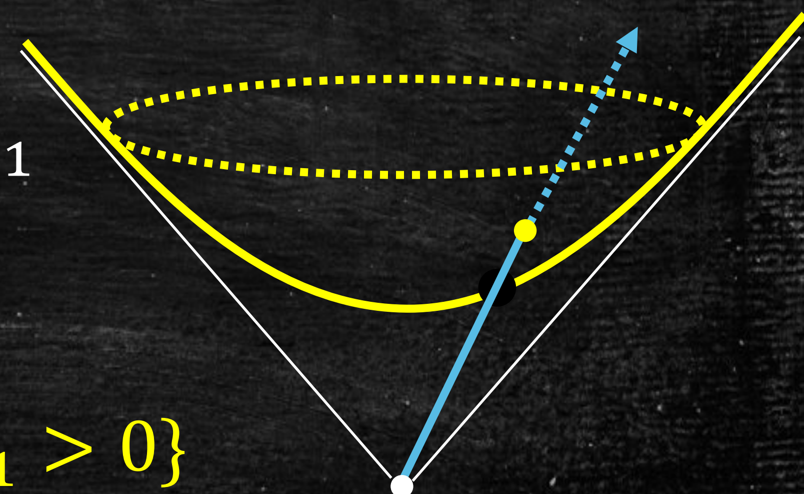

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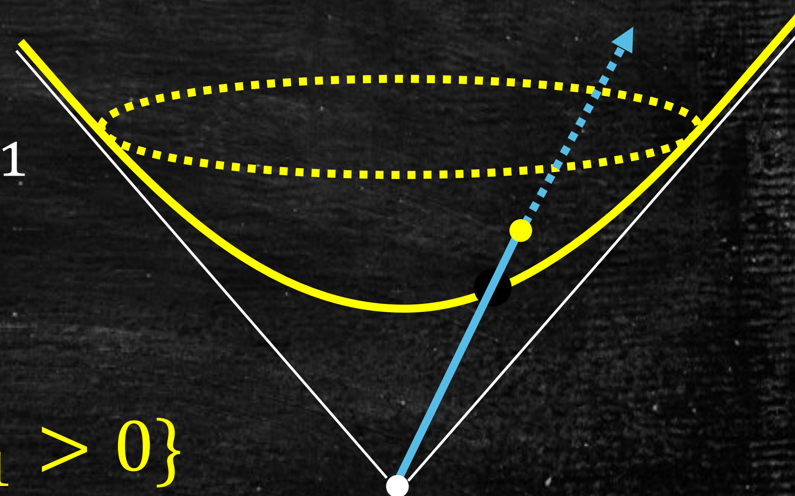

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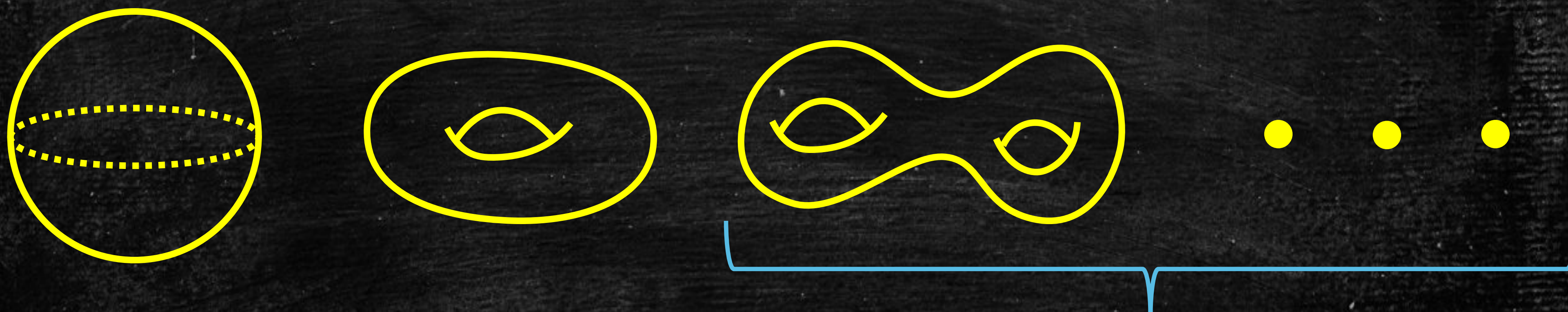

Spherical geometry

Euclidean geometry

Hyperbolic geometry

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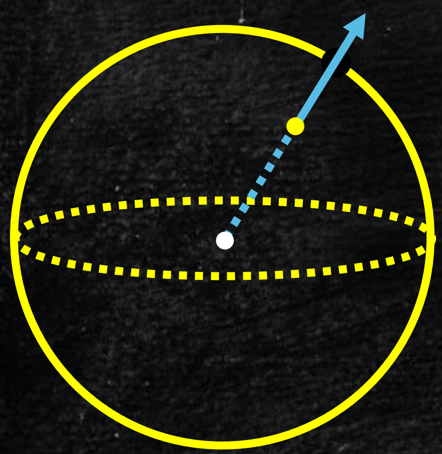

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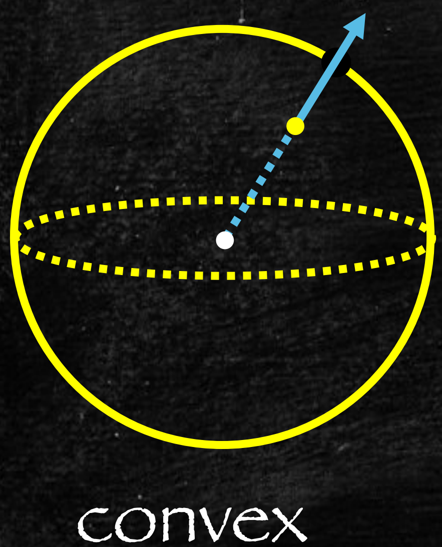

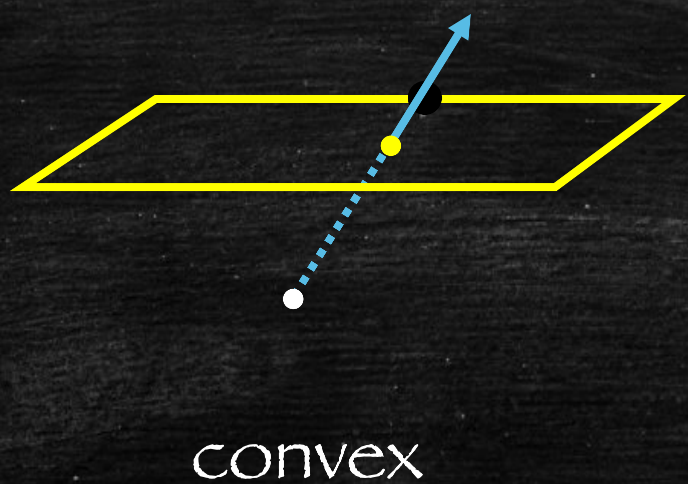

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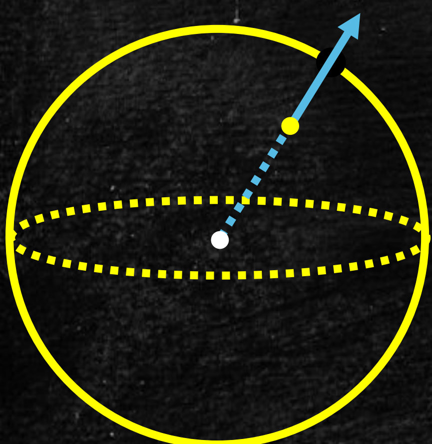

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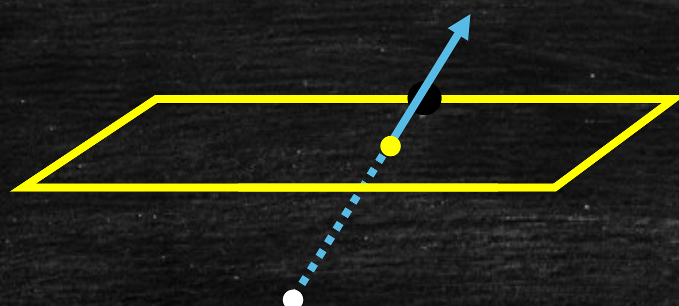

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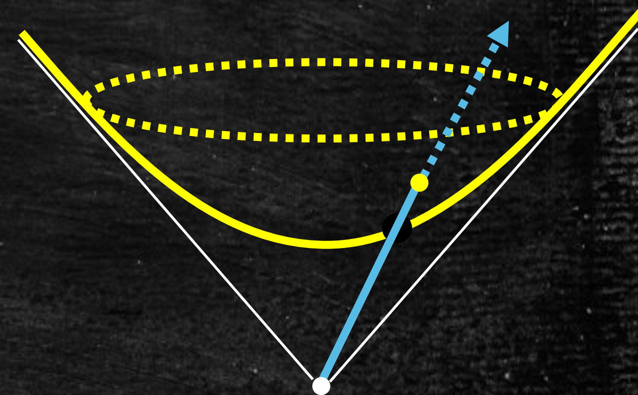

properly convex

Question

Let M be a compact n-manifold ($n > 3$) with a union of tori as boundary such that the interior of M admits a finite volume hyperbolic structure. Do almost all Dehn fillings of M admit a properly convex projective structure?

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A n-dimensional manifold M admits a properly convex projective structure if M is homeomorphic to Ω/Γ, where Ω is a properly convex subset of $\mathbb{S}^n$ and Γ is a discrete subgroup of $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$ acting properly discontinuously on Ω.

Coxeter group

A Coxeter system is a pair (S, M) of a finite set S and a symmetric matrix $M = (M_{st})_{s,t \in S}$ such that $M_{ss} = 1$ and other $M_{st} \in \{2, 3, \cdots, m, \cdots, \infty\}$.

