Light-cone spreading of perturbations in the Heisenberg spin chain

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Outline

- Chaos in classical systems.
- Propagation of chaos in many-body systems.
- The out-of-time-ordered-commutator (OTOC) as a probe of quantum chaos and of ballistic propagation of information.
- Numerical results for classical Heisenberg spin-chain — Das et al, Phys. Rev. Lett. **121**, 024101 (2018).
- Numerical results for classical oscillator chains (Harmonic, Toda, FPU, Phi-4) — AD (in preparation)

Chaos in classical systems

A butterfly fluttering its wings in Beijing causes a tornado in Kansas!!

Classical Chaos - Sensitivity to initial conditions

Consider a dynamical system with deterministic time-evolution

$$\frac{dz_x}{dt} = f_x(\mathbf{z}) \ , \quad x = 1, 2, \ldots, N \ .$$

Consider two different initial conditions

$$z^A(0) \ \text{and} \ \ z^B(0) = z^A(0) + \delta z(0) \ .$$

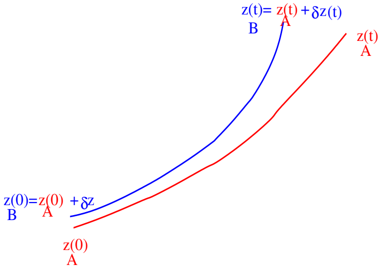
After time t let

$$\delta z(t) = z^B(t) - z^A(t)$$

Definition of chaos — at large t

$$\lim_{\delta z(0) \to 0} \frac{|\delta z(t)|}{|\delta z(0)|} \sim e^{\lambda_{\mathbf{z}} t}$$

with $\lambda > 0$ — LYAPUNOV EXPONENT



Propagation of chaos

Suppose that the initial perturbation is localized in space

$$\delta z_x(0) = \epsilon \delta_{x,0} \ .$$

We expect that the differences in phase-space variables at any point in space should eventually diverge with time. Thus

$$\lim_{\delta z_0(0)\to 0} \frac{|\delta z_x(t)|}{|\delta z_0(0)|} \sim e^{\lambda_{\mathbf{z}}(x,t)t} \ .$$

Some questions:

- How long does it take before the perturbation is felt at point x ?
- How does the spatio-temporal evolution of the perturbation take place?

Spread of perturbations in a Hamiltonian system

Consider a chain of coupled oscillators (e.g coupled pendula).

$z_x = (q_x,\ p_x)$ — (position, momentum) of x^{th} particle. $[x = -N/2, \ldots, N/2]$.

- Disturb system locally.
- The spread of the perturbation can be expressed as a Poisson bracket:
  $$\frac{\partial p_x(t)}{\partial p_0(0)} = -\{p_x(t), q_0(0)\} \qquad \{A, B\} = \sum_x \frac{\partial A}{\partial q_x}\frac{\partial B}{\partial p_x} - \frac{\partial A}{\partial p_x}\frac{\partial B}{\partial q_x}$$
- Linear response $\langle \partial p_x(t)/\partial p_0(0)\rangle_{eq} = -\langle\{p_x(t), q_0(0)\}\rangle = \beta\langle p_x(t)p_0(0)\rangle$
  —- Equilibrium correlation functions
- Chaos and "non-linear response" — $\left\langle \left(\frac{\partial p_x(t)}{\partial p_0(0)}\right)^2 \right\rangle$

Chaos in quantum systems

- $z \to \psi$ — No chaos since linear dynamics.
- Usual characterization of quantum chaos: level-spacing statistics
- Another possible characterizer of quantum chaos: replace Poisson bracket by commutator. Hence look at

$$-\langle [q_x(t), p_0(0)]^2 \rangle / \hbar^2 \ ,$$

  $< ... >$ represents expectation over pure or thermal state.
- No long time divergence but perhaps the intermediate-time growth shows an exponential growth regime and one can extract a Lyapunov exponent.
- Ehrenfest time and scrambling time.
- Ballistic spread characterized by butterfly velocity, Lieb-Robinson velocity.

