Renormalization group study of the interplay between Majorana fermions and Kondo effect in quantum dots

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Outline

- Kondo model and its renormalization flows
- Z_2 topological order and Majorana edge modes
- Interplay between Majorana fermions and Kondo effect
- Interplay between Majorana-Kramers and Kondo effect
- Ongoing projects and Future Directions

Kondo Effect: Strong Correlation Physics

- Kondo effect is the prototypical quantum many-body phenomenon in which magnetic impurity couples to Fermi sea antiferromagnetically and leads to the formation of quantum many-body Kondo singlet.
- Kondo effect is ubiquitous: First discovered in magnetic alloys. Recently realized in quantum dots.
- Kondo effect has also been studied in graphene, topological insulators and superconductors, Dirac and Weyl semimetals.
- Single channel Kondo effect leads to metallic phase while as multi-channel Kondo effect leads to Non-Fermi ground states.
- Kondo effect from non-magnetic degrees of freedom: Charge Kondo effect,orbital Kondo effect.

Kondo model

$$H = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{kk'} J_{kk'} S.s_{kk'}$$

$$s_{kk'} = \sum_{\sigma\sigma'} c^\dagger_{k\sigma} (\tau)_{\sigma\sigma'} c_{k'\sigma'} \tag{1}$$

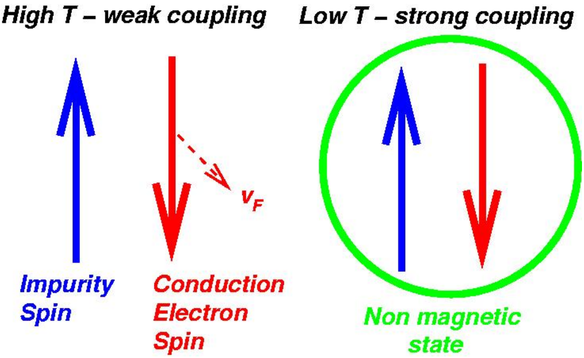


Image Source:Wikipedia

Renormalization flows of Kondo model

$$\frac{dJ_{\pm}}{DlnD} = -J_z^2 J_{\pm}\rho_0 \quad \frac{dJ_z}{lnD} = -2\rho_0 J_{\pm}$$

Kondo scale is the scaling invariant of the RG flow of Kondo model.

$$T_k \sim De^{-\frac{1}{J\rho}}$$

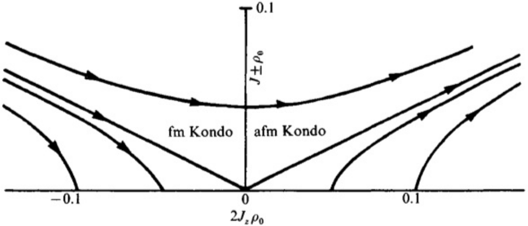
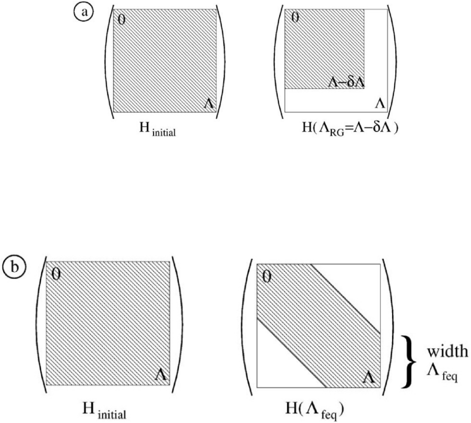


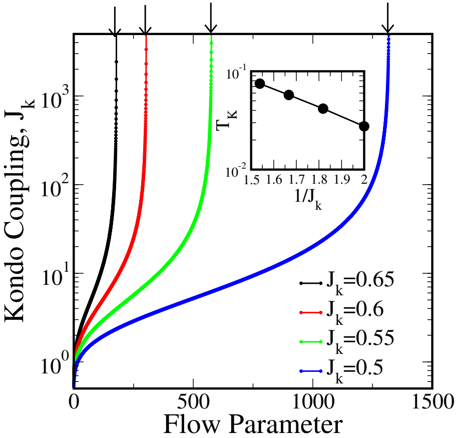
Figure 3.7 Scaling trajectories for the anisotropic s-d model, calculated in lowest order perturbation theory (Anderson, 1970).

