Densities for the Navier–Stokes equations with noise

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ABSTRACT. One of the most important open problem for the Navier–Stokes equations concerns regularity of solutions. There is an extensive literature devoted to the problem, both in the non–random and randomly forced case. Existence of densities for the distribution of the solution is a probabilistic form of regularity and the course addresses some attempts at understanding, characterizing and proving existence of such mathematical objects.

While the topic is somewhat specific, it offers the opportunity to introduce the mathematical theory of the Navier–Stokes equations, together with some of the most recent results, as well as a good selection of tools in stochastic analysis and in the theory of (stochastic) partial differential equations.

In the first part of the course we give a quick introduction to the equations and discuss a few results in the literature related to densities and absolute continuity. We then present four different methods to prove existence of a density with respect to the Lebesgue measure for the law of finite dimensional functionals of the solutions of the Navier-Stokes equations forced by Gaussian noise.

Each of the four methods has some advantages, as well as disadvantages. The first two methods provide a qualitative result, while the other two provide quantitative estimates and ensure a bit of regularity.

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Preface

These pages are the lecture notes for the Winter school on stochastic analysis and control of fluid flow which will take place at the School of Mathematics of the Indian Institute of Science Education and Research in Thiruvananthapuram from December 3 to 20, 2012. I would like to thank the organisers, the professors S. Dharmatti, R. K. George, U. Manna, A. K. Nandakumaran and M. P. Rajan for their kind invitation.

The subject of this work has recently come to my attention and I have chosen to present it during the Winter school in Thiruvananthapuram because if on the one hand it deals with fairly specific topics in the theory of Navier–Stokes equations with Gaussian noise, on the other hand it gives the chance to present and apply several classical tools of stochastic analysis. I hope this selection of tools and results will be of interest for all the participants to the school.

Introduction

Understanding the laws of solutions of the Navier–Stokes equations is a major goal, due to the paramount importance of the questions of uniqueness and regularity for this problem [Fef06]. The stochastic PDE allows for different formulations of uniqueness and the weakest concerns the distribution of the solutions.

In this lectures we deal with the Navier–Stokes equations with additive Gaussian noise, which is white in time and coloured in space. It is well known that as such, the status of our knowledge on the problem is quite similar to the case of the equations without noise (i. e. deterministic). On the other hand the recent literature has shown that some information can be obtained if the noise is sufficiently non–degenerate and this already indicates that the noise is indeed effective in improving our understanding of the problem.

The main subject of these notes, the existence of densities of the laws of the solutions, is part of this program. Existence of densities can be considered as a sort of smoothness property of the solution, albeit a purely probabilistic one, hence of interest for the theory of Navier–Stokes equations.

The first difficulty is that the solution belongs to an infinite dimensional space, where no standard "flat" reference measure is available. It is tempting to use a Gaussian measure which is related to the problem, such as the one given by the covariance of the driving force, but this idea apparently fails already at the level of the two–dimensional problem.

The idea we broadly use in these notes is to reduce the problem back to finite dimension, where the Lebesgue measure is available, by means of finite dimensional approximations. A standard mathematical tool to study densities of random variables is the Malliavin calculus, but unfortunately it seems hopeless to use Malliavin calculus for our problem. Indeed, it is not even possible to prove that the solutions are Malliavin differentiable. The reason for this is immediately apparent once one notices that the equation satisfied by the Malliavin derivative is essentially the linearisation of Navier–Stokes, and any piece of information on that equation could be used with much more proficiency for proving well–posedness. Other methods are necessary and the presentation of a few of them is the main aim of these notes.

The notes are organised as follows. The first chapter is a warm–up, we state the notations that will be used throughout the work, together with a few basic standard results. The second chapter is divided into two parts. In the first part we provide some motivations for the analysis of densities that hopefully should convince the reader that these notes are worth keep reading. In the second part we consider the problem of the reference measure in infinite dimensions. There are also some results of existence of (infinite dimensional) densities for some stochastic PDE, and a result of existence and smoothness of densities for approximations of 3D Navier–Stokes.

In the third chapter we finally enter into the main subject of these notes. We briefly introduce Malliavin calculus and we show how it can be used to prove existence of densities. These ideas are then used to study the problem in the two–dimensional case. The three–dimensional case is analysed as well for a special class of solutions, those having the Markov property. This is motivated by the fact that such solutions can be reduced, by localisation, to smooth processes.

The fourth chapter is devoted to the problem of equivalence and singularities of Gaussian measures and the Girsanov theorem. This stuff is classical, but it is useful to introduce the second method for proving existence of densities for 3D Navier–Stokes with noise.

The fifth chapter introduces a new method for dealing with problems which are not amenable to Malliavin calculus. A suggestive description is that we do a fractional integration by parts and measure fractional derivatives in Besov spaces.

Finally, in the sixth chapter we provide an alternative proof to the regularity result of the fifth chapter by means of the Fokker–Planck equation. This is a fairly classical approach, and Malliavin calculus was conceived to provide a probabilistic counterpart to the analytical techniques used for the Fokker–Planck equation.

We recall a list of recommended papers. Some of the content of these notes is taken from the following papers and books.

- Franco Flandoli, *An introduction to 3D stochastic fluid dynamics*, SPDE in hydrodynamic: recent progress and prospects, Lecture Notes in Math., vol. 1942, Springer, Berlin, 2008, Lectures given at the C.I.M.E. Summer School held in Cetraro, August 29–September 3, 2005, Edited by Giuseppe Da Prato and Michael Röckner, pp. 51–150.
- Robert S. Liptser and Albert N. Shiryaev, *Statistics of random processes. I*, expanded ed., Applications of Mathematics (New York), vol. 5, Springer-Verlag, Berlin, 2001, General theory, Translated from the 1974 Russian original by A. B. Aries, Stochastic Modelling and Applied Probability.

- Jonathan C. Mattingly and Étienne Pardoux, *Malliavin calculus for the stochastic 2D Navier-Stokes equation*, Comm. Pure Appl. Math. **59** (2006), no. 12, 1742–1790.
- Arnaud Debussche and Marco Romito, *Existence of densities for the 3D Navier-Stokes equations driven by Gaussian noise*, 2012, arXiv:1203.0417.

CHAPTER 1

Setting of the problem

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This is a technical chapter, where we introduce the equations, the most standard results and the notation we shall use throughout the paper. in the last part of the chapter we wil give a formulation of the equations in components with respect to a basis of eigenvalues of the Stokes operator A, and the Galerkin approximation based on this expansion.

There are several good references for the mathematical theory of the Navier–Stokes equations, for instance the books of Temam [**Tem77**, **Tem95**, **Tem97**], of Constantin and Foias [**CF88**], the series of books of P. L. Lions [**Lio96**, **Lio98**], or the recent developments in the book of Lemarié–Rieusset [**LR02**]. The connection between the equations and the phenomenological theory of turbulence is the main theme of the book by Frisch [**Fri95**] (see also [**FMRT01**]).

Standard references for stochastic PDE are [**DPZ92**, **Wal86**], see also [**Roz90**, **PR07**]. For the Navier–Stokes equations driven by noise the interested reader can look at the lecture notes of Flandoli [**Fla08**], Debussche [**Debar**] or the books by Kuksin [**Kuk06**, **KS12**].

1.1. The Navier–Stokes system

We shall work with the Navier–Stokes equation either with periodic boundary conditions on the d–dimensional torus $\mathbb{T}_d = [-\pi, \pi]^d$ or with Dirichlet boundary conditions on a bounded domain $D \subset \mathbf{R}^d$ with smooth enough (Lipschitz most of the time, but for our purposes more smoothness does not change the qualitative value of the results) boundary,

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \\ \operatorname{div} u = 0, \end{cases} \tag{1.1}$$

where $u = u(t,x) \in \mathbf{R}^d$, with $x \in D$ or $x \in \mathbb{T}_d$, is the velocity field, $p = p(t,x) \in \mathbf{R}$ is the pressure field and ν is the viscosity. The equations and the quantities involved have precise physical meanings, but we will not give detail in this direction and keep the above system as a mathematical model, which anyway retains all the difficulties of the model [Fef06].

1.1.1. Function spaces and the Leray projection. As it is costumary after the work of J. L. Lions, we introduce the function spaces where we shall study the equations. Although some properties and result may be true also in higher dimensions, it is understood that in the following we work either in dimension $d = 2$ or $d = 3$.

Dirichlet boundary conditions. Consider the space of test functions $\mathscr{T}$ as the sub–space of $C_c^\infty(D, \mathbf{R}^d)$ of smooth vector fields with zero divergence. Denote by H the closure of $\mathscr{T}$ with respect to the L^2 norm and by V the closure of $\mathscr{T}$ with respect to the H^1 norm. It turns out that

$$H = \{u \in L^2(D, \mathbf{R}^d) : \operatorname{div} u = 0,\ u \cdot \vec{n} = 0\},$$

where $\vec{n}$ is the normal vector to the boundary, $\operatorname{div} u$ is understood as distributional derivative and $u \cdot \vec{n}$ as a trace. Likewise,

$$V = \{u \in H_0^1(D, \mathbf{R}^d) : \operatorname{div} u = 0\}.$$

Under our assumptions on the domain, the Poincaré inequality holds,

$$\|u\|_H \leqslant \|\nabla u\|_{L^2}$$

In a similar way, one can define Sobolev spaces with more derivatives. We shall do this in a different way in the nest section.

Periodic boundary conditions. This time we define the space of test functions $\mathscr{T}$ as the sub–space of $C^\infty(\mathbf{R}^d, \mathbf{R}^d)$ of smooth 2π–periodic (in all d coordinate directions) with mean zero,

$$\int_{\mathbb{T}_d} u(y)\, dy = 0,$$

and again H and V as the closure of $\mathscr{T}$ with respect to the L^2 and H^1 norm respectively (computed on $\mathbb{T}_d$). A characterization of these spaces is

$$H = \{u \in L^2(\mathbb{T}_d, \mathbf{R}^d) : \operatorname{div} u = 0,\ u \cdot \vec{n}|_{\Gamma_i} + u \cdot \vec{n}|_{\Gamma_{i'}} = 0, i = 1, \dots, d\}$$

and

$$V = \{u \in H^1(\mathbb{T}_d, \mathbf{R}^d) : \operatorname{div} u = 0,\ u|_{\Gamma_i} = u|_{\Gamma_{i'}}, i = 1, \dots, d\}$$

where Γ_i, $\Gamma_{i'}$ are opposite faces of $\mathbb{T}_d$. Likewise, the Poincaré inequality holds.

Before turning to the next section, we wish to give an explanation on the "mean zero" condition. Let m_u be the mean of u on $\mathbb{T}_d$ and assume that u is solution of (1.1). By integrating the equations we see that by integration by parts, and due to the periodic boundary conditions, all terms with spatial derivatives disappear. The only term which is not apparent is the non–linearity, but a little bit of algebra shows that, due to the divergence zero, $(u \cdot \nabla)u = \operatorname{div}(u \otimes u)$. Hence the equation for m_u singles out to

$$\frac{d}{dt} m_u = 0,$$

that is m_u remains constant. In fact, physically speaking, it corresponds to the velocity of the center of mass, hence the velocity of the fluid as a rigid body. If in a reference system moving with the center of mass, this velocity is zero and we can (almost) safely assume that $m_u = 0$.

The Leray projection. Denote by Π_L the projection of $L^2(\mathbb{T}_d, \mathbf{R}^d)$ (or $L^2(D, \mathbf{R}^d)$ if with Dirichlet boundary conditions) onto H. Then $L^2(\mathbb{T}_d, \mathbf{R}^d)$ decomposes in the direct sum of H and its orthogonal $H^\perp$ which turns out to be the space of gradients,

$$H^\perp = \{\nabla p : p \in H^1(\mathbb{T}_d) \text{ and periodic}\}.$$

This is a key point for the reconstruction of the pressure in the existence theorems.

The Leray projection has a simple expression in the periodic case. Indeed consider the (complex) Fourier exponentials

$$f_k(y) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{ik \cdot y}, \qquad y \in \mathbb{T}_d,$$

with $k \in \mathbf{Z}^d_\star = \mathbf{Z}^d \setminus \{0\}$. Every $u \in H$ can be expanded as

$$u = \sum_{k \in \mathbf{Z}^d_\star} u_k f_k$$

with $f_k \in \mathbf{C}^d$ and $f_k \cdot k = 0$ (this is the divergence–free constraint). A few computations show that for every $k \in \mathbf{Z}^d_\star$,

$$(\Pi_L u)_k = u_k - \frac{u_k \cdot k}{|k|^2} k.$$

1.1.2. The Stokes operator. Define the Stokes operator $A = \Pi_L(-\Delta)$, so that A is the realisation of $-\Delta$ with the corresponding boundary conditions and the divergence–free constraint. The operator A is positive self–adjoint with compact inverse, hence there exists an orthonormal basis $(e_n)_{n\in\mathbf{N}}$ of H of eigenvectors of A.

Let $(\lambda_n)_{n\in\mathbf{N}}$ be the corresponding eigenvalues listed in non–decreasing order and repeated according to their multiplicity. Define for each $a \in \mathbf{R}$ the space

$$V_a = \Big\{u \in \mathscr{T}' : \sum_{n\in\mathbf{N}} \lambda_n^{2a} u_n^2 < \infty\Big\},$$

where $(u_n)_{n\in\mathbf{N}}$ are the coefficients of u with respect to the basis $(e_n)_{n\in\mathbf{N}}$. It turns out that $V_0 = H$, $V_1 = V$, $V_{-1} = V'$, where V' is the topological dual of V (in general, for $a > 0$, $V_{-a} = V_a'$) and for m integer, V_m is the closure of $\mathscr{T}$ with respect to the $W^{m,2}$–norm.