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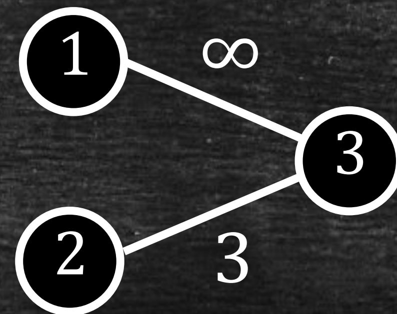
The Coxeter graph of W is the labeled graph such that
the vertices are elements of S,
$\exists$ edge connecting two vertices $s, t \in S \iff M_{st} \neq 2$,
the label of the edge $\overline{st}$ is $M_{st} \in \{3, \cdots, m, \cdots, \infty\}$.

Example

$$M = \begin{pmatrix} 1 & 2 & \infty \\ 2 & 1 & 3 \\ \infty & 3 & 1 \end{pmatrix}$$



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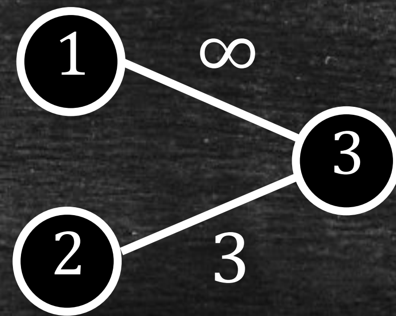

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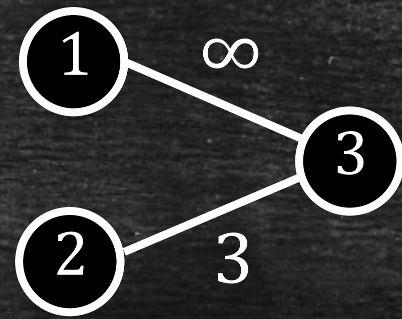

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<u>Theorem</u> (Margulis-Vinberg, 2000) If W is an irreducible Coxeter group, then W is either spherical, affine or large.

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put $\theta_e \in \{\pi/m \mid m = 2,3,\cdots,\infty\}$ (labeled n-polytope)

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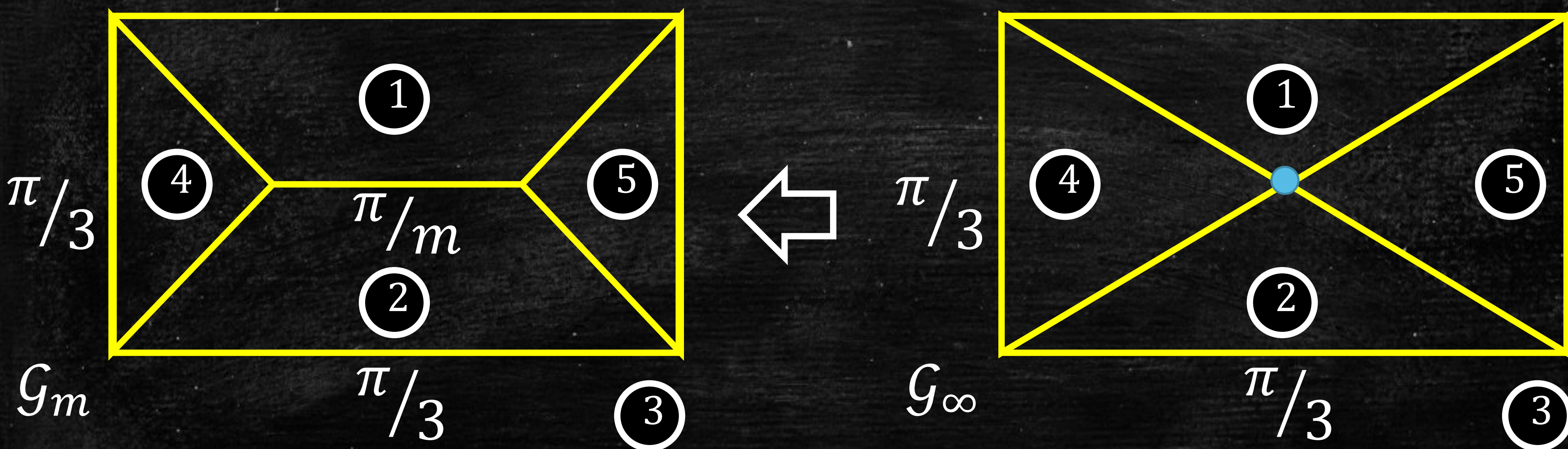

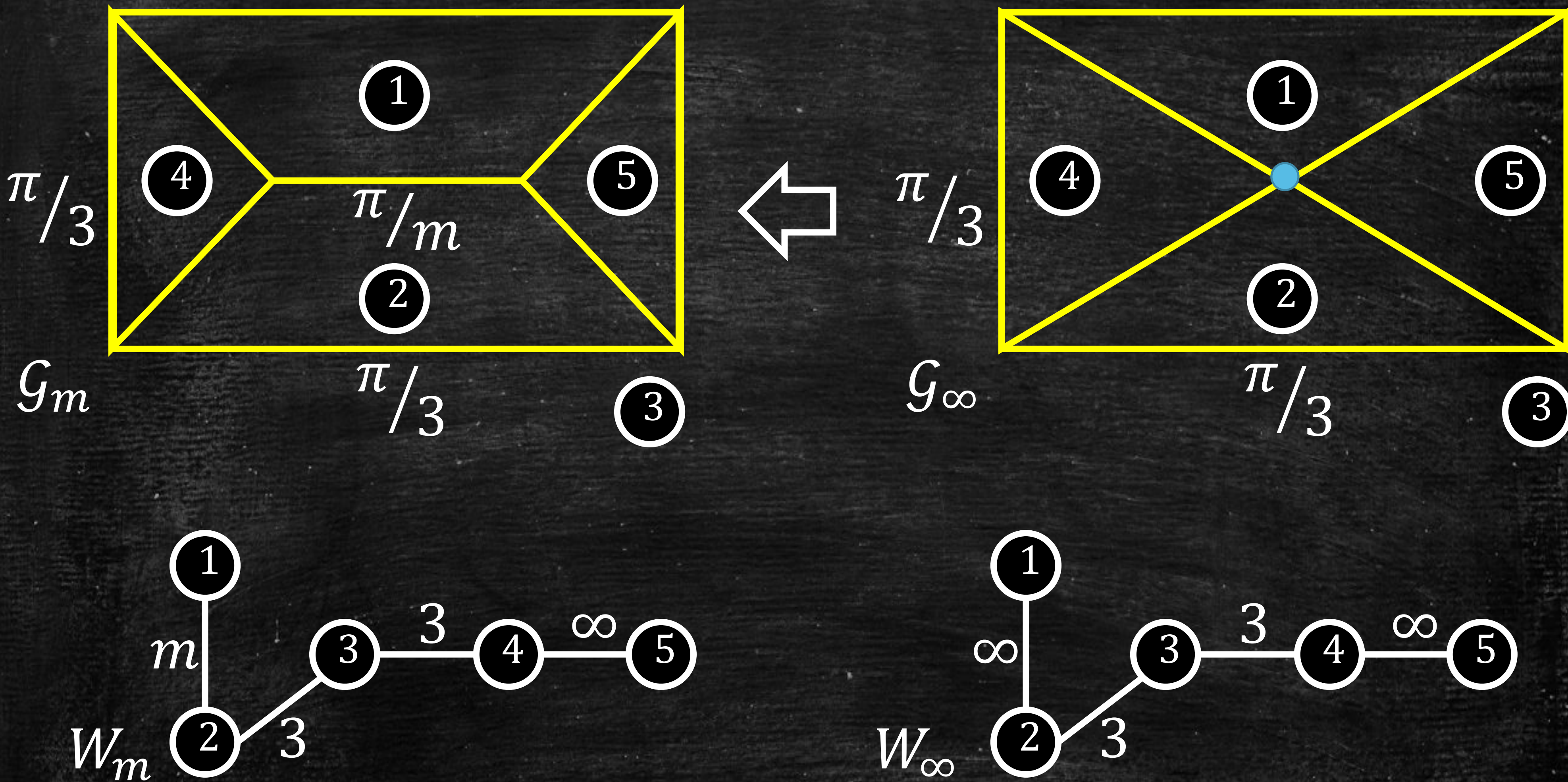

1
2
3
4
5
$\pi/3$
π/m
$\pi/3$
$\mathcal{G}_m$
$\pi/3$
$\pi/3$
$\mathcal{G}_\infty$
m
3
3
∞
W_m
∞
3
3
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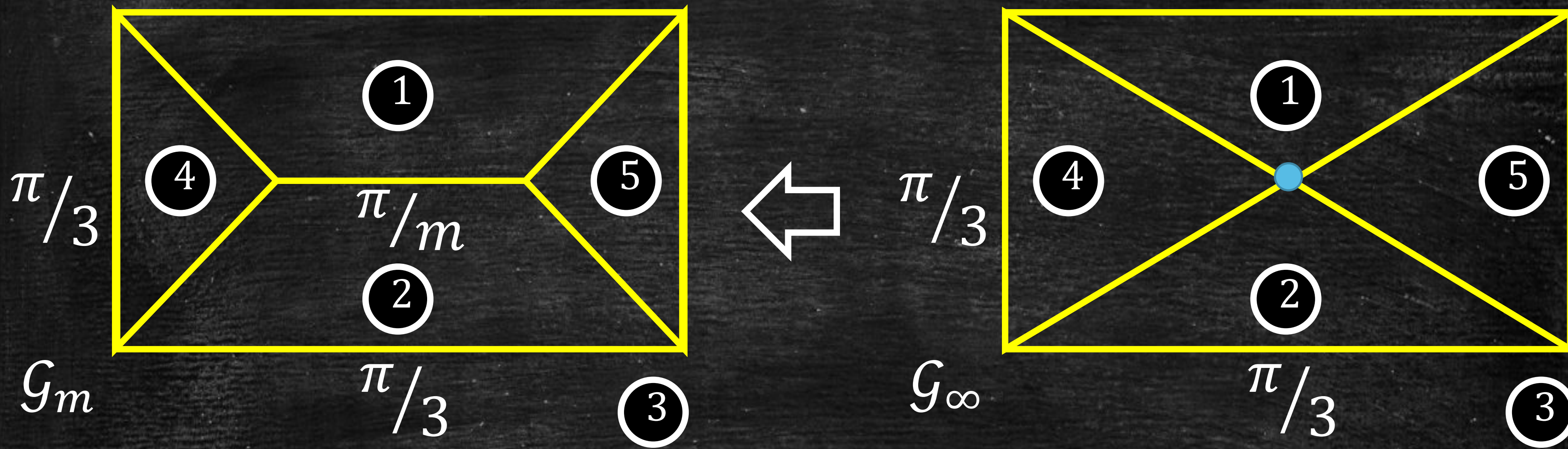
W_∞



