Introduction to Particle Physics

Sreerup Raychaudhuri

TIFR

Lecture 6

Electroweak Unification

Nonabelian Gauge Theory

Consider a scalar multiplet $\Phi(x)$ of length n, i.e.

$$\Phi(x) = \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \vdots \\ \varphi_n(x) \end{pmatrix}$$

where each $\varphi_i(x)$ $(i = 1, \ldots, n)$ is a complex scalar field.

Construct the 'free' Lagrangian density

$$\mathcal{L} = (\partial^\mu \Phi)^\dagger \partial_\mu \Phi - M^2 \Phi^\dagger \Phi$$

This is just a shorthand for n mass-degenerate free scalar fields, i.e.

$$\mathcal{L} = \sum_{i=1}^{n} \left(\partial^\mu \varphi_i^* \, \partial_\mu \varphi_i - M^2 \varphi_i^* \varphi_i \right)$$

Now consider a global SU(N) gauge transformation

$$\Phi(x) \rightarrow \Phi^{'}(x) = \mathbb{U}\Phi(x)$$

where $\mathbb{U}$ is a SU(N) matrix, i.e. $\mathbb{U}^{\dagger}\,\mathbb{U} = \mathbb{1}$ and $\det \mathbb{U} = +1$, where

$$\mathbb{U} = \begin{pmatrix} U_{11} & \cdots & U_{1n} \\ \vdots & & \vdots \\ U_{n1} & \cdots & U_{nn} \end{pmatrix}$$

n and N are different (in general)

The number of free (real) parameters in this SU(N) matrix is

$$p = 2N^2 - N - 2\,{}^{N}C_2 - 1 = N^2 - 1$$

We can write this SU(N) transformation in the form $\mathbb{U} = e^{-ig\vec{\theta}.\vec{\mathbb{T}}}$

where the $\vec{\theta} = (\theta_1, \cdots, \theta_p)$ are free (real) parameters

and the $\vec{\mathbb{T}} = (\mathbb{T}_1, \cdots, \mathbb{T}_p)$ are the generators of SU(N)

$$\vec{\theta}.\vec{\mathbb{T}} = \sum_{a=1}^{p} \theta_a \mathbb{T}_a$$

Under this gauge transformation

$$\Phi(x) \rightarrow \Phi'(x) = \mathbb{U}\,\Phi(x)$$

$$\Phi^\dagger(x) \rightarrow \Phi'^\dagger(x) = \Phi^\dagger(x)\,\mathbb{U}^\dagger$$

The Lagrangian density transforms to

$$\mathcal{L} \rightarrow \mathcal{L}' = (\partial^\mu \Phi')^\dagger \partial_\mu \Phi' - M^2 \Phi'^\dagger \Phi'$$

$$= (\partial^\mu \mathbb{U}\Phi)^\dagger \partial_\mu \mathbb{U}\Phi - M^2 \Phi^\dagger \mathbb{U}^\dagger \mathbb{U}\Phi \qquad \text{global}$$

$$= (\partial^\mu \Phi)^\dagger \mathbb{U}^\dagger \mathbb{U} \partial_\mu \Phi - M^2 \Phi^\dagger \mathbb{U}^\dagger \mathbb{U}\Phi \qquad \text{unitary}$$

$$= (\partial^\mu \Phi)^\dagger \partial_\mu \Phi - M^2 \Phi^\dagger \Phi$$

$$= \mathcal{L}$$

Thus, this system of n mass-degenerate free scalar fields possesses a SU(N) global gauge symmetry — with p conserved currents/charges.

The next step is to convert this to a SU(N) local gauge symmetry, i.e.

$$\Phi(x) \;\rightarrow \Phi^{'}(x) \;= \mathbb{U}(x)\,\Phi(x)$$
$$\Phi^{\dagger}(x) \rightarrow \Phi^{'\dagger}(x) = \Phi^{\dagger}(x)\,\mathbb{U}^{\dagger}(x)$$

As in the nonAbelian case, the Lagrangian density will no longer remain gauge invariant...

$$\mathcal{L} \rightarrow \;\mathcal{L}' = (\partial^{\mu}\Phi^{'})^{\dagger}\partial_{\mu}\Phi' - M^2\Phi^{'\dagger}\Phi'$$

$$= (\partial^{\mu}\mathbb{U}\,\Phi)^{\dagger}\partial_{\mu}\mathbb{U}\,\Phi - M^2\Phi^{\dagger}\mathbb{U}^{\dagger}\mathbb{U}\,\Phi \quad \text{local}$$

$$= \left(\mathbb{U}\partial_{\mu}\Phi + \partial_{\mu}\mathbb{U}\Phi\right)^{\dagger}\left(\mathbb{U}\partial_{\mu}\Phi + \partial_{\mu}\mathbb{U}\Phi\right) - M^2\Phi^{\dagger}\mathbb{U}^{\dagger}\mathbb{U}\Phi$$

$$= \left[\left(\mathbb{1}\partial_{\mu} + \mathbb{U}^{\dagger}\partial_{\mu}\mathbb{U}\right)\Phi\right]^{\dagger}\mathbb{U}^{\dagger}\mathbb{U}\left(\mathbb{1}\partial_{\mu} + \mathbb{U}^{\dagger}\partial_{\mu}\mathbb{U}\right)\Phi - M^2\Phi^{\dagger}\mathbb{U}^{\dagger}\mathbb{U}\Phi \;\; \text{unitary}$$

$$= \left[\left(\mathbb{1}\partial_{\mu} + \mathbb{U}^{\dagger}\partial_{\mu}\mathbb{U}\right)\Phi\right]^{\dagger}\left(\mathbb{1}\partial_{\mu} + \mathbb{U}^{\dagger}\partial_{\mu}\mathbb{U}\right)\Phi - M^2\Phi^{\dagger}\Phi \qquad \neq \mathcal{L}$$

<u>Solution:</u> define a covariant derivative $\mathbb{D}_\mu = \mathbb{1}\partial_\mu + ig\mathbb{A}_\mu(x)$

where the $\mathbb{A}_\mu(x)$ is a $n \times n$ matrix of gauge fields, i.e.

$$\mathbb{A}^\mu = \begin{pmatrix} a_{11}^\mu & \cdots & a_{1n}^\mu \\ \vdots & & \vdots \\ a_{n1}^\mu & \cdots & a_{nn}^\mu \end{pmatrix}$$

Not all of these need to be independent... ($\mathbb{A}^\mu$ is Hermitian...)

We require the covariant derivative $\mathbb{D}_\mu \Phi$ to transform exactly like Φ, i.e.

$$\mathbb{D}_\mu \Phi \to \mathbb{D}'_\mu \Phi' = \mathbb{U}\, \mathbb{D}_\mu \Phi$$

for then, if we rewrite the Lagrangian density as

$$\mathcal{L} = (\mathbb{D}^\mu \Phi)^\dagger \mathbb{D}_\mu \Phi - M^2 \Phi^\dagger \Phi$$

it will be trivially gauge invariant.

How do we ensure that $\mathbb{D}_\mu \Phi \rightarrow \mathbb{D}'_\mu \Phi' = \mathbb{U}\, \mathbb{D}_\mu \Phi$?