1.1.3. The non–linear operator. Define the operator

$$B(u_1, u_2) = \Pi_L\big((u_1 \cdot \nabla)u_2\big)$$

and $B(u) = B(u, u)$. The most fundamental property of the non–linear operator B is

$$\langle u_1, B(u_2, u_3)\rangle_H = -\langle u_3, B(u_2, u_1)\rangle_H, \tag{1.2}$$

which in particular for $u_1 = u_3$ yields $\langle u_1, B(u_2, u_1)\rangle_H = -\langle u_1, B(u_2, u_1)\rangle_H$, hence $\langle u_1, B(u_2, u_1)\rangle_H = 0$. The theory of weak solutions for Navier–Stokes since the work of Leray [Ler34] is based on this property.

Several inequalities are satisfied by the operator B, the most basic being

$$\langle u_1, B(u_2, u_3)\rangle \leqslant c\|u_1\|_V\|u_2\|_V\|u_3\|_V$$

where in the formula above the left–hand side is understood as a duality between V and V'. In other words, $B : V \times V \to V'$. It is useful to check the spaces where B is bi–linear continuous. For instance the following result [Tem95, Lemma 2.1] analyses Hilbert–Sobolev spaces.

LEMMA 1.1. *If a_1, a_2, a_3 are non–negative real numbers such that $a_1+a_2+a_3 \geqslant \frac{d}{2}$ (and $a_1+a_2+a_3 > \frac{d}{2}$ if at least one is equal to $d/2$), then there exists $c = c(a_1, a_2, a_3)$ such that*

$$|\langle u_1, B(u_2, u_3)\rangle| \leqslant c\|u_1\|_{V_{a_1}}\|u_2\|_{V_{a_2}}\|u_3\|_{V_{a_3+1}}$$

An extension to a slightly larger set of parameters is proved via Fourier series expansion in the periodic case [FR08, Rom11a]. Here we state it for dimension $d = 2, 3$,

$$\|B(u)\|_{V_\alpha} \leqslant c_\alpha\|u\|^2_{V_{\theta(\alpha)}}, \tag{1.3}$$

for $\alpha > -1$ and $\alpha \neq \frac{d}{2}-1$ (which corresponds to the critical value for the Sobolev embeddings), where $\theta(\alpha) = \left(\frac{d}{4}+\frac{\alpha+1}{2}\right)\vee(\alpha+1)$. To get an extension to the whole space or bounded domain, one can use the mapping properties of Besov spaces for the product of functions, see for instance [**RS96**][1]. A useful special case of the above inequality 1.3, which we will use in a future chapter, is

$$\|A^{\frac{1}{2}}B(u_1,u_2)\|_H \leqslant c\|Au_1\|_H\|Au_2\|_H, \qquad u_1,\ u_2 \in D(A). \tag{1.4}$$

1.1.4. The abstract form. With all the positions at hands, (1.1) can be recast in an abstract form as an ordinary differential equation on an infinite dimensional space as

$$\frac{du}{dt} + \nu A u + B(u,u) = 0,$$

and existence theorems are stated for the initial value problem

$$\begin{cases} \dfrac{du}{dt} + \nu A u + B(u,u) = 0, \\ u(0) = u_0 \in H. \end{cases} \tag{1.5}$$

The fourth unknown, the pressure, is then reconstructed from u. This may be a delicate matter which we will not delve into.

Let us show how (1.2) provides the energy inequality, and hence, up to technical details, existence of weak solutions. Compute the derivative in time of the energy of the solution, then by (1.2),

$$\begin{aligned}\frac{d}{dt}\|u(t)\|_H^2 = 2\langle u(t), u'(t)\rangle_H = \\ = -2\nu\langle u(t), Au(t)\rangle - 2\langle u(t), B(u(t),u(t))\rangle = \\ = -2\nu\langle u(t), Au(t)\rangle = -2\nu\|u(t)\|_V^2,\end{aligned}$$

since A is self–adjoint. Hence we obtain the main *a–priori* estimate

$$\|u(t)\|_H^2 + 2\nu\int_0^t \|u(s)\|_V^2\, ds = \|u(0)\|_H^2 \tag{1.6}$$

(which is only an inequality in dimension three). With the above–mentioned technical details this yields the following result.

THEOREM 1.2. *For every initial condition $u_0 \in H$ there exists at least a global weak solution u of (1.5), namely a functional u such that*

- $u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;V)$,
- *for every $\psi \in D(A)$ and $t \geqslant 0$,*

$$\langle u(t) - u_0, \psi\rangle_H + \int_0^t \langle u(s), A\psi\rangle_H\, ds - \int_0^t \langle u(s), B(u(s),\psi)\rangle\, ds = 0,$$

[1] Kudos to D. Blömker for this reference.

- $u : [0,\infty) \to H$ *is continuous when* H *is endowed of the weak topology and* $u(0) = u_0$.

The technical details that connect the energy inequality to existence of weak solutions are, typically, a good approximation procedure (see Section 1.3 below for instance) and good compactness theorems.

In dimension $d = 2$ weak solutions are also unique. Indeed let u_1, u_2 be two solutions with a common initial condition and $w = u_1 - u_2$. Then

$$\frac{d}{dt}w + \nu A w + B(u_1, w) + B(w, u_2) = 0$$

and the energy estimate (one of the two non–linear terms does not disappear!) yields

$$\frac{d}{dt}\|w\|_H^2 + 2\nu\|w\|_V^2 + 2\langle w, B(w, u_2)\rangle = 0$$

and, by the Ladyzhenskaya inequality $\|\cdot\|_{L^4}^2 \leqslant c\|\cdot\|_H\|\cdot\|_V$ and Young's inequality,

$$\langle w, B(w, u_2)\rangle \leqslant \|u_2\|_V\|w\|_{L^4}^2 \leqslant \nu\|w\|_V^2 + \frac{c}{\nu}\|u_2\|_V^2\|w\|_H^2.$$

Gronwall's lemma and the a–priori estimate (1.6) imply that $w \equiv 0$.

One can also prove that, again in dimension $d = 2$, the weak solution is also smooth. In dimension $d = 3$ this is not the case and a *strong solution* on an interval $[0, T]$ is a solution $u \in L^\infty([0, T]; V) \cap L^2([0, T]; D(A))$. Strong solution are unique in the class of strong solutions, but also in the class of weak solutions that satisfy the energy inequality (1.6). This last result is known as weak–strong uniqueness and we shall see a stochastic version in Theorem 3.8 of Chapter 3.

1.2. The stochastic PDE

Let $\mathcal{C} \in \mathscr{L}(H)$ be a non–negative self–adjoint trace-class operator and let $(W_t)_{t\geqslant 0}$ be a cylindrical Wiener process on H on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$. The stochastic version of (1.5) is

$$du + (\nu A u + B(u, u))\, dt = \mathcal{C}^{\frac{1}{2}}\, dW, \tag{1.7}$$

where this time u is an H–valued stochastic process. The addition of the probabilistic framework adds a new level of complexity and we may have weak and strong solutions in the probabilistic sense. For this and other motivation, a "probabilistic weak" solution will be called *martingale solution*.

DEFINITION 1.3 (weak martingale solution). Given a probability measure μ on H, a weak martingale solution with initial distribution μ is a couple of processes $(u_t, W_t)_{t\geqslant 0}$ on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$ such that

- $u \in C(0,\infty; H_w) \cap L^2(0,\infty; V)$, $\mathbb{P}$–a. s., where H_w is the space H endowed with the weak topology,

- for every $\psi \in D(A)$ and $t \geqslant 0$,

$$\langle u(t) - u_0, \psi\rangle_H + \int_0^t \langle u(s), A\psi\rangle_H \, ds - \int_0^t \langle u(s), B(u(s), \psi)\rangle \, ds = \langle \mathcal{C}^{\frac{1}{2}} W_t, \psi\rangle,$$

- the law of $u(0)$ is μ.

In stochastic analysis *strong solution* is a random field that solves the equation corresponding to a *given* driving Wiener process. In different words, for a strong solution the driving noise is part of the data, while for martingale solutions is part of the solution. In the case of strong solutions, for a matter of measurability properties which are not detailed, an application of Doob's lemma tells us that there is a measurable map such that, roughly speaking, $u = \Phi(W)$.

In dimension $d = 2$ the same proof of uniqueness we have given in the previous section yields path–wise uniqueness that, together with existence of martingale solutions, ensures existence of a (probabilistically) strong weak solution. Existence of strong solutions is an open problem in dimension $d = 3$.

THEOREM 1.4. *Given a probability measure on* H *with finite second moment, there exists at least a weak martingale solution with initial distribution* μ.

As in the case of the non–random problem, the proof of this result is based on an approximation procedure, together with some compactness criteria, which are applied at the level of the law of the trajectories.

A standard approximation is, again, the Galerkin method, although for instance other approximation methods can give additional properties such as a local energy balance [**Rom10**].

Itô formula on reader's favourite approximation applied to $\|u(t)\|_H^2$ provides (in the limit) the stochastic version of the energy balance we have computed in the previous section. Here we explain a different estimate, based on an idea of [**Tem95**], which provides a moment of the solutions in $D(A)$. For further details and improved estimates see [**Oda06**, **Rom08**].

First apply Itô's formula on $\|u(t)\|_V^2$ (this time the non–linear term does not disappear),

$$d\|u\|_V^2 = -2\nu\|Au\|_H^2 \, dt - 2\langle u, B(u)\rangle \, dt + 2\langle Au, \mathcal{C}^{\frac{1}{2}} dW\rangle + \sigma_1^2 \, dt,$$

with $\sigma_1^2 = \operatorname{Tr}(A\mathcal{C})$ and we need a bit more smoothing from $\mathcal{C}$ (although with additive noise the requirement can be weakened). Then, Itô's formula on the function $F(u) = (1 + \|u(t)\|_V^2)$ yields

$$dF(u) = -\frac{d\|u\|_V^2}{(1 + \|u\|_V^2)^2} + \frac{\|A\mathcal{C}u\|_H^2}{(1 + \|u\|_V^2)^3} \, dt,$$

hence

$$2\nu\frac{\|Au\|_H^2}{(1+\|u\|_V^2)^2}\,dt \leqslant dF(u)\,dt + 2\frac{\langle u, B(u)\rangle_V}{(1+\|u\|_V^2)^2}\,dt + \sigma_1^2\frac{1}{(1+\|u\|_V^2)^2}\,dt$$
$$+ 2\frac{\langle Au, \mathcal{C}^{\frac12}dW\rangle}{(1+\|u\|_V^2)^2}.$$

Integrate on $[0,t]$, use Lemma 1.1 and Hölder's inequality to estimate the non–linearity and take expectation to get

$$\nu\mathbb{E}\Big[\int_0^t \frac{\|Au\|_H^2}{(1+\|u\|_V^2)^2}\,ds\Big] \leqslant 1+\sigma_1^2 t + \frac{c}{\nu^3}\mathbb{E}\Big[\int_0^t (1+\|u\|_V^2)\,ds\Big].$$

The right–hand side is finite by the energy estimate, hence another application of Hölder's inequality finally yields

$$\mathbb{E}\Big[\int_0^t \|Au\|_H^{\frac23}\,ds\Big] \leqslant 1+\sigma_1^2 t + c_\nu\mathbb{E}\Big[\int_0^t (1+\|u\|_V^2)\,ds\Big]. \tag{1.8}$$

1.3. Galerkin approximations

In this last section we give additional details on one of the possible approximations of (1.7), namely the Galerkin approximations, in view of the next chapters. With obvious changes, the following considerations may apply also to problem (1.5).

Let $e_1, e_2 \dots$ be the basis of eigenvectors of A, with corresponding eigenvalues $\lambda_1, \lambda_2, \dots$ and set for every $N \geqslant 1$, $H_N = \mathrm{span}[e_1, \dots, e_N]$. Let $\pi_N : H \to H_N$ be the projection of H onto H_N and set

$$A_N = \pi_N A \pi_N, \qquad B_N(\cdot,\cdot) = \pi_N B(\pi_N\cdot, \pi_N\cdot), \qquad \mathcal{C}_N = \pi_N\mathcal{C}\pi_N.$$

The N^{th} Galerkin approximation is given by the following equation

$$du^N + \big(\nu A_N u^N + B_N(u^N, u^N)\big)\,dt = \pi_N\mathcal{C}^{\frac12}dW, \tag{1.9}$$

This is a stochastic differential equation on $H_N \approx \mathbf{R}^N$ which admits a unique strong solution for every initial condition in H_N. A thorough analysis of this equations can be found in [Fla08]. Here we only remark that by applying the Itô formula to $\|u^N\|_{H_N}^2$ (where norm and scalar product of $H_N \approx \mathbf{R}^N$ are chosen so that they corresponds to the norm and scalar product of H),

$$\|u^N(t)\|_{H_N}^2 + 2\int_0^t \|u^N\|_{V_N}^2\,ds = \|u^N(0)\|_{H_N}^2 + 2\int_0^t \langle u^N, \pi_N\mathcal{C}^{\frac12}dW_s\rangle_{H_N} + \mathrm{Tr}(\mathcal{C}_N)t,$$

where $V_N = \pi_N V$. The stochastic integral is a martingale and hence one can get the estimate

$$\mathbb{E}\Big[\sup_{[0,T]}\|u^N(t)\|_{H_N}^2 + \int_0^T \|u^N\|_{V_N}^2\,ds\Big] \leqslant c,$$

with a bounding number c independent from N. It turns out that every limit point of the sequence of laws $(\mathbb{P}^N)_{N\geqslant 1}$, where each u^N has distribution $\mathbb{P}^N$, satisfies the same bound. With similar methods, one can also obtain the following estimate, again uniform in N, that we will use in the following chapters,

$$\mathbb{E}\Big[\sup_{[0,T]}\|u^N(t)\|_H^p\Big] \leqslant c_p(1+\|u^N(0)\|_H^p), \tag{1.10}$$

for every $p \geqslant 1$ and $T > 0$, where c_p depends only on p, T and the trace of $\mathcal{C}$.

