Hydrodynamics and Gravity

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- It has long been argued that $U(N)$ gauge theories reduce to effectively classical systems in the t’ Hooft large N limit.
- $\mathcal{N} = 4$ Yang Mills theories are $U(N)$ gauge theoeries. These theories are conformally invariant; they define a line of fixed points labeled by a continuous coupling constant λ. In 1997 Maldacena identified the corresponding large N classical systems.
- While the classical equations identified by Maldacena are unfamliar (and appear complicated) at finite λ, they simplify dramatically at large λ. In this limit they reduce to the equations of Einstein (IIB super) gravity on spacetimes that asymptote to $AdS_5 \times S^5$.

- The gravitational description of field theory dynamics is unfamiliar partly because it applies only at very strong coupling.
- Even at strong coupling, however, we have some qualitative expectations of local QFTs. For instance they are expected to equilibriate at every finite temperature.
- What is the gravitational description of this thermal state? Answer (Witten): an asymptotically *AdS* black brane.
- This answer is universal in the following sense. Every 2 derivative theory of gravity interacting with other fields of spin ≤ 2 admits a consistent truncation to Einstein's equations with a negative cosmological constant. Black brane solutions lie in this universal sector.

- In appropriately chosen units, Einstein's equations with a negative cosmological constant in $d+1$ dimensions are

$$R_{MN} - \frac{R}{2} g_{MN} = \frac{d(d-1)}{2} g_{MN} : \quad : M, N = 1 \ldots d+1$$

- The black brane at temperature T and velocity u_μ are a d parameter set of exact solutions of these equations

$$ds^2 = \frac{dr^2}{r^2 f(r)} + r^2 \mathcal{P}_{\mu\nu} dx^\mu dx^\nu - r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu$$

$$f(r) = 1 - \left(\frac{4\pi T}{d\ r} \right)^d ; \quad \mathcal{P}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

- These solutions have a horizon at $r = \frac{4\pi T}{d}$. The thermal nature of these solutions follows from well known properties of event horizons.

- Einstein's equations allow us to study deviations from thermal equilibrium. Natural first question: what is the spectrum of linearized fluctuations about thermal equilibrium?.
- If we impose the requirement of regularity of the future horizon, the anwer is given by gravitational 'quasinormal modes'. Discrete infinity of such modes labeled by integers. For the n^{th} mode $\omega = \omega_n(k)$. Frequency complex corresponding to decay.
- It follows from conformal invariance that $\omega_n(0) = \frac{f(n)}{T}$. $f(n) \neq 0$ except for the 4 Goldstone modes corresponding to variations of T and u_μ. Infact Policastro Starinets and Son demonstrated that the dispersion relation for these Goldstone modes at small k takes the form predicted by fluid dynamics -(shear and sound waves) provided $\frac{\eta}{s} = \frac{1}{4\pi}$

- So we now have a first hint that black branes mimic the behaviour of thermal QFTs for dynamical, not just static purposes.
- Can we take this further? It is well known that the effective dynamical description of field theories in *local* thermal equilibrium are the equations of fluid dynamics.
- Fluid dynamics is a derivative expansion: it works provided all variations are slow (compared to a dynamical relaxation time) but does not require amplitude variations to be small.
- Thus the AdS/CFT correspondence appears to implythat the equations of asymptotically *AdS* gravity reduce to (relativistic generalizations of) the Navier Stokes equations at the *full nonlinear level* in an appropriate long distance expansion. Is this exciting suggestion true?

- It is useful to isolate the trivial from the nontrivial parts of this suggestion. The equations of uncharged fluid dynamics take the form $\partial^\mu T_{\mu\nu} = 0$. These equation follows simply from symmetries that are true on both sides of the duality, and are trivially automatic on both sides.
- However $\partial^\mu T_{\mu\nu} = 0$ are d equations for $\frac{d(d+1)}{2} - 1$ variables. They constitute a well defined dynamical system only if you are able to express $T^{\mu\nu}$ as a function of d variables. This 'constitutive relationship' is the nontrivial assertion of fluid dynamics.
- How could this work in gravity? Suggestion: lets use the collective coordinate method (or Goldstone philosophy) on the d parameter set of exact black brane solutions.