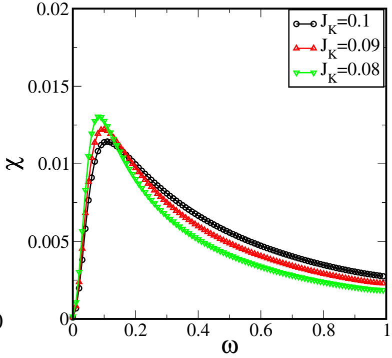
Flow Equation Renormalization Method

$$\frac{dH}{dl} = [\eta, H] \quad \eta = [H_0, H_v] \tag{2}$$



Flow equations for Kondo Model





Anderson Impurity Model

Anderson(1961): Fate of local moments in alloys with magnetic impurities

$$H = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{k\sigma} V_k (c^\dagger_{k\sigma} d_\sigma + H.C.) + \sum_{\sigma} \epsilon_d d^\dagger_\sigma d_\sigma + U n_{d\uparrow} n_{d\downarrow}$$

- Single impurity Anderson model (SIAM) is one of the most important models for strongly correlated electron systems.
- It not only describes the physics of quantum impurity system but has been extensively used to study quantum transport in quantum dots.
- Within dynamical mean field theory (DMFT), lattice models like the Hubbard model get mapped to this model.

Relation between Anderson and Kondo models

- SIAM and Kondo model were introduced to understand two different phenomena.
- Schrieffer and Wolff showed that they are related to each other via unitary transformation....Schrieffer-Wolff transformation (SWT).
- SWT integrates out high energy charge excitations and SIAM gets transformed to Kondo model (and other terms).

$$H' = U^{\dagger} H U \quad H' = e^{S} H e^{-S} \quad [S, H_0] = -H_v$$

- Kondo model is the low energy effective Hamiltonian of SIAM which is obtained once the charge excitations are integrated out.
- Strong coupling physics of SIAM is determined by Kondo model.

Z_2 topological order and Majorana doubling in Kitaev Chain Model

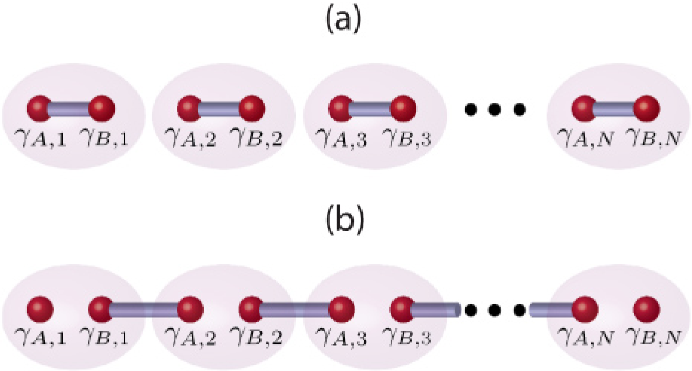
- Majorana fermions(actually **Majorana Zero Modes**) have attracted lot of attention in condensed matter physics community.
- Majorana fermions are the promising candidates for topological quantum computing because of their non-abelian anyonic statistics.
- Majorana fermions occur in quantum Hall fluids, topological superconductors, quantum spin liquids, Multi-channel Kondo models.
- Existence of Majorana fermions is signature of topological order.
- Kitaev chain model can be obtained from Transverse field Ising model(TFIM).
- Why there is topological order in Kitaev chain and not in TFIM?
- Some attempts to answer this question: Greiter *et al*,Cobanerra *et al*
- However they have just explored the duality between the models and not explained the emergence of topological order in Kitaev chain.

Ref: Annals of Physics, **351**,1026(2014), Phys. Rev. B.**87**, 0411705(2013)

Majorana Edge Modes in Kitaev Chain

Kitaev introduced p-wave chain model:

$$H = -t\sum_{i=0}^{N-1}(c_i^\dagger c_{i+1} + h.c.) + \triangle\sum_{i=0}^{N-1} c_i^\dagger c_{i+1}^\dagger + h.c. - \mu\sum_{i=0}^{N} c_i^\dagger c_i$$



Ref:Jason Alicea Rep. Prog. Phys.75 (2012)

Transverse Field Ising Model

$$H = -J\sum_{i=1}^{N-1}\sigma_i^x\sigma_{i+1}^x - h_z\sum_{i=1}^{N}\sigma_i^z \tag{3}$$

This model has Z_2 symmetry due to which the global symmetry operator commutes with Hamiltonian.