Let $(q_n)_{n\in\mathbf{N}}$ be a basis of eigenvectors of $\mathcal{C}$ and let $(\sigma_n^2)_{n\in\mathbf{N}}$ be the corresponding eigenvalues in non–increasing order and repeated according to their multiplicity. Then for a sequence $(\beta_n)_{n\in\mathbf{N}}$ of independent one–dimensional Brownian motions,

$$\mathcal{C}^{\frac{1}{2}}W_t = \sum_{n=1}^{\infty}\sigma_n\beta_n(t)q_n.$$

Set $c_{mn} = \langle q_m, e_n\rangle$ (this is the matrix for the change of basis), then

$$\pi_N\mathcal{C}^{\frac{1}{2}}W_t = \sum_{n=1}^{N}\Big(\sum_{m=1}^{\infty}\sigma_m\beta_m(t)c_{mn}\Big)e_n.$$

Let now $u_1,\dots,u_N$ be the components of u^N, namely $u_n = \langle u^N, e_n\rangle$, and set $b_{nlm} = \langle e_n, B(e_l, e_m)\rangle$ (hence by property (1.2) $b_{nlm}+b_{mln} = 0$ and $b_{nmn} = 0$). With these positions, equation (1.9) reads in components as

$$du_n + \Big(\nu\lambda_n u_n + \sum_{l,m=1}^{N} b_{nlm}u_l u_m\Big)\,dt = \sum_{m=1}^{\infty} c_{mn}\sigma_m\,d\beta_m.$$

1.3.1. The periodic setting. In the case of periodic boundary conditions we have an explicit form for both the eigenvalues and eigenvectors of A. In order to make things even simpler we assume that the covariance $\mathcal{C}$ and the Stokes operator A commute. Hence without loss of generality $q_n = e_n$ and the matrix $c_{mn} = \delta(m-n)$.

The Fourier exponentials

$$f_k = e^{ik\cdot x}, \qquad k\in\mathbf{Z}^d$$

provide a (complex) basis for $L^2(\mathbb{T}_d)$ which is very convenient, for instance the divergence–free constraints reads $k\cdot u_k = 0$ if $u = \sum_k u_k f_k$. The non–linearity computed on $u = \sum_k u_k f_k$ and $v = \sum_k v_k f_k$ gives

$$B(u,v) = i\sum_{k\in\mathbf{Z}^d_\star}\Big(\sum_{m+n=k}(u_m\cdot n)\Pi_{L,k}v_n\Big)f_k,$$

where $\Pi_{L,k}$ is the k–component of the Leray projection. For a series of reasons (it gives complex valued coefficients, if $u = \sum u_k f_k$ is real valued, then $\overline{u_k} = u_{-k}$, etc.) it is also useful to look for a real basis. As in the one–dimensional case, a real basis can be given it terms of trigonometric functions. Set

$$\mathbf{Z}^d_+ = \{k \in \mathbf{Z}^d : k_d > 0\} \cup \{k \in \mathbf{Z}^d : k_{d-1} > 0, k_d = 0\} \cup \dots$$
$$\dots \cup \{k \in \mathbf{Z}^d : k_1 > 0, k_2 = \dots = k_d = 0\},$$

$\mathbf{Z}^d_- = -\mathbf{Z}^d_+$ and $\mathbf{Z}^d_\star = \mathbf{Z}^d_+ \cup \mathbf{Z}^d_-$, and

$$e_k = \begin{cases} \sin k \cdot x, & k \in \mathbf{Z}^d_+, \\ \cos k \cdot x, & k \in \mathbf{Z}^d_-, \end{cases} \tag{1.11}$$

then $(e_k)_{k \in \mathbf{Z}^d_\star}$ is a (orthogonal) basis of $L^2(\mathbb{T}_d)$–functions with zero mean (see for instance [MB02]). To make it an orthonormal basis, we should multiply each e_k by the constant $a_d = \sqrt{2}(2\pi)^{-\frac{d}{2}}$, but we will not do it for simplicity. If $d = 2$ and $E_k = \frac{k^\perp}{|k|} e_k$ for every $k \in \mathbf{Z}^2_\star$, then $(E_k)_{k \in \mathbf{Z}^2_\star}$ is a basis of H, where $k^\perp = (-k_2, k_1)$. In the three–dimensional case there is no obvious choice for the two eigenvectors orthogonal to k.

The equations in Fourier modes, two dimension. Here we follow the construction of [MP06] (see also [EM01]). In dimension two it is more convenient to work with the *vorticity*, $\xi = \operatorname{curl} u$. Without noise, the equations satisfied by the vorticity is, by differentiating each term of (1.1),

$$\frac{\partial \xi}{\partial t} + u \cdot \nabla \xi = \nu \Delta \xi.$$

If $u = \sum_k u_k e_k$ and $\xi = \sum_k \xi_k e_k$, then $\xi_k = u_k \cdot k^\perp$. Since u is divergence–free (which reads in Fourier modes as $k \cdot u_k = 0$), the map $u \mapsto \xi$ is invertible (this is the *Biot–Savart law*) and

$$u_k = \frac{k^\perp}{|k|^2} \xi_k.$$

The equation for the vorticity in Fourier modes is then given as follows,

$$\dot{\xi}_k + \nu |k|^2 \xi_k + \frac{1}{2} \sum_{m,n,k \in \mathscr{I}_+} (m^\perp \cdot n) \Big(\frac{1}{|m|^2} - \frac{1}{|n|^2} \Big) \xi_m \xi_n + \\ - \frac{1}{2} \sum_{m,n,k \in \mathscr{I}_-} (m^\perp \cdot n) \Big(\frac{1}{|m|^2} - \frac{1}{|n|^2} \Big) \xi_m \xi_n = \sigma_k \dot{\beta}_k, \tag{1.12}$$

where $\mathscr{I}_+$ is the set of indices

$$\{(m,n,k) \in \mathbf{Z}^2_+ \times \mathbf{Z}^2_- \times \mathbf{Z}^2_+ \cup \mathbf{Z}^2_- \times \mathbf{Z}^2_+ \times \mathbf{Z}^2_+ \cup \mathbf{Z}^2_+ \times \mathbf{Z}^2_+ \times \mathbf{Z}^2_- : m + n + k = 0\}$$
$$\cup \{(m,n,k) \in \mathbf{Z}^2_+ \times \mathbf{Z}^2_- \times \mathbf{Z}^2_+ \cup \mathbf{Z}^2_- \times \mathbf{Z}^2_- \times \mathbf{Z}^2_- \cup \mathbf{Z}^2_+ \times \mathbf{Z}^2_+ \times \mathbf{Z}^2_- : k = m - n\}$$
$$\cup \{(m,n,k) \in \mathbf{Z}^2_- \times \mathbf{Z}^2_+ \times \mathbf{Z}^2_+ \cup \mathbf{Z}^2_- \times \mathbf{Z}^2_- \times \mathbf{Z}^2_- \cup \mathbf{Z}^2_+ \times \mathbf{Z}^2_+ \times \mathbf{Z}^2_- : k = n - m\},$$

and

$$\mathscr{I}_- = \{(m,n,k) \in \mathbf{Z}^2_- \times \mathbf{Z}^2_+ \times \mathbf{Z}^2_+ \cup \mathbf{Z}^2_+ \times \mathbf{Z}^2_- \times \mathbf{Z}^2_+ \cup \mathbf{Z}^2_- \times \mathbf{Z}^2_- \times \mathbf{Z}^2_- : k = m+n\}.$$

The equations in Fourier modes, three dimension. Here we follow the computations of [**Rom04**]. Write the velocity in terms of the real basis $(e_k)_{k\in\mathbf{Z}^3_\star}$,

$$\sum_{k\in\mathbf{Z}^3_\star} u_k e_k$$

and recall that the condition on the divergence of u reads $u_k \cdot k = 0$ and that the Leray projection acts on each mode k as

$$\Pi_{L,k} u_k = \Big(u_k - \frac{k\cdot u_k}{|k|^2} k\Big).$$

where we have interpreted it as a map from $\mathbf{R}^3$ to $\mathbf{R}^3$. Given $k \in \mathbf{Z}^3_\star$, define the map $b_k : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ as

$$b_k(a,b) = (k\cdot a)\Pi_{L,k} b + (k\cdot b)\Pi_{L,k} a,$$

then the Navier–Stokes equations read as

$$\dot u_k + \nu|k|^2 u_k + \frac12 \sum_{\substack{m,n<0\\ m+n+k=0}} b_k(u_m,u_n) - \frac12 \sum_{\substack{m,n>0\\ m+n=k}} b_k(u_m,u_n) + \sum_{\substack{m,n>0 \text{ or } m,n<0\\ m-n=k}} b_k(u_m,u_n) = \sigma_k \dot\beta_k$$

for $k \in \mathbf{Z}^3_+$, and

$$\dot u_k + \nu|k|^2 u_k + \sum_{\substack{m<0,n>0\\ m+n+k=0}} b_k(u_m,u_n) + \sum_{\substack{m<0,n>0\\ m-n=k}} b_k(u_m,u_n) - \sum_{\substack{m<0,n>0\\ m+n=k}} b_k(u_m,u_n) = \sigma_k \dot\beta_k$$

for $k \in \mathbf{Z}^3_-$, where for simplicity we have written $m > 0$ as a shorthand for $m \in \mathbf{Z}^3_+$, and $m < 0$ for $m \in \mathbf{Z}^3_-$. In the two equations above the $(\beta_k)_{k\in\mathbf{Z}^3_\star}$ is a sequence of *two–dimensional* independent standard Brownian motions and each σ_k is a 3×2 matrix with real entries such that $k^T \cdot \sigma_k = 0$. This condition is the mode–by–mode version of the assumption that the covariance has range on divergence–free vector fields.

CHAPTER 2

Some motivations and related results

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This chapter has a two–fold aim. On the one hand we present some motivations that should support the interest in the analysis of densities of the time–marginal laws of the solutions of (1.1). On the other hand we we are faced with the infinite dimension and the problem of having a reasonable reference measure with respect to which we should look for densities. This problem can be partially solved only in some cases and Section 2.2.2 should convince the reader that our problem is not included among them.

This is one of the reasons we turn to finite dimensional functionals and finite dimensional projections of the solution in the chapters that follow. In the last part of this chapter we show that finite dimensional *approximations* of (1.1), namely the Galerkin approximations of section 1.3, have densities with respect to the Lebesgue measure. The result holds even for highly degenerate noise.

2.1. Motivations

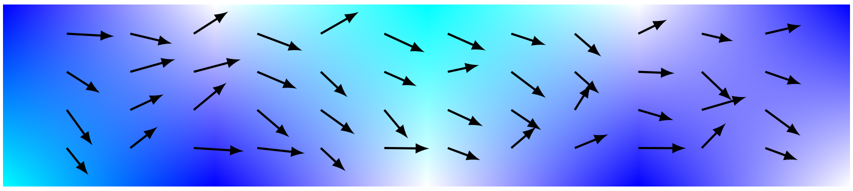
This section aims to convince the reader about the interest in studying existence of densities for solutions of stochastic PDEs and in particular existence of densities for finite dimensional projections.

It is a standard statement from any basic course in probability that the most important facts about a random variable are carried by its law. Knowing that the random variable has a density, as well as having some information on the regularity of this density, adds up to our knowledge and it is clear that looking for densities is a problem worth to be studied even without the (more or less rigorous) motivations we give below.

We will see later in Chapter 5 that an explicit density is the crucial part of a proof of weak uniqueness for stochastic differential equations.

2.1.1. Motivation from sampling the velocity of a real fluid. This section is intentionally rather vague, due to the fact that it gives some heuristic ideas on a topic that is not part of the mathematical analysis of fluids evolution.

Most of the real–life experiments to evaluate the velocity of a fluid are based on a finite number of samples in a finite number of points (Eulerian point of view), or by tracing some particles (smoke, etc...) moving according to the fluid velocity (Lagrangian point of view). The literature on experimental fluid dynamics is huge. Here we refer for instance to [Tav05] for some examples of design of experiments.



Let us focus on the Eulerian point of view. To simplify, consider a torus (similar consideration for a finite box with homogeneous Dirichlet boundary conditions), then sampling the velocity field means measuring the velocity in some space points $y_1, \dots, y_d$,

$$u \rightsquigarrow (u(t, y_1), \dots, u(t, y_d))$$

and a bit of Fourier series manipulations shows that this is a "projection", since

$$u(t, y) = \sum_k u_k(t) e_k(y) = \sum_k \langle u(t), e_k \rangle \, e^{ik\cdot y} = \Big\langle u(t), \sum_k e^{ik\cdot y} \, e_k \Big\rangle$$

where $e_k(y) = e^{ik\cdot y}$ is the Fourier basis. Unfortunately our theory will allow this kind of projections only under the assumptions of Chapter 3, leaving out the methods of Chapters 5, 6 and 7.