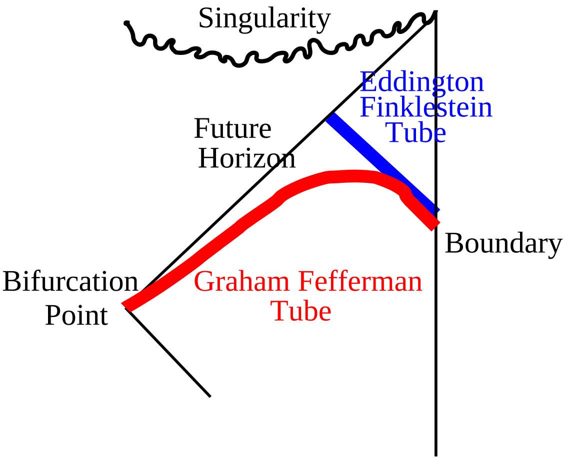
- We look for solutions that 'locally' approximate black branes but with space varying velocities and temperatures. More precisely we search for bulk solution tubewise approximated by black branes. But along which tubes?
- Naive guess: lines of constant x^μ in Schwarschild (Graham Fefferman) coordinates, i.e. metric approximately

$$ds^2 = \frac{dr^2}{r^2 f(r)} + r^2 \mathcal{P}_{\mu\nu}(x) dx^\mu(x) dx^\nu(x) - r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu$$

$$f(r) = 1 - \left(\frac{4\pi T(x)}{d\ r}\right)^d; \quad \mathcal{P}_{\mu\nu} = g_{\mu\nu}(x) + u_\mu(x) u_\nu(x)$$

- Does not seem useful. Appears to be a bad starting point for perturbation theory. Also has several interpretative difficulties.

Penrose diagram



- Causality suggests the use of tubes centered around ingoing null geodesics. In particular we try

$$ds^2 = g^{(0)}_{MN} dx^M dx^N = -2u_\mu(x) dx^\mu dr + r^2 \mathcal{P}_{\mu\nu}(x) dx^\mu dx^\nu - r^2 f(r, T(x)) u_\mu(x) u_\nu(x) dx^\mu dx^\nu$$

- Metric generally regular but not solution to Einstein's equations. However solves equations for constant $u^\mu, T, g_{\mu\nu}$. Consequntly appropriate starting point for a perturbative soln of equations in the parameter $\epsilon(x)$.

- That is we set

$$g_{MN} = g^{(0)}_{MN}(\epsilon x) + \epsilon g^{(1)}_{MN}(\epsilon x) + \epsilon^2 g^{(2)}_{MN}(\epsilon x) \ldots$$

  and attempt to solve for $g^{(n)}_{MN}$ order by order in ϵ.
- Perturbation expansion surprisingly simple to implement. Nonlinear partial differential equation $\rightarrow \frac{d(d+1)}{2}$ ordinary differential equations, in the variable r at each order and each boundary point.

- It turns out that all equations can be solved analytically (and rather simply). Upon solving the equations we find that the perturbative procedure spelt out above can be implimented at n^{th} order *only* when an integrability condition of the form $\partial_\mu T^{\mu\nu}_{n-1}(u^\mu(x), T(x))$ where $T^{\mu\nu}_{n-1}(u^\mu(x), T(x))$ is a specific function of temperature and velocities and their spacetime derivatives to $(n-1)^{th}$ order. This function is determined directly from Einstein’s equations by the perturbative procedure.
- For every $u^\mu(x)$ and $T(x)$ that satisfies this Fluid Dynamical equation we have a solution to Einstein’s equations. The map from fluid dynamics to gravity is locally invertable assuming regularity of the future event horizon.