Classical version of the OTOC

Let $(\mathbf{Q},\mathbf{P}) = (\delta\mathbf{q},\delta\mathbf{p})$ be the perturbation of an initial condition $(\mathbf{q},\mathbf{p})$.
Time evolution is given by:

$$\dot{q}_x = p_x \ , \quad \dot{p}_x = f_x$$

$$\dot{Q}_x = P_x, \quad \dot{P}_x = \sum_{y=1}^{N} \frac{\partial f_x}{\partial q_y} Q_y = \sum_{y=1}^{N} M_{xy}(\mathbf{q}) Q_y \quad , x = 1,2,\ldots,N.$$

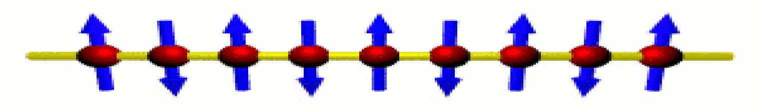
Solve with initial condition $(\mathbf{q},\mathbf{p})$ chosen from an equilibrium distribution and $Q_x = 0, \ P_x = \delta_{x,0}$, $x = 0,1,\ldots,N-1$.
Compute

$$C(x,t) = -\langle\{p_x(t), q_0\}\rangle = \langle P_x(t)\rangle, - - - - \text{Correlations and linear response}$$

$$D(x,t) = \langle\{p_x(t), q_0\}^2\rangle = \langle P_x^2(t)\rangle - - - - \text{OTOC}$$

— Chaos propagation in the Heisenberg spin chain with precession dynamics.

Classical Heisenberg spin chains



Heisenberg spins $\mathbf{S} = (S^x, S^y, S^z)$ — unit three-dimensional vectors. Hamiltonian given by

$$H = -\sum_{\ell=1}^{N} \mathbf{S}_\ell . \mathbf{S}_{\ell+1} \ .$$

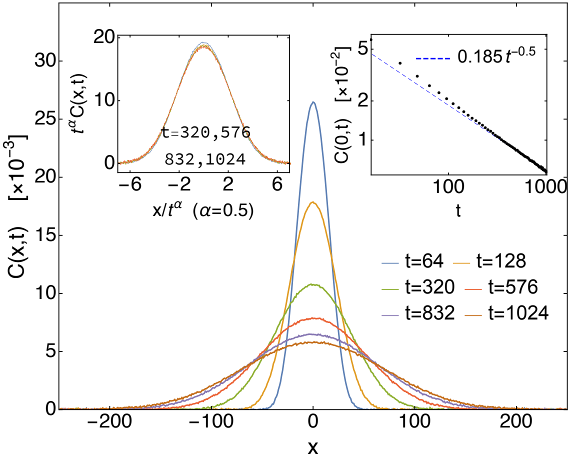
Equations of motion [Structure of usual Hamiltonian dynamics]

$$\dot{\mathbf{S}}_\ell = \mathbf{S}_\ell \times \mathbf{B}_\ell^{\mathrm{eff}} = \{\mathbf{S}_\ell, H\} \ .$$

Linear response at high temperatures

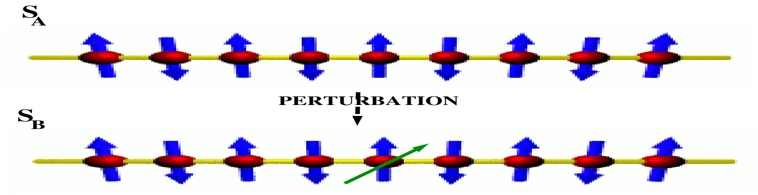
Expect diffusive hydrodynamic equations.

Spreading of a spin perturbation $\quad C_{ss}(x,t) = \langle \mathbf{S}_x(t).\mathbf{S}_0(0)\rangle_{eq}$.



Energy correlations are similar.

Chaos and ballistic propagation in the *XXX* Heisenberg chain at high temerature



- Consider finite perturbation $\delta\mathbf{S}_0(0) = \epsilon\hat{\mathbf{n}}$ and look at $\delta\mathbf{S}_x(t) = \mathbf{S}_x^A(t) - \mathbf{S}_x^B(t)$.

$$D_\epsilon(x,t) = \langle[\delta\mathbf{S}_x(t)]^2\rangle/2 = 1 - \langle\mathbf{S}_x^A(t).\mathbf{S}_x^B(t)\rangle$$

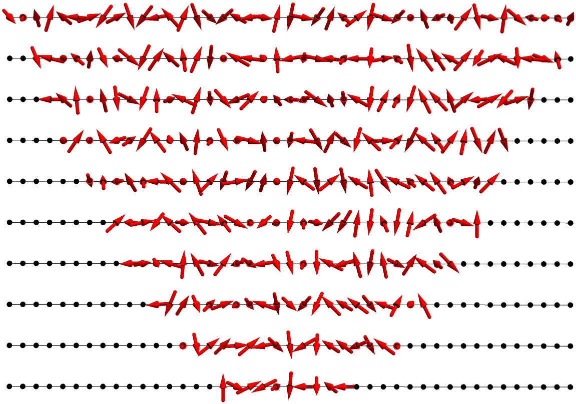
.