2.1.2. Motivations from some previous work. The main interest in studying densities comes from some previous work, done partially in collaboration with F. Flandoli on the existence of Markov solutions for stochastic PDEs. If either the stochastic PDE has multiple weak solutions or if weak uniqueness is open, the standard proofs for existence of a Markov process fail. A smart idea of Krylov [Kry73] allows to prove existence of Markov processes for stochastic differential equations without weak uniqueness. The idea is then adapted in [FR08] for stochastic PDEs, in particular the Navier–Stokes equations.

Let us briefly explain the underlying ideas, we will be more precise in section 3.3.1. Assume one can prove existence of at least one global weak solution of the problem under examination for each initial condition in the state space H. The laws of these solution are probability measures on the path space $\Omega = C([0,\infty);\mathcal{H})$, where $\mathcal{H}$ is a suitable space larger than H, for instance the domain of a sufficiently large negative power of A. A natural filtration $(\mathscr{F}_t)_{t\geqslant 0}$ can be defined on Ω, given for each t by the Borel sets of $C([0,t];\mathcal{H})$.

A Markov solution (or selection) is a family $(\mathbb{P}_x)_{x\in H}$ of probability measures on Ω such that for each $x \in H$ $\mathbb{P}_x$ is a (suitably defined, see Definitions 3.3 and 3.4) weak martingale solution of Navier–Stokes equations (1.1) with initial condition x, and such that the Markov property holds, namely for (almost) each time $t > 0$,

$$\mathbb{P}_x|_{\mathscr{F}_t} = \mathbb{P}_{\xi_t}, \qquad \mathbb{P}_x - a.s.,$$

where $\xi_t : \Omega \to \mathcal{H}$ is the canonical process, defined as $\xi_t(\omega) = \omega(t)$, and $\omega \mapsto \mathbb{P}_x|_{\mathscr{F}_t}(\omega)$ is a version of the regular conditional probability distribution of $\mathbb{P}_x$ given $\mathscr{F}_t$.

The knowledge of Markov selections is crucial in the problem of uniqueness. In [**FR08**] we proved the following strong uniqueness result, under suitable assumptions on the covariance (which essentially amount to non–degeneracy and enough smoothing).

THEOREM 2.1. *Assume there are an initial condition x_0, a time $t_0 > 0$ and a solution $\widehat{\mathbb{P}}_{x_0}$ to the martingale problem starting at x_0, such that*

$$\widehat{\mathbb{P}}_{x_0}[\tau_\infty \geqslant t_0] = 1,$$

where τ_∞ is the blow–up time (that is, the lifespan of the smooth solution). Then pathwise uniqueness holds for every initial condition.

A thorough and general analysis of the blow–up time has been later done in section 5 of [**Rom11c**].

A corresponding weak uniqueness result can also be given. Roughly speaking, the theorem below says that if two irreducible Markov processes coincide over a small time interval, then they are equal.

THEOREM 2.2. *Consider two arbitrary Markov selections $(\mathbb{P}^1_x)_{x\in H}$ and $(\mathbb{P}^2_x)_{x\in H}$. If there are an initial condition x_0 and a time $t_0 > 0$ such that $\mathbb{P}^1_{x_0} = \mathbb{P}^2_{x_0}$ on $[0,t_0]$, then $\mathbb{P}^1_x = P^2_x$ for all initial conditions x.*

The theorem above has been later extended in [**Rom08**]. Assume each Markov solution has a unique invariant measure (as it is the case for the 3D Navier–Stokes with suitable noise [**DPD03**, **FR08**]). Uniqueness of invariant measures implies uniqueness of the processes, and in the end the invariant measure carries all the important information on the solution.

THEOREM 2.3. *Consider two arbitrary Markov selections* $(\mathbb{P}^1_x)_{x\in H}$ *and* $(\mathbb{P}^2_x)_{x\in H}$ *and assume that each selection has an invariant measure* μ_i, $i = 1, 2$. *If* $\mu_1 = \mu_2$, *then* $\mathbb{P}^1_x = P^2_x$ *for all initial conditions* x.

The most fundamental result related to Markov solutions, in view of this lecture notes, states that if one considers two invariant measures which are equilibrium states of two different Markov solutions, then the two probability measures, albeit not equal, are *equivalent measures* [**Rom08**]. Actually, the result also holds if one considers the law of two different solutions at some time [**FR07**].

THEOREM 2.4. *Consider two arbitrary Markov kernels* $P_i(t, x, \cdot)_{t\geqslant 0, x\in H}$, $i = 1, 2$. *Then for every* x, y *and every* $s, t > 0$ *the measures* $P_1(t, x, \cdot)$ *and* $P_2(s, y, \cdot)$ *are equivalent measures.*

Moreover, if each Markov kernel has a (unique) invariant measure μ_i, $i = 1, 2$, *then* μ_1 *and* μ_2 *are equivalent measures.*

It is thus of paramount importance to understand these densities, as on the one hand this can provide an approach to prove uniqueness, or it may help to understand th difference between two solutions with the same initial condition.

The equivalence result has also an interesting interpretation in terms of quantification of the uncertainty related to the fluid flow governed by the equations[1]. Indeed, any information collected through the equations which has probability one, is true regardless of the equilibrium state (or of the solution) one considers. In different words, the almost sure information obtained are reliable *regardless* of the equilibrium analysed and this validate the robustness of our model, even in the case non–uniqueness would be true.

This also opens an intermediate problem, since an L^p–like estimate on the densities between different invariant measures would further strenghten this interpretation. Indeed, with a more quantitative estimate one could extend the reliability to events which are of small (or large) probability and which would remain of small (large) probability regardless of the equilibrium state one considers.

EXAMPLE 2.5 (On reliability of numerics for non unique problems). Let me recall one of my favourite arguments on numerics for problem without uniqueness. Consider the classical problem

$$\dot{x} = 2\sqrt{x},$$

with initial condition $x(0) = 0$. A simple computation shows that the one–step explicit Euler approximation x_h of the above problem yields $x_h(nh) = 0$ for every time–step. On the other hand the one–step implicit Euler approximation yields $x_h(nh) = (nh)^2$, namely the maximal solution. The reader should

[1]Many thanks to P. Constantin for suggesting me this interpretation of the equivalence result of the invariant measures.

consider that this subject is under analysis by several authors and the above example is clearly not exhaustive!

At this stage we do not have any result in the directions explained just above. A first step in the analysis is provided in the next chapters 3, 5, 6, 7, in which we show that suitable finite dimensional functionals of weak solutions of 3D Navier–Stokes admit densities with respect to the Lebesgue measure.

2.2. Infinite–dimensional densities

As we will see in Chapter 5 with details and with several examples, equivalence of probability measures and in the overall existence of densities in the infinite dimensional setting is not a common situation, so to say.

The first main problem is the lack of a good *flat* reference measure, as is the Lebesgue measure in finite dimension. Indeed, some (if not most) of the times the main problem is to understand which is the right reference measure.

The most simple example where a reference measure naturally appears is given by a gradient flow perturbed by a suitable noise. The problem admits an invariant measure of exponential type, which is explicitly known.

A natural generalization to this are problems where, for instance, all one–dimensional time marginals are equivalent to the invariant measure. The difference with the previous case is that there is no explicit expression for the invariant measure. We shall see an example of this in Chapter 3. One of our motivations previously stated in this chapter, more precisely Theorem 2.4, is partly a non–trivial consequence of this fact.

The most direct way to obtain a density is through Girsanov's theorem. We will see here a few not very detailed examples here, and we postpone a thorough discussion on the topic to Chapter 5.

2.2.1. Gradient flows. Our first example is a stochastic PDE where a meaningful reference measure, namely the invariant measure, does exist and can be explicitly given. Moreover for each initial condition the law of the process at any positive time is equivalent to this measure.

To understand how this works we start first with its finite dimensional counterpart. We will see that the inifinite dimensional case is a direct generalisation.

The finite–dimensional case. Consider the following one dimensional stochastic differential equation,

$$dX = -f(X)\,dt + \sqrt{2}\,dW. \tag{2.1}$$

The measure $\mu = \frac{1}{Z}\,e^{-F(x)}\,dx$ is an invariant measure of the above diffusion, where $F' = f$ and $Z = \int e^{-F(x)}\,dx$ is the normalising constant. Under reasonable minimal growth assumptions on F, the total mass Z is finite and μ is a probability measure. Moreover, by asking a bit more on F, one can even ensure that μ is the unique invariant probability measure.

To see that μ is invariant, the quickest way is to verify that its density satisfies the Fokker–Planck equation associated to (2.1). We will give more details on this tooic in Chapter 7 and here we give only a quick glimpse of the idea. Assume, as it actually is, that X_t has a density $p(t,x)$ with respect to the Lebesgue measure, then for $\phi \in C^2_b(\mathbf{R})$, by the Itô formula,

$$d\phi(X_t) = -f(X_t)\phi'(X_t)\,dt + \sqrt{2}\phi'(X_t)\,dW_t + \phi''(X_t)\,dt,$$

hence by integration by parts,

$$\begin{aligned}\frac{d}{dt}\mathbb{E}[\phi(X_t)] &= \mathbb{E}[\phi''(X_t) - f(X_t)\phi'(X_t)] = \\ &= \int\big(\phi''(x) - f(x)\phi'(x)\big)p(t,x)\,dt = \int\big(p_x x + (fp)_x\big)\phi\,dx.\end{aligned}$$

We notice that if $p(t,x) = \frac{1}{Z}e^{-F(x)}$, then $p_x x + (fp)_x = 0$, $\mathbb{E}[\phi(X_t)]$ remains constant and μ is invariant.

The same holds in larger but finite dimension if one is careful enough to ensure the existence of a primitive of the drift,

$$dX_t = -\nabla F(X_t)\,dt + \sqrt{2}\,dW,$$

where W is a d–dimensional Wiener process and $F : \mathbf{R}^d \to \mathbf{R}^d$. The invariant measure is again $\mu = \frac{1}{Z}e^{-F(x)}\,dx$ with Z normalising constant.

A last remark is that what we have said breaks down, at least at the level of getting explicit formulas, as long as the driving noise has a non–trivial covariance (i. e., different from identity). The problem still admits an invariant measure, but we cannot write it down explicitly.

The infinite dimensional case. Consider now the following stochastic PDE on $[0,1]$ with, for instance, homogeneous Dirichlet boundary conditions,

$$du = \Delta u - DF(u) + \sqrt{2}\,dW, \tag{2.2}$$

where W is a cylindrical Wiener process on $L^2(0,1)$, hence $\dot{W}$ is space–time white noise, and $F : L^2(0,1) \to \mathbf{R}$.

Let ν be the Gaussian measure $\nu = \bigoplus_k N(0, \frac{1}{2\lambda_k}$, where $(\lambda_k)_{k\geqslant 1}$ are the eigenvalues of $(-\Delta)$ with the given boundary conditions, that is ν is the law of the stationary solution

$$z_\star(t) = \sqrt{2}\int_{-\infty}^{t} e^{\Delta(t-s)}\,dW_s$$

of the linear equation

$$dz = \Delta z + \sqrt{2}\,dW,$$

with the same boundary conditions of (2.2). Then $\mu = \frac{1}{Z}e^{-F(x)}\nu$ is the invariant measure of (2.2) and the distribution of $u(t)$ for each initial condition in $L^2(0,1)$ is equivalent to μ.

To see that this last fact holds the idea is to prove that the transition semigroup generated by the equation (2.2) is *strong Feller*, that is the semigroup maps bounded measurable functions into continuous functions. This can be proved for instance through the *Bismut–Elworthy–Li formula* [**Bis81**, **EL94**]. A theorem of Khas'minskii (see for instance [**DPZ96**, Proposition 4.1.1]) ensures then that the transition probabilities are equivalent. Notice that the strong Feller property is equivalent [**Sei01**, **Hai09**] to continuity in total variation of transition probabilities.

For more details on this type of problems, one can refer to Chapter 11 of [**DPZ96**].

2.2.2. The Girsanov transformation. We stress again that we are going to be sloppy in the mathematics that follows. A more detailed introduction to the topic will be done in Chapter 5, to whom we refer for all details.