Explicit Results at second order

We have explicitly implemented our perturbation theory to second order.

$$
\begin{aligned}
ds^2 = & -2u_\mu dx^\mu (dr + r\, A_\nu dx^\nu) + r^2 g_{\mu\nu} dx^\mu dx^\nu \\
& - \left[\omega_\mu{}^\lambda \omega_{\lambda\nu} + \frac{1}{d-2} \mathcal{D}_\lambda \omega^\lambda{}_{(\mu} u_{\nu)} - \frac{1}{d-2} \mathcal{D}_\lambda \sigma^\lambda{}_{(\mu} u_{\nu)} \right. \\
& \left. + \frac{\mathcal{R}}{(d-1)(d-2)} u_\mu u_\nu \right] dx^\mu dx^\nu \\
& + \frac{1}{(br)^d} (r^2 - \frac{1}{2} \omega_{\alpha\beta} \omega^{\alpha\beta}) u_\mu u_\nu dx^\mu dx^\nu \\
& + 2(br)^2 F(br) \left[\frac{1}{b} \sigma_{\mu\nu} + F(br) \sigma_\mu{}^\lambda \sigma_{\lambda\nu} \right] dx^\mu dx^\nu \ldots
\end{aligned}
$$

Explicit Results at second order

$$
\begin{aligned}
&- 2(br)^2 \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1} P_{\mu\nu} K_1(br) - \frac{u_\mu u_\nu}{(br)^{d-2}} \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{(d-1)} K_2(br) \\
&+ \frac{2\, L(br)}{(br)^{d-2}} \left[P_\mu^\lambda \mathcal{D}_\alpha \sigma^\alpha{}_\lambda u_\nu + P_\nu^\lambda \mathcal{D}_\alpha \sigma_\lambda{}^\alpha u_\mu \right] dx^\mu dx^\nu \\
&- 2(br)^2 H_1(br) \left[u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} + \sigma_\mu{}^\lambda \sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1} P_{\mu\nu} \right. \\
&\qquad \left. + C_{\mu\alpha\nu\beta} u^\alpha u^\beta \right] dx^\mu dx^\nu \\
&+ 2(br)^2 H_2(br) \left[u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} + \omega_\mu{}^\lambda \sigma_{\lambda\nu} - \sigma_\mu{}^\lambda \omega_{\lambda\nu} \right] dx^\mu dx^\nu
\end{aligned}
$$

Explicit results at second order

Where

$$F(br) \equiv \int_{br}^{\infty} \frac{y^{d-1}-1}{y(y^d-1)} dy \ ; \ L(br) \equiv \int_{br}^{\infty} \xi^{d-1} d\xi \int_{\xi}^{\infty} dy \ \frac{y-1}{y^3(y^d-1)}$$

$$H_2(br) \equiv \int_{br}^{\infty} \frac{d\xi}{\xi(\xi^d-1)} \int_1^{\xi} y^{d-3} dy \left[1+(d-1)yF(y)+2y^2F'(y)\right]$$

$$K_1(br) \equiv \int_{br}^{\infty} \frac{d\xi}{\xi^2} \int_{\xi}^{\infty} dy \ y^2 F'(y)^2 \ ; \ H_1(br) \equiv \int_{br}^{\infty} \frac{y^{d-2}-1}{y(y^d-1)} dy$$

$$K_2(br) \equiv \int_{br}^{\infty} \frac{d\xi}{\xi^2} \Bigg[1-\xi(\xi-1)F'(\xi)-2(d-1)\xi^{d-1} + \Big(2(d-1)\xi^d-(d-2)\Big)\int_{\xi}^{\infty} dy \ y^2F'(y)^2\Bigg]$$