<u>Theorem</u> (Andreev, 1970) If $m \in \{7,8,\cdots,\infty\}$,
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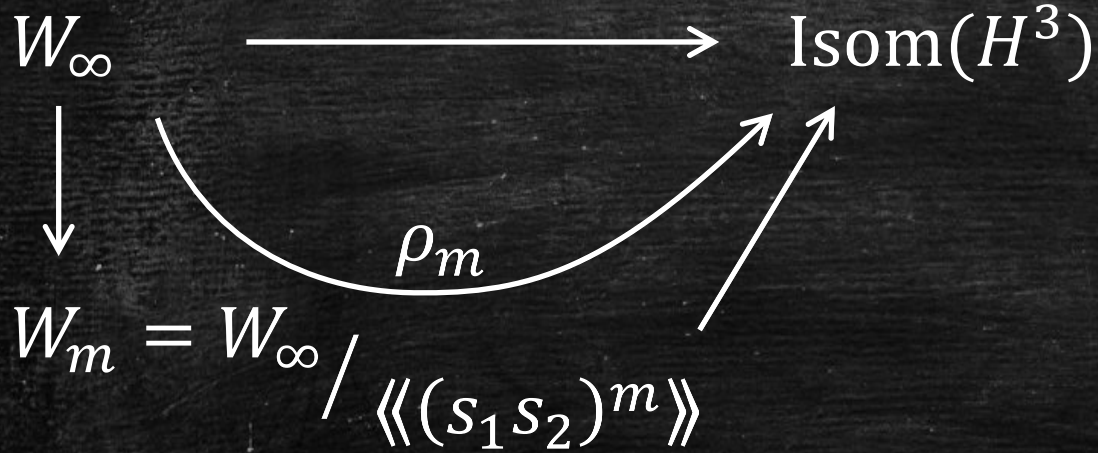

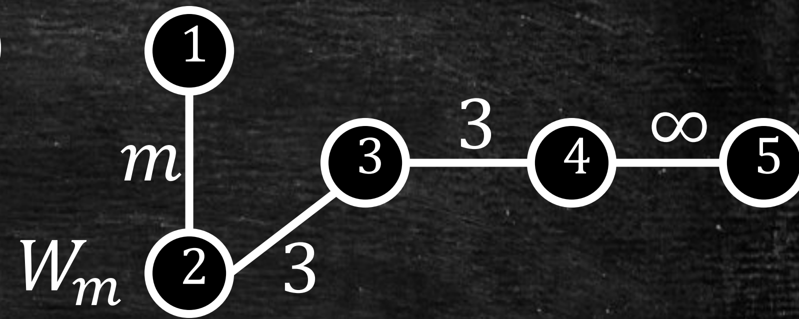

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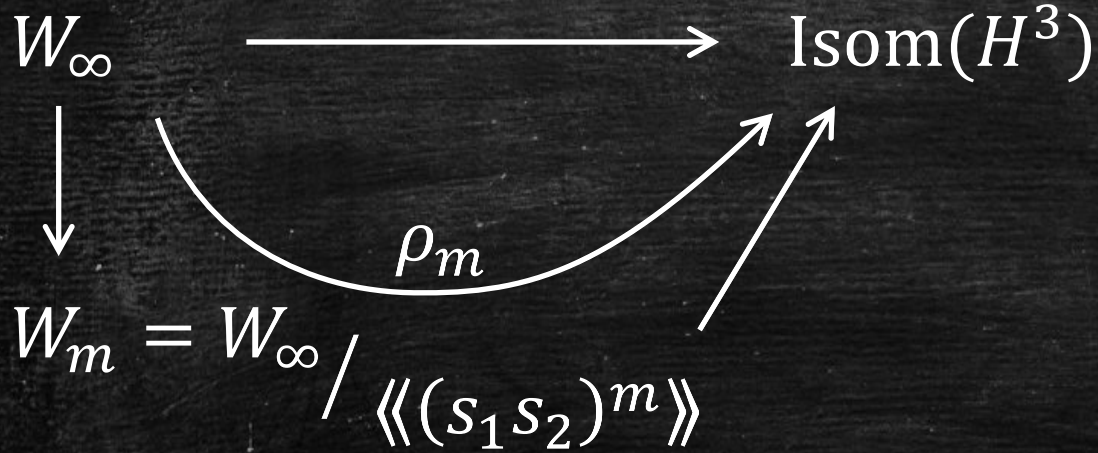

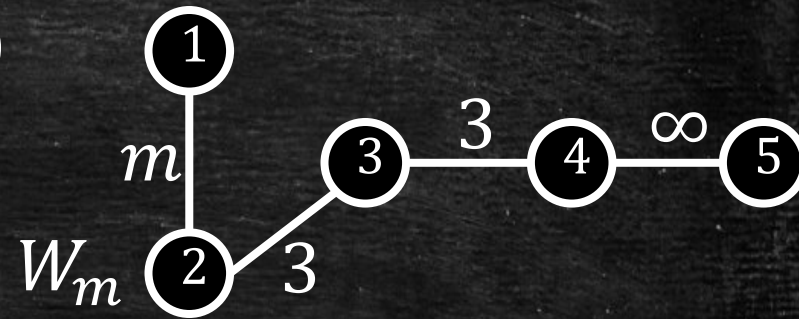

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1) $P_m \cap H^3$ is a fundamental domain for Γ_m.

2) If $6 < m < \infty$, then $P_m \subset H^3$ and Γ_m is a uniform lattice in $\mathrm{Isom}(H^3)$. If $m = \infty$, then $P_\infty \cap \partial H^3 = \{\ \}$ and Γ_∞ is a non-uniform lattice.

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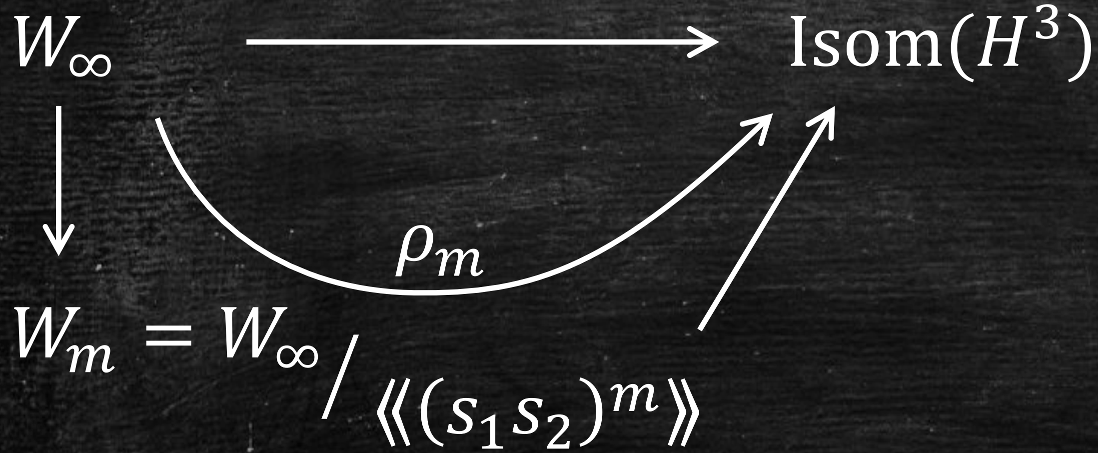

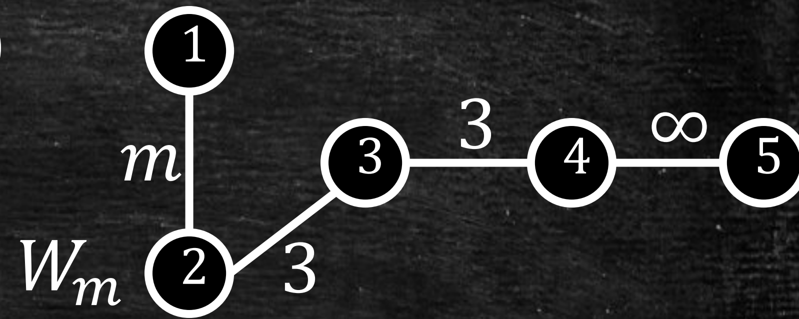

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Corollary $\forall$ neighborhood $\mathcal{U}$ of $\rho_\infty \in \mathrm{Hom}(W_\infty, \mathrm{Isom}(H^3))$
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Theorem (Garland-Raghunathan, 1970)

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Let's change the target Lie group of representations from $\mathrm{Isom}(H^d)$ to $\mathrm{SL}^{\pm}_{d+1}(\mathbb{R})$.