Consider the following prototypical (at least for us) equation

$$du + (Au + B(u))\,dt = \mathcal{C}^{\frac{1}{2}}\,dW,$$

where B is a bi–linear operator. Let us write the equation as

$$du + Au\,dt = \mathcal{C}^{\frac{1}{2}}\big(dW - \mathcal{C}^{-\frac{1}{2}}B(u)\,dt\big)$$

and use the Girsanov transformation of measure to have that, under a new probability $\widetilde{\mathbb{P}}$ which is absolutely continuous with respect to the old one $\mathbb{P}$, u coincides, on a finite time–span interval $[0,T]$, with the solution z of

$$dz + Az\,dt = \mathcal{C}^{\frac{1}{2}}\,d\widetilde{W},$$

where

$$\widetilde{W}_t = W_t - \int_0^t B(u_s)\,ds$$

is a Brownian motion on $[0,T]$ under $\widetilde{\mathbb{P}}$. The new probability $\widetilde{\mathbb{P}}$ is given through a density by

$$\widetilde{\mathbb{P}}(d\omega) = \exp\Big(-\int_0^T \langle \mathcal{C}^{-\frac{1}{2}}B(u), dW\rangle - \frac{1}{2}\int_0^T \|\mathcal{C}^{-\frac{1}{2}}B(u)\|_H^2\,ds\Big)\mathbb{P}(d\omega).$$

A classical condition which ensures that the above considerations are rigorous is the Novikov condition,

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^T \|\mathcal{C}^{-\frac{1}{2}}B(u)\|_H^2\,ds\Big)\Big] < \infty,$$

but we can quickly realise that in our example this cannot be possible. Indeed, $B(u)$ is quadratic, hence $\|B(u)\|^2$ is quartic and in general we do not expect u to be any better than the solution z to the linear problem obtained dropping the non–linearity (which is Gaussian). If we assume that both u and z are strong solutions of their respective equations, the Novikov condition can be replaced by the much weaker condition

$$\mathbb{P}\Big[\int_0^T \|\mathcal{C}^{-\frac{1}{2}}B(u)\|_H^2\,ds < \infty\Big] = 1.$$

Let us apply these computation to the Navier–Stokes in dimension 2. Assume for simplicity that $\mathcal{C} = A^{-\alpha}$, hence by (1.3),

$$\int_0^T \|\mathcal{C}^{-\frac{1}{2}}B(u)\|_H^2\,ds \leqslant c_\alpha \int_0^T \|u\|_{V_{\theta(\alpha)}}^4\,ds.$$

On the other hand, we expect u to behave, at least for what concerns the space regularity, as z, due to the fact that $u - z$ is more regular than z. A simple computation [**DPZ92**] shows that z is bounded in V_β with $\beta < \alpha + 1 - \frac{d}{2}$, hence $\beta < \alpha$. In conclusion we should have $\theta(\alpha) < \alpha$, which is plainly impossible.

A simple solution to overcome this difficulty is to increase the effect of the dissipation (see [**MS05**], see also [**Fer08**]), namely by replacing $-\Delta$ with $(-\Delta)^\gamma$ (or A with A^γ) for some γ large enough. With this replacement the above considerations are rigorous and Girsanov's theorem applies, although the replacement in a way changes the physics of dissipation.

A more interesting solution, introduced implicitly in [**MS05**] (see also [**MS08**]) and explained more explicitly in [**Wat10**], improves the above situation. It is odd that at the same time a completely different method proposed in [**DPD04**] leads to similar results. Let us explain the method in [**MS05**, **Wat10**], which is simpler and more direct. The idea is to show equivalence at the level of one–dimensional time marginal distributions, rather than at the level of trajectories, as given by the Girsanov transformation (do not forget that for a Markov process equality of the one dimensional marginals is enough to ensure equality of the laws on the full trajectories).

Here is the idea. Consider u as above, fix $t > 0$ and write the mild formulation,

$$u(t) = e^{-A(t-s)}\,u(0) - \int_0^t e^{-A(t-s)}\,B(u)\,ds + \int_0^t e^{-A(t-s)}\,\mathcal{C}^{\frac{1}{2}}dW.$$

By a simple change of variable in time,

$$\int_0^t e^{-A(t-s)}\,B(u)\,ds = \int_0^t e^{-\frac{1}{2}A(t-s)}\,e^{-\frac{1}{2}A(t-s)}\,B(u)\,ds = \int_0^t e^{-A(t-s)}\,G_t(s)\,ds,$$

where $G_t(s) = 2\,e^{-A(t-s)}\,B(u_{2s-t})\mathbb{1}_{[t/2,t]}$. Now, let y be the process

$$y(s) = e^{-A(s-r)}\,u(0) - \int_0^s e^{-A(s-r)}\,G_t(r)\,dr + \int_0^s e^{-A(s-r)}\,\mathcal{C}^{\frac{1}{2}}dW, \qquad s \in [0,t],$$

then $u(t)$ and $y(t)$ have the same distribution. On the other hand, y is path–equivalent to the linear problem if

$$\mathbb{P}\Big[\int_0^T \|\mathcal{C}^{-\frac{1}{2}}\,e^{-A(t-s)}\,B(u)\|_H^2\,ds < \infty\Big] = 1,$$

which is a much weaker condition due to the regularizing effect of the semigroup. In a way this condition is also optimal, as it essentially says that $B(u)$ is in the Cameron–Martin space of the Gaussian measure distribution of $z(t)$. Unfortunately this is not enough. Again (1.3) yields that $\|\mathcal{C}^{-\frac{1}{2}}\,e^{-A(t-s)}\,B(u)\|_H^2 < \infty$ if $\theta(\alpha) < \alpha + 1$, which is again impossible.

We finally remark that different ways to approach similar problems have been presented in [**BDPR96**, **DPD04**].

2.2.3. The 2D Navier–Stokes equations with delta–correlated noise. There is a special case for equation in fluid dynamics where one can exhibit an explicit invariant measure as in the previous sections. In this section we sketch a few ideas from [**DPD02**], where they are able to show that the Gaussian measure

$$\mu = \bigotimes_{k\in\mathbf{Z}^2_\star} \mathcal{N}\Big(0, \frac{1}{2\nu|k|^2}\Big)$$

is the invariant measure of the two–dimensional Navier–Stokes equations with periodic boundary conditions on the torus and driven by space–time white noise. The measure μ can be written formally as

$$\mu = \frac{1}{Z}\,e^{-\nu\|x\|_V^2}\,dx$$

with Z a normalizing constant and "dx" an infinite dimensional flat measure. Here the viscosity plays the role of the inverse temperature. The fact that μ is an invariant measure can be heuristically explained by the equality

$$\langle Au, B(u,u)\rangle = 0,$$

which complements (1.2) and which holds only in this special case (two dimensions, periodic boundary conditions).

The main difficulty here is that the roughness of the noise is such that the solution is not regular enough and should be interpreted as element of some suitable dual space. On the other hand a minimal condition to give meaning to the equations is $u \in L^2$, since for a test function ϕ,

$$\langle \phi, (u\cdot\nabla)u\rangle = -\langle u, (u\cdot\nabla)\phi\rangle = -\sum_{ij}\int u_i u_j \partial_j \phi_i.$$

A similar problem arises in stochastic quantization and it is solved with renormalization [**MR99**]. A similar idea is applied in this case, with the additional advantage that the non–linear terms need not be modified, since the renormalization constant disappears by diffeentiation. Indeed, for divergence free vector fields,

$$(u \cdot \nabla)u = \operatorname{div}(u \otimes u).$$

To be more precise, let π_N be the projection onto the space H_N generated by the modes of frequency at most N, define

$$\rho_N = \Big(\sum_{0<|k|\leqslant N} \frac{k_1^2}{|k|^4}\Big)^{\frac{1}{2}}$$

and

$$: \pi_N u \otimes \pi_N u := \begin{pmatrix} :(\pi_N u_1)^2: & \pi_N u_1 \pi_N u_2 \\ \pi_N u_1 \pi_N u_2 & :(\pi_N u_2)^2: \end{pmatrix},$$

where

$$:(\pi_N u_i)^2 := (\pi_N u_i)^2 - \rho_N^2.$$

Hence

$$B(\pi_N u) = \Pi_L \operatorname{div}(\pi_N u \otimes \pi_N u) = \Pi_L \operatorname{div}(: \pi_N u \otimes \pi_N u :).$$

It turns out that the above quantities admit a limit as $N \to \infty$ in $V_{-\epsilon}$ for every $\epsilon > 0$ and the limit quantity of $B(\pi_N \cdot)$ has all polynomial moments finite with respect to μ in $V_{-1-\epsilon}$. It is clear that there is nothing special here with finite dimensional projections and any other reasonable approximation would work as well.

Their first main result is that there is a stationary solution with law μ. By means of the stationary solution then they can prove existence of a global strong solution (i.e. same noise) for μ–a. e. initial condition. The idea, that we do not detail here, is that one can first show existence of a unique local solution by a fixed point theorem, and then use the stationary solution to propagate existence to all times.

2.3. Densities for the Galerkin approximations of Navier–Stokes

In this section we come back to the Galerkin approximations presented in Section 1.3 to give a sketch of the ideas that show that the solutions of the finite dimensional approximations have a smooth probability density with respect to the Lebesgue measure. This result goes beyond its intrinsic interest and will be used in the following chapters as a starting point for our proofs.

With the same notations of Section 1.3, fix $N \geqslant 1$ and consider the stochastic differential equation (1.9),

$$du^N + \big(\nu A_N u^N + B_N(u^N, u^N)\big)\,dt = \pi_N \mathcal{C}^{\frac{1}{2}} dW.$$

Assume for simplicity that $\mathcal{C}$ is diagonal in the basis of eigenvalues of A, hence we can write the above equation in components as

$$du_n + \Big(\nu\lambda_n u_n + \sum_{l,m=1}^{N} b_{nlm} u_l u_m\Big)\,dt = \sigma_n\,d\beta_n, \qquad n = 1,\dots,N. \tag{2.3}$$

The proof of smoothness of the result we state here is based on Malliavin calculus, although at this stage we will use it only as a *black–box* through the results of [**Nor86**]. A more detailed discussion on the topic is postponed to Chapter 3. We notice also that as a by–product of this approach we can allow that some of the coefficients σ_n of the noise may be zero (degenerate diffusion).

Let us define

$$X_0(u) = -\Big(\nu\lambda_n u_n + \sum_{l,m=1}^{N} b_{nlm} u_l u_m\Big)_{n=1,\dots,N},$$

and for every $x \in \{1,\dots,N\}$, $X_n(u) = \sigma_n\delta_n$, where $\delta_n = (\delta_{n1},\dots,\delta_{mn})$ is such that $\delta_{nm} = 1$ if $m = n$ and 0 otherwise. With these positions, the above system of stochastic differential equations becomes

$$du = X_0(u)\,dt + \sum_{n=1}^{N} X_n(u)\,d\beta_n. \tag{2.4}$$

To each vector field X_n, $n = 0,\dots,N$, we can associate the corresponding element in the Lie–algebra of operators

$$\mathfrak{X}_n = \sum_{j=1}^{n} X_{nj}(u)\frac{\partial}{\partial u_j},$$

and it turns out that the generaor of the diffusion defined by (2.4) (or equivalently by (2.3)) is

$$\mathscr{L} = \mathfrak{X}_0 + \frac{1}{2}\sum_{n=1}^{N}\mathfrak{X}_n^2.$$

We can define a *bracket* operation on vector fields (or on the corresponding differential operators) as

$$[X,Y] = \sum_{n=1}^{N}\Big(Y\frac{\partial X}{\partial u_n} - X\frac{\partial Y_n}{\partial u_n}\Big) = \mathfrak{Y}X - \mathfrak{X}Y.$$

The result [**Nor86**] we shall use is as follows. Consider a system such as (2.4), assume that the vector fields $X_0,\dots,X_N$ are C^∞ and that the vector fields

$$X_1,\dots,X_N, \quad [X_i,X_j]_{0\leqslant i,j\leqslant N}, \quad [X_i,[X_j,X_l]_{0\leqslant i,j,l\leqslant N}, \quad \dots \tag{2.5}$$

evaluated at each $u \in \mathbf{R}^N$ span the whole $\mathbf{R}^N$, then the solution of (2.4) has a smooth density at each $t > 0$ for all initial conditions. This is the probabilistic version of the celebrated Hörmander theorem for hypoelliptic operators [**Hör67**]. Roughly speaking, even though the generator $\mathscr{L}$ defined above is not elliptic, one can still prove regularizing properties. A proof of the theorem with probabilistic arguments has been the subject of intense research starting with [**Mal78**, **Bis81**, **Str81**, **Str83**] and has been the motivating origin of Malliavin calculus.

Turning back to the definition of our vector fields $X_0, \dots, X_N$, we see that $X_1, \dots, X_N$ are constant and X_0 is quadratic, hence a good number of the above brackets are zero. If we focus on the constant vector fields,

$$X_1, \dots, X_N, \quad [X_i, [X_j, X_0]_{0 \leqslant i,j \leqslant N}, \quad \dots \tag{2.6}$$

for instance

$$[X_l, [X_m, X_0] = \sum_k (b_{klm} + b_{kml}) \frac{\partial}{\partial u_k},$$

and if the above vector fields generate $\mathbf{R}^N$, then the result will hold for any initial condition. We state this result in the form of a theorem for future reference.

THEOREM 2.6. *Consider a stochastic differential equation of the form* (2.4)*, and assume that the vector fields* (2.5) *generate* $\mathbf{R}^N$ *when evaluated at each* $u \in \mathbf{R}^N$ *(or the constant vector fields* (2.6) *generate* $\mathbf{R}^N$*). Then the solutions of the stochastic differential equation has a smooth density with respect to the Lebesgue measure on* $\mathbf{R}^N$.

2.3.1. The periodic case. An obvious case in which the vector fields in (2.6) generate $\mathbf{R}^N$ is when $\sigma_n \neq 0$ for all $n = 1, \dots, N$, and to obtain $\mathbf{R}^N$ it is sufficient to consider only $X_1, \dots, X_N$ and no brackets.

On the other hand when there are some $\sigma_n = 0$, or when $\mathcal{C}$ does not commute with A, in general the condition on the brackets (2.5) or (2.6) is hard to verify. There is though a case where one can write explicitly the brackets in (2.6) (as well as (2.5)), and this is the case of periodic boundary conditions. As we have seen in Section 1.3.1, we know explicitly the eigenvectors e_n as well as the coefficients B_{nlm} of the non–linear operator.

