Introduction to Particle Physics

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Lecture 2

Relativistic Quantum Mechanics

Recall how we arrived at the Schrödinger equation.

Assume a wave solution $\psi = \psi_0 e^{i(Et - \vec{p}.\vec{x})/\hbar}$

Implicit that $E = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$

Leads to operator equivalences: $E\psi = i\hbar\, \partial_t \psi$ and $\vec{p}\psi = -i\hbar\, \vec{\nabla}\psi$

Now consider the energy-momentum relation

$$\frac{1}{2m}\vec{p}^2 = E$$

Make these operators on a wavefunction ψ...

$$\frac{1}{2m}\vec{p}^2\psi = E\psi \quad \Rightarrow \quad -\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\, \partial_t \psi$$

Schrödinger equation for a free particle (not really a derivation)

<u>Relativistic case</u>:

Assume a wave function and operator equivalences as before ($\hbar = 1$)

$$E\psi = i\partial_t\psi \quad \text{and} \quad \vec{p}\psi = -i\vec{\nabla}\psi$$

Relativistic energy-momentum relation:

$$\vec{p}^2 + M^2 = E^2$$

Make these operators on a wavefunction φ...

$$(\vec{p}^2 + M^2)\varphi = E^2\varphi \quad \Rightarrow (\Box + M^2)\varphi = 0$$

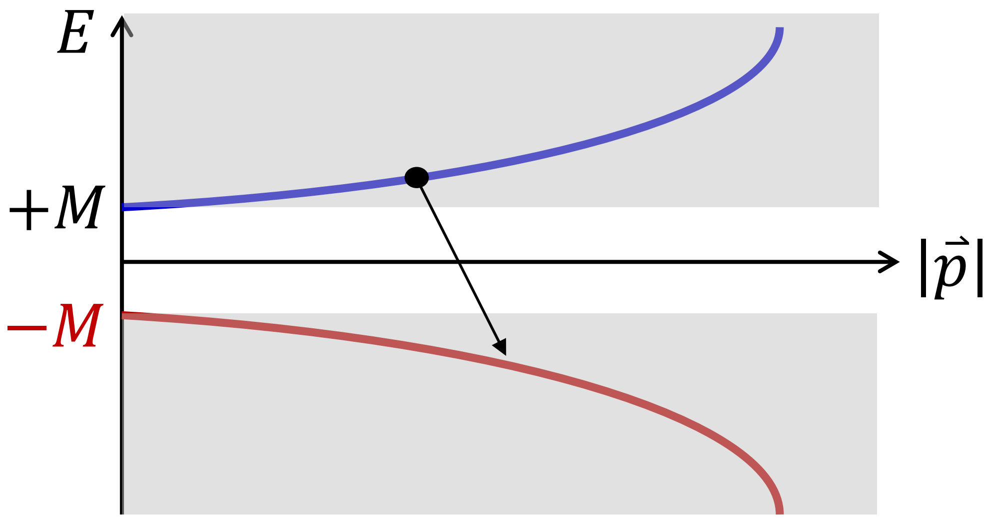
Klein-Gordon equation for a free particle (not really a derivation)

Solutions: $\varphi = \varphi_0\, e^{i(Et-\vec{p}.\vec{x})} = \varphi_0 e^{ip.x}$

$\Box = \partial_t^2 - \nabla^2$ d'Alembertian operator, $p = (E, \vec{p})$ and $x = (t, \vec{x})$

Problem of negative energy states:

$$E^2 = \vec{p}^2 + M^2 \;\Rightarrow\; E = +\sqrt{\vec{p}^2 + M^2} \text{ or } E = -\sqrt{\vec{p}^2 + M^2}$$