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A projective pre-Coxeter polytope is a pair $(P,(\sigma_s)_{s\in S})$ where P is a polytope in the projective sphere $\mathbb{S}^n$

$$P = \bigcap_{s\in S} \{\, [v] \in \mathbb{S}^n \mid \alpha_s(v) \le 0 \,\}$$

and for each facet s of P, a reflection

$$\sigma_s = \mathrm{Id} - \alpha_s \otimes b_s \quad (\, \alpha_s(b_s) = 2 \,)$$

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A pre-Coxeter polytope is a Coxeter polytope if

$$P \cap \gamma . P = \emptyset \quad \forall \gamma \in \Gamma \backslash \{\mathrm{Id}\}$$

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Theorem (Vinberg) A pre-Coxeter polytope is a Coxeter polytope if and only if the Cartan matrix

$$A = (A_{st})_{s,t\in S} = (\alpha_s(b_t))_{s,t\in S}$$

satisfies the following:

1) $A_{ss} = 2$; for $s \neq t$, $A_{st} \leq 0$; $A_{st} = 0 \Leftrightarrow A_{ts} = 0$;

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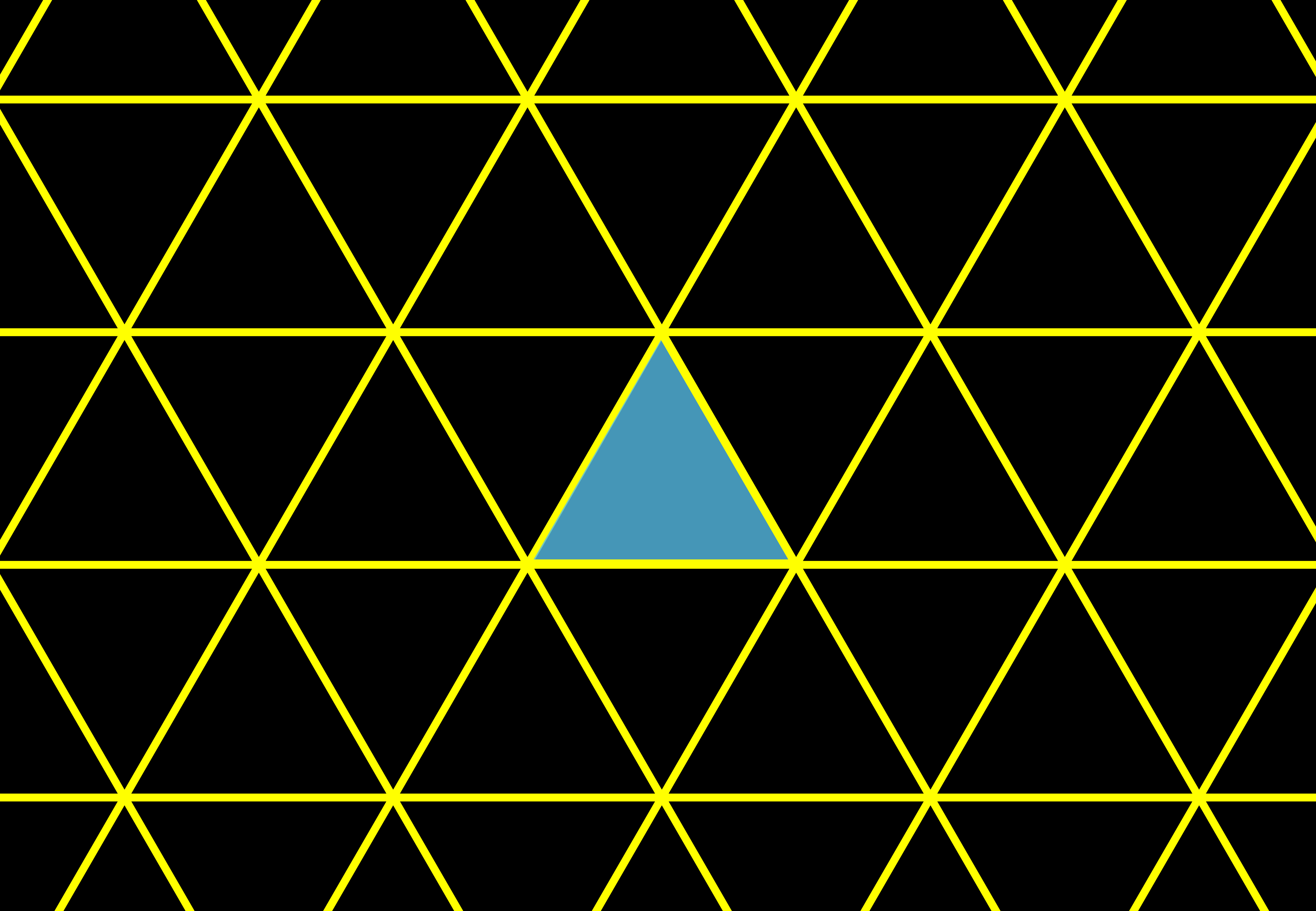
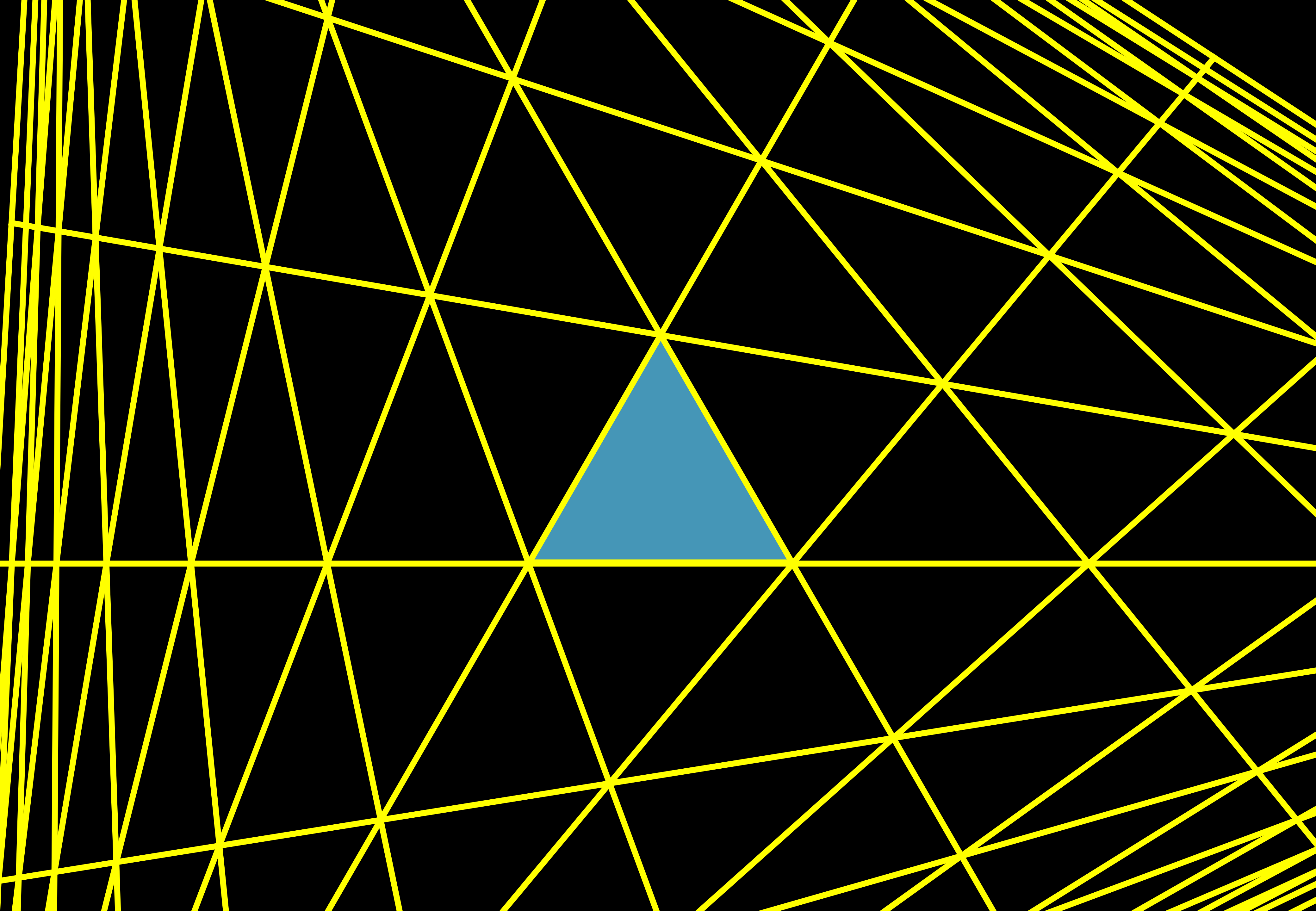
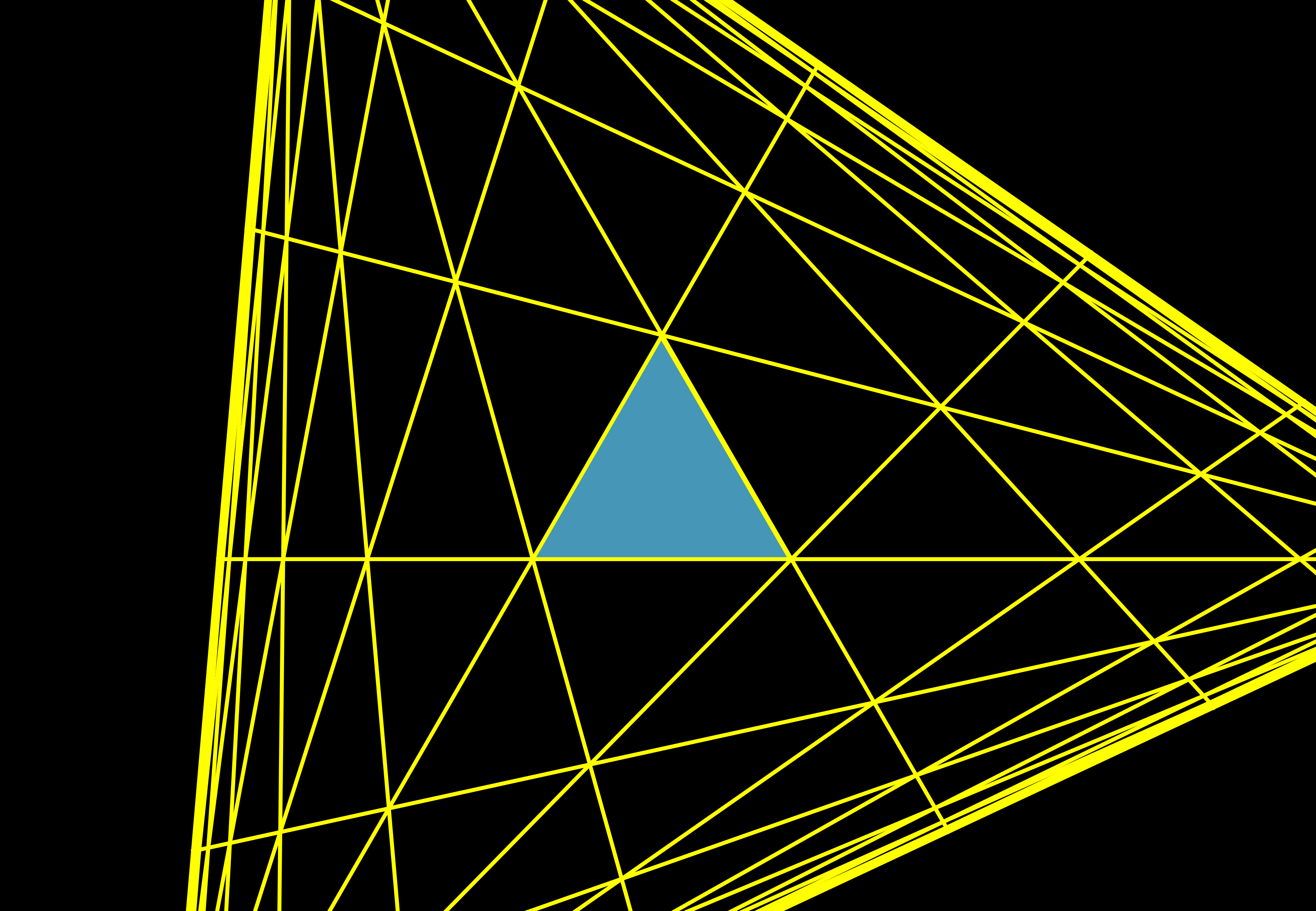
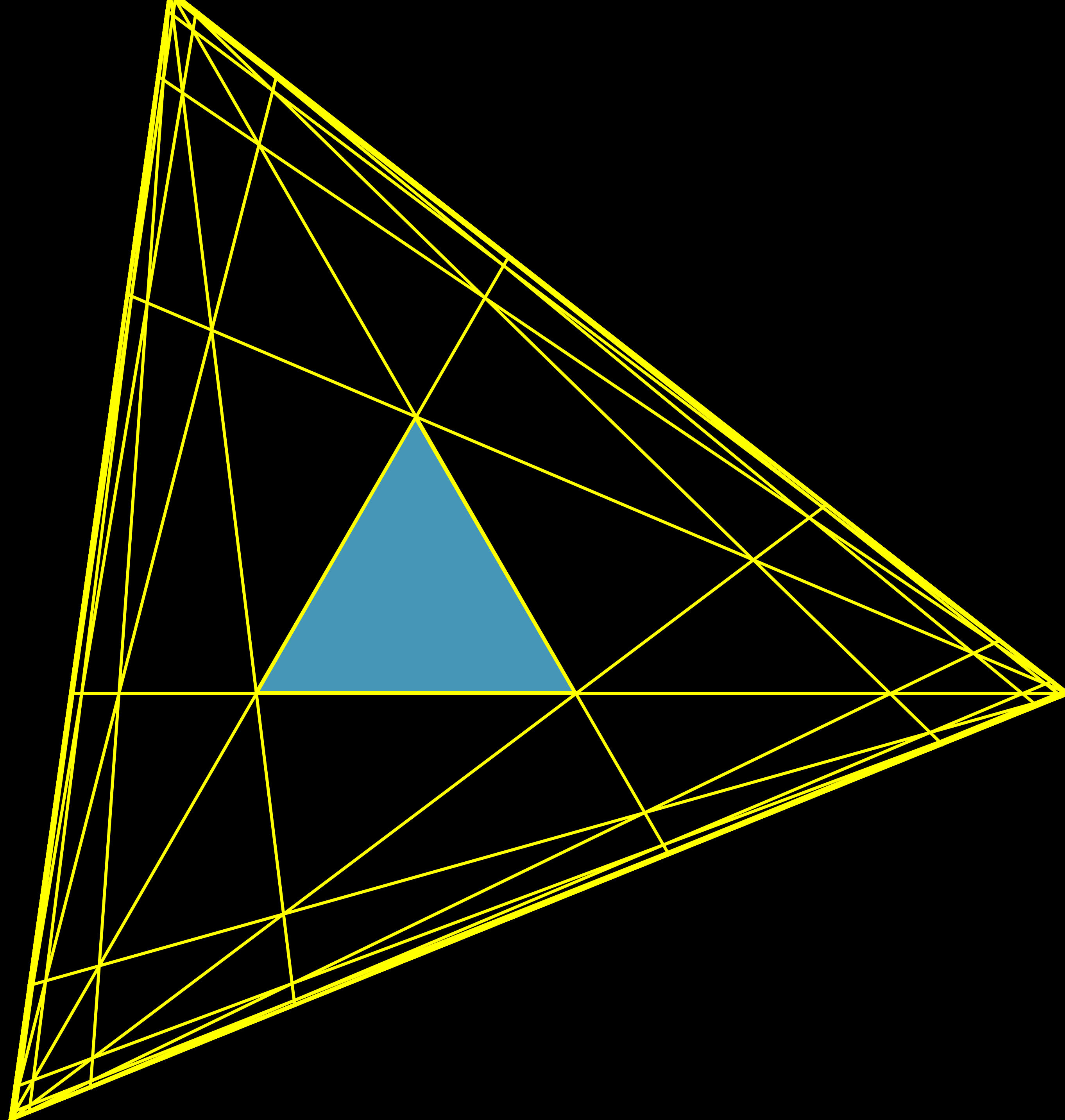
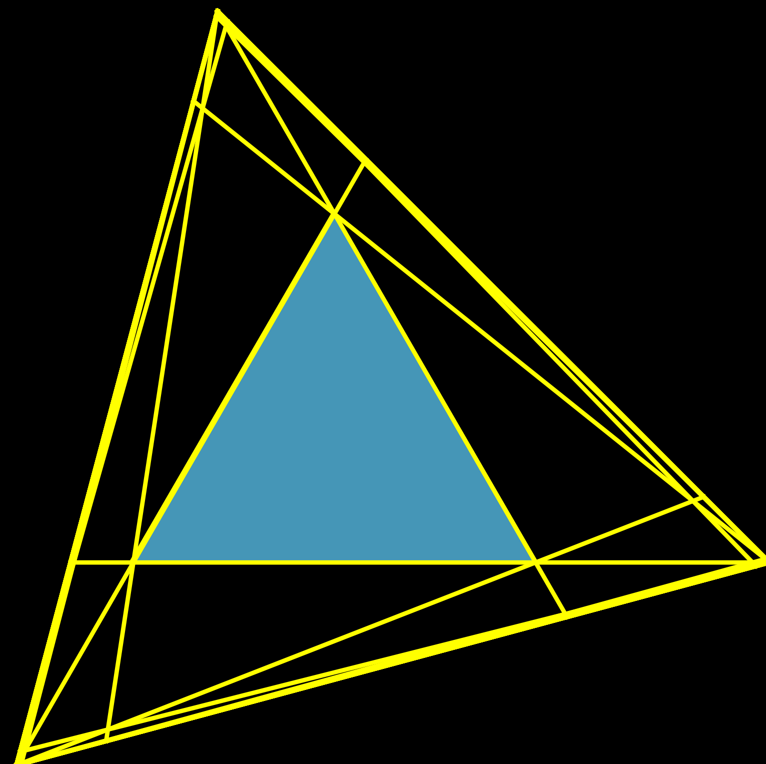