$$\left[\prod_i \sigma_i^z, H\right] = 0 \tag{4}$$

Jordan-wigner Transformation maps spin operators into fermion operators.

$$c_i = \sigma_i^\dagger(\prod_{j=1}^{i-1}\sigma_i^z) \quad c_i^\dagger = \sigma_i^-((\prod_{j=1}^{i-1}\sigma_i^z) \tag{5}$$

$$H = -J\sum_{i=0}^{N-1}(c_i^\dagger c_{i+1} + h.c.) - J\sum_{i=0}^{N-1}c_i^\dagger c_{i+1}^\dagger + h.c. - 2h\sum_{i=0}^{N}c_i^\dagger c_i$$

Algebra of Majorana Doubling

- Kitaev found Majorana edge modes in his model.
- Each state in Kitaev chain spectrum has degenerate partner due to parity symmetry like as time reversal symmetry leads to Kramers pairs.
- Lee and Wilzeck(PRL **111**(2013)) showed that in Kitaev chain there are more symmetries which lead to doubled spectrum.

$$\{1, \gamma_1 = a_1, \gamma_2 = a_2, \gamma_3 = a_3, \gamma_{12} = a_1 a_2, \gamma_{23} = a_2 a_3, \gamma_{31} = a_3 a_1, \gamma_{123} = a_1 a_2 a_3\}$$

Hamiltonian for three Majorana fermions:

$$H_m = -i(\alpha b_1 b_2 + \beta b_2 b_3 + \gamma b_3 b_1)$$

Naively one would take it for spin Hamiltonian but there are subtle differences:

$$\Gamma \equiv -i b_1 b_2 b_3$$
$$\Gamma^2 = 1 \quad [\Gamma, b_j] = 0 \quad [\Gamma, H_m] = 0 \quad \{\Gamma, P\} = 0$$

$\{\Gamma, P\} = 0$ leads to the even-odd pair for each energy value.

Topological Order and Majorana Mode operators

- **Fermionic zero modes are one of the very important signatures of topological order.**
- **Fermionic mode operators give a neat way to find topological order in a given Hamiltonian.**

A fermionic zero mode is an operator Γ such that

- Commutes with Hamiltonian:$[H, \Gamma] = 0$
- anticommutes with parity:$\{P, \Gamma\} = 0$
- has finite "normalization" even in the $L \to \infty$ limit:$\Gamma^\dagger \Gamma = 1$.

We find that the same Majorana mode operator which leads to the spectrum doubling also leads to the topological order in Kitaev chain model.
Majorana mode operator is not present for the spin Hamiltonian and hence there is no topological order over there.

Topological Order and Yang-Baxter equation

Braid group generators satisfy:

$$T_i T_j = T_j T_i \quad | i - j | > 1$$
$$T_i T_j T_i = T_j T_i T_j \quad | i - j | = 1 \tag{6}$$

Ivanov showed that Majorana fermions give representation of braid group.

$$\tau_i = exp(\frac{\pi}{4}\gamma_{i+1}\gamma_i) = \frac{1}{\sqrt{2}}(1 + \gamma_{i+1}\gamma_i) \tag{7}$$

Braiding of Majorana fermions happens only in topological phase.

$$H = 2it\sum_{i=0}^{N-1}\gamma_{1,i}\gamma_{2,i+1} \tag{8}$$

We propose a new characterization of topological order: *Topological order is there if we get solutions of Yang-Baxter equation*.
Ref:Ivanov,PRL, **86** (2001) Kauffman-Lomanaco, NJP **4**(2002)
Kauffman-Lomanaco,arxiv:1603.07827

Conclusions

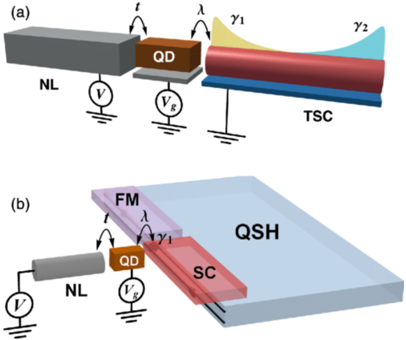
- Showed that topological order in Kitaev chain model arises from the same symmetry operators which lead to its doubled spectrum.
- There is no topological order in TFIM because there are no such fermionic mode operators.
- Showed that how topological order in Majorana fermion systems is related to topological entanglement as given in Yang-Baxter equation.