Classically, a positive energy particle cannot cross the *energy gap*

A quantum particle can jump to negative energies $\Rightarrow$ catastrophe

Problem of negative probability density:

$$(\Box + M^2)\varphi = 0 \quad \Rightarrow (\partial_t^2 - \nabla^2 + M^2)\varphi = 0$$

$$\Rightarrow (\nabla^2 - M^2)\varphi = \partial_t^2 \varphi$$

$$(\nabla^2 - M^2)\varphi^* = \partial_t^2 \varphi^*$$

Multiply by φ^* and φ and subtract

$$\varphi^* \nabla^2 \varphi - \varphi \nabla^2 \varphi^* = \varphi^* \partial_t^2 \varphi - \varphi \partial_t^2 \varphi^*$$

$$\Rightarrow \vec{\nabla}.\left(\varphi^* \vec{\nabla} \varphi - \varphi \vec{\nabla} \varphi^*\right) = \partial_t (\varphi^* \partial_t \varphi - \varphi \partial_t \varphi^*)$$

$$\Rightarrow \vec{\nabla}.\left(-\varphi^* \vec{\nabla} \varphi + \varphi \vec{\nabla} \varphi^*\right) + \partial_t (\varphi^* \partial_t \varphi - \varphi \partial_t \varphi^*) = 0$$

$\Rightarrow \vec{\nabla}.\vec{\mathrm{J}} + \partial_t \rho = 0$ equation of continuity

probability density: $\rho = -\frac{i}{2}(\varphi^* \partial_t \varphi - \varphi \partial_t \varphi^*)$

Q. Is the probability density guaranteed to be positive definite?

$$\rho = -\frac{i}{2}(\varphi^*\partial_t\varphi - \varphi\partial_t\varphi^*)$$

Take $\varphi = \varphi_0\, e^{i(Et-\vec{p}.\vec{x})}$, then $\rho = E\varphi^*\varphi$

which is positive for $E > 0$ and negative for $E < 0$

We cannot interpret negative probability physically, just as one cannot interpret negative energy (for a free particle) physically...

$\Rightarrow$ something must be wrong somewhere!

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$\Rightarrow$ something must be wrong somewhere!

- The Klein-Gordon equation is wrong; must look for a new equation (Dirac 1928) $\Rightarrow$ led to the Dirac equation
- We are interpreting the Klein-Gordon equation in the wrong way (Pauli & Weisskopf 1933) $\Rightarrow$ led to quantum field theory

<u>Dirac approach</u>:

Go back to the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\,\partial_t\psi \qquad\qquad -\frac{\hbar^2}{2m}\nabla^2\psi^* = -i\hbar\,\partial_t\psi^*$$

Multiply by ψ^* and ψ and subtract

$$-\frac{\hbar^2}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) = i\hbar\,(\psi^*\partial_t\psi + \psi\partial_t\psi^*)$$

$$\Rightarrow \frac{i\hbar}{2m}\,\vec{\nabla}.\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) = \partial_t(\psi^*\psi)$$

$$\Rightarrow \vec{\nabla}.\vec{\mathrm{J}} + \partial_t\rho = 0$$ equation of continuity

probability density: $\rho = \psi^*\psi \;\geq 0$

Problem arose because we had $E^2 \to -\partial_t^2$ instead of $E \to i\partial_t$

Let us write a linear energy-momentum relation in the form

$$E = \vec{\alpha}.\vec{p} + \beta M$$

where $\vec{\alpha}$ and β are constants such that $E^2 = \vec{p}^2 + M^2$ is regained.

This requires the $\vec{\alpha}$ and β to satisfy the following relations:

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = \mathbb{1}$$

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \quad \forall\, i, j = 1,3$$

$$\alpha_i \beta + \beta \alpha_i = 0 \quad \forall\, i = 1,3$$

Clearly they cannot be real/complex numbers...

Because they do not commute, it is natural to try some matrices

Dirac-Pauli representation: 4x4 matrices (smallest)

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Wave function must be a column vector of length 4 , i.e. $\Psi(x) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$

Thus,

$$(\vec{\alpha}.\vec{p} + \beta M)\Psi = E\Psi$$

Putting back the operator equivalences

$$\left(-i\vec{\alpha}.\vec{\nabla} + \beta M\right)\Psi(x) = i\partial_t \Psi(x)$$

This is known as the *Dirac equation* for a free particle.

Like the Schrödinger equation, this is a fundamental equation.

Dirac Hamiltonan is $H = -i\vec{\alpha}.\vec{\nabla} + \beta M \;\Rightarrow\; \vec{\alpha}$ and β must be Hermitian.