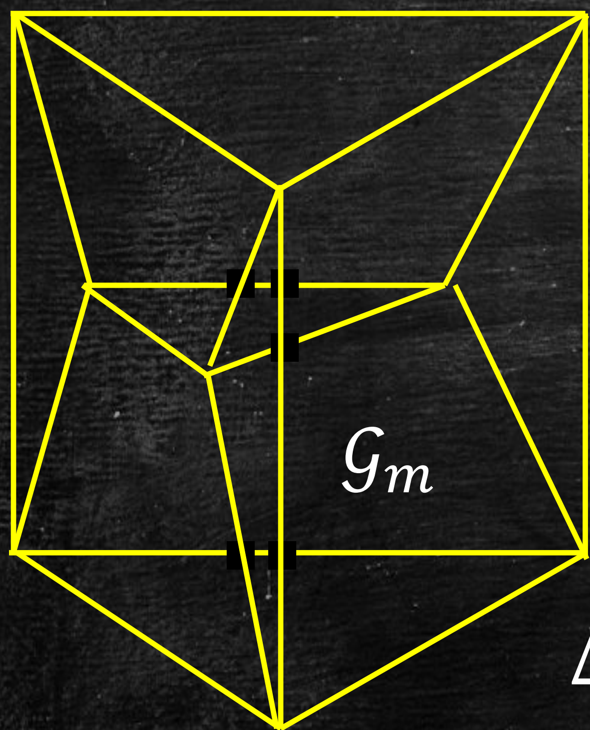




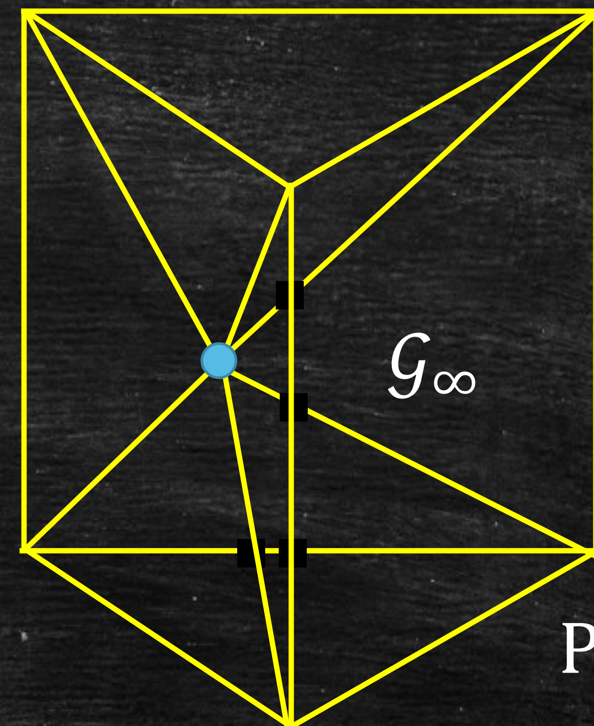




Schlegel diagram



$\Delta_2 \times \Delta_2$



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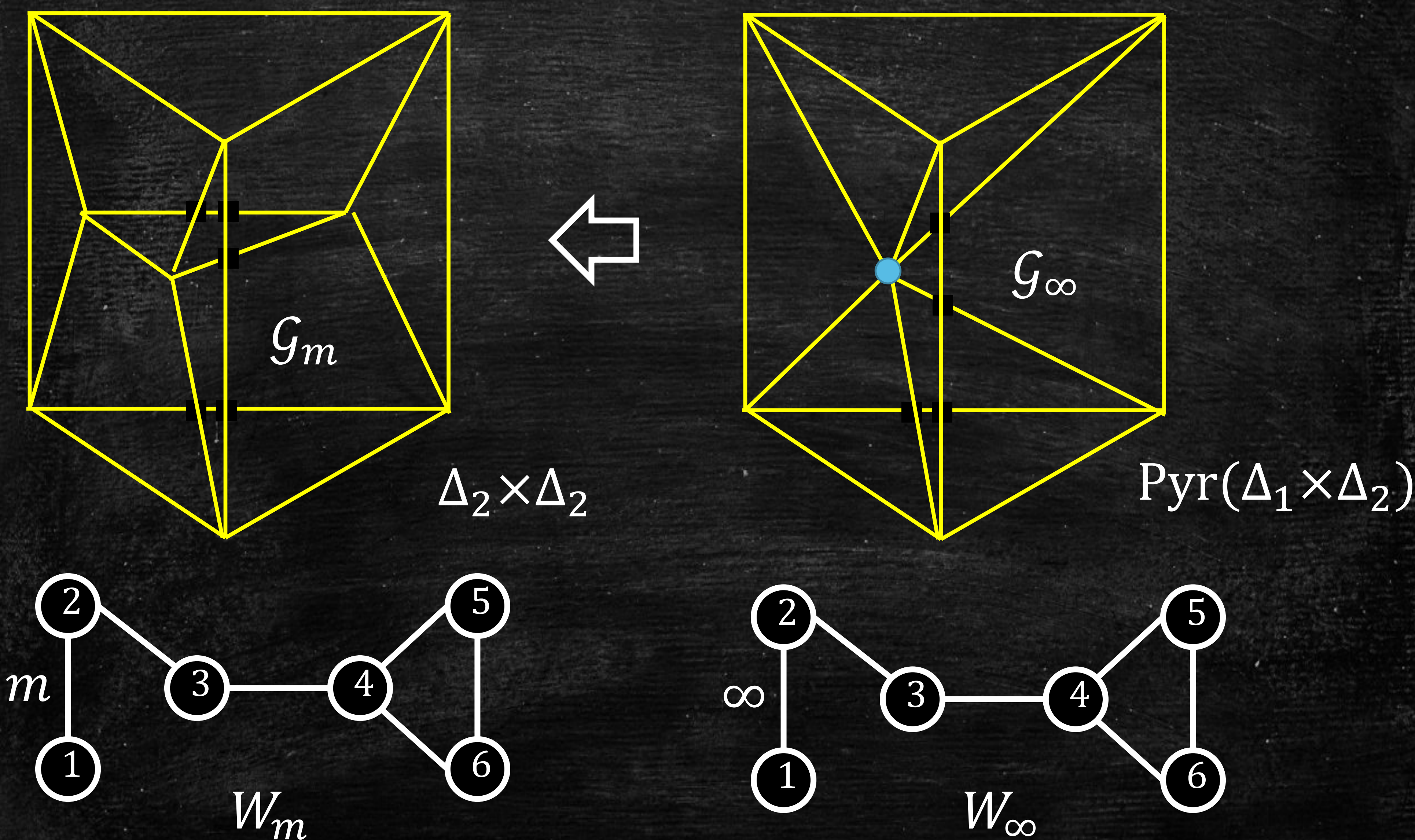

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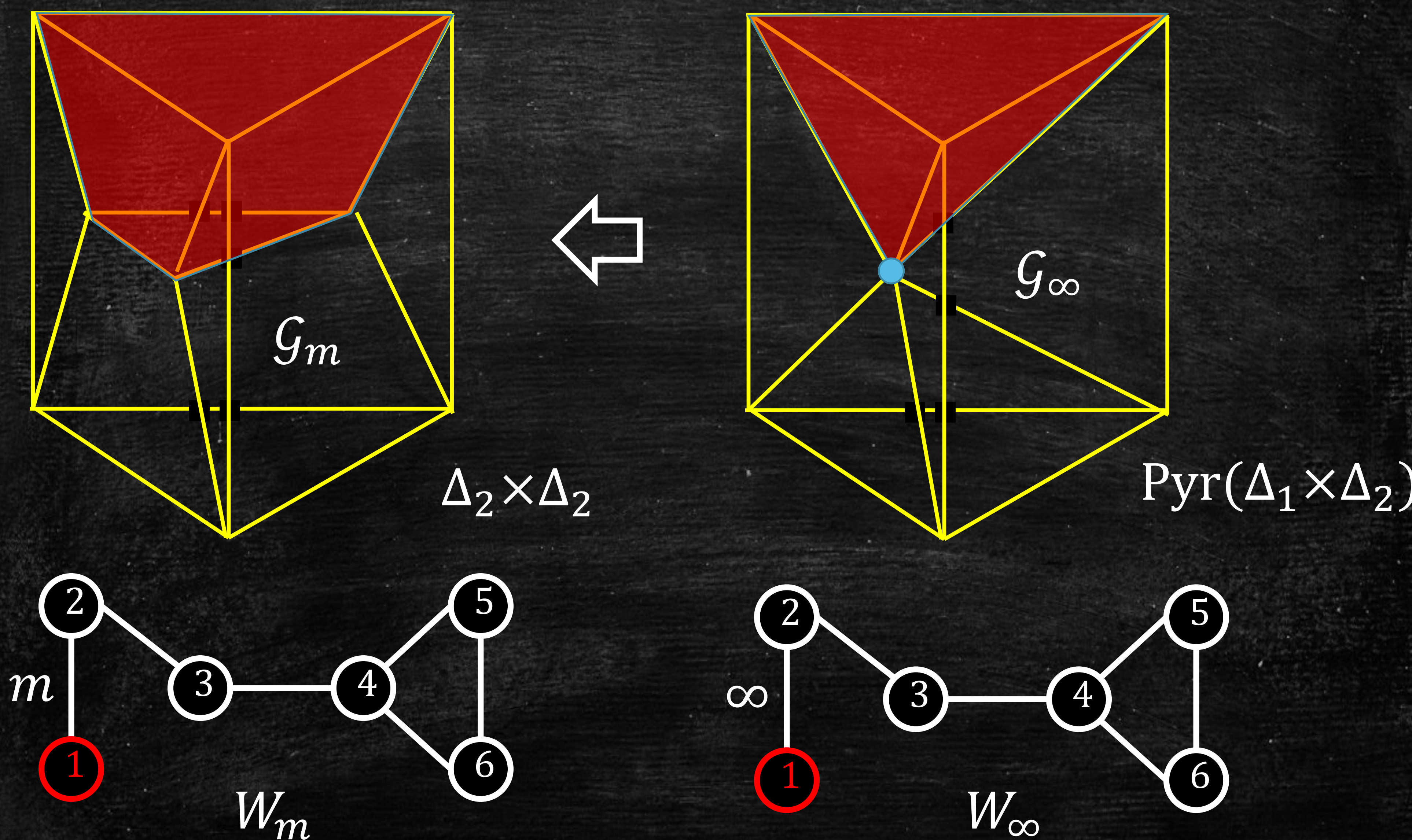