Renormalization group study of the interplay between Kondo effect and Majorana fermions in quantum dots

- Given the significance of Majorana fermions,there is need for unambiguous ways of their detection and control.
- Kondo effect in quantum dots give very feasible way of detection of Majorana fermions.
- Stability of Majorana fermions in presence of Interactions and realistic set-ups.
- Do Majorana fermions lead to new fixed point?
- Stability of Kondo fixed point in NL-QD-TSC setup?
- Particle-hole asymmetry: Competition between Majorana and Kondo physics
- Signatures of Majorana fermions in spin susceptibility...Experimental Probes?

TSC-QD-NL Hamiltonian

$$H = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{\sigma} \epsilon_d d^\dagger_\sigma d_\sigma + U n_{d\uparrow} n_{d\downarrow} + \sum_{\sigma} t_\sigma (c^\dagger_{k\sigma} d_\sigma + h.c.) + \sum_{\sigma} i\lambda_\sigma \gamma (d_\sigma + d^\dagger_\sigma)$$



Ref:Phys. Rev X, **4**, 031051(2014)

Earlier Studies

- NL-QD-TSC Hamiltonian has been studied by other research groups as well.
- Golub *et al* have done perturbative renormalization and found that Kondo effect remains stable to Majorana couplings.
- Lee *et al* have done NRG study of the model and they also find that Kondo effect remain stable though due to the emergent Zeeman field Kondo peak gets shifted.
- However Lutchyn's group have reached opposite conclusion. Based on their perurbative RG, DMRG and slave boson mean field theory calculations they propose that Majorans fermions lead to new fixed point.
- There was no unanimous understanding of the interplay between Majorana fermions and Kondo effect which motivated our work.

Ref:Golub,PRL,**107**(2011);Lee,PRB,**87**(2013);Lutchyn,PRX,**4**(2014)

Majorana-Kondo Model

We are interested in the Kondo regime of the model so using projection operator method we project the Hamiltonian to the singly occupied subspace and obtain the effective Hamiltonian which we call as Majorana-Kondo model.

$$H_{eff} = \underbrace{\sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + J_k \sum_{kk'} S.s_{kk'}}_{\text{Kondo model}} - \underbrace{h_z S^z}_{\text{Zeeman term}} + \underbrace{J_1(i\gamma)(c_{k\uparrow} + c^\dagger_{k\uparrow})}_{\text{Andreev term}} +$$

$$J_2(i\gamma)(c_{k\uparrow} + c^\dagger_{k\uparrow})S^z + J_3(i\lambda)\gamma(c^\dagger_{k\downarrow} S^+ + c_{k\downarrow} S^-)$$

$$J_k = t^2\zeta_+ \quad h_Z = \lambda^2\zeta_- \quad J_2 = t\lambda\zeta_+ = J_3$$

$$\zeta_\mp = \frac{1}{\epsilon_0} \mp \frac{1}{U - \epsilon_0}$$

Majorana-Kondo model:Particle-Hole symmetry

Particle-hole(PH) symmetry plays a very important role for Majorana-Kondo model. At PH symmetric point, Zeeman term and Andreev scattering term vanish. There is emergent time reversal symmetry as well.
Switching to Majorana representation for lead electrons leads to a simple form of the Hamiltonian which gives insight about the Majorana terms.

$$c_{0\uparrow} + c_{0\uparrow}^{\dagger} = \eta_z \quad c_{0\downarrow} + c_{0\downarrow}^{\dagger} = \eta_x \quad i(c_{0\downarrow}^{\dagger} - c_{0\downarrow}) = \eta_y$$

$$\begin{aligned} H =& J_2(i\gamma)\left((c_{k\uparrow} + c_{k\uparrow}^{\dagger})S^z + (c_{k\downarrow}^{\dagger}S^+ + c_{k\downarrow}S^-)\right) + J_k s_0.S \\ H =& iJ_2\sum_{\alpha}\gamma S^{\alpha}\eta_{\alpha} + J_k s_0.S \end{aligned}$$

Majorana couplings favor the formation of parity singlet with impurity spin which gives competition to to Kondo singlet formation.