Second order boundary stress tensor

The dual stress tensor corresponding to this metric is given by $(4\pi T = b^{-1}d)$

$$
\begin{aligned}
T_{\mu\nu} &= p\left(g_{\mu\nu} + d u_\mu u_\nu\right) \\
&\quad - 2\eta\left[\sigma_{\mu\nu} - \tau_\pi u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} - \tau_\omega\left(\sigma_\mu{}^\lambda \omega_{\lambda\nu} - \omega_\mu{}^\lambda \sigma_{\lambda\nu}\right)\right] \\
&\quad + \xi_\sigma\left[\sigma_\mu{}^\lambda \sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1} P_{\mu\nu}\right] + \xi_C C_{\mu\alpha\nu\beta} u^\alpha u^\beta
\end{aligned}
$$

$$
p = \frac{1}{16\pi G_{d+1} b^d} \quad ; \quad \eta = \frac{s}{4\pi} = \frac{1}{16\pi G_{d+1} b^{d-1}}
$$

$$
\tau_\pi = (1 - H_1(1))b \quad ; \quad \tau_\omega = H_1(1)b \quad ; \quad \xi_\sigma = \xi_C = 2\eta b
$$

- Note that gravity reduces to fluid dynamics with particular (holographically determined) values for dissipative parameters. As we have seen the schematic form of the fluid stress tensor is

$$T_{\mu\nu} = aT^d(g_{\mu\nu} + du_\mu u_\nu) + bT^{d-1}\sigma_{\mu\nu} + T^{d-2}\sum_{i=1}^{5} c_i S^i_{\mu\nu}$$

- a is a thermodynamic parameter. b is related to the viscosity: we find $\eta/s = 1/(4\pi)$. c_i coefficients of the five traceless symmetric Weyl covariant two derivative tensors are second order transport coefficients. Value disagree with the predictions of the Israel Stewart formalism.
- Recall that results universal. Should yield correct order of magnitude estimate of transport coefficients in any strongly coupled CFT.

- Our solutions are singular at $r = 0$. Quite remarkably it is possible (under certain conditions) to demonstrate that these solutions have event horizons and to explicitly determine the event horizon manifold order by order in the derivative expansion. This horizon shields the $r = 0$ singularity from the boundary.
- Our control over the event horizon, together with the classic area increase theorem of general relativity, can be used to derive an ‘entropy current’ for our fluid flows that is local and has positive divergence.

Entropy Current at second order

Explicitly this entropy current is given to second order by

$$4\,G_{d+1}\,b^{d\,-\,1}\,J^{\mu}_{S} = [\,1 + b^2\,(\,A_1\,\sigma^{\alpha\beta}\,\sigma_{\alpha\beta} + A_2\,\omega^{\alpha\beta}\,\omega_{\alpha\beta} + A_3\,\mathcal{R}\,)\,]\,u^{\mu} + b^2\,[\,B_1\,\mathcal{D}_{\lambda}\,\sigma^{\mu\lambda} + B_2\,\mathcal{D}_{\lambda}\,\omega^{\mu\lambda}\,]$$

where

$$A_1 = \frac{2}{d^2}(d+2) - \frac{K_1(1)d + K_2(1)}{d}\,, \qquad A_2 = -\frac{1}{2d}, \qquad B_2 = \frac{1}{d-2}$$

$$B_1 = -2A_3 = \frac{2}{d\,(d-2)}$$

Charged Fluid Dynamics

- The story described above was developed over three years ago. It has since been generalized in many directions. The two most interesting generalizations both require the addition of a Maxwell field (in addition to the graviton) to the bulk.
- The addition of a bulk Maxwell field endows the boundary theory with a conserved global charge. Equilibrium states in such a system are labeled by a charge density together with the energy density and velocity; in the bulk these equilibrium configurations are given by charged AdS-Reisnner Nordstorm black branes.

- Following the general procedure described in this talk, it has been established that the AdS Einstein -Maxwell equations reduce, in an appropriate long wavelength limit, to the equations of charged relativistic hydrodynamics.
- The procedure yields expressions for the stress tensor and charge currents as a function of local temperatures, velocities and chemical potentials.
- We find a surprise here even at first order in the derivative expansion. In addition to the usual diffusive currents, in $d = 4$ we find a term in the charge current proportional to $\epsilon_{\mu\nu\rho\sigma}\omega^{\nu\rho}u^{\sigma}$. This is important because this term was ignored by Landau and Lifshitz and perhaps all authors subsequently.