By adjusting the transformation of the gauge field matrix $\mathbb{A}^\mu$ ...

$$\mathbb{D}_\mu \Phi \rightarrow \mathbb{D}'_\mu \Phi' = \left(\mathbb{1}\partial_\mu + ig\mathbb{A}'_\mu\right)\mathbb{U}\Phi$$

$$= \partial_\mu(\mathbb{U}\Phi) + ig\mathbb{A}'_\mu \mathbb{U}\Phi$$

$$= \mathbb{U}\left(\partial_\mu \Phi\right) + \left(\partial_\mu \mathbb{U}\right)\Phi + ig\mathbb{A}'_\mu \mathbb{U}\Phi$$

$$= \mathbb{U}\left(\partial_\mu \Phi\right) + \mathbb{U}\mathbb{U}^\dagger\left(\partial_\mu \mathbb{U}\right)\Phi + ig\mathbb{U}\mathbb{U}^\dagger \mathbb{A}'_\mu \mathbb{U}\Phi$$

$$= \mathbb{U}\left[\mathbb{1}\partial_\mu + \mathbb{U}^\dagger \partial_\mu \mathbb{U} + ig\mathbb{U}^\dagger \mathbb{A}'_\mu \mathbb{U}\right]\Phi$$

If this is to be the same as

$$\mathbb{D}_\mu \Phi = \left(\mathbb{1}\partial_\mu + ig\mathbb{A}_\mu\right)\Phi$$

we must have $ig\mathbb{A}_\mu = ig\mathbb{U}^\dagger \mathbb{A}'_\mu \mathbb{U} + \mathbb{U}^\dagger \partial_\mu \mathbb{U}$

Rewrite

$$ig\mathbb{A}_\mu = ig\mathbb{U}^\dagger \mathbb{A}'_\mu \mathbb{U} + \mathbb{U}^\dagger \partial_\mu \mathbb{U}$$

as

$$ig\mathbb{U}^\dagger \mathbb{A}'_\mu \mathbb{U} = ig\mathbb{A}_\mu - \mathbb{U}^\dagger \partial_\mu \mathbb{U}$$

or,

$$ig\mathbb{A}'_\mu = ig\mathbb{U}\mathbb{A}_\mu \mathbb{U}^\dagger - (\partial_\mu \mathbb{U})\mathbb{U}^\dagger$$

Note that $\mathbb{U}\mathbb{U}^\dagger = \mathbb{1}$ leads to $(\partial_\mu \mathbb{U})\mathbb{U}^\dagger + \mathbb{U}(\partial_\mu \mathbb{U}^\dagger) = 0$

i.e.

$$ig\mathbb{A}'_\mu = ig\mathbb{U}\mathbb{A}_\mu \mathbb{U}^\dagger + \mathbb{U}(\partial_\mu \mathbb{U}^\dagger) = ig\mathbb{U}\mathbb{A}_\mu \mathbb{U}^\dagger + \mathbb{U}(\partial_\mu \mathbb{U}^\dagger)\mathbb{U}\mathbb{U}^\dagger$$

or, finally,

$$\mathbb{A}'_\mu = \mathbb{U}\left[\mathbb{A}_\mu - \frac{i}{g}(\partial_\mu \mathbb{U}^\dagger)\mathbb{U}\right]\mathbb{U}^\dagger$$

<u>Quick check</u>: suppose $N = 1$ and $n = 1$, i.e. U(1) gauge symmetry

Then $\mathbb{U} = e^{-ig\theta}$ and $\mathbb{A}_\mu = A_\mu$.

Now,

$$\mathbb{A}'_\mu = \mathbb{U}\left[\mathbb{A}_\mu - \frac{i}{g}(\partial_\mu \mathbb{U}^\dagger)\mathbb{U}\right]\mathbb{U}^\dagger$$

assumes the form

$$\begin{aligned} A'_\mu &= e^{-ig\theta}\left[A_\mu - \frac{i}{g}(\partial_\mu e^{+ig\theta})e^{-ig\theta}\right]e^{+ig\theta} \\ &= e^{-ig\theta}\left[A_\mu - \frac{i}{g}(ig\partial_\mu\theta\, e^{+ig\theta})e^{-ig\theta}\right]e^{+ig\theta} \\ &= A_\mu + \partial_\mu\theta \end{aligned}$$

which is what we had derived for the U(1) case.

How many independent fields do we require in the $\mathbb{A}_\mu$ matrix?

$$\mathbb{A}'_\mu = \mathbb{U}\left[\mathbb{A}_\mu - \frac{i}{g}(\partial_\mu \mathbb{U}^\dagger)\mathbb{U}\right]\mathbb{U}^\dagger$$

Since $\mathbb{U} = e^{-ig\vec{\theta}.\vec{\mathbb{T}}}$ i.e. $\mathbb{U}$ has p free parameters, $\mathbb{A}_\mu$ should have p independent fields. This encourages us to expand

$$\mathbb{A}^\mu(x) = \sum_{a=1}^{p} A_a^\mu(x)\,\mathbb{T}_a = \overrightarrow{A^\mu}\,.\vec{\mathbb{T}}$$

One can now work out the transformation properties of the $A_a^\mu(x)$ fields in terms of the parameters $\vec{\theta} = (\theta_1, \cdots, \theta_p)$.

(Will do this for specific cases...)

We can also use this expression

$$\mathbb{A}^{\mu}(x) = \sum_{a=1}^{p} A_a^{\mu}(x)\, \mathbb{T}_a = \overrightarrow{A^{\mu}} . \vec{\mathbb{T}}$$

to write out the interaction terms in the Lagrangian density...

$$\mathcal{L} = (\mathbb{D}^{\mu}\Phi)^{\dagger}\mathbb{D}_{\mu}\Phi - M^2\Phi^{\dagger}\Phi$$

$$= [(\mathbb{1}\partial^{\mu} + ig\mathbb{A}^{\mu})\Phi]^{\dagger}(\mathbb{1}\partial_{\mu} + ig\mathbb{A}_{\mu})\Phi - M^2\Phi^{\dagger}\Phi$$

$$= (\partial^{\mu}\Phi)^{\dagger}\partial_{\mu}\Phi - M^2\Phi^{\dagger}\Phi \qquad \text{free scalar}$$

$$+ ig\left[(\partial^{\mu}\Phi)^{\dagger}\mathbb{A}_{\mu}\Phi - \Phi^{\dagger}\mathbb{A}^{\mu}\partial_{\mu}\Phi\right] \qquad \text{gauge-scalar interaction}$$

$$+ \; g^2\Phi^{\dagger}\mathbb{A}^{\mu}\mathbb{A}_{\mu}\Phi \qquad \text{seagull terms}$$

We should complete the Lagrangian density by adding a kinetic term for the gauge fields...

$$\mathbb{F}_{\mu\nu} = -\frac{i}{g}\left[\mathbb{D}_\mu, \mathbb{D}_\nu\right] = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + ig\left[\mathbb{A}_\mu, \mathbb{A}_\nu\right]$$

Now, we have

$$\begin{aligned} \mathbb{D}_\mu \Phi \to \mathbb{D}'_\mu \Phi' &= \mathbb{U}\ \mathbb{D}_\mu \Phi \\ &= \mathbb{U}\ \mathbb{D}_\mu \mathbb{U}^\dagger\ \mathbb{U}\Phi \\ &= \mathbb{U}\ \mathbb{D}_\mu \mathbb{U}^\dagger\ \Phi' \qquad \Rightarrow\ \mathbb{D}'_\mu = \mathbb{U}\ \mathbb{D}_\mu \mathbb{U}^\dagger \end{aligned}$$

Thus,

$$\mathbb{F}_{\mu\nu} \to \mathbb{F}'_{\mu\nu} = -\frac{i}{g}\left[\mathbb{D}'_\mu, \mathbb{D}'_\nu\right] = -\frac{i}{g}\left[\mathbb{U}\ \mathbb{D}_\mu \mathbb{U}^\dagger, \mathbb{U}\ \mathbb{D}_\nu \mathbb{U}^\dagger\right] = \mathbb{U}\ \mathbb{F}_{\mu\nu} \mathbb{U}^\dagger$$

To get gauge invariance, we have to take the trace...