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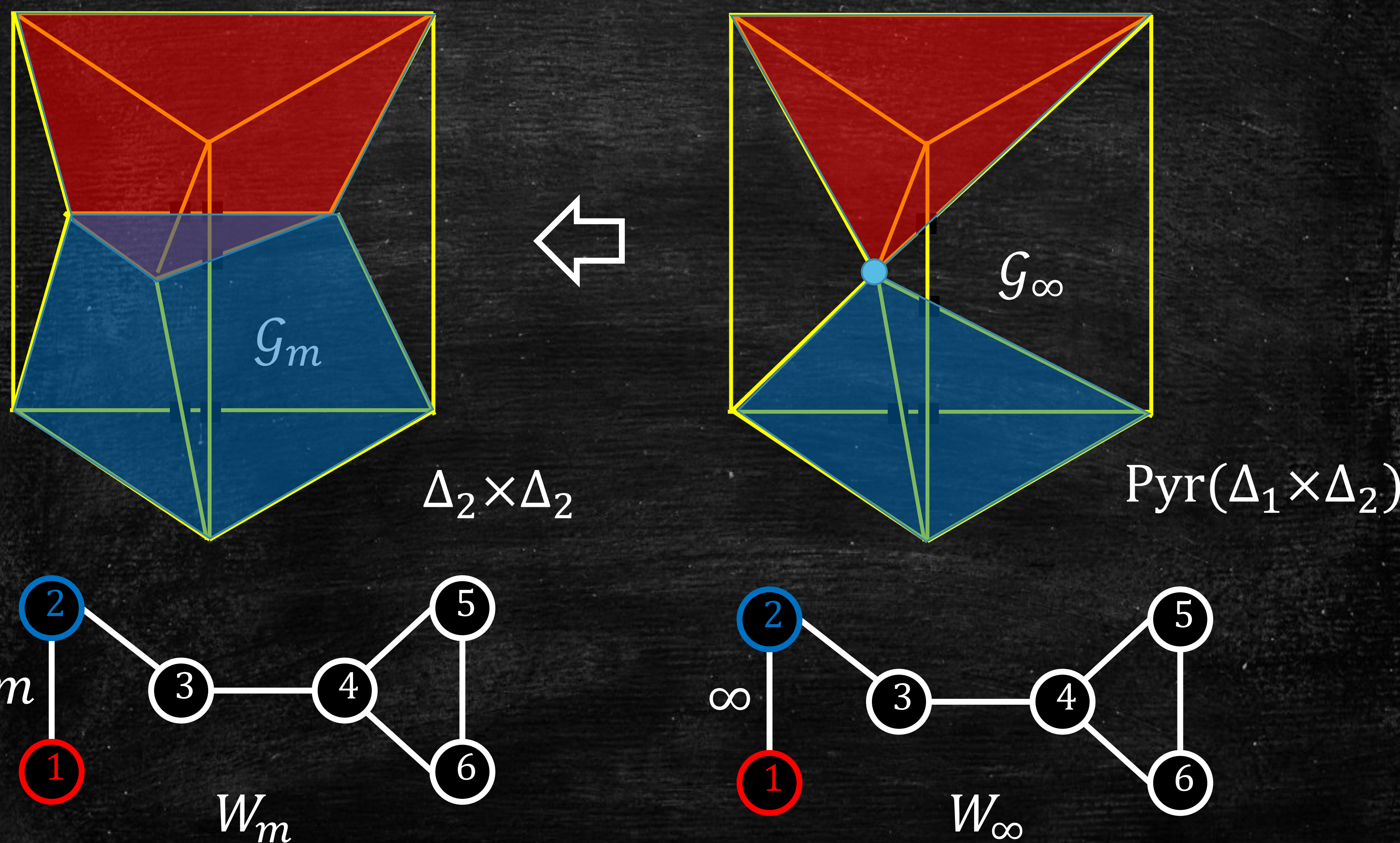

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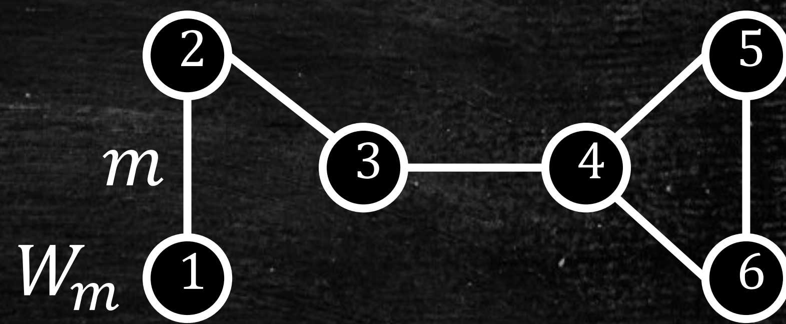

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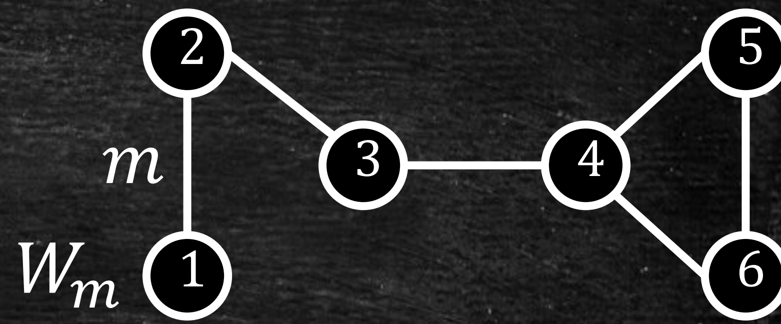

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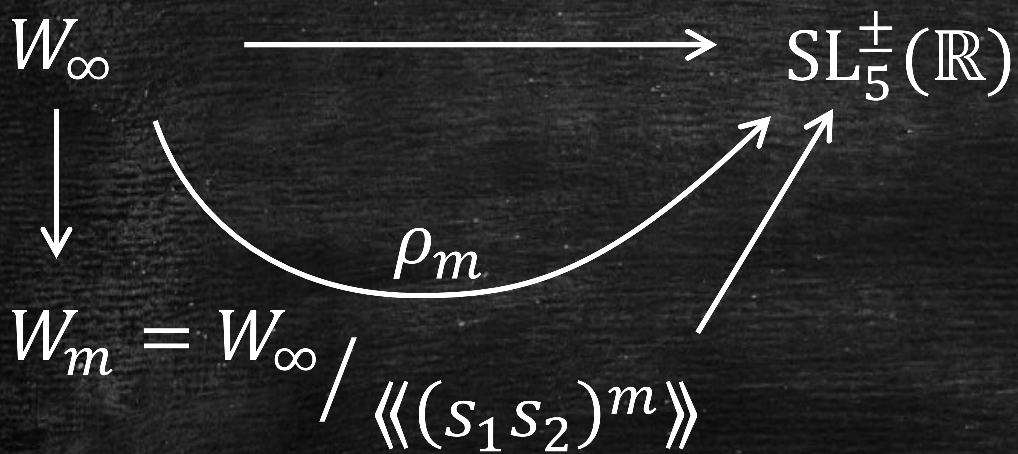

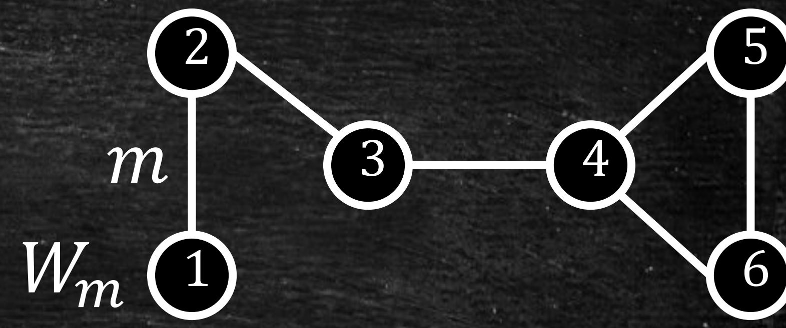

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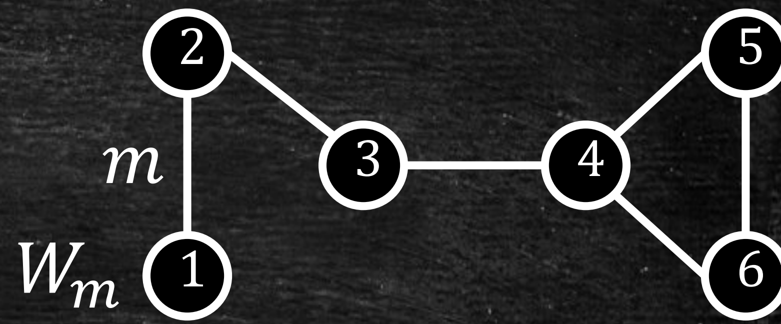

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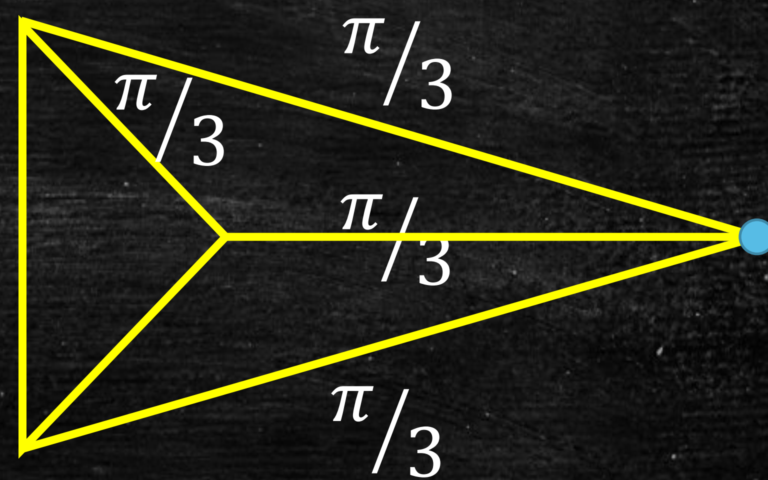