- Clearly $D_\epsilon(x,t) \to 1$ at large times
- However in the limit $\epsilon \to 0$

$$D(x,t) = \left\langle \left(\frac{\partial\mathbf{S}_x(t)}{\partial\mathbf{S}_0(0)}\right)^2 \right\rangle = \frac{D_\epsilon(x,t)}{\epsilon^2} \sim e^{2\lambda t}$$

Spreading of perturbation for single realization

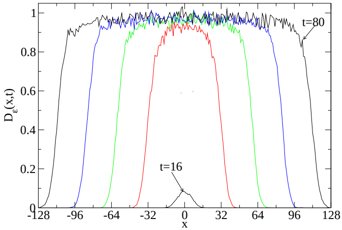
Space-time evolution of localized perturbation.

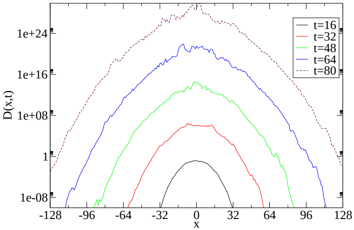


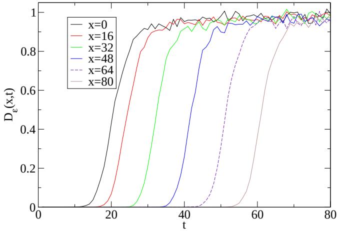
Spreading of perturbation for single realization

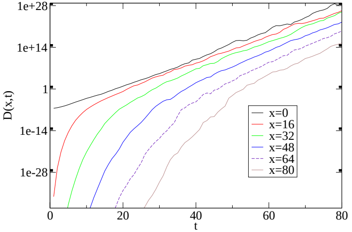
$D_\epsilon = \langle [\delta \mathbf{S}_x(t)]^2 \rangle$

$D = \lim_{\epsilon \to 0} \langle [\delta \mathbf{S}_x(t)]^2 \rangle / \epsilon^2$



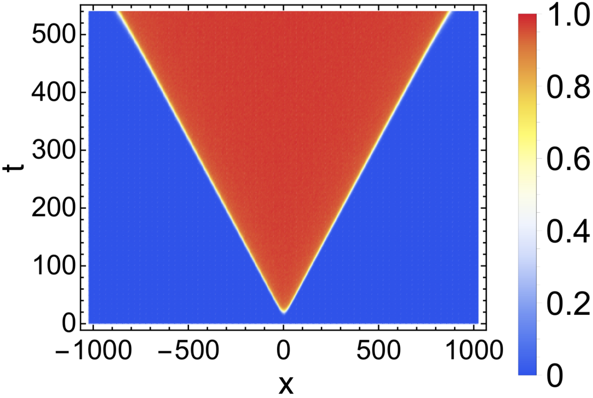






Light-cone evolution of $D_\epsilon(x, t)$

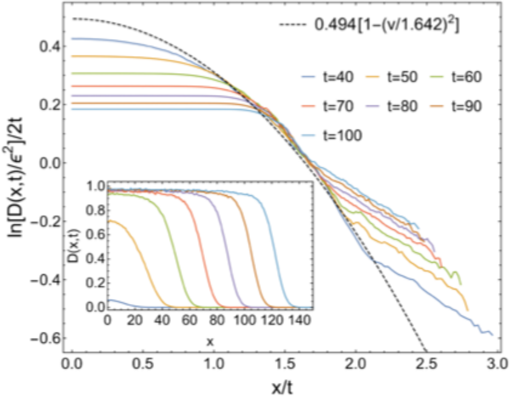
System of $N = 2048$ spins, average over random initial conditions.



The signal decays exponentially with distance away from the light cone $x = v_b t$, with $v \approx 1.67$.