Particle-Hole asymmetry: Kondo vs Majorana physics

- At PH symmetric point,impurity spin is strongly fluctuating $\langle S \rangle = 0$ This can be also understood from the fact that there is time reversal symmetry as well.
- Away from PH symmetry impurity spin gets polarized due to the Zeeman field. PH asymmetry is a relevant perturbation for Majorana-Kondo model while as for Kondo model PH asymmetry is not relevant perturbation.
- Particle-hole asymmetry parameter $\eta = 1 + \frac{2\epsilon_d}{U}$
- Kondo Model: $\langle S \rangle = 0$ both for $\eta = 0$ and $\eta \neq 0$
  Majorana-Kondo model $\langle S \rangle = 0$ for $\eta = 0$
  $\langle S \rangle \neq 0$ for $\eta \neq 0$
- The relevance of the particle-hole asymmetry for the Majorana-Kondo model is signature of Majorana fermions.
- It is reminiscent of the Kondo model with gapped hosts(like as Superconductor)

Ref:Phys. Rev X,**4**,031051(2014)

Numerical solution: PHS case

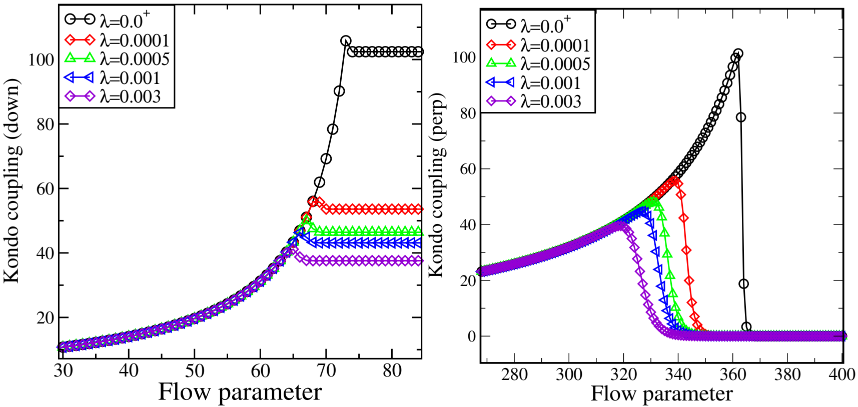


Figure : In left panel, longitudinal Kondo coupling is plotted versus flow parameter. In right panel, transverse Kondo coupling is plotted versus flow parameter. Other parameters are: U= 6.0, ϵ_d= -3.0, t= 3.0.

Kondo Scale: PHS

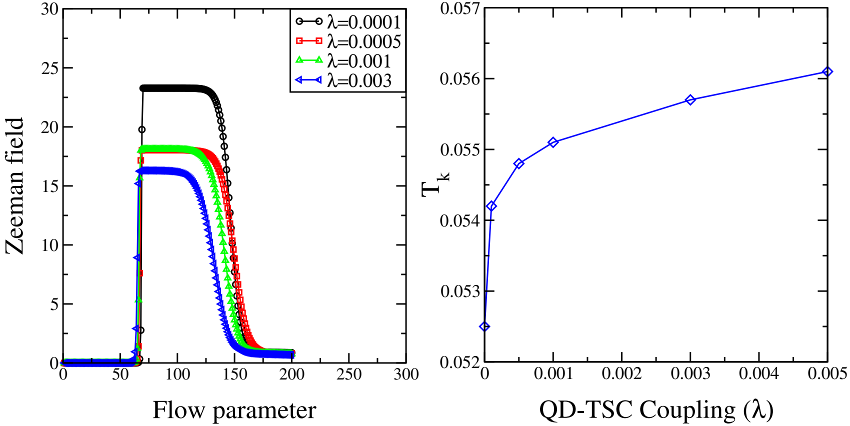


Figure : In the left panel, flow of Zeeman field has been plotted. In the right panel, Kondo scale has been plotted as a function of λ, coupling of quantum Dot to the topological superconductor.