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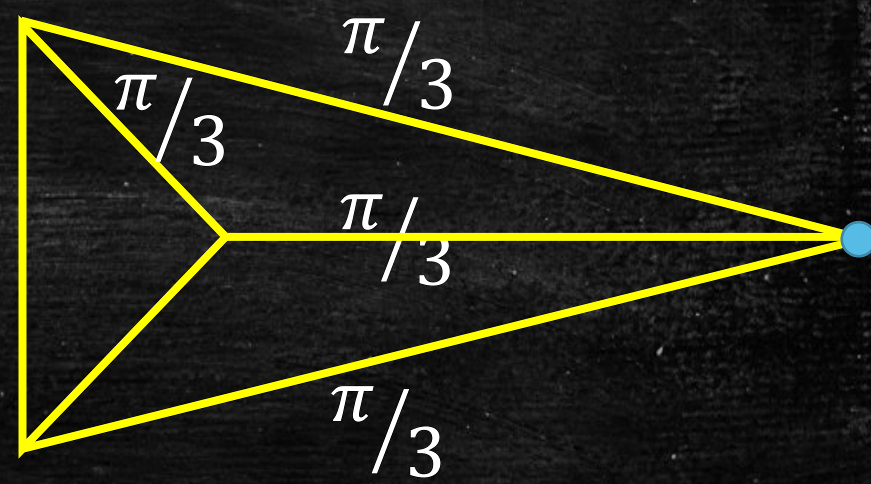

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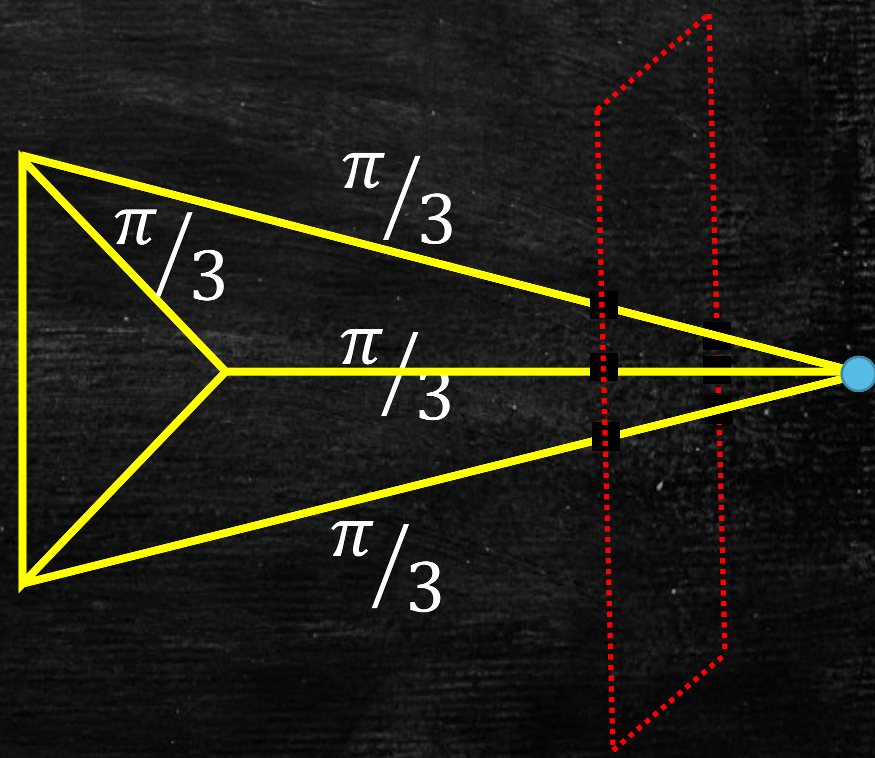

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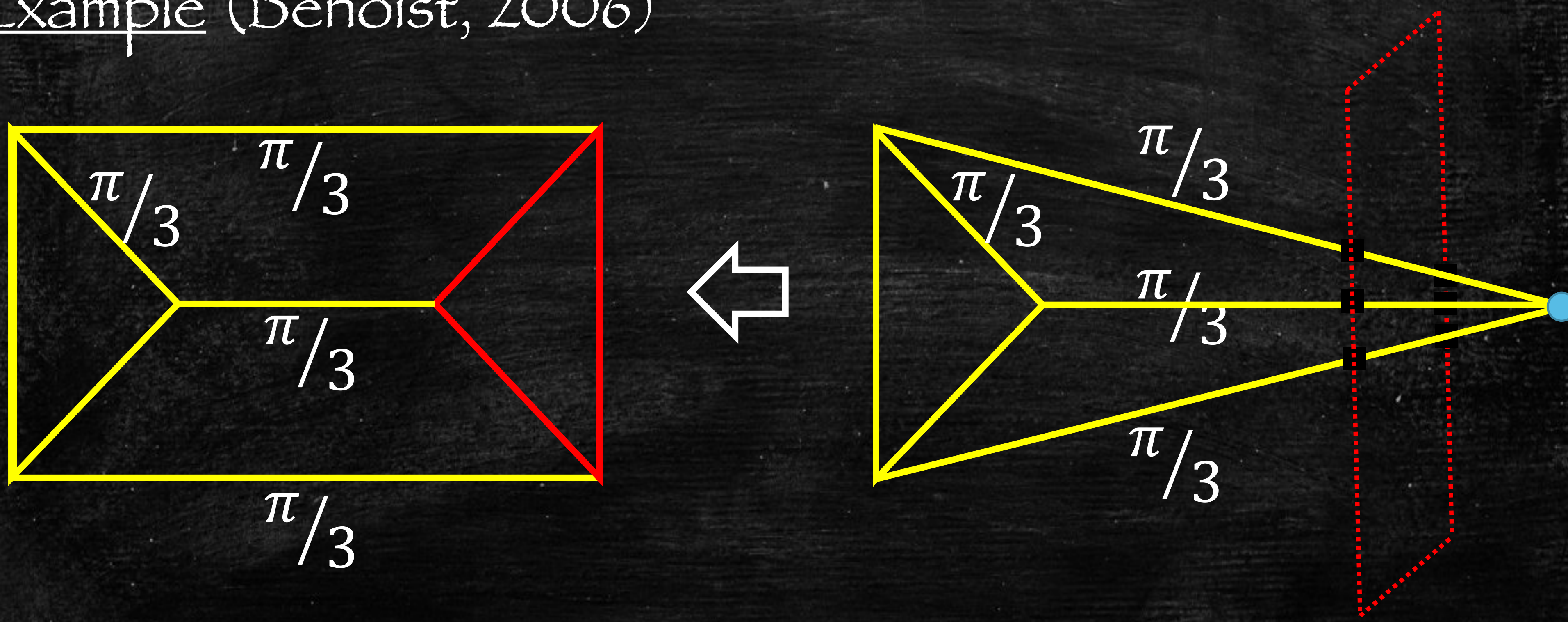

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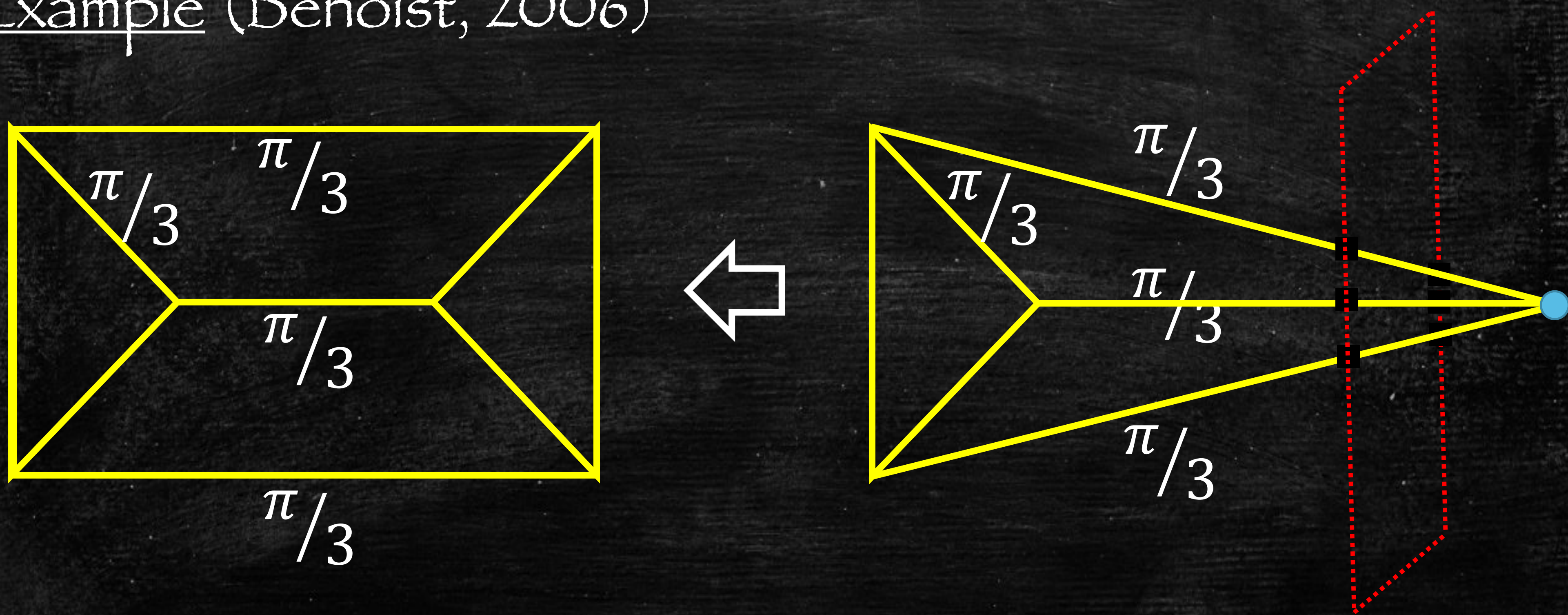

Thank you!