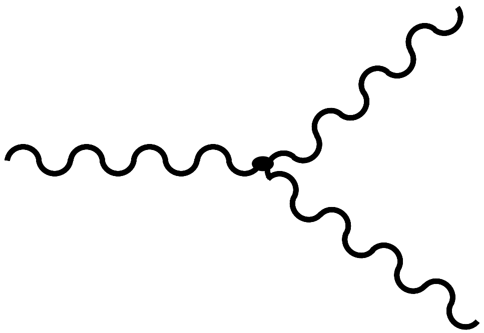
The full Lagrangian density is now

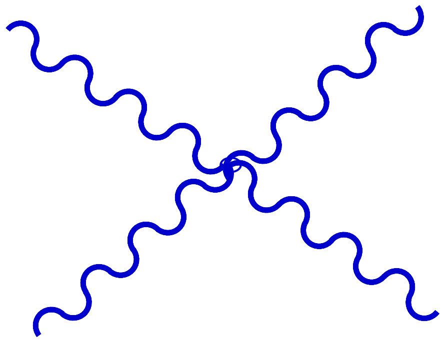
$$\mathcal{L} = (\mathbb{D}^{\mu}\Phi)^{\dagger}\mathbb{D}_{\mu}\Phi - M^2\Phi^{\dagger}\Phi - \frac{1}{2}\mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}\right]$$

Since $\mathbb{F}_{\mu\nu} = \left(\partial_{\mu}\mathbb{A}_{\nu} - \partial_{\nu}\mathbb{A}_{\mu}\right) + ig\left[\mathbb{A}_{\mu},\mathbb{A}_{\nu}\right]$

$$\mathbb{F}^{\mu\nu} = (\partial^{\mu}\mathbb{A}^{\nu} - \partial^{\nu}\mathbb{A}^{\mu}) + ig[\mathbb{A}^{\mu},\mathbb{A}^{\nu}]$$

Leads to triple gauge vertices and quadruple gauge vertices





absent in an Abelian gauge theory, e.g. QED

SU(2) Gauge Theory

Recall that for weak interactions we needed three gauge bosons, the

$$W_\mu^+, W_\mu^-, W_\mu^0$$

This seems to indicate a gauge theory with three generators

and the obvious one to take is an SU(2) gauge theory.

All of the above formalism will work, except that now we must take the generators as

$$\mathbb{T}_1 = \tfrac{1}{2}\sigma_1 \qquad \mathbb{T}_2 = \tfrac{1}{2}\sigma_2 \qquad \mathbb{T}_3 = \tfrac{1}{2}\sigma_3$$

obeying the Lie algebra

$$[\mathbb{T}_a, \mathbb{T}_b] = i\varepsilon_{abc}\,\mathbb{T}_c$$

The full Lagrangian for this is

$$\mathcal{L} = (\partial^\mu \Phi)^\dagger \partial_\mu \Phi - M^2 \Phi^\dagger \Phi + ig\left[(\partial^\mu \Phi)^\dagger \mathbb{A}_\mu \Phi - \Phi^\dagger \mathbb{A}^\mu \partial_\mu \Phi\right]$$

$$+ g^2 \Phi^\dagger \mathbb{A}^\mu \mathbb{A}_\mu \Phi - \tfrac{1}{2}\mathrm{Tr}\left[\mathbb{F}_{\mu\nu} \mathbb{F}^{\mu\nu}\right]$$

$$\Phi = \begin{pmatrix} \varphi_\mathrm{A} \\ \varphi_\mathrm{B} \end{pmatrix}$$

where

$$\mathbb{A}^\mu = A_1^\mu \mathbb{T}_1 + A_2^\mu \mathbb{T}_2 + A_3^\mu \mathbb{T}_3$$

We can also expand

$$\mathbb{F}^{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + ig\left[\mathbb{A}_\mu, \mathbb{A}_\nu\right]$$

$$= F_1^{\mu\nu} \mathbb{T}_1 + F_2^{\mu\nu} \mathbb{T}_2 + F_3^{\mu\nu} \mathbb{T}_3$$

where

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g\varepsilon_{abc} A_b^\mu A_c^\nu$$

Mass generation:

To break this symmetry spontaneously, we now replace the scalar mass term by a potential

$- M^2\Phi^\dagger\Phi \rightarrow -\mathrm{V}(\Phi)$

$$\mathrm{V}(\Phi) = -M^2\Phi^\dagger\Phi + \lambda\left(\Phi^\dagger\Phi\right)^2$$

i.e. this is a theory with n massless scalars and some self-interactions

As before, if we define a real field

$$\Phi^\dagger(x)\Phi(x) \equiv \eta(x)^2$$

then we can write the potential as

$$\mathrm{V}(\eta) = -M^2\eta^2 + \lambda\eta^4$$

with a local maximum at $\eta = 0$; local minima at $\eta = v/\sqrt{2} = \sqrt{M^2/2\lambda}$

These local minima correspond to

$$\Phi^\dagger \Phi = \eta^2 = \frac{M^2}{2\lambda}$$

Recall that

$$\Phi = \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix} = \begin{pmatrix} \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \\ \frac{\varphi_3 + i\varphi_4}{\sqrt{2}} \end{pmatrix}$$

so that $\Phi^\dagger \Phi = |\varphi_A|^2 + |\varphi_B|^2 = \frac{1}{2}(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2)$

i.e.

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 = \frac{M^2}{\lambda}$$

Equation of a 4-sphere – only one of these points can be the vacuum

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Equation of a 4-sphere – only one of these points can be the vacuum

Hidden Symmetry!!

Vacuum manifold in a U(1) gauge theory is a circle

- The scalar field is

$$\varphi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}$$

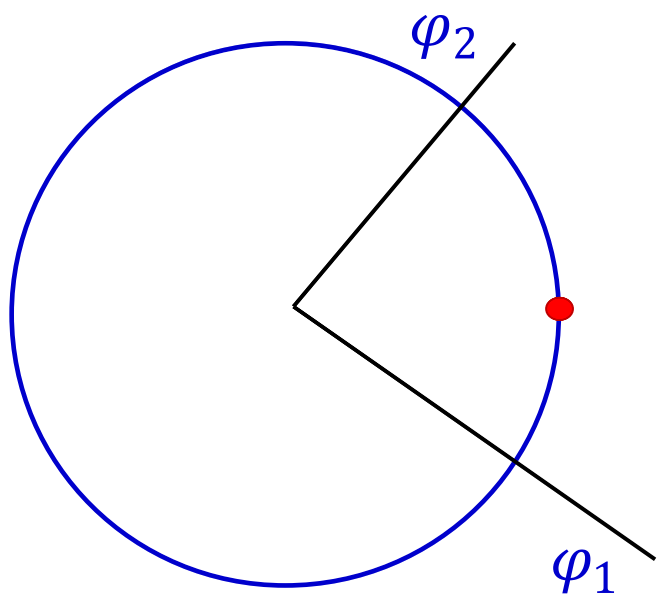
- Traditional to orient the axes in the φ-space such that only the φ_1 has a vacuum expectation value

$$\varphi_0 \equiv \langle \varphi_1 \rangle = v$$

i.e.

$$\langle \varphi \rangle = \frac{v}{\sqrt{2}}$$

- Now shift $\varphi = \langle \varphi \rangle + \varphi'$



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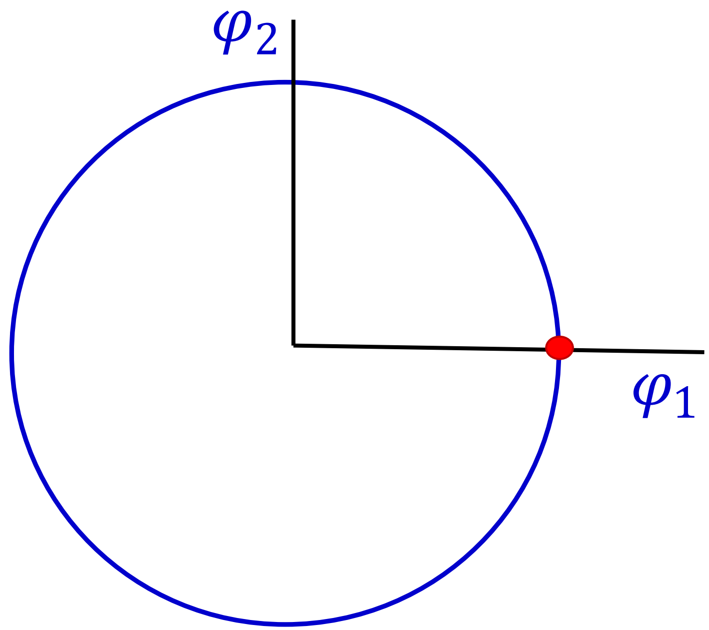