Check the probability density:

$$\Psi^\dagger \times [\qquad -i\vec{\alpha}.\vec{\nabla}\Psi + M\beta\Psi = i\partial_t\Psi$$

$$i\,\vec{\nabla}\Psi^\dagger.\vec{\alpha} + M\Psi^\dagger\beta = -i\partial_t\Psi^\dagger \qquad] \times \Psi$$

Multiply by $\Psi^\dagger$ and Ψ and subtract

$$-i\left(\Psi^\dagger\vec{\alpha}.\vec{\nabla}\Psi + \vec{\nabla}\Psi^\dagger.\vec{\alpha}\Psi\right) = i\left(\Psi^\dagger\partial_t\Psi + \partial_t\Psi^\dagger\Psi\right)$$

$$\Rightarrow \vec{\nabla}.\left(\Psi^\dagger\vec{\alpha}\,\Psi\right) + \partial_t\left(\Psi^\dagger\Psi\right) = 0$$

$\Rightarrow \vec{\nabla}.\vec{J} + \partial_t\rho = 0$ equation of continuity

probability density: $\rho = \Psi^\dagger\Psi \;\geq 0$

Dirac succeeded in solving the probability problem...

...but what about the negative energy states?

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Try to solve the Dirac equation: $\Psi(x) = u(\vec{p})e^{i(\omega t - \vec{k}.\vec{x})}$

Substitute in the Dirac equation: $\left(-i\vec{\alpha}.\vec{\nabla} + \beta M\right)\Psi = i\partial_t \Psi$

Leads to the equation: $\left(-\vec{\alpha}.\vec{k} + \beta M\right)u(\vec{p}) = -\omega u(\vec{p})$

If $u = \binom{\varphi}{\chi}$, this equation becomes

$$-\begin{pmatrix} 0 & \vec{\sigma}.\vec{k} \\ \vec{\sigma}.\vec{k} & 0 \end{pmatrix}\binom{\varphi}{\chi} + M\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\binom{\varphi}{\chi} = \omega\binom{\varphi}{\chi}$$

i.e. $-\vec{\sigma}.\vec{k}\chi + M\varphi = \omega\varphi \;\Rightarrow -\vec{\sigma}.\vec{k}\chi = (\omega - M)\varphi$

$-\vec{\sigma}.\vec{k}\varphi - M\chi = \omega\chi \;\Rightarrow -\vec{\sigma}.\vec{k}\varphi = (\omega + M)\chi$

$$\vec{\sigma}.\vec{a}\;\vec{\sigma}.\vec{b} = \vec{a}.\vec{b} + i\vec{a}\times\vec{b}.\vec{\sigma}$$

For consistency:

$$\vec{\sigma}.\vec{k}\,\chi = -(\omega - M)\varphi \qquad\qquad \vec{\sigma}.\vec{k}\varphi = -(\omega + M)\chi$$

Multiply from the left by $\vec{\sigma}.\vec{k}$:

$$\vec{\sigma}.\vec{k}\;\vec{\sigma}.\vec{k}\chi = -(\omega - M)\vec{\sigma}.\vec{k}\varphi \qquad\qquad \vec{\sigma}.\vec{k}\;\vec{\sigma}.\vec{k}\varphi = -(\omega + M)\vec{\sigma}.\vec{k}\chi$$

$$\vec{k}^2\chi = (\omega - M)(\omega + M)\chi \qquad\qquad \vec{k}^2\varphi = (\omega + M)(\omega - M)\varphi$$

i.e. $\vec{k}^2 = (\omega - M)(\omega + M) = \omega^2 - M^2$

or, $\omega^2 = \vec{k}^2 + M^2$

i.e. $\omega = \pm\sqrt{\vec{k}^2 + M^2}$ recall $\hbar = 1$

i.e. $E = \pm\sqrt{\vec{p}^2 + M^2}$ $\Rightarrow$ problem of negative energies persists

Again Dirac found an unconventional solution to the problem...

...by the discovery that the Dirac equation describes spin-½ particles

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Dirac Hamiltonan: $H = i\vec{\alpha}.\vec{p} + \beta M$

If we consider the angular momentum operator $\vec{L}$ then

$$\left[H, \vec{L}\right] = \vec{\alpha} \times \vec{p}$$

How can a free particle violate angular momentum conservation?