The ideas we discuss here were first explained in [**EM01**] for the 2D Navier–Stokes equations with noise and here we present the three dimensional version of [**Rom04**]. Recall that one can write the problem with respect to the real basis (1.11) as

$$\dot{u}_k + \nu|k|^2 u_k + \frac{1}{2} \sum_{m,n \in \mathscr{I}_k} (-1)^{e_k(m,n)} b_k(u_m, u_n) = \sigma_k \dot{\beta}_k, \qquad |k|_\infty \leqslant N, \tag{2.7}$$

for suitable sets $\mathscr{I}_k$ and suitable exponents $e_k(m,n)$ which have been given explicitly in Section 1.3.1, and where $|k|_\infty = \max(|k_1|, |k_2|, |k_3|)$. Set $Z_N = \{k \neq 0 : |k|_\infty \leqslant N\}$.

Let $\mathcal{N}$ be the set of indices (*modes*) corresponding to those directions forced by the noise, that is the vector fields X_k for k such that $\sigma_k \neq 0$, in the notation of the previous section. Assume that $\mathcal{N}$ is symmetric, namely that $-\mathcal{N} = \mathcal{N}$. Let $A(\mathcal{N})$ be the set of indices k such that $|k|_\infty \leqslant N$ that correspond to the directions generated by the brackets, as explained in the first part of this section.

To be more precise in the definition of the set $A(\mathcal{N})$, let us clarify which is the state space of (2.7). Indeed, each u_k in (2.7) satisfies $k \cdot u_k = 0$ by the incompressibility condition. The state space of $(u_k)_{k\in Z_N}$ is

$$\mathcal{S} = \bigoplus_{k\in Z_N} k^\perp.$$

Let us denote by $\mathcal{S}_k$ the two–dimensional subspace of $\mathcal{S}$ corresponding to $k^\perp$. In the following we will often identify elements of $k^\perp$, which are vectors in $\mathbf{R}^3$, with elements of $\mathcal{S}_k$, which are elements of $\mathbf{R}^{\#(Z_N)}$ with most of the coordinates zero in the standard basis.

With these positions, we write $k \in A(\mathcal{N})$ if $\mathcal{S}_k$ is in the subspace generated by the constant brackets. It turns out that the elements of $A(\mathcal{N})$ obey to a simple set of algebraic rules, as stated by the following proposition (Lemma 4.2 of [**Rom04**]).

PROPOSITION 2.7. *Consider the Galerkin approximations of the 3D Navier–Stokes equations* (2.7)*. Let $\mathcal{N}$ and $A(\mathcal{N})$ be defined as above, and assume that $\mathcal{N}$ is non–empty and symmetric. Then the following properties hold.*

- *if $m \in A(\mathcal{N})$, then $-m \in A(\mathcal{N})$.*
- *if $m, n \in A(\mathcal{N})$, $|m+n|_\infty \leqslant N$, $|m| \neq |n|$ and m, n are linearly independent, then $m + n \in A(\mathcal{N})$.*

PROOF. We give the proof only of one case to give the idea. Let $m, n \in \mathbf{Z}^3_-$ and set $k = m + n$. Fix $x \in m^\perp$ and $y \in n^\perp$ and let $M_1 \in \mathcal{S}_m$ and $M_2 \in \mathcal{S}_{-m}$ be the vectors in $\mathcal{S}$ having the same non–zero coefficients of x, and define similarly $N_1 \in \mathcal{S}_m$ and $N_2 \in \mathcal{S}_{-m}$ with y. Lemma 4.1 of [**Rom04**] yields that $[[X_0, M_1], N_2] + [[X_0, M_2], N_1]$ is the element of $\mathcal{S}_k$ having the non–zero coefficients equal to those of $b_k(x, y)$, and $[[X_0, M_1], N_1] - [[X_0, M_2], N_2]$ is the element of $\mathcal{S}_k$ associated to $b_k(x, y)$.

Hence, if $m, n \in A(\mathcal{N})$, to show that $\pm k \in A(\mathcal{N})$ it is sufficient to prove that $\{b_k(x, y) : x \in m^\perp, y \in n^\perp\}$ spans $k^\perp$. Let $a, b \in k^\perp$ such that k, a, b is an orthogonal basis of $\mathbf{R}^3$ and $m, n \in \mathrm{span}[k, a]$. Write $x = x_1 k + x_2 a + x_3 b$ and $y = y_1 k + y_2 a + y_3 b$, then

$$b_k(x, y) = |k|^2(x_1 y_2 + x_2 y_1)a + |k|^2(x_1 y_3 + x_3 y_1)b.$$

The incompressibility condition reads

$$k \cdot m\, x_1 + a \cdot m\, x_2 = 0, \qquad k \cdot n\, y_1 + a \cdot n\, y_2 = 0.$$

The above system can be solved since m and n are not parallel, hence $a \cdot m = -a \cdot n \neq 0$ and

$$x_2 = -\frac{k \cdot m}{a \cdot m} x_1, \qquad y_2 = -\frac{k \cdot n}{a \cdot n} y_1.$$

By replacing the above equality in $b_k(x, y)$, we get

$$b_k(x, y) = -\frac{|k|^2}{a \cdot m}\big[(|m|^2 - |n|^2) x_1 y_1 \, a - (x_1 y_3 + x_3 y_1) \, b\big].$$

Now choose $x_1 = 0$ and $x_3 = y_1 = y_3 = 1$ to obtain b, and $x_1 = y_1 = 1$ and $x_3 = y_3 = 0$ ot obtain (a vector parallel to) a. □

REMARK 2.8. In the proof of the above result, if $m \parallel n$, then $m, n \parallel k$ and $b_k(x, y) = 0$ when $x \in m^\perp$, $y \in n^\perp$, and nothing can be done. On the other hand, if $m \not\parallel n$ but $|m| = |n|$, the procedure in the proof above yields that if $m, n \in A(\mathcal{N})$, then the one–dimensional space $\mathrm{span}[m, n]^\perp$ is in the space spanned by the constant vector fields. This is an important difference with respect to the 2D case of [**EM01**].

From this result and the bit of additional work suggested by the previous remark, one gets the following result, which provides a minimal set of forced modes that ensure smoothness of the densities. Other sets are equally possible and the rule of thumb is that to have $A(\mathcal{N}) = \{k \neq 0 : |k|_\infty \leqslant N\}$, one essentially needs that the subgroup of $(\mathbf{Z}^3, +)$ generated by $\mathcal{N}$ is $\mathbf{Z}^3$.

THEOREM 2.9. *Assume that $\mathcal{N}$ is symmetric and that $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are in $\mathcal{N}$, then the solution of (2.7) has a smooth density with respect to the Lebesgue measure at any positive time.*

CHAPTER 3

Malliavin calculus

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In this chapter we introduce the first method to show existence of densities. The method is based on Malliavin calculus, which may be considered the standard way to prove existence of densities. As we shall see in Chapter 6, the knowledge we have of the Navier–Stokes equations in three dimensions does not allow to prove Malliavin differentiability, so we shall need to work on some kind of approximation. This will be possible with the weak–strong uniqueness principle (Theorem 3.8), that states that for a short random time every weak solution from a regular initial condition is a strong solution. This implies equivalence of laws between weak and strong solution, at least for a special class of solution, those satisfying the Markov property.

In the first part of the chapter we give a (very) short introduction to Malliavin calculus and a short review of [**MP06**], where the stochastic calculus of variation is used to get densities of finite dimensional projections in the two dimensional case. Then we turn to the three dimensional case, we review the theory developed for the existence of Markov solutions, we apply Malliavin calculus to strong solutions and finally we use equivalence to conclude.

3.1. Densities with the Malliavin calculus

In this section we give a very short introduction to the Malliavin calculus, and we show how to use it to prove existence of a density for random variables. We follow mainly [Bal03], although the interested reader can also look at the standard reference [Nua06]. An introduction of Malliavin calculus with a view to stochastic PDEs is given in [SS05].

3.1.1. A very short introduction to the Malliavin calculus. Let $(W_t)_{t\geqslant 0}$ be a one dimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and consider a functional

$$F = F\big((W_t)_{t\in[0,1]}\big)$$

of Brownian paths. The goal is to define a derivative of W with respect to the probability variable $\omega \in \Omega$, or more precisely with respect to Brownian increments,

$$\mathcal{D}_s F \approx \frac{\partial F}{\partial(W_{s+h} - W_s)}.$$

A way to gove meaning to the above formula is to start with a special set of functionals where the formula makes sense and hope the set we have chosen is large enough to allow the extension of the definition to a reasonably large class of functionals.

Define the dyadic times $t^n_k = \frac{k}{2^n}$ for $k = 0, \dots, 2^n$ and set $\Delta^n_k = W_{t^n_{k+1}} - W_{t^n_k}$. The special set of functionals at level n is

$$\mathcal{S}_n = \big\{F = f(\Delta^n_0, \dots, \Delta^n_{2^n-1}) : f \in C^\infty_p(\mathbf{R}^{2^n})\big\}.$$

By a simple telescopic sum argument it is easy to see that the set $\mathcal{S}_n$ is equal to the set of functionals $F = f(W_0, \dots, W_{t^n_k}, \dots, W_1)$, with $f \in C^\infty_p(\mathbf{R}^{2^n}$. Let $\mathcal{S}_\infty = \bigcup_n \mathcal{S}_n$, then $\mathcal{S}_\infty \subset L^2(\Omega)$ (actually in every $L^p(\Omega)$), due to the fact that Gaussian measures have all the moments finite.

Define the *Malliavin derivative* of $F \in \mathcal{S}_\infty$ as the process

$$\mathcal{D}_s F = \sum_{k=1}^{2^n-1} \frac{\partial f}{\partial \Delta^n_k}(\Delta^n_0, \dots, \Delta^n_{2^n-1}) \mathbb{1}_{[t_k, t_{k+1}]}(s).$$

Define the set of simple processes

$$\mathcal{P}_n = \Big\{\sum_{k=0}^{2^n-1} U_k \mathbb{1}_{[t_k, t_{k+1})} : U_k \in \mathcal{S}_n\Big\},$$

and $\mathcal{P}_\infty = \bigcup_n \mathcal{P}_n$. By its definition it is clear that the Malliavin derivative maps $\mathcal{S}_n$ into $\mathcal{P}_n$.

The second important object we define is the *Skorokhod integral*. Given $U \in \mathcal{P}_n$, the Skorokhod integral is

$$\delta U = \sum_{k=0}^{2^n-1} U_k \Delta_k^n - \frac{1}{2^n} \sum_{k=0}^{2^n-1} \frac{\partial U_k}{\partial \Delta_k^n},$$

and clearly $\delta U \in \mathcal{S}_n$. There are two good reasons to call "integral" the Skorokhod integral. The first is that if U is an adapted process, then $\partial_{\Delta_k^n} U_k = 0$ and δU is the Itô stochastic integral $\int U_s \, dW_s$ of U. The second good reason is the fundamental duality relation that connects the Skorokhod integral and the Malliavin derivative,

$$\mathbb{E}\Big[\int_0^1 U_s \mathcal{D}_s F \, ds\Big] = \mathbb{E}[F \delta U], \qquad F \in \mathcal{S}_n, \ U \in \mathcal{P}_n.$$

The next step is to extend the Malliavin derivative and the Skorokhod integral to a larger class of random variables and processes. Since $\mathcal{S}_\infty \subset L^2(\Omega)$ and $\mathcal{P}_\infty \subset L^2(\Omega; L^2(0,1))$ are dense, the duality formula ensures that $\mathcal{D} : \mathcal{S}_\infty \to \mathcal{P}_\infty$ and $\delta : \mathcal{P}_\infty \to \mathcal{S}_\infty$ are closeable. The Malliavin derivative extends in $L^2(\Omega)$ to its domain of definition $\mathbb{D}^{1,2}$, which is given as

$$\mathbb{D}^{1,2} = \Big\{ F \in L^2(\Omega) : \|F\|_{1,2}^2 := \mathbb{E}[|F|^2] + \mathbb{E}\Big[\int_0^1 |\mathcal{D}_s F|^2 \, ds\Big] < \infty \Big\}, \tag{3.1}$$

in the sense that $\mathbb{D}^{1,2}$ is the closure of $\mathcal{S}_\infty$ with respect to the norm $\|\cdot\|_{1,2}$. Likewise for $p \geqslant 1$ define

$$\|F\|_{1,p}^p := \mathbb{E}[|F|^p] + \mathbb{E}\Big[\Big(\int_0^1 |\mathcal{D}_s F|^2 \, ds\Big)^{\frac{p}{2}}\Big]$$

and $\mathbb{D}^{1,p}$ as the closure of $\mathcal{S}_\infty$ with respect to the norm $\|\cdot\|_{1,p}$. Higher order derivatives can be defined by iterating the definition we have given and $\mathbb{D}^{k,p}$ is the closure with respect to the norm

$$\|F\|_{k,p}^p := \mathbb{E}[|F|^p] + \sum_{j=1}^k \mathbb{E}\Big[\Big(\int_0^1 \dots \int_0^1 |\mathcal{D}^{(j)}_{s_1 \dots s_j} F|^2 \, ds_1 \dots ds_j\Big)^{\frac{p}{2}}\Big].$$

The whole set of definitions given above have a strong analogy with the construction of Sobolev spaces with weak derivatives. An extension of the Malliavin derivative to the multi–dimensional case is possible and straightforward.