- Now shift $\varphi = \langle \varphi \rangle + \varphi'$

Vacuum manifold in a SU(2) gauge theory is a four-sphere

- The scalar field is

$$\Phi = \begin{pmatrix} \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \\ \frac{\varphi_3 + i\varphi_4}{\sqrt{2}} \end{pmatrix}$$

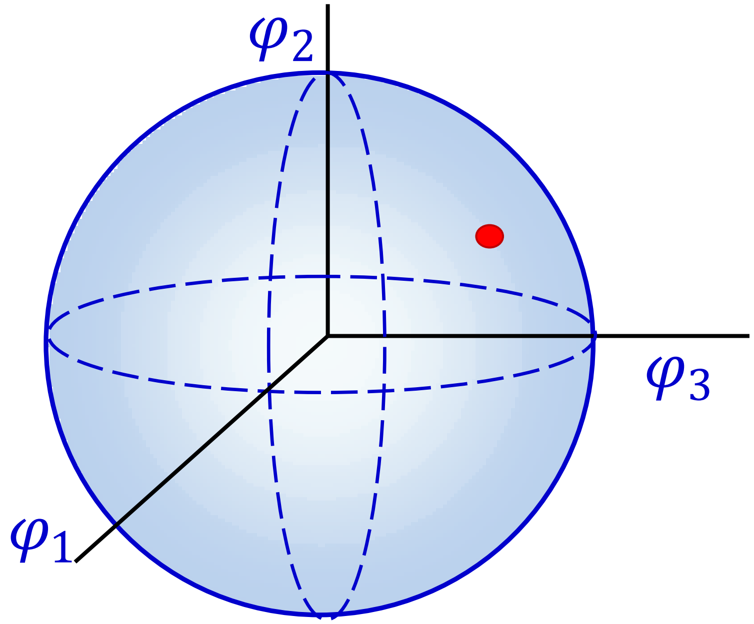
- Traditional to orient the axes in the φ-space such that only the φ_3 has a vacuum expectation value

$$\langle \varphi_3 \rangle = v$$

i.e.

$$\langle \Phi \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

- Now shift $\Phi = \langle \Phi \rangle + \Phi'$



(The φ_4 axis is not shown...)

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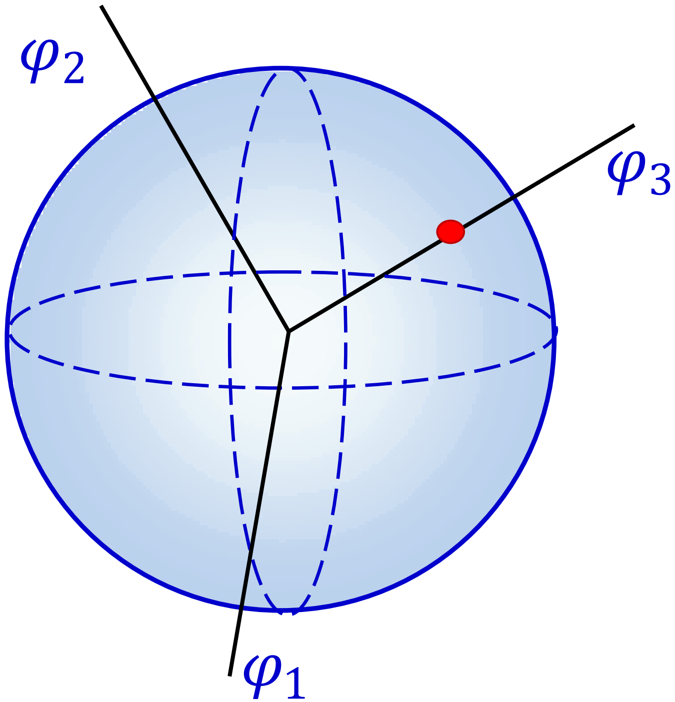

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(The φ_4 axis is not shown...)

- Now shift $\Phi = \langle\Phi\rangle + \Phi'$

<u>Seagull term</u>:

$$\mathcal{L}_{\text{sg}} = g^2 \Phi^\dagger \mathbb{A}^\mu \mathbb{A}_\mu \Phi \rightarrow g^2 (\langle\Phi\rangle + \Phi')^\dagger \mathbb{A}^\mu \mathbb{A}_\mu (\langle\Phi\rangle + \Phi')$$
$$= g^2 \langle\Phi\rangle^\dagger \mathbb{A}^\mu \mathbb{A}_\mu \langle\Phi\rangle + \cdots$$

We thus get a mass term for the gauge bosons, viz.

$$\mathcal{L}_{\text{mass}} = g^2 \langle\Phi\rangle^\dagger \mathbb{A}^\mu \mathbb{A}_\mu \langle\Phi\rangle = g^2 (\mathbb{A}^\mu \langle\Phi\rangle)^\dagger \left(\mathbb{A}_\mu \langle\Phi\rangle\right)$$

Expand this...

$$\mathbb{A}_\mu = A_{\mu 1}\mathbb{T}_1 + A_{\mu 2}\mathbb{T}_2 + A_{\mu 3}\mathbb{T}_3 = \tfrac{1}{2}\left(A_{\mu 1}\sigma_1 + A_{\mu 2}\sigma_2 + A_{\mu 3}\sigma_3\right)$$
$$= \begin{pmatrix} \frac{A_{\mu 3}}{2} & \frac{A_{\mu 1} - iA_{\mu 2}}{2} \\ \frac{A_{\mu 1} + iA_{\mu 2}}{2} & -\frac{A_{\mu 3}}{2} \end{pmatrix} \equiv \begin{pmatrix} \frac{W_\mu^0}{2} & \frac{W_\mu^+}{\sqrt{2}} \\ \frac{W_\mu^-}{\sqrt{2}} & -\frac{W_\mu^0}{2} \end{pmatrix}$$

$$\mathbb{A}_\mu \langle \Phi \rangle = \begin{pmatrix} \frac{W_\mu^0}{2} & \frac{W_\mu^+}{\sqrt{2}} \\ \frac{W_\mu^-}{\sqrt{2}} & -\frac{W_\mu^0}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{v}{2} W_\mu^+ \\ -\frac{v}{2\sqrt{2}} W_\mu^0 \end{pmatrix}$$

and

$$(\mathbb{A}^\mu \langle \Phi \rangle)^\dagger = \begin{pmatrix} \frac{v}{2} W^{\mu -} & -\frac{v}{2\sqrt{2}} W^{\mu 0} \end{pmatrix}$$

Thus,

$$\mathcal{L}_{\text{mass}} = g^2 (\mathbb{A}^\mu \langle \Phi \rangle)^\dagger (\mathbb{A}_\mu \langle \Phi \rangle) = \left(\frac{g^2 v^2}{4} W_\mu^+ W^{\mu -} + \frac{g^2 v^2}{4} W_\mu^0 W^{\mu 0} \right)$$

$$= M_W^2 W_\mu^+ W^{\mu -} + \tfrac{1}{2} M_W^2 W_\mu^0 W^{\mu 0}$$

where $M_W = \frac{1}{2} g v$

In a hidden U(1) gauge theory: $\varphi = \langle\varphi\rangle + \varphi'$

$$\frac{\varphi_1 + i\varphi_2}{\sqrt{2}} = \frac{v}{\sqrt{2}} + \frac{\varphi'_1 + i\varphi'_2}{\sqrt{2}} = \frac{(\varphi'_1 + v) + i\varphi'_2}{\sqrt{2}}$$

When substituted into the potential, this leads to a correct-sign mass for φ'_1 (massive scalar) and keeps φ'_2 massless (Goldstone boson)

In a hidden SU(2) gauge theory: $\Phi = \langle\Phi\rangle + \Phi'$

$$\begin{pmatrix} \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \\ \frac{\varphi_3 + i\varphi_4}{\sqrt{2}} \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} \frac{\varphi'_1 + i\varphi'_2}{\sqrt{2}} \\ \frac{\varphi'_3 + i\varphi'_4}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{\varphi'_1 + i\varphi'_2}{\sqrt{2}} \\ \frac{(\varphi'_3 + v) + i\varphi'_4}{\sqrt{2}} \end{pmatrix}$$

When substituted into the potential, this leads to a correct-sign mass for φ'_3 (massive scalar) and keeps $\varphi'_{1,2,4}$ massless (Goldstone bosons)

We now have to worry about <u>three</u> Goldstone bosons

The Higgs mechanism works here too...