Flow equations: PH asymmetric case

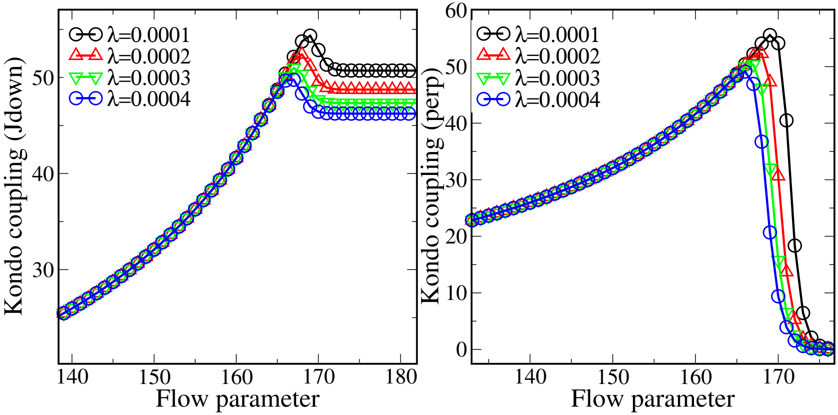


Figure : Flow of the Kondo couplings for the asymmetric case of Majorana-Kondo model: In the left panel, longitudinal Kondo coupling is plotted versus the flow parameter. In the right panel, transverse Kondo coupling versus flow parameter is plotted. Other parameters are: $U = 6.0$, ϵ_d = -3.1, t= 3.0

Effect on Kondo Scale

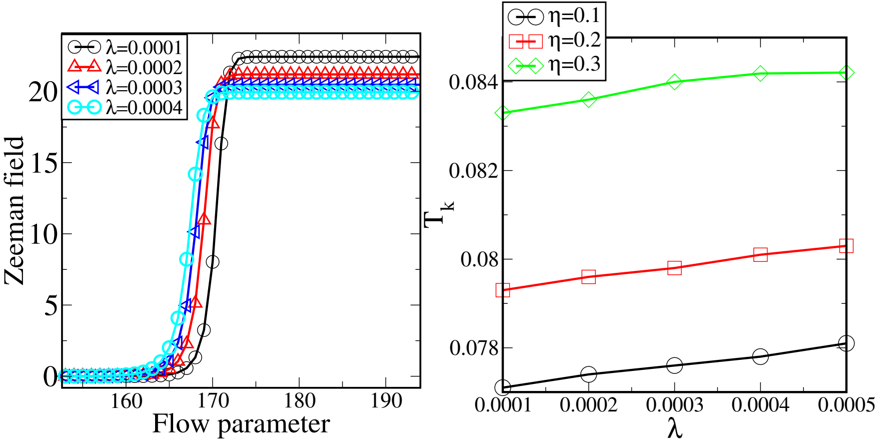


Figure : Plots for asymmetric Majorana-Kondo model: In the left panel, the flow of the emergent Zeeman field has been plotted. In right panel, Kondo scale has been plotted as function of λ, coupling of quantum dot to topological superconductor, for different asymmetry parameters.

Flow equations for the impurity Spin

- One of the important features of flow equation method is that it gives access to study the renormalization flows of the observables also.
- For the Kondo model, spin is the observable and under the unitary flow, spin operator undergoes renormalization.
- From the RG flow of spin operator, we can calculate dynamic spin susceptibility.

$$S^z(l) = h^z(l)S^z + \frac{M(l)}{2} + \sum_{pq} \gamma_{pq}(l)(:c^\dagger_{p\uparrow}c_{q\downarrow}: S^- + :c^\dagger_{q\downarrow}c_{p\uparrow}: S^+)$$
$$+ \sum_p \zeta_p(l)(i\gamma)(c_{p\uparrow} + c^\dagger_{p\uparrow})S^z) + \sum_p \eta_p(l)(\gamma(c^+_{p\downarrow}S^+ + c_{p\downarrow}S^-) \tag{9}$$

$$\frac{dS^z}{dl} = [\eta(l), S^z(l)] \tag{10}$$

Spin Susceptibility: PHS

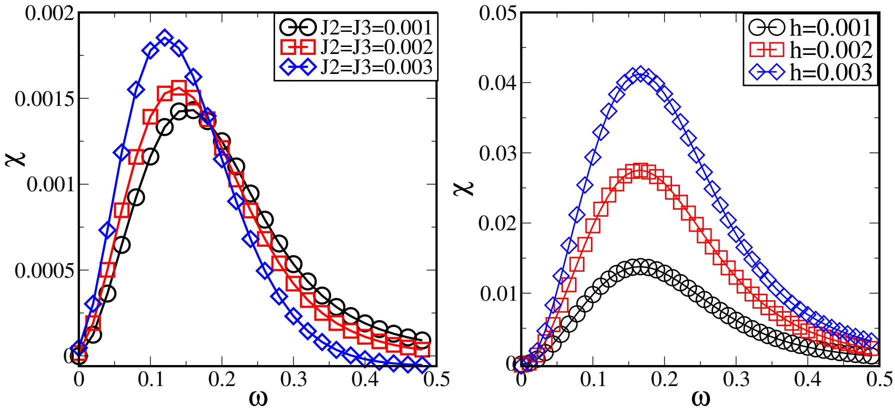


Figure : Dynamical spin susceptibility plots for Majorana-Kondo model at particle-hole symmetric point: In left panel, spin susceptibility has been plotted on frequency axis for different J_2 coupling constants. In right panel, spin susceptibility has been plotted for increasing values of Zeeman field.