Only if $\vec{L}$ is not the total angular momentum, but there is some other component...

Take the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ such that $\left[H, \vec{J}\right] = 0$

Then

$$\left[H, \vec{S}\right] = -\vec{\alpha} \times \vec{p}$$

We need to construct an operator $\vec{S}$ such that: $[H, \vec{S}] = -\vec{\alpha} \times \vec{p}$

Easily done: take $\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ (eigenvalues $\pm$½ $\Rightarrow$ spin)

Obviously, the actual conserved quantity is $\vec{S}.\vec{p}$, for

$$[H, \vec{S}.\vec{p}] = -(\vec{\alpha} \times \vec{p}).\vec{p} = 0$$

Traditionally, we define the *helicity* as $\eta = \frac{2\vec{S}.\vec{p}}{|\vec{p}|} = \vec{\Sigma}.\hat{p}$

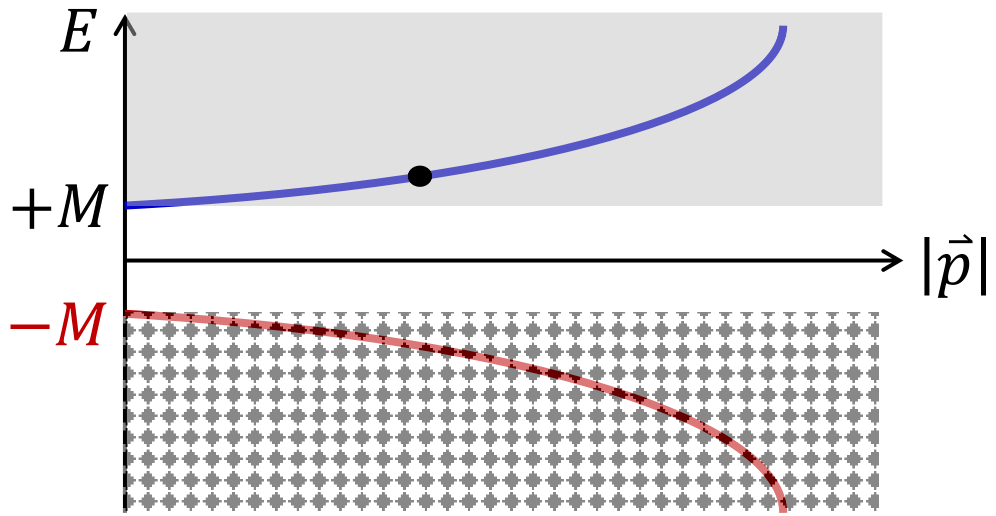
This has eigenvalues ± 1. Projection of spin along the motion.

The <u>four</u> solutions of the Dirac equation can then be classified as

$$E = +\sqrt{\vec{p}^2 + M^2}\,;\ \ \eta = +1 \quad ; \quad E = +\sqrt{\vec{p}^2 + M^2}\,;\ \ \eta = -1$$

$$E = -\sqrt{\vec{p}^2 + M^2}\,;\ \ \eta = +1 \quad ; \quad E = -\sqrt{\vec{p}^2 + M^2}\,;\ \ \eta = -1$$

Back to the problem of negative energies:

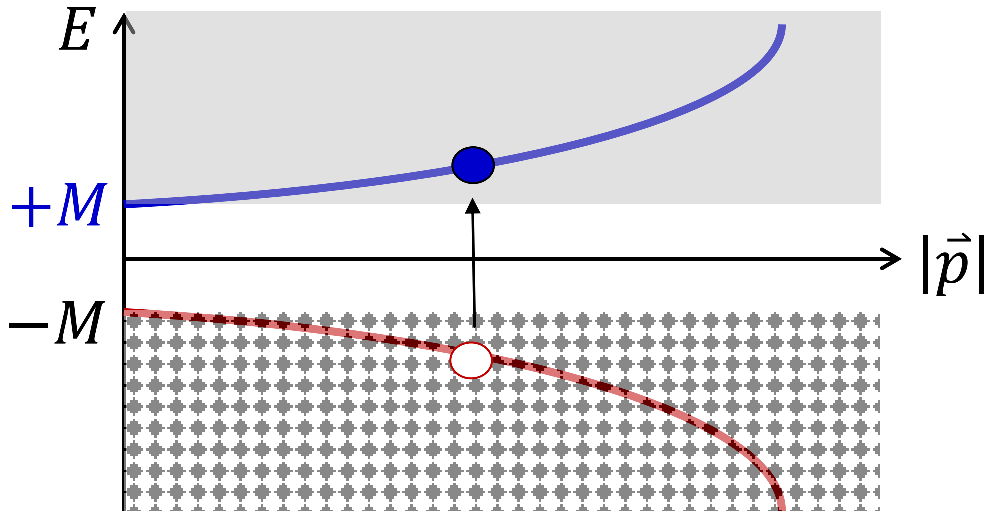


Dirac sea hypothesis:

Dirac suggested that the negative energy states are already occupied by Dirac particles, which are *invisible when in a negative energy state*.

Spin-½ particles are fermions, so transitions from positive to negative energy states are not permitted by Fermi-Dirac statistics.

But negative energy particles can be knocked out of their negative energy states into positive energy states, leaving a ‘hole’ behind....



This ‘hole’ will appear as an *antiparticle* i.e. same mass and spin, but opposite charge (to keep the Universe neutral)

Thus, Dirac predicted (1928) the existence of the *positron*....

The positron was discovered by Anderson in cosmic ray showers...

Same e/m but opposite sign – exactly as predicted by Dirac...

Is the Dirac equation covariant under Lorentz transformations?

$$(-\vec{\alpha}.\vec{p} + \beta M)u(\vec{p}) = -Eu(\vec{p})$$

$$\Rightarrow (E\mathbb{1} - \vec{\alpha}.\vec{p} + \beta M)u(\vec{p}) = 0$$

multiply from the left by β

$$(E\beta - \beta\vec{\alpha}.\vec{p} + M\mathbb{1})u(\vec{p}) = 0$$

Define $\beta = \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $\vec{\gamma} = \beta\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$

Then, writing four vectors $\gamma^\mu = (\gamma^0, \vec{\gamma})$ and $p^\mu = (E, \vec{p})$

$$(\gamma^\mu p_\mu + M\mathbb{1})u(\vec{p}) = 0$$

If we go back to ordinary spacetime, we get the ‘covariant’ form

$$(i\gamma^\mu \partial_\mu - M\mathbb{1})\Psi(x) = 0$$

Under a Lorentz transformation, $x^{\mu} \rightarrow x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

and $\Psi(x) \rightarrow \Psi'(x) = M(\Lambda)\Psi(x)$,

where $M(\Lambda)$ satisfies: $M(\Lambda)^{-1}\gamma^{\mu} M(\Lambda) = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$
(spinor transformation)

The γ^{μ} obey the *Dirac algebra*, viz.

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}\mathbb{1} \qquad \forall \mu, \nu$$

We can also define $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ which satisfies

$$\gamma^{\mu}\gamma_5 = -\gamma_5\gamma^{\mu} \qquad \forall \mu$$

Define $\not{a} = \gamma^{\mu} a_{\mu}$ (Feynman slash notation)

$$\left(i\not{\partial} - M\right)\Psi = 0$$

Bilinear covariants:

We can construct the *Dirac adjoint* $\overline{\Psi(x)} = \Psi^\dagger(x)\gamma^0$

which transforms as $\overline{\Psi(x)} \to \overline{\Psi'(x)} = \overline{\Psi(x)}\, M(\Lambda)^{-1}$

It follows that we can construct the *bilinear covariants*

Scalar	S	$= \overline{\Psi(x)}\,\Psi(x)'$	$S \to S' = S$
Pseudoscalar	P	$= \overline{\Psi(x)}\,\gamma_5\Psi(x)$	$P \to P' = P \det\Lambda$
Vector	V^μ	$= \overline{\Psi(x)}\,\gamma^\mu\Psi(x)$	$V^\mu \to V'^\mu = \Lambda^\mu{}_\nu V^\nu$
Pseudovector	A^μ	$= \overline{\Psi(x)}\,\gamma^\mu\gamma_5\Psi(x)$	$A^\mu \to A'^\mu = \det\Lambda\,\Lambda^\mu{}_\nu A^\nu$
Tensor	$T^{\mu\nu}$	$= \overline{\Psi(x)}\,\sigma^{\mu\nu}\Psi(x)$	$T^{\mu\nu} \to T'^{\mu\nu} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta T^{\alpha\beta}$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^{\nu\mu}]$