Exactly as before: parametrise $\Phi(x) = \; e^{i\vec{\xi}(x).\vec{\mathbb{T}}}\begin{pmatrix} 0 \\ \eta(x) \end{pmatrix}$ (polar form)

Consider the unbroken (i.e. gauge invariant) Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}\right] + \left(\mathbb{D}^{\mu}\Phi\right)^{\dagger}\mathbb{D}_{\mu}\Phi - V(\Phi)$$

where $V(\varphi) = -M^2\Phi^{\dagger}\Phi + \lambda\left(\Phi^{\dagger}\Phi\right)^2$

At this level, we are free to make any gauge choice we wish...

Make a gauge transformation

$$\Phi(x) \to U(x)\Phi(x) = e^{-ig\vec{\theta}(x).\vec{\mathbb{T}}}\Phi(x) = e^{i\left[g\vec{\theta}(x)-\vec{\xi}(x)\right].\vec{\mathbb{T}}}\begin{pmatrix} 0 \\ \eta(x) \end{pmatrix}$$

We might as well choose a special gauge, since the gauge symmetry is going to be broken anyway...

Choose the three gauge functions $\vec{\theta}(x)$ such that

$$g\vec{\theta}(x) - \vec{\xi}(x) = \vec{0}$$

This is called the unitary gauge.

In this gauge, $\Phi(x) = \Phi_\eta(x) = \begin{pmatrix} 0 \\ \eta(x) \end{pmatrix}$ and the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}] + \left(\mathbb{D}^\mu\Phi_\eta\right)^\dagger\mathbb{D}_\mu\Phi_\eta - V(\eta)$$

where $V(\eta) = -M^2\eta^2 + \lambda\eta^4$

The ground state is still at $v/\sqrt{2}$ so we must shift

$$\eta = \frac{v}{\sqrt{2}} + \eta'$$

This will lead to

1. $\mathcal{L}_{mass} = M_W^2 W_\mu^+ W^{\mu-} + \frac{1}{2} M_W^2 W_\mu^0 W^{\mu 0}$ with $M_W = \frac{1}{2} g v$

2. $V\left(\frac{v}{\sqrt{2}} + \eta'\right) = +\frac{1}{2} 4M^2\eta^2 + \cdots$ i.e. $M_\eta = 2M$

3. and there are no Goldstone bosons...

if we had kept the $\vec{\xi}(x)$ they would have been the Goldstone bosons

These three degrees of freedom reappear in the longitudinal polarisations of the three W^+, W^- and W^0.

This will lead to

1. $\mathcal{L}_{mass} = M_W^2 W_\mu^+ W^{\mu-} + \frac{1}{2} M_W^2 W_\mu^0 W^{\mu 0}$ with $M_W = \frac{1}{2} g v$

2. $V\left(\frac{v}{\sqrt{2}} + \eta'\right) = +\frac{1}{2} 4 M^2 \eta^2 + \cdots$ i.e. $M_\eta = 2M$

3. and there are no Goldstone bosons...

if we had kept the $\vec{\xi}(x)$ they would have been the Goldstone bosons

These three degrees of freedom reappear in the longitudinal polarisations of the three W^+, W^- and W^0.



The gauge field matrix expands to

$$\mathbb{A}_\mu = A_{\mu 1}\mathbb{T}_1 + A_{\mu 2}\mathbb{T}_2 + A_{\mu 3}\mathbb{T}_3$$

Now,

$$W_\mu^+ = \frac{A_{\mu 1} - iA_{\mu 2}}{\sqrt{2}} \qquad W_\mu^- = \frac{A_{\mu 1} + iA_{\mu 2}}{\sqrt{2}} \qquad W_\mu^0 = A_{\mu 3}$$

$$\Rightarrow \quad A_{\mu 1} = \frac{1}{\sqrt{2}}\left(W_\mu^+ + W_\mu^-\right) \qquad A_{\mu 2} = \frac{i}{\sqrt{2}}\left(W_\mu^+ - W_\mu^-\right) \qquad A_{\mu 3} = W_\mu^0$$

i.e.

$$\begin{aligned}\mathbb{A}_\mu &= \frac{1}{\sqrt{2}}\left(W_\mu^+ + W_\mu^-\right)\mathbb{T}_1 + \frac{i}{\sqrt{2}}\left(W_\mu^+ - W_\mu^-\right)\mathbb{T}_2 + W_\mu^0\,\mathbb{T}_3 \\ &= \frac{1}{\sqrt{2}}(\mathbb{T}_1 + i\mathbb{T}_2)W_\mu^+ + \frac{1}{\sqrt{2}}(\mathbb{T}_1 - i\mathbb{T}_2)W_\mu^- + W_\mu^0\,\mathbb{T}_3 \\ &\equiv W_\mu^+\mathbb{T}_+ + W_\mu^-\mathbb{T}_- + W_\mu^0\,\mathbb{T}_3 \quad \text{where } \mathbb{T}_\pm = \frac{1}{\sqrt{2}}(\mathbb{T}_1 \pm i\mathbb{T}_2)\end{aligned}$$

Inclusion of fermions

If fermions are to interact with the W^+, W^- and W^0 bosons, they must transform as doublets under $SU(2)_W$, just like the scalar doublet $\Phi(x)$

Consider a fermion doublet (we could do a similar thing for SU(N) ...)

$$\Psi = \begin{pmatrix} \psi_\mathrm{A} \\ \psi_\mathrm{B} \end{pmatrix}$$

where the ψ_A and ψ_B are two mass-degenerate Dirac fermions.

Construct the ‘free’ Lagrangian density

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu \partial_\mu \Psi - m\,\overline{\Psi}\Psi$$

where $\overline{\Psi} = (\bar{\psi}_A \quad \bar{\psi}_B)$.

Sum of two free Dirac fermion Lagrangian densities, with equal masses.

Now, under a global $SU(2)_W$ gauge transformation, if

$$\Psi(x) \to \Psi'(x) = \mathbb{U}\Psi(x)$$

then

$$\bar{\Psi}(x) \to \bar{\Psi}'(x) = \bar{\Psi}(x)\mathbb{U}^\dagger$$

It follows that the Lagrangian density

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m\overline{\Psi}\Psi$$

must be invariant under global $SU(2)_W$ gauge transformations.

As before, we try to upgrade this to a local $SU(2)_W$ gauge invariance, by writing

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu \mathbb{D}_\mu \Psi - m\overline{\Psi}\Psi - \frac{1}{2}\mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\,\mathbb{F}^{\mu\nu}\right]$$

where $\mathbb{D}_\mu = \mathbb{1}\partial_\mu + ig\mathbb{A}_\mu(x)$ as before. Invariance is now guaranteed.