A fundamental property that holds for the Malliavin derivative is the *chain rule*. The property can be easily derived by standard properties of the usual derivative for simple functionals and extends to the general case by approximation. Let $\phi \in C^1_p(\mathbf{R})$ and $F \in \mathbb{D}^{1,2}$, then $\phi(F) \in \mathbb{D}^{1,2}$ and

$$\mathcal{D}\phi(F) = \phi'(F)\mathcal{D}F \tag{3.2}$$

The third important object we aim to define is the *Malliavin covariance matrix*. Given $F = (F_1, \dots, F_d) \in \mathbb{D}^{1,2}$, define

$$\mathcal{M}^F_{ij} = \langle \mathcal{D}F_i, \mathcal{D}F_j \rangle_{L^2(0,1)} = \int_0^1 (\mathcal{D}_s F_i)(\mathcal{D}_s F_j)\, ds, \qquad i, j = 1, \dots, d. \tag{3.3}$$

For instance, if F is Gaussian, say $F = QW_t$, where Q is a $d \times d$ matrix and $(W_t)_{t\geqslant 0}$ is d–dimensional Brownian motion, then F is a simple functional and $\mathcal{D}^\ell_s F_i = Q_{i\ell}\mathbb{1}_{[0,t]}(s)$[1] hence $\mathcal{M}^F_{ij} = t\sum_{\ell=1}^d Q_{i\ell}Q_{j\ell} = t(QQ^\star)_{ij}$. Since it will play a role in the sequel, we notice that the random variable $F = QW_t$ has a density with respect to the Lebesgue measure if and only if Q is invertible, and hence if and only if the Malliavin matrix is invertible. A crucial step in proving invertibility of the Malliavin matrix is the estimate of the smallest eigenvalue of $\mathcal{M}^F$.

Let us make a couple of examples. Let $F = \int_0^1 h(s)\, dW_s$, then $\mathcal{D}_s F = h(s)$ and, if h is vector valued,

$$\mathcal{M}^F_{ij} = \int_0^1 h_i(s)h_j(s)\, ds = \mathbb{E}[F_iF_j].$$

So in general if $F = f\big(\int_0^1 h(s)^1\, dW_s, \dots, \int_0^1 h(s)^d\, dW_s\big)$ with $f \in C^\infty_p(\mathbf{R}^d)$, then

$$\mathcal{D}_s F = \sum_{k=1}^d \frac{\partial f}{\partial x_k}\Big(\int_0^1 h(s)^1\, dW_s, \dots, \int_0^1 h(s)^d\, dW_s\Big) h^k_s$$

Actually the above class of functionals and the above formula may provide a different starting point for the definition of the Malliavin derivative. The operator $\mathcal{D}$ is closeable on this set of functionals and the final outcome is the same derivative operator $\mathcal{D}$ on the same domain $\mathbb{D}^{1,2}$. We remark for completeness that a third possible definition of Malliavin derivative can be given through Wiener chaos expansion, see [**Nua06**].

We use the above formula to give an heuristic interpretation of the Malliavin derivative as a derivative with respect to the Wiener process variable. The idea is to make variations in the Wiener process, those suggested by the Cameron–Martin theorem 5.4. Let $F = \int_0^1 h(s)\, dW_s$, then we can write F as a function of the process W as $F = \Phi((W_t)_{t\in[0,1]})$. Now consider the new random variable obtained by perturbing the Wiener process along an absolute continuous direction

$$W^\epsilon_t = W_t + \epsilon \int_0^1 v(s)\, ds$$

[1]To make the whole thing a bit clearer, we remark that the index ℓ in the derivative $\mathcal{D}^\ell_s F_i$ refers to the Malliavin derivative with respect to the ℓ component of the Wiener process increments.

and the derivative in ϵ at $\epsilon = 0$ yields,

$$\frac{\Phi(W + \epsilon \int \nu\,ds) - \Phi(W)}{\epsilon} = \int_0^1 h(s)\nu(s)\,ds = \langle \mathcal{D}F, \nu\rangle_{L^2(0,1)},$$

that is the Malliavin derivative of F in the direction ν.

This interpretation makes reasonably easy to compute derivatives of, for instance, solutions to stochastic differential equations. Let

$$dX = b(X)\,dt + \sigma(X)\,dW,$$

then a variation of the noise $W_\epsilon = W + \epsilon \int \nu\,ds$ induces a new stochastic equation

$$dX_\epsilon = b(X_\epsilon)\,dt + \sigma(X_\epsilon)\,(dW + \epsilon\nu\,ds),$$

hence

$$d\Big(\frac{X_\epsilon - X}{\epsilon}\Big) = \frac{b(X_\epsilon) - b(X)}{\epsilon} + \frac{\sigma(X_\epsilon) - \sigma(X)}{\epsilon}\,dW + \sigma(X^\epsilon)\nu\,ds$$

and in the limit the Malliavin derivative $\mathcal{D}_\nu X_t$ of X_t in the direction ν satisfies the following equation,

$$d\mathcal{D}_\nu X = b'(X)\mathcal{D}_\nu X\,ds + \sigma'(X)\mathcal{D}_\nu X\,dW_s + \sigma(X)\nu\,ds,$$

that can be also recast as

$$\mathcal{D}_s X_t = \sigma(X_s) + \int_s^t \sigma'(X_r)\mathcal{D}_s X_r\,dW_r + \int_s^t b'(X_r)\mathcal{D}_s X_r\,dr, \qquad s \leqslant t$$

and clearly $\mathcal{D}_s X_t = 0$ for $s > t$, since X does not depend on the future.

3.1.2. Densities and integration by parts. **[This part has not been completed yet]**

3.2. The two dimensional Navier–Stokes equations

Consider the Navier–Stokes equations with noise on the two–dimensional torus. Existence of densities for finite dimensional approximations was proved in [EM01]. Later in [MP06] the existence of densities was proved for finite dimensional projections of the full infinite dimensional process. We remark that the results of [MP06] have been a crucial step towards the fundamental results of [HM06], see also [HM11]. Detailed discussions can be found in [Mat08].

To see the assumptions of [MP06], we use the notation of Section § 1.3.1. The driving forcing is given as

$$W_t = \sum_{k\in\mathcal{Z}} \beta_k e_k,$$

where $(e_k)_{k\in\mathbf{Z}^2_\star}$ is the basis defined in (1.11) and $(\beta_k)_{k\geqslant 1}$ are independent one–dimensional standard Brownian motions. The set $\mathcal{Z}$ is finite and for instance one can choose $\mathcal{Z} = \{(1,0), (-1,0), (1,1), (-1,1)\}$ and the projection of the solution onto a finite number of Fourier modes has a smooth density with respect to the

Lebesgue measure. The kind of $\mathcal{Z}$ that yields the above results is completely characterized by some algebraic rules similar to those of Proposition 2.7 (see also Remark 2.8).

To cut the long story of [**MP06**] very short, the existence of a density follows from Theorem 2.1.2 of [**Nua06**] together with a representation of the Malliavin matrix by means of a backward partial differential equation with random coefficients. Smoothness follows from an estimate of the small eigenvalues of the projection of the Malliavin matrix.

[This part has not been completed yet]

3.3. The three–dimensional Navier–Stokes equations

In this last section we wish to apply the ideas of Malliavin calculus to the three–dimensional case. As already mentioned, since we do not have a unique (strong/weak) solution, we need to specify the kind of solutions we are going to consider. The special property that will characterise our solutions is that upon conditioning at a given time, the regular conditional probability distribution of a solution is a family of solutions (with appropriate initial condition). An abstract theorem of [**FR08**], adapted to the infinite dimensional setting from [**Kry73**] allows then to show the existence of Markov families of solutions.

Under suitable assumptions on the covariance, we can prove continuity with respect to the state–space variable of Markov kernels, and this is the key property to prove the equivalence of Markov kernels of these solutions with those of "smooth solutions".

The final step is to prove the existence of densities for the "smooth solutions", and this is done my means of Malliavin calculus. We are in a much simpler situation than in Section 3.2, since to prove continuity of Markov kernels we need an invertible covariance. In order to extend these results to degenerate covariances new ideas are necessary to avoid continuity and this is the subject of a work in progress.

The theory of Markov solutions and its developments is taken from [**FR06**, **FR08**, **FR07**, **Rom08**, **Rom09**, **Rom11a**], while the existence of densities is the first method presented in [**DR12**]

3.3.1. Markov solutions. Set $\Omega = C([0,\infty); D(A)')$, let $\mathscr{B}$ be the Borel σ-field on Ω and let $\xi : \Omega \to D(A)'$ be the canonical process on Ω, that is the coordinate process $\xi_t(\omega) = \omega(t)$. Define the filtration $\mathscr{B}_t = \sigma(\xi_s : 0 \leqslant s \leqslant t)$, which can be seen as the Borel σ-field of $C[0,t]; D(A)')$ if this space is seen as a subspace of Ω.

Assume the covariance $\mathcal{C}$ is a trace class operator on H. In this framework we give a slight modification of Definition 1.3 of weak martingale solution, which is anyway equivalent [**Fla08**]. For $\varphi \in D(A)$ consider the process $(M_t^\varphi)_{t\geqslant 0}$ on

Ω defined for $t \geqslant 0$ as

$$M_t^\varphi = \langle \xi_t - \xi_0, \varphi\rangle_H + \nu \int_0^t \langle \xi_s, A\varphi\rangle_H \, ds - \int_0^t \langle B(\xi_s, \varphi)_H, \xi_s\rangle \, ds.$$

DEFINITION 3.1 (Weak martingale solutions). Given a probability measure μ on H, a probability measure $\mathbb{P}_\mu$ on $(\Omega, \mathscr{B})$ is a weak martingale solution of Navier–Stokes equations with initial condition μ if

- the marginal at time $t = 0$ of $\mathbb{P}_\mu$ is equal to μ,
- $\mathbb{P}_\mu[L^2_{loc}([0,\infty); H)] = 1$,
- for each $\varphi \in D(A)$ the process $(M_t^\varphi, \mathscr{B}_t)_{t\geqslant 0}$ on $(\Omega, \mathscr{B}, \mathbb{P}_\mu)$ is a square integrable continuous martingale with quadratic variation $t\|\mathcal{C}^{\frac12}\varphi\|_H^2$.[2]

This definition assigns a minimal regularity to the trajectories of $\mathbb{P}_\mu$, the one that allows to write down the equations (the martingales M^φ here). Since in principle (in contrast to limit points of Galerkin solutions, as in Section 1.3), we cannot prove rigorously the energy inequality, the main tool for weak solutions of Navier–Stokes since the seminal work of Leray [**Ler34**], we will add it as part of the definition. This can be done in two ways, while keeping the conditional property explained above.

Super–martingale energy solutions. The first notion of solution is given in terms of super–martingales [**FR08**]. Before proceeding, we need to introduce a slightly different variant of super–martingale. This is required by the lack of continuity in time with values in H (endowed with the strong topology) of Navier–Stokes trajectories.

DEFINITION 3.2 (a. s. super-martingale). An adapted process $(\theta_t, \mathscr{B}_t, \mathbb{P})_{t\geqslant 0}$ is an *a. s. super–martingale* if $\mathbb{E}^{\mathbb{P}}|\theta_t| < \infty$ for all $t \geqslant 0$ and there is a Lebesgue null set $T_\theta \subset (0,\infty)$ such that

$$\mathbb{E}^{\mathbb{P}}[\theta_t \,|\, \mathscr{B}_s] \leqslant \theta_s$$

holds for every $s \notin T_\theta$ and every $t \geqslant s$. The set T_θ will be called the set of *exceptional times* of θ.

A short overview of properties of a. s. super–martingales is given in [**FR08**, Appendix B]. Define for each $n \geqslant 1$ the process $(E_t^n)_{t\geqslant 0}$ on Ω defined as

$$E_t^n = \|\xi_t\|_H^{2n} - \|\xi_0\|_H^{2n} + 2n\nu \int_0^t \|\xi_s\|_H^{2n-2}\|\xi_s\|_V^2 \, ds - n(2n-1)\sigma_{\mathcal{C}}^2 \int_0^t \|\xi_s\|_H^{2n-2} \, ds,$$

where we have denoted by $\sigma_{\mathcal{C}}$ the trace of the operator $\mathcal{C}$ in H. Each process E^n is essentially the result of Itô's formula applied to $\|\xi_t\|_H^{2n}$ but for the martingale term.

[2] Hence a one–dimensional Brownian motion, by the Lévy characterization

DEFINITION 3.3 (super–martingale energy solution). A weak martingale solution $\mathbb{P}_\mu$ of Navier–Stokes equations is a super–martingale energy martingale solution if

- $P[L^\infty_{loc}([0,\infty);H) \cap L^2_{loc}([0,\infty);V)] = 1$,
- for each $n \geqslant 1$, the process $(E^n_t)_{t\geqslant 0}$, defined $\mathbb{P}_\mu$–a. s. on $(\Omega,\mathscr{B})$ is $\mathbb{P}_\mu$–integrable and $(E^n_t,\mathscr{B}_t,\mathbb{P})_{t\geqslant 0}$ is an a. s. super-martingale.

Almost sure energy solutions. The definition we have given through super–martingales is flexible and can be extended to the case of multiplicative noise [**GRZ09**]. For additive noise we can give an alternative definition (which has been used in [**Rom08**] to prove exponential convergence to the unique invariant measure).