The explicit solutions of the Dirac equation are written:

$E \downarrow \quad \eta \rightarrow$	+1 (rest frame)	-1 (rest frame)
+ve	$u^{(1)}(p) = \frac{1}{\sqrt{E+M}} \begin{pmatrix} E+M \\ 0 \\ p_z \\ p_+ \end{pmatrix}$	$u^{(2)}(p) = \frac{1}{\sqrt{E+M}} \begin{pmatrix} 0 \\ E+M \\ p_- \\ -p_z \end{pmatrix}$
-ve	$v^{(1)}(p) = \frac{1}{\sqrt{-E+M}} \begin{pmatrix} p_z \\ p_+ \\ -E+M \\ 0 \end{pmatrix}$	$v^{(2)}(p) = \frac{1}{\sqrt{-E+M}} \begin{pmatrix} p_- \\ -p_z \\ 0 \\ -E+M \end{pmatrix}$

$$p_{\pm} = p_x \pm i p_y$$

Orthogonality relations:

$$\overline{u^{(a)}(p)}\, u^{(b)}(p) = 2M\delta^{ab} \qquad \overline{u^{(a)}(p)}\, v^{(b)}(p) = 0$$

$$\overline{v^{(a)}(p)}\, u^{(b)}(p) = 0 \qquad \overline{v^{(a)}(p)}\, v^{(b)}(p) = -2M\delta^{ab}$$

Completeness relations:

$$\sum_{a=1}^{2} u^{(a)}(p)\, \overline{u^{(a)}(p)} = \not{p} + M$$

$$\sum_{a=1}^{2} v^{(a)}(p)\, \overline{v^{(a)}(p)} = -\not{p} + M$$

Dirac matrix identities:

$$\gamma^{\mu}\gamma_{\mu} = 4\,\mathbb{1}$$

$$\gamma^{\mu}\gamma^{\alpha}\gamma_{\mu} = -2\gamma^{\alpha}$$

$$\gamma^{\mu}\gamma^{\alpha}\gamma^{\beta}\gamma_{\mu} = 4g^{\alpha\beta}\,\mathbb{1}$$

$$\gamma^{\mu}\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma_{\mu} = -2\gamma^{\gamma}\gamma^{\beta}\gamma^{\alpha}$$

Trace identities:

$$\mathrm{Tr}[\gamma^{\mu}] = 0$$

$$\mathrm{Tr}[\gamma^{\mu}\gamma^{\nu}\ \dots] = 0 \quad \text{for any odd number of gamma matrices}$$

$$\mathrm{Tr}[\gamma^{\mu}\gamma^{\nu}] = 4g^{\mu\nu}$$

$$\mathrm{Tr}\left[\gamma^{\mu}\gamma^{\nu}\gamma_{5}\right] = 0$$

$$\mathrm{Tr}\left[\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta}\right] = 4\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}\right)$$

$$\mathrm{Tr}\left[\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta}\gamma_{5}\right] = -4i\varepsilon^{\mu\nu\alpha\beta}$$

Parity P

basically a Lorentz transformation

Under parity, $x^0 \rightarrow x'^0 = x^0$ and $\vec{x} \rightarrow \vec{x}' = -\vec{x}$

$$\Lambda \rightarrow P = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Under parity $\Psi(x) \rightarrow \Psi'(x) = M(P)\Psi(x)$

where $M(P)^{-1}\gamma^{\mu} M(P) = P^{\mu}{}_{\nu}\gamma^{\nu}$

It can then be shown that $M(P) = \gamma^0$, i.e.

$$\Psi(x^0, \vec{x}) \rightarrow \Psi'(x^0, -\vec{x}) = \gamma^0 \Psi(x^0, \vec{x})$$