Spin Susceptibility: PH Asymmetry

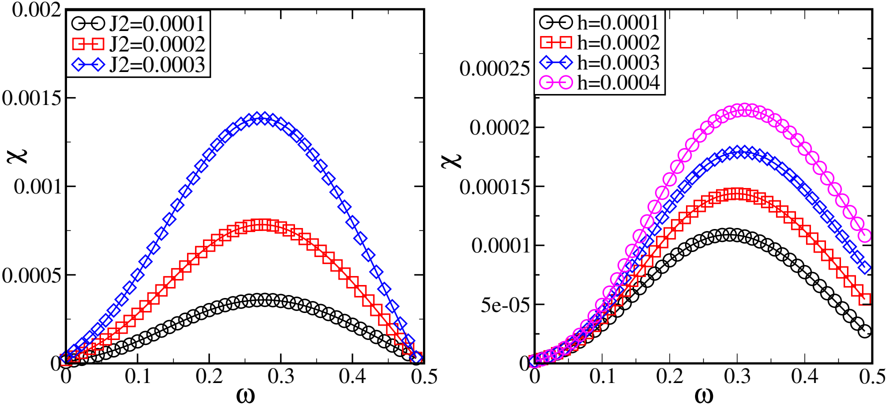


Figure : Dynamic spin susceptibility for asymmetric Majorana-Kondo model: In the left panel, spin susceptibility has been plotted on the frequency axis for increasing values of J_2 coupling. In right panel, spin susceptibility has been plotted for increasing values of Zeeman field.

Conclusions

- We have studied Majorana-Kondo model using Wegner's flow equation method. We have calculated flow equations both for zero and finite temperature.
- We find that particle-hole asymmetry is a relevant perturbation for Majorana-Kondo model and hence we have explored the model both in PH symmetric and asymmetric cases.
- Relevance of PH asymmetry is one important signature of Majorana fermion in Kondo physics.
- Away from the PH symmetry there is an emergent Zeeman field which breaks time reversal symmetry.
- At PH symmetry we find that Kondo effect is very stable to Majorana couplings.
- Away from PH symmetry Kondo scale gets suppressed.
- We found that dynamical spin susceptibility gets enhanced due to Majorana couplings.
- Since spin susceptibility can be probed in experiments we propose that it can be used for the detection of Majorana fermions.

Renormalization flows of Majorana-Kramers-Kondo model

For the case of time reversal symmetric topological superconductor, there is a pair of Majorana modes at each edge and hence the resulting effective Hamiltonian for NL-QD-TRITOPS has been called Majorana-Kramers-Kondo model(MKKM).

$$
\begin{aligned}
H_{MKKM} = & \sum_{k\sigma} \epsilon_k c^{\dagger}_{k\sigma} c_{k\sigma} + \sum_{kk'} J(k,k') S.s_{kk'} - J_0(k) i \gamma_{\uparrow}\gamma_{\downarrow} S^y + \\
& \sum_k J_1(k) i(\gamma_{\uparrow}(c_{k\uparrow} + c^{\dagger}_{k\uparrow}) - \gamma_{\downarrow}(c_{k\downarrow} + c^{\dagger}_{k\downarrow})) + \\
& \sum_k J_2(k) i(\gamma_{\uparrow}(c_{k\uparrow} + c^{\dagger}_{k\uparrow}) + \gamma_{\downarrow}(c_{k\downarrow} + c^{\dagger}_{k\downarrow})) S^z + \\
& \sum_k J_3(k) i(\gamma_{\uparrow}(c^{\dagger}_{k\downarrow} S^+ + c_{k\downarrow} S^- - \gamma_{\downarrow}(c_{k\uparrow} S^{\dagger}) + c^{\dagger}_{k\uparrow} S^-))
\end{aligned}
\tag{11}
$$

where $J(k,k') = t^2\zeta_{1+}$, $J_0 = \lambda^2\zeta_{1+}$, $J_1 = t\lambda\zeta_{-}$, $J_2 = J_3 = t\lambda\zeta_{2+}$