Expand the covariant derivate and get the full Lagrangian density

$$\mathcal{L} = \underbrace{i\bar{\Psi}\partial_\mu\Psi - m\overline{\Psi}\Psi}_{\text{free fermion}} \underbrace{-\tfrac{1}{2}\mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}\right]}_{\text{'free' gauge}} - \underbrace{g\bar{\Psi}\gamma^\mu\mathbb{A}_\mu\Psi}_{\text{interaction term}}$$

Expand the interaction term...

$$\begin{aligned}\mathcal{L}_{\text{int}} &= -g\bar{\Psi}\gamma^\mu\mathbb{A}_\mu\Psi \\ &= -g\bar{\Psi}\gamma^\mu\left(W_\mu^+\mathbb{T}_+ + W_\mu^-\mathbb{T}_- + W_\mu^0\,\mathbb{T}_3\right)\Psi \\ &= -g\bar{\Psi}\gamma^\mu\mathbb{T}_+\Psi\, W_\mu^+ - g\bar{\Psi}\gamma^\mu\mathbb{T}_-\Psi\, W_\mu^- - g\bar{\Psi}\gamma^\mu\mathbb{T}_3\Psi\, W_\mu^0 \\ &\equiv -gj_+^\mu\, W_\mu^+ - gj_-^\mu\, W_\mu^- - gj_0^\mu\, W_\mu^0\end{aligned}$$

$j_\pm^\mu = \bar{\Psi}\gamma^\mu\mathbb{T}_\pm\Psi$ are 'charged' currents

$j_0^\mu = \bar{\Psi}\gamma^\mu\mathbb{T}_3\Psi$ is a 'neutral' current

Write the currents explicitly:

- $j_+^\mu = \bar{\Psi}\gamma^\mu \mathbb{T}_+ \Psi = \bar{\Psi}\gamma^\mu \frac{1}{\sqrt{2}}(\mathbb{T}_1 + i\mathbb{T}_2)\Psi$

$$= \frac{1}{\sqrt{2}}(\bar{\psi}_A \quad \bar{\psi}_B)\gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_\mathrm{A} \\ \psi_\mathrm{B} \end{pmatrix} = \frac{1}{\sqrt{2}}\bar{\psi}_A\gamma^\mu\psi_\mathrm{B}$$

- $j_-^\mu = \bar{\Psi}\gamma^\mu \mathbb{T}_- \Psi = \bar{\Psi}\gamma^\mu \frac{1}{\sqrt{2}}(\mathbb{T}_1 - i\mathbb{T}_2)\Psi$

$$= \frac{1}{\sqrt{2}}(\bar{\psi}_A \quad \bar{\psi}_B)\gamma^\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi_\mathrm{A} \\ \psi_\mathrm{B} \end{pmatrix} = \frac{1}{\sqrt{2}}\bar{\psi}_B\gamma^\mu\psi_\mathrm{A}$$

- $j_0^\mu = \bar{\Psi}\gamma^\mu \mathbb{T}_3 \Psi$

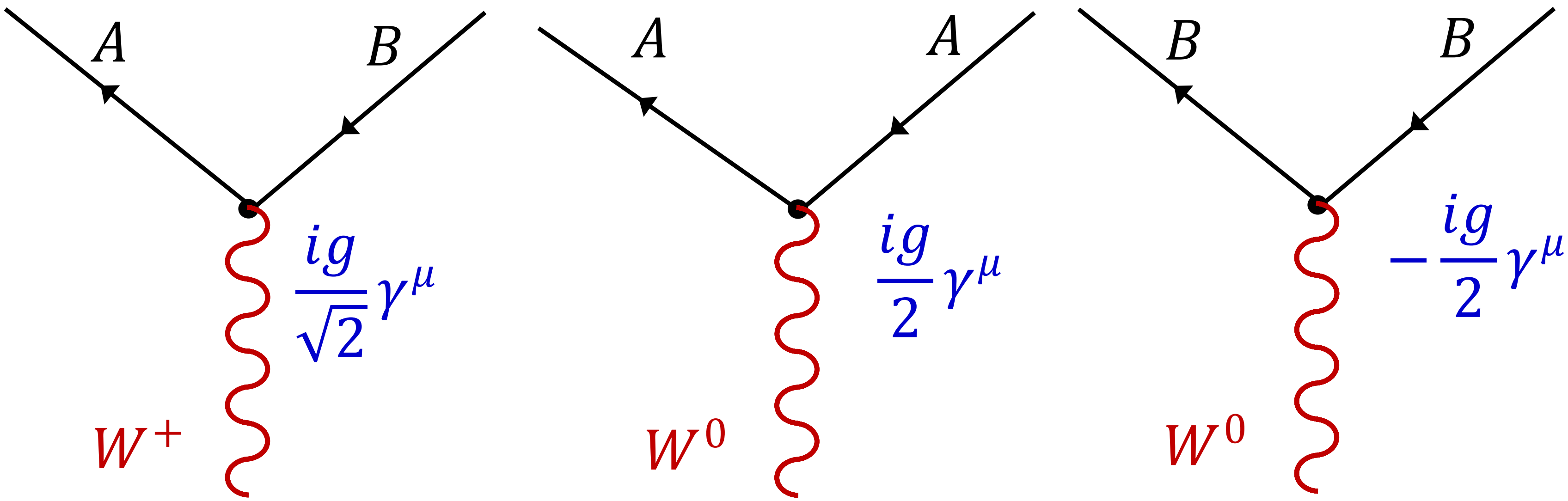
$$= \frac{1}{2}(\bar{\psi}_A \quad \bar{\psi}_B)\gamma^\mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \psi_\mathrm{A} \\ \psi_\mathrm{B} \end{pmatrix} = \frac{1}{2}(\bar{\psi}_A\gamma^\mu\psi_\mathrm{A} - \bar{\psi}_B\gamma^\mu\psi_\mathrm{B})$$

$$\mathcal{L}_{\text{int}} = -gj_{+}^{\mu} W_{\mu}^{+} - gj_{-}^{\mu} W_{\mu}^{-} - gj_{0}^{\mu} W_{\mu}^{0}$$

$$= -\frac{g}{\sqrt{2}}\bar{\psi}_A\gamma^{\mu}\psi_{\mathrm{B}} W_{\mu}^{+} - \frac{g}{\sqrt{2}}\bar{\psi}_B\gamma^{\mu}\psi_{\mathrm{A}} W_{\mu}^{-}$$ c.c. interactions

$$-\frac{g}{2}(\bar{\psi}_A\gamma^{\mu}\psi_{\mathrm{A}} - \bar{\psi}_B\gamma^{\mu}\psi_{\mathrm{B}}) W_{\mu}^{0}$$ n.c. interactions

This leads to vertices



Comparing with the IVB hypothesis for the $W_\mu^\pm$, we should be able to identify

$$\begin{pmatrix}\psi_\mathrm{A}\\ \psi_\mathrm{B}\end{pmatrix} = \begin{pmatrix}p\\ n\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}\psi_\mathrm{A}\\ \psi_\mathrm{B}\end{pmatrix} = \begin{pmatrix}\nu_e\\ e\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}\psi_\mathrm{A}\\ \psi_\mathrm{B}\end{pmatrix} = \begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix}$$

Q. Can we identify the W_μ^0 with the photon (forgetting the mass)?

If the $W_\mu^\pm$ are charged, we will have, under $\mathrm{U(1)_{em}}$

$$W_\mu^+ \to W_\mu^{'+} = e^{-ie\theta}\, W_\mu^+ \qquad\qquad W_\mu^- \to W_\mu^{'-} = e^{+ie\theta}\, W_\mu^-$$

Now, if the term $\bar{\psi}_A \gamma^\mu \psi_\mathrm{B}\, W_\mu^+$ is to remain invariant, we must assign charges $q_A e$ and $q_B e$ to the A and B, s.t. the term transforms as

$$\bar{\psi}_A \gamma^\mu \psi_\mathrm{B}\, W_\mu^+ \to e^{-ie\theta + iq_A e\theta - iq_B e\theta}\, \bar{\psi}_A \gamma^\mu \psi_\mathrm{B}\, W_\mu^+$$

To keep the Lagrangian neutral, we require $q_A - q_B = -1$

But if we look at the W_μ^0 vertices, and consider them to be QED vertices, we must identify

$$\frac{g}{2} = -q_A e \qquad \text{and} \qquad -\frac{g}{2} = -q_B e$$

i.e. $q_A = -q_B$.