We can now assign a better meaning to $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

Define two matrix operators

$$P_+ = \tfrac{1}{2}\left(\mathbb{1} + \gamma_5\right) \quad \text{and} \quad P_- = \tfrac{1}{2}\left(\mathbb{1} - \gamma_5\right)$$

and two sub-wavefunctions

$$\Psi_+ = P_+\Psi \quad \text{and} \quad \Psi_- = P_-\Psi$$

Now, note the following

$$P_+\Psi_+ = \Psi_+ \qquad P_-\Psi_- = \Psi_-$$

$$P_+\Psi_- = 0 \qquad P_-\Psi_+ = 0$$

$$\Psi = \Psi_+ + \Psi_-$$

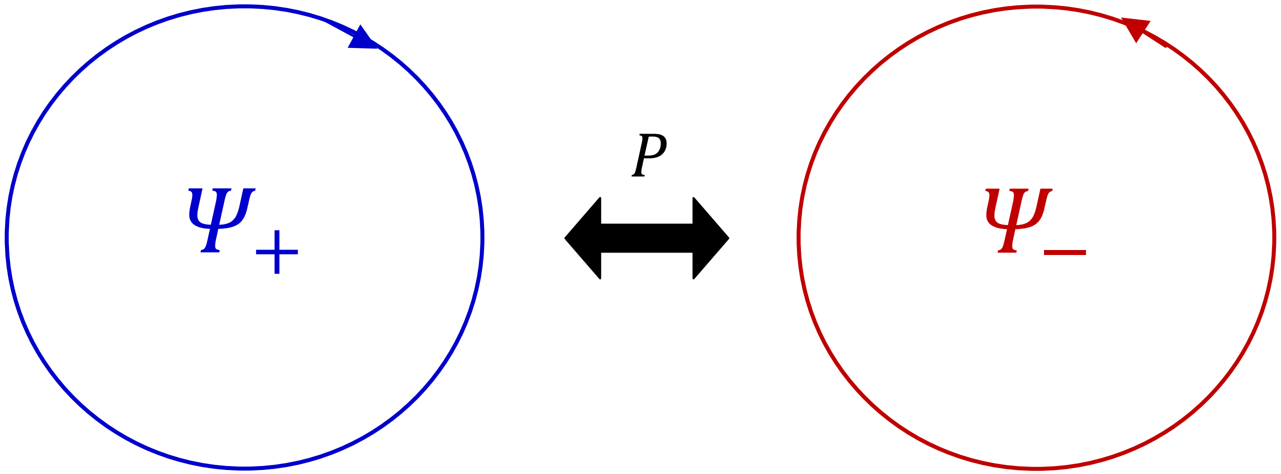
Clearly the P_+ and P_- project out two orthogonal components of Ψ

Now, under parity, $\Psi \to \Psi' = \gamma^0 \Psi$

Now, $(\Psi')_+ = P_+ \Psi' = P_+ \gamma^0 \Psi = \gamma^0 P_- \Psi = \gamma^0 \Psi_- = (\Psi_-)'$

and $(\Psi')_- = P_- \Psi' = P_- \gamma^0 \Psi = \gamma^0 P_+ \Psi = \gamma^0 \Psi_+ = (\Psi_+)'$

These states are interchanged by parity....