Evolution of $D_\epsilon(x,t)$

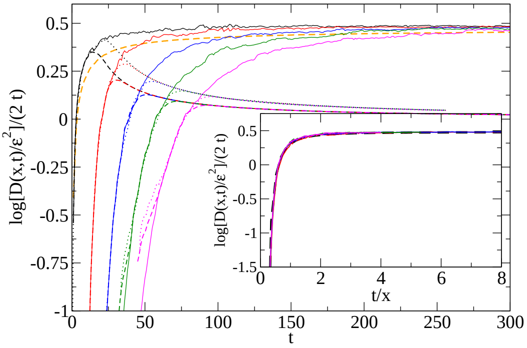
Plot of $D_\epsilon(x,t)$ profile at different times (non-linear dynamics with finite ϵ).



From numerics, the front behaves as $D_\epsilon(x,t) \sim \epsilon^2 \; e^{2\lambda t[1-(x/v_b t)^2]}$

Comparison between $D(x,t)$ and $D_\epsilon(x,t)$

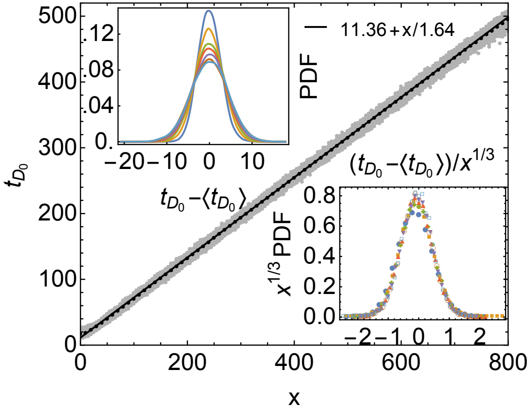
Growth of $D(x,t)$ profile at different positions – from linear and non-linear dynamics.



From numerics, the signal grows as $D(x,t) \sim \epsilon^2 \; e^{2\lambda t[1-(x/v_b t)^2]}$, with $\lambda \approx 0.5$, $v_b \approx 1.67$.

Behaviour of $D(x, t)$

Estimating the butterfly speed - At what time does the signal reach some threshold value D_o at a given point x.



$x^{1/3}$ broadening of the front.

Connection to KPZ surface growth

- Exponential growth within the light-cone (Butterfly effect)
- Ballistic front (Butterfly velocity)
- Saturation at long times (for finite perturbation)
- Dynamics of growing surface from linearized equations - relations to KPZ — A. S. Pikovsky and J. Kurths (1994)

Consider linearized dynamics. Define —

$$h(x,t) = \lim_{\epsilon \to 0} \log[\delta \mathbf{S}^2(x,t)/2\epsilon^2]/2$$

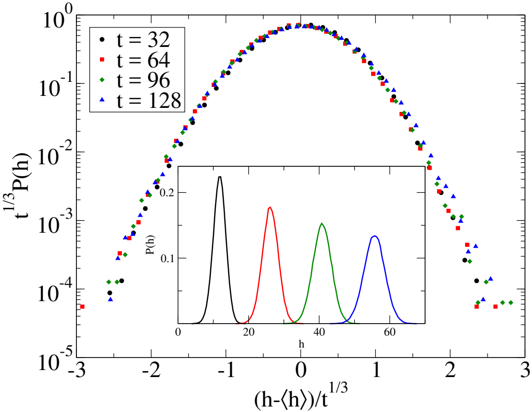
Our observations:

$$h(x,t) = at - bx^2/t + t^{1/3}\eta$$

Numerically $a \approx \lambda_L$ and $b \approx \lambda_L/v_b^2$.
This suggests that $h(x,t)$ is like the KPZ height field.
Distribution of η — Tracy-Widom ?

Connections to KPZ

Distribution of height at $x = 0$ at different times.



Clear $t^{1/3}$-scaling but no signs yet of Tracy-Widom.

Conclusions

- Investigated the classical version of "Out-of-Time-Ordered-Commutator" in the Heisenberg spin chain.

  —Ballistic front propagation in a diffusive system, exponential growth, broadening of front and possible connections to KPZ universality.

- $D(x,t) \sim e^{2\mu(x/t)t}, \quad \mu(v) = \lambda[1-(v/v_b)^2]$ — velocity-dependent Lyapunov exponent [R. J. Deissler and K. Kaneko (1987), S. Lepri, A. Politi, and A. Torcini (1996)]

  Integrable systems: $\mu(v) \sim -(1-v/v_b)^{3/2}$. [vanishes within light cone, e.g Harmonic chain; Toda lattice]

  Broad discussion/review: Khemani, Huse, Nahum (2018).

- Possible connections to the KPZ surface growth model.
- What physical observable ? Information propagation, Loschmidt echo,...
- OTOC in oscillator chains — connections to transport.