Superfluidity

- Another generalization is to the study of superfluid hydrodynamics. Superfluidity arises in systems with a conserved global charge when the equilibrium state includes a condensate of an operator of nonzero global charge.
- The bulk dual description of the equilibrium state of a superfluid is a charged black brane emmersed in charged scalar 'hair'. Such hairy black brane solutions exist in certain AdS models, and have been intensively studied. Once again repeating the procedure above on these solutions results in the equations of Landau-Tiza superfluidity, with one new term that was missed by previous studies, and a new structure for parity odd fluids.

- As we have described above, computations within gravity have revealed that the ‘standard’ equations of hydrodynamics are incomplete in certain situations (some allowed terms have been missed).
- It is clear from this observation that a complete and systematic ‘theory of hydrodynamics’ is at present lacking.
- We will list three general principles that constrain the equations of fluid dynamics and whose analysis may eventually lead to a complete ‘theory’ of fluid dynamics.

The Entropy positivity constraint

- Fluid constitutive relations must be consistent with existence of an entropy current whose divergence is positive in every fluid flow in every consitent background, including in curved space.
- This principle turns out to be surprisingly constraining. At first order in ordinary relativistic charged fluid dynamics it sets two of the five symmetry allowed constitutive parameters to zero. At second order in uncharged fluid dynamics it kills 5 out of the 15 parameters . At first order in parity invariant superfluid dynamics it kills 26 of the 47 parameters.

The Correlator symmetry constraint

- The equations of fluid dynamics may be used to compute n point (retarded) correlation functions at long distances.
- One may prove on general field theoretic grounds that (for instance) the retarded two point function must obey the symmetry constraint

$$G_{ab}(\omega, k) = (-1)^{\eta_a+\eta_b} G_{ba}(\omega, -k)$$

- These conditions are automatic in perfect fluid dynamics but impose constraints on higher order constitutive relations. We are currently investigating how powerful these constraints are.

The zero mode thermodynamical constraint

- Long distance correlators computed out of fluid dynamics must also be consistent with thermodynamics. For instance

$$< J_0(0) J_0(0) >= \partial_\mu q$$

- Such relations are often automatic from thermodynamical identities. Some relations of the above sort, however, impose constraints on constitutive parameters. For instance the equation written above implies the Einstein relation
- Key question: What are the full set of constraints one must impose. What is the relation between these (and possibly other) sets of constraints.

Consequences for gravity

- The second and third constraints listed above are kinematical in AdS/CFT. However the first constraint is more interesting, and has only been proved for 2 derivative Einstein gravity in the bulk.
- It is possible that the kinematical constraints completely imply the first set of constraints. This would suggest that the area increase theorem of gravity can be generalized to an entropy increase theorem for arbitrary higher derivative gravity, on purely kinematical grounds.
- Another possibility, however, is that the constraints from entropy positivity are cannot be derived from kinematical considerations. Their validity could conceivably constrain higher derivative corrections to Einstein's equations.

- Within the fluid gravity correspondence, it is possible to show that there exist nontrivial gravitational configurations in which the entropy production vanishes in Einstein gravity
- It is possible that requirement that entropy remain either constant, or increase in such configurations will yield nontrivial constraints on possible higher derivative corrections to Einstein’s equation.
- This suggests the exciting - though perhaps unlikely - possibility of an infinite number of completely new purely low energy constraints on structure of higher derivative corrections to Einstein gravity whose origin lies in the thermodyanical nature of gravity in configurations with a horizon.

Conclusions

- Asymptotically AdS_{d+1} gravity reduces, in the long wavelength limit, to the equations $d+1$ dimensional Navier Stokes equations with gravitationally determined dissipative parameters.
- The gravitationaly determined fluid dynamical system has no free parameters, and so enjoys a degree of universality. Values of these parameters are interesting and sometimes surprising.
- This map allows us to reword open problems in fluid dynamicsas problems in gravity. Potential for new insights?
- It would be very interesting to extend this connection past the large N (or classical) limit.