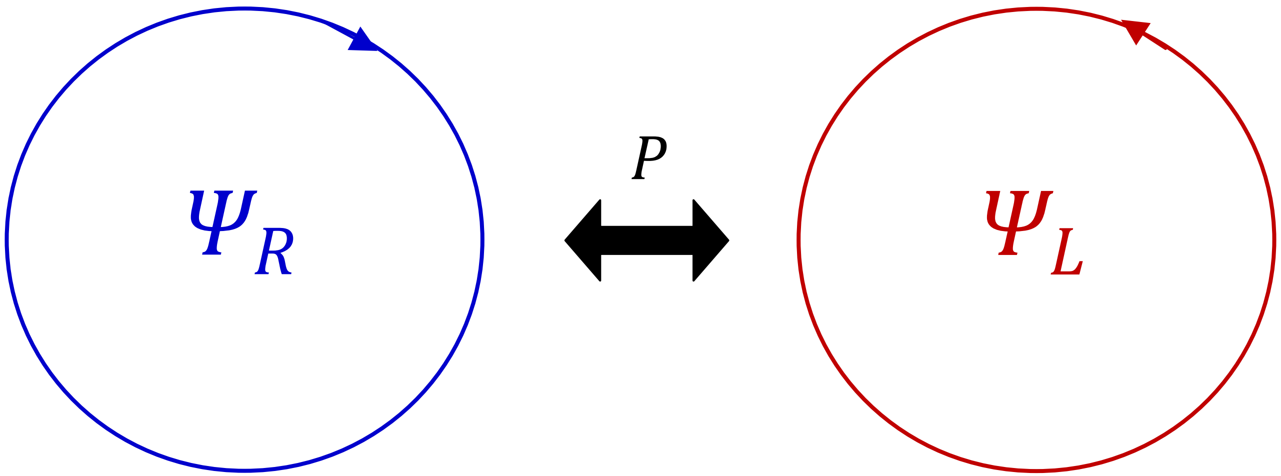
Now, under parity, $\Psi \rightarrow \Psi' = \gamma^0 \Psi$

Now, $(\Psi')_R = P_R \Psi' = P_R \gamma^0 \Psi = \gamma^0 P_L \Psi = \gamma^0 \Psi_L = (\Psi_L)'$

and $(\Psi')_L = P_L \Psi' = P_L \gamma^0 \Psi = \gamma^0 P_R \Psi = \gamma^0 \Psi_R = (\Psi_R)'$

These states are interchanged by parity....

$$P_R = \tfrac{1}{2}\left(\mathbb{1} + \gamma_5\right) \quad \text{and} \quad P_L = \tfrac{1}{2}\left(\mathbb{1} - \gamma_5\right)$$



Must correspond to left- and right-handed projections: *chirality*

Time Reversal T

again, basically a Lorentz transformation

Under time reversal, $x^0 \rightarrow x'^0 = -x^0$ and $\vec{x} \rightarrow \vec{x}' = \vec{x}$

...construct $T^{\mu}{}_{\nu}$... under time reversal $\Psi(x) \rightarrow \Psi'(x) = M(T)\Psi(x)$

where $M(T)^{-1}\gamma^{\mu}M(T) = T^{\mu}{}_{\nu}\gamma^{\nu}$

It can then be shown that $M(T) = i\gamma^1\gamma^3$.

But in quantum mechanics $H\psi = i\partial_t\psi$, so, even if $THT^{-1} = H$, the right side changes sign... i.e. we also require to change from ψ to ψ^*...

Thus, for the Dirac equation, we have

$$\Psi(x^0, \vec{x}) \rightarrow \Psi'(-x^0, \vec{x}) = i\gamma^1\gamma^3\Psi^*(x^0, \vec{x})$$

Charge Conjugation C

Not a spacetime symmetry, but an internal symmetry

Define the operator $C = i\gamma^2\gamma^0$ and the wavefunction

$$\Psi(x) \to \Psi^c(x) = C\overline{\Psi(x)}^t$$

where ^{t} stands for transpose.

If we take the Dirac equation through these changes, it remains the same, i.e.

if $\left(i\gamma^\mu \partial_\mu + M\mathbb{1}\right)\Psi(x) = 0$

then $\left(i\gamma^\mu \partial_\mu + M\mathbb{1}\right)\Psi^c(x) = 0$

Dirac equation has *charge conjugation invariance.*

However, if we have a charged Dirac particle, we replace

$$p^\mu \to p^\mu - eA^\mu$$

where $A^\mu = (\varphi, \vec{A})$ is the electromagnetic four potential.

The charge-coupled Dirac equation has the form

$$\left(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - M\mathbb{1}\right)\Psi(x) = 0$$

Under charge conjugation, it changes to

$$\left(i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu - M\mathbb{1}\right)\Psi^c(x) = 0$$

Thus, the $\Psi^c(x)$ wavefunction describes the antiparticle.

Obviously the neutral Dirac equation is invariant under C, P, CP, T, CPT

$\Rightarrow$ discrete symmetries of the Dirac equation