Numerical solution of flow equations of MKKM

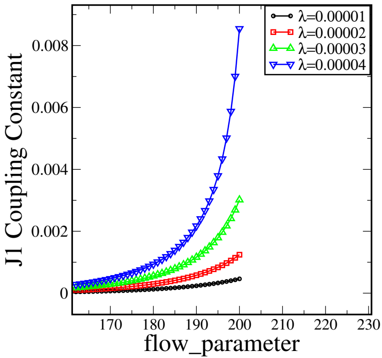


Figure : In left panel, J1 coupling constant is plotted versus flow parameter. In right panel, J2 coupling constant is plotted versus flow parameter. Other parameters are: U= 6.0, ϵ_d= -3.0, t= 4.0.

Flow of Kondo couplings

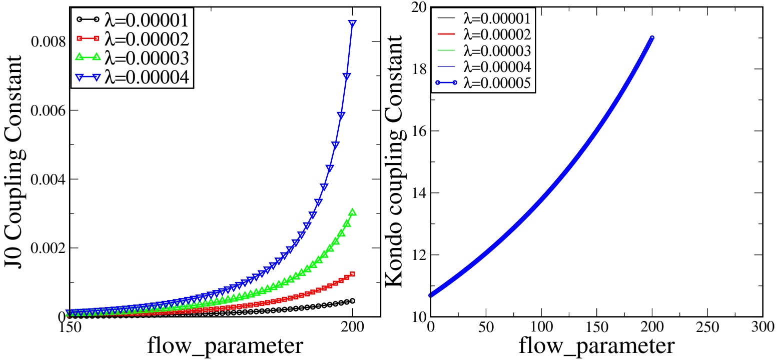


Figure : In the left panel, flow of J0 coupling constant has been plotted. In the right panel, flow of Kondo coupling constant has been plotted. Other parameters are same as given in Figure 6.1

Flow of Spin operator

$$C(\omega) = \frac{\pi}{4} \sum_p \left(\tilde{\gamma}^2_{\epsilon_p, \epsilon_p + \omega} n_f(\epsilon_p)(1 - n_f(\epsilon_p + \omega)) + n_f(\epsilon_p + \omega)(1 - n_f(\epsilon_p)) \right) \quad (12)$$

This correlation is related to dynamical spin susceptibility via fluctuation-dissipation theorm. The tilde sign denotes that γ co-efficient is in the limit $l \to \infty$.

$$\chi(\omega) = tanh(\frac{\omega}{2T})C(\omega) \quad (13)$$

To compute dynamic susceptibility we need to solve the flow equations for the spin observable numerically.

Dynamic spin susceptibility

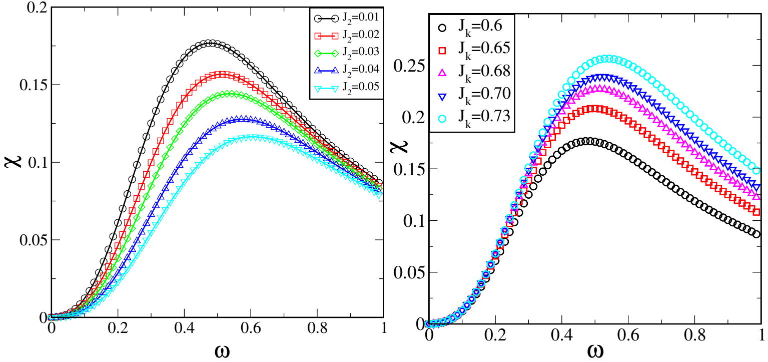


Figure : Dynamical spin susceptibility plots for Majorana-Kramers-Kondo model : In left panel, spin susceptibility has been plotted on frequency axis for different J_2 coupling constants. In right panel, spin susceptibility has been plotted for increasing values of Kondo coupling constant.

Ongoing Projects

1. Renormalization group studies of Majorana-Kondo model and its variants.
2. Flow equation renormalization study of Many-body localization in topologically ordered systems. Effects of disorder in Kitaev chain model.
3. Floquet driven systems and their non-equilibrium quantum dynamics... Schrieffer-Wolff transformation in Floquet space. Flow equation method for Floquet driven systems.
4. Local quantum criticality in quantum impurity systems with non-trivial density of states..Pseudogap Kondo model, Kitaev-Kondo model, Kondo effect in Dirac and Weyl semimetals.