Now solve the equations: $q_A - q_B = -1$ and $q_A = -q_B$ ...

result is

$$q_A = -q_B = -\frac{1}{2}$$

Two alternatives:

- A and B cannot be the Fermi-IVB particles (defeats whole effort...)
- W_μ^0 cannot be the photon... (already hinted by the mass)

Electroweak unification

Why not just include the U(1)$_{em}$ group as a direct product with the SU(2)$_W$ group?

The transformation matrix on a fermion of charge qe will then look like

$$\mathbb{U} = e^{-ig\vec{\theta}.\vec{\mathbb{T}} - iqe\,\theta'\,\mathbb{T}'}$$

where $\mathbb{T}'$ is the generator of U(1)$_{em}$ and the direct product means that

$$[\mathbb{T}', \mathbb{T}_a] = 0 \quad \forall a$$

The gauge field matrix should expand to

$$g\mathbb{A}_\mu = gW_\mu^+\mathbb{T}_+ + gW_\mu^-\mathbb{T}_- + gW_\mu^0\,\mathbb{T}_3 + qeA_\mu\mathbb{T}'$$

and give us interaction terms as before...

i.e., to the interaction terms with the W boson we must now add interaction terms with the photon:

$$\mathcal{L}_{\text{int}} = -\frac{g}{\sqrt{2}}\bar{\psi}_A\gamma^\mu\psi_\text{B}\,W_\mu^+ - \frac{g}{\sqrt{2}}\bar{\psi}_B\gamma^\mu\psi_\text{A}\,W_\mu^-$$

$$-\frac{g}{2}\bar{\psi}_A\gamma^\mu\psi_\text{A}\,W_\mu^0 + \frac{g}{2}\bar{\psi}_B\gamma^\mu\psi_\text{B}\,W_\mu^0$$

$$-\,q_A e\,\bar{\psi}_A\gamma^\mu\psi_\text{A}A_\mu - \,q_B e\,\bar{\psi}_B\gamma^\mu\psi_\text{B}A_\mu$$

Working back, we can write this as

$$\mathcal{L}_{\text{int}} = -(\bar{\psi}_A \quad \bar{\psi}_B)\gamma^\mu \begin{pmatrix} \frac{g}{2}\,W_\mu^0 + q_A e A_\mu & \frac{g}{\sqrt{2}}\,W_\mu^+ \\ \frac{g}{\sqrt{2}}\,W_\mu^- & -\frac{g}{2}\,W_\mu^0 + q_B e A_\mu \end{pmatrix}\begin{pmatrix}\psi_\text{A} \\ \psi_\text{B}\end{pmatrix}$$

$$= -\overline{\Psi}\left(g\overrightarrow{A^\mu}\,.\,\vec{\mathbb{T}} + eA_\mu\mathbb{T}'\right)\Psi \quad \text{where} \quad \mathbb{T}' = \begin{pmatrix} q_A & 0 \\ 0 & q_B \end{pmatrix}$$

This generator of $U(1)_{em}$ can be rewritten

$$\mathbb{T}' = \begin{pmatrix} q_A & 0 \\ 0 & q_B \end{pmatrix} = \frac{q_A + q_B}{2}\mathbb{1} + \frac{q_A - q_B}{2}\mathbb{T}_3$$

If we remember that $q_A - q_B = -1$, then

$$\mathbb{T}' = (2q_A + 1)\mathbb{1} - \frac{1}{2}\mathbb{T}_3$$

Paradox!

$$[\mathbb{T}', \mathbb{T}_a] \neq 0 \quad \text{for} \quad a = 1,2$$

This generator of $U(1)_{em}$ can be rewritten

$$\mathbb{T}' = \begin{pmatrix} q_A & 0 \\ 0 & q_B \end{pmatrix} = \frac{q_A + q_B}{2} \mathbb{1} + \frac{q_A - q_B}{2} \mathbb{T}_3$$

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Paradox!

$$[\mathbb{T}', \mathbb{T}_a] \neq 0 \quad \text{for} \quad a = 1,2$$

Glashow (1961) :

We cannot treat weak interactions and electromagnetism as separate (direct product) gauge theories $\Rightarrow$ electroweak unification

SU(2)$_W$xU(1)$_Y$ model

Introduce a new U(1)$_y$ which is different from U(1)$_{em}$ and exists as a direct product with the SU(2)$_W$...

The gauge transformation matrix will become

$$\mathbb{U} = e^{-ig\vec{\theta}.\vec{\mathbb{T}}+ig'\theta'\mathbb{T}'}$$

where $\mathbb{T}' = \frac{y}{2}\mathbb{1}$, which, by construction, will commute with all the $\vec{\mathbb{T}}$

We now expand the gauge field matrix as

$$g\mathbb{A}_\mu = gW_\mu^+\mathbb{T}_+ + gW_\mu^-\mathbb{T}_- + gW_\mu^0\,\mathbb{T}_3 - g'B_\mu\mathbb{T}'$$

B_μ is a new gauge field and y is a new quantum number which is clearly same for both the A and B component of the fermion doublet.

We now construct the gauge-fermion interaction term as before

$$\mathcal{L}_{\text{int}} = -g\bar{\Psi}\gamma^{\mu}\mathbb{A}_{\mu}\Psi$$

$$= -\bar{\Psi}\gamma^{\mu}\left(gW_{\mu}^{+}\mathbb{T}_{+} + gW_{\mu}^{-}\mathbb{T}_{-} + gW_{\mu}^{0}\,\mathbb{T}_{3} - g'B_{\mu}\mathbb{T}'\right)\Psi$$

Expanding as before

$$\mathcal{L}_{\text{int}} = -(\bar{\psi}_A \quad \bar{\psi}_B)\gamma^{\mu}\begin{pmatrix} \frac{g}{2}\,W_{\mu}^{0} - \frac{g'y}{2}B_{\mu} & \frac{g}{\sqrt{2}}\,W_{\mu}^{+} \\ \frac{g}{\sqrt{2}}\,W_{\mu}^{-} & -\frac{g}{2}\,W_{\mu}^{0} - \frac{g'y}{2}B_{\mu} \end{pmatrix}\begin{pmatrix}\psi_{\text{A}} \\ \psi_{\text{B}}\end{pmatrix}$$

$$= -\frac{g}{\sqrt{2}}\bar{\psi}_A\gamma^{\mu}\psi_{\text{B}}\,W_{\mu}^{+} - \frac{g}{\sqrt{2}}\bar{\psi}_B\gamma^{\mu}\psi_{\text{A}}\,W_{\mu}^{-}$$

$$-\bar{\psi}_A\gamma^{\mu}\psi_{\text{A}}\left(\frac{g}{2}\,W_{\mu}^{0} - \frac{g'y}{2}B_{\mu}\right) + \bar{\psi}_B\gamma^{\mu}\psi_{\text{B}}\left(\frac{g}{2}\,W_{\mu}^{0} + \frac{g'y}{2}B_{\mu}\right)$$

Glashow (1961): *for some reason*, the W_μ^0 and B_μ mix, i.e. the physical states are orthonormal combinations (demanded by gauge kinetic terms) of the W_μ^0 and B_μ ...

$$\begin{pmatrix} W_\mu^0 \\ B_\mu \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \qquad c = \cos\omega, \; s = \sin\omega$$

In terms of this, the neutral current terms come out to be

$$\mathcal{L}_{\text{nc}} = -\bar{\psi}_A \gamma^\mu \psi_{\text{A}} \left(\frac{g}{2} W_\mu^0 - \frac{g' y}{2} B_\mu \right) + \bar{\psi}_B \gamma^\mu \psi_{\text{B}} \left(\frac{g}{2} W_\mu^0 + \frac{g' y}{2} B_\mu \right)$$

$$= -\frac{1}{2} \bar{\psi}_A \gamma^\mu \psi_{\text{A}} \left[(gc - g'ys) Z_\mu - (gs + g'yc) A_\mu \right]$$

$$-\frac{1}{2} \bar{\psi}_B \gamma^\mu \psi_{\text{B}} \left[(gc - g'ys) Z_\mu + (gs - g'yc) A_\mu \right]$$

If we now wish to identify A_μ with the photon, we require to set

$$-\frac{1}{2}(gs + g'yc) = q_A e \qquad \frac{1}{2}(gs - g'yc) = q_B e$$

Solving for g and g' we get

$$-gs = (q_A - q_B)e \qquad -g'yc = (q_A + q_B)e$$

Recall that $q_A - q_B = -1$. It follows that

$$e = gs \qquad e = -g'c\frac{y}{q_A + q_B}$$

Choose $-y = q_A + q_B$. Then

$$e = g\sin\omega \qquad g' = g\tan\omega$$

Note that ω is some arbitrary angle... it must be nonzero, else $e = 0$

We can also obtain

$$q_A = \frac{1}{2} + \frac{y}{2} \qquad\qquad q_B = -\frac{1}{2} + \frac{y}{2}$$

Now, these $\pm\frac{1}{2}$ are precisely the eigenvalues of the $\mathbb{T}_3$ operator

i.e. we can write a general relation

$$q = t_3 + \frac{y}{2}$$

Looks exactly like the Gell-Mann-Nishijima relation...

Call t_3 the weak isospin and y the weak hypercharge

This gauge theory works pretty well and can give the correct couplings of all the gauge bosons... up to the angle ω, which is not determined by the fermion sector...

Determination of ω :

Back to the gauge boson mass term...

$$\mathcal{L}_{\text{mass}} = g^2(\mathbb{A}^\mu\langle\Phi\rangle)^\dagger(\mathbb{A}_\mu\langle\Phi\rangle) = (g\mathbb{A}^\mu\langle\Phi\rangle)^\dagger(g\mathbb{A}_\mu\langle\Phi\rangle)$$

For the Glashow theory, we must include the U(1)$_y$ field in the gauge field matrix, i.e.

$$g\mathbb{A}_\mu = gW_\mu^+\mathbb{T}_+ + gW_\mu^-\mathbb{T}_- + gW_\mu^0\,\mathbb{T}_3 - g'B_\mu\mathbb{T}'$$

$$= \begin{pmatrix} \frac{g}{2}W_\mu^0 - \frac{g'Y}{2}B_\mu & \frac{g}{\sqrt{2}}W_\mu^+ \\ \frac{g}{\sqrt{2}}W_\mu^- & -\frac{g}{2}W_\mu^0 - \frac{g'Y}{2}B_\mu \end{pmatrix}$$

where Y is the hypercharge of the Φ field.

Thus,

$$g\mathbb{A}_\mu\langle\Phi\rangle = \begin{pmatrix} \frac{g}{2}\,W_\mu^0 - \frac{g'Y}{2}B_\mu & \frac{g}{\sqrt{2}}\,W_\mu^+ \\ \frac{g}{\sqrt{2}}\,W_\mu^- & -\frac{g}{2}\,W_\mu^0 - \frac{g'Y}{2}B_\mu \end{pmatrix}\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{gv}{2}\,W_\mu^+ \\ -\frac{gv}{2\sqrt{2}}\left(g\,W_\mu^0 + g'YB_\mu\right) \end{pmatrix}$$

and

$$(g\mathbb{A}^\mu\langle\Phi\rangle)^\dagger = \begin{pmatrix} \frac{gv}{2}W^{\mu -} & -\frac{gv}{2\sqrt{2}}\left(g\,W^{\mu 0} + g'YB^\mu\right) \end{pmatrix}$$

Multiplying these

$$\mathcal{L}_{\text{mass}} = \left(\frac{gv}{2}\right)^2 W_\mu^+ W^{\mu -} + \left(\frac{v}{2\sqrt{2}}\right)^2 (g\, W^{\mu 0} + g^{'} Y B^\mu)(g\, W_\mu^0 + g^{'} Y B_\mu)$$

Consider only the neutral bosons:

$$(g\, W^{\mu 0} + g^{'} Y B^\mu)(g\, W_\mu^0 + g^{'} Y B_\mu)$$

$$= g^2\, W^{\mu 0}\, W_\mu^0 + g g^{'} Y\, W^{\mu 0} B_\mu + g g^{'} Y B^\mu\, W_\mu^0 + (g^{'} Y)^2 B^\mu B_\mu$$

One cannot have mass terms of the form $W^{\mu 0} B_\mu$ and $B^\mu\, W_\mu^0$ in a viable field theory, since our starting point is always a theory with free fields.

Thus, it is essential to transform to orthogonal states

$$\begin{pmatrix} W_\mu^0 \\ B_\mu \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \qquad c = \cos\omega,\ \ s = \sin\omega$$

and choose ω to cancel out cross terms...

Rewrite the neutral boson mass terms as

$$(g\,W^{\mu 0} + g^{'}YB^{\mu})(g\,W_{\mu}^{0} + g^{'}YB_{\mu})$$

$$= g^2\,W^{\mu 0}\,W_{\mu}^{0} + gg^{'}Y\,W^{\mu 0}B_{\mu} + gg^{'}YB^{\mu}\,W_{\mu}^{0} + (g^{'}Y)^2B^{\mu}B_{\mu}$$

$$= (\,W^{\mu 0} \quad B^{\mu})\begin{pmatrix} g^2 & gg'Y \\ gg'Y & (g^{'}Y)^2 \end{pmatrix}\begin{pmatrix} W_{\mu}^{0} \\ B_{\mu} \end{pmatrix}$$

The diagonalising matrix will be

$$\begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$$

where

$$\tan\omega = \frac{g'Y}{g}$$

How to determine Y ?

Write out the interaction terms for the gauge bosons with the scalar doublet. One finds that once again, to match the couplings to the charges of the W bosons, we get the Gell-Mann-Nishijima relation, i.e.

$$q = t_3 + \frac{Y}{2}$$

Now, the lower component φ_B develops a vacuum expectation value, so it must be neutral, i.e.

$$0 = -\frac{1}{2} + \frac{Y}{2} \quad \Rightarrow Y = 1$$

It follows that

Weinberg angle

$$\tan\omega = \frac{g'}{g} = \tan\theta_W$$

Eigenvalues of the mass matrix:

$$\begin{pmatrix} g^2 & gg' \\ gg' & g'^2 \end{pmatrix}$$

Determinant = 0 ; trace = $g^2 + g'^2$, i.e.

$$M_A = 0$$

and

$$M_Z^2 = 2\left(\frac{v}{2\sqrt{2}}\right)^2 (g^2 + g'^2) = \left(\frac{gv}{2}\right)^2 \left(1 + \frac{g'^2}{g^2}\right) = M_W^2 (1 + \tan^2 \theta_W)$$

$$= M_W^2 \sec^2 \theta_W$$

$$\Rightarrow \qquad M_Z = \frac{M_W}{\cos \theta_W}$$

Determination of parameters:

$$\frac{e^2}{4\pi} = \alpha \approx \frac{1}{137} \qquad M_Z = \frac{M_W}{\cos\theta_W}$$

$$e = g\sin\theta_W \qquad g' = g\tan\theta_W$$

Experimental measurements show that

$$M_W \approx 80.4 \text{ GeV} \quad \text{and} \quad M_Z \approx 91.2 \text{ GeV}$$

It follows that $\cos\theta_W = M_W/M_Z \approx 0.8816 \Rightarrow \theta_W \approx 28^o.17$

We can now calculate: $e = \sqrt{4\pi\alpha} \approx 0.303$

$$g = e/\sin\theta_W \approx 0.642$$

$$g' = g\tan\theta_W \approx 0.344$$