Let $(\sigma_k^2)_{k\in\mathbf{N}}$ be the system of eigenvectors of the covariance $\mathcal{C}$ and let $(g_k)_{k\in\mathbf{N}}$ be a corresponding complete orthonormal system of eigenfunctions. Define for every $k \in \mathbf{N}$ the process $\beta_k(t) = \sigma_k^{-1} M_t^{g_k}$. Under a weak martingale solution $\mathbb{P}$, $(\beta_k)_{k\in\mathbf{N}}$ is a sequence of independent one dimensional Brownian motions, thus the process

$$W(t) = \sum_{k=0}^{\infty} \sigma_k \beta_k(t) g_k$$

is a $\mathcal{C}$-Wiener process and $z(t) = W(t) - \nu\int_0^t A\,e^{-\nu A(t-s)}\,W(s)\,ds$ is the associated Ornstein–Uhlenbeck process starting at 0, that is the solution of

$$dz + \nu Az\,dt = \mathcal{C}^{\frac{1}{2}}\,dW, \qquad z(0) = 0. \tag{3.4}$$

Define the process $v(t,\cdot) = \xi_t(\cdot) - z(t,\cdot)$. Since $M_t^\varphi = \langle W(t),\varphi\rangle$ for every test function φ, it follows that v is a weak solution of the following PDE with random coefficients,

$$\partial_t v + \nu Av + B(v+z, v+z) = 0, \qquad \mathbb{P}-\text{a. s.},$$

with initial condition $v(0) = \xi_0$. The energy balance functional associated to v is given as

$$\mathcal{E}_t(v,z) = \frac{1}{2}\|v_t\|_H^2 + \nu\int_0^t \|v_r\|_V^2\,dr - \int_0^t \langle z_r, B(v_r+z_r, v_r)\rangle\,dr.$$

DEFINITION 3.4 (almost sure energy solutions). A weak martingale solution $\mathbb{P}_\mu$ of Navier–Stokes equations is an almost sure energy martingale solution if

- $\mathbb{P}_\mu[v \in L^\infty_{loc}([0,\infty);H) \cap L^2_{loc}([0,\infty);V)] = 1$,
- there is a Lebesgue null set $T_{\mathbb{P}_\mu} \subset (0,\infty)$ such that

  $$\mathbb{P}_\mu[\mathcal{E}_t(v,z) \leqslant \mathcal{E}_s(v,z)] = 1$$

  for all $s \notin T_{\mathbb{P}_\mu}$ and all $t \geqslant s$, *Existence of Markov solutions.* A Markov family $(\mathbb{P}_x)_{x\in H}$ of solutions of Navier–Stokes equations is a family of weak martingale solutions such that for every $x \in H$ $\mathbb{P}_x$ has initial condition δ_x and the *almost sure Markov property* holds: for every $x \in H$ there is a Lebesgue null-set $T_x \subset (0,\infty)$ such that for every $\phi \in C_b(H)$, every $t \geqslant 0$ and all $s \notin T_x$,

$$\mathbb{E}^{\mathbb{P}_x}[\phi(\xi_{t+s})|\mathscr{B}_s] = \mathbb{E}^{\mathbb{P}_{\xi_s}}[\phi(\xi_t)], \qquad \mathbb{P}_x - \text{a. s.}$$

THEOREM 3.5. *There exists a family $(\mathbb{P}_x)_{x\in H}$ of energy martingale solutions (either super–martingale or almost sure energy solutions) such that $\mathbb{P}_x[\xi_0 = x] = 1$ for every $x \in H$ and the almost sure Markov property holds.*

The almost sure Markov property can be stated in terms of Markov kernels. A map $P(\cdot,\cdot,\cdot) : [0,\infty) \times H \times \mathscr{B} \to [0,1]$ is a a. s. Markov kernel of transition probabilities if

- $P(\cdot,\cdot,\Gamma)$ is Borel measurable for every $\Gamma \in \mathscr{B}$,
- $P(t,x,\cdot)$ is a probability measure on $\mathscr{B}$ for every $(t,x) \in [0,\infty) \times H$,
- the Chapman–Kolmogorov equation

$$P(s+t,x,\Gamma) = \int P(s,y,\Gamma)\, P(t,x,dy)$$

  holds for every $x \in H$, $t \geqslant 0$ and for every s outside a Lebesgue null set in $(0,\infty)$.

The connection between the two definitions is that for every $x \in H$ there is a solution $\mathbb{P}_x$ with initial condition x such that $P(t,x,\Gamma) = \mathbb{P}_x(\xi_t \in \Gamma)$ for all $t \geqslant 0$.

It is worth mentioning that the first proof of existence of a Markov family, as well as of strong Feller (see below) is in [**DPD03**, **DO06**], using the (backward) Kolmogorov operator. In either approach each Markov family is the unique solution of a suitable martingale problem [**Rom11b**, **DPD08**].

Continuity in total variation. Given a Markov family $(\mathbb{P}_x)_{x\in H}$ (or a Markov kernel $P(\cdot,\cdot\cdot)$), define the transition semigroup $(\mathcal{P}_t)_{t\geqslant 0}$ as

$$\mathcal{P}_t\phi(x) = \mathbb{E}^{\mathbb{P}_x}[\phi(\xi_t)] = \int \phi(y)\, P(t,x,dy).$$

for every $\phi \in \mathcal{B}(H)$. Clearly, $\mathcal{P}$ is a semigroup up to a Lebesgue null set of times (which depend on the point x).

In order to say something on the smoothing property of $\mathcal{P}$, we need a further assumption on the covariance.

ASSUMPTION 3.6. There are $\epsilon > 0$ and $\delta \in (1, \frac{3}{2}]$ such that

- $\mathrm{Tr}(A^{1+\epsilon}\mathcal{C}) < \infty$,
- $\mathcal{C}^{-\frac{1}{2}}A^{-\delta} \in \mathscr{L}(H)$.

For example, $\mathcal{C} = A^{-\alpha}$ with $\alpha \in (\frac{5}{2}, 3]$ satisfies the above assumptions.

The range of parameters in the above assumption can be made a bit larger [**Rom11a**] and the assumption of invertibility can be slightly relaxed [**RX11**, **ADX12**], but the bulk of the result remains essentially the same, so we keep the simple setting above.

Under the above assumption we have that

- the family $(\mathcal{P}_t)_{t\geqslant 0}$ is a true transition semigroup (no exceptional times) in $D(A)$ (in particular our a. s. Markov family is a true Markov process on $D(A)$),
- for every $\phi \in \mathcal{B}(D(A))$, $\mathcal{P}_t\phi \in C_b(D(A))$.
- each Markov family admits a unique invariant measure, and convergence is exponentially fast [**DPD03**, **DO06**, **Oda07**, **Rom08**].

The second property is known *strong Feller* property and can be essentially re–stated as continuity in total variation of the map $x \mapsto P(t, x, \cdot)$. One can have have almost Lipschitz regularity in the Markov selections framework [**FR07**] and full differentiability via the Kolmogorov equation [**DPD03**, **DO06**]. We mention only the following consequence of the strong Feller property, since it will turn out to be useful later in this chapter. We notice that since $\xi_t \to x$, $\mathbb{P}_x$–a. s., it follows that $\mathcal{P}_t\varphi(x) \to \varphi(x)$ as $t \downarrow 0$ for every continuous function φ, that is $(\mathcal{P}_t)_{t\geqslant 0}$ is *stochastically continuous*.

LEMMA 3.7. *For every Borel set* $\Gamma \subset D(A)$ *and every* $t > 0$,

$$P(t, x, \Gamma) \leqslant \liminf_{\epsilon\downarrow 0} P(t+\epsilon, x, \Gamma)$$

PROOF. We observe preliminarly that if G is open and $x \in G$, then $P(\epsilon, x, G) \to 1$. This is an immediate consequence of the definition of stochastically continuous, since $\mathbb{1}_G$ is lower semi–continuous.

By Chapman–Kolmogorov, for every open set $G \subset D(A)$ with $x \in G$,

$$\begin{aligned} P(t+\epsilon, x, \Gamma) = \int P(t, y, \Gamma)\, P(\epsilon, x, dy) \geqslant \\ \geqslant \int_G P(t, y, \Gamma)\, P(\epsilon, x, dy) \geqslant P(\epsilon, x, G) \inf_{y\in G} P(t, y, \Gamma). \end{aligned}$$

By our preliminary remark, $P(\epsilon, x, G) \to 0$, hence

$$\inf_{y\in G} P(t, y, \Gamma) \leqslant \liminf_{\epsilon\downarrow 0} P(t+\epsilon, x, \Gamma).$$

By the strong Feller property, $y \mapsto P(t, y, \Gamma)$ is continuous, hence the supremum over all open sets containing x of $\inf_{y\in G} P(t, y, \Gamma)$ is $P(t, x, \Gamma)$. □

3.3.2. Reduction to the local smooth solution. This section contains the key result for the existence of densities. The first part states the weak–strong uniqueness principle, which shows that every weak solution coincides with the local smooth solution up to the first blow–up time. The second part shows that due

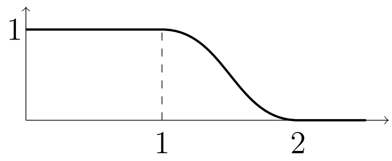

FIGURE 1. The cut–off function χ

to this, the Markov kernels of each weak solution and of the smooth solution "essentially" have equivalent finite dimensional marginals.

The weak–strong uniqueness principle. Let $\chi \in C^\infty(\mathbf{R})$ be a function such that $0 \leqslant \chi \leqslant 1$, $\chi(s) = 1$ for $s \leqslant 1$ and $\chi(s) = 0$ for $s \geqslant 2$, and set for every $R > 0$, $\chi_R(s) = \chi(\frac{s}{R})$. Set

$$B_R(v) = \chi_R(\|Av\|_H^2)B(v) \tag{3.5}$$

and denote by $u_R(\cdot; x)$ the solution of

$$du_R + \big(\nu A u_R + B_R(u_R)\big)\,dt = \mathcal{C}^{\frac{1}{2}}\,dW. \tag{3.6}$$

with initial condition $x \in D(A)$. Existence and uniqueness of a strong solution as well as several regularity properties of this problem are proved in [FR08, Theorem 5.12]. Among these we recall, due to invertibility of the covariance, *irreducibility*, namely that for every $x \in D(A)$, every $t > 0$ and every open set $G \subset D(A)$,

$$P_R(t, x, G) > 0, \tag{3.7}$$

that is for $t > 0$ the measure $P_R(t, x, \cdot)$ is supported on the whole $D(A)$. We shall give more details on this property in Chapter 4, where we give a direct proof of a related fact. Here we notice that the property is strictly related to controllability properties of the deterministic, controlled Navier–Stokes equations and for further details we refer to [Fla97] or [FR08, Proposition 6.1, Lemma C.2, Lemma C.3], and for the strongest result to [Shi06, Shi07].

Denote by $P_R(\cdot,\cdot,\cdot)$ and $\mathbb{P}_x^R$ the associated transition probabilities and laws of the solutions. To be convinced that the above problem has a unique regular solution, consider that the idea here is that when the solution of the above equation becomes too big (for instance close to a singularity, if there is any), then the non–linearity (which is the responsible of emergence of singularities) is killed. It is an indirect way to apply localisation. Indeed, if one defines

$$\tau_R = \inf\{t \geqslant 0 : \|Au_R(t)\|_H^2 \geqslant R\},$$

then it follows that $\tau_R > 0$ with probability one if $\|Ax\|_H^2 < R$ and that the following *weak–strong uniqueness* principle holds.

THEOREM 3.8 (Weak-strong uniqueness). *Let* $x \in D(A)$ *and* $\mathbb{P}_x$ *be any (either super–martingale or almost sure) energy martingale solution starting at* x. *Then* $\mathbb{P}_x[\tau_R > 0] = 1$ *for* $\|Ax\|_H < R$ *and*

$$u_R(t;x)\mathbb{1}_{\{\tau_R \geqslant t\}} = \xi_t \mathbb{1}_{\{\tau_R \geqslant t\}}, \qquad \mathbb{P}_x - a.\ s.$$

for every $t \geqslant 0$. *In particular for every* $t \geqslant 0$ *and every bounded continuous function* $\varphi : H \to \mathbf{R}$,

$$\mathbb{E}^{\mathbb{P}_x^R}[\varphi(\xi_t)\mathbb{1}_{\{\tau_R \geqslant t\}}] = \mathbb{E}^{\mathbb{P}_x}[\varphi(\xi_t)\mathbb{1}_{\{\tau_R \geqslant t\}}].$$

The weak–strong uniqueness principle is subtler than localisation, since it incorporates the idea, due to Prodi [**Pro59**] and Serrin [**Ser63**], that if two solutions of Navier–Stokes have the same initial condition and *at least one* of the two is regular enough[3] then the two solutions coincide. In different words, regularity implies uniqueness.

Equivalence with the local smooth solution. For the following result, we need the strong Feller property to hold, so we work under Assumption 3.6.

LEMMA 3.9. *Let* $f : D(A) \to \mathbf{R}^d$ *be a measurable function. Assume that for every* $x \in D(A)$, $t > 0$ *and* $R \geqslant 1$ *the image measure* $f_{\#}P_R(t, x, \cdot)$ *of the transition density* $P_R(t, x, \cdot)$ *corresponding to problem* (3.6) *is absolutely continuous with respect to the Lebesgue measure* $\mathscr{L}_d$ *on* $\mathbf{R}^d$. *Then the probability measure* $f_{\#}P(t, x, \cdot)$ *is absolutely continuous with respect to* $\mathscr{L}_d$ *for every* $x \in H$ *every* $t > 0$ *and every Markov solution* $(\mathbb{P}_x)_{x\in H}$.

PROOF. Fix a Markov solution $(\mathbb{P}_x)_{x\in H}$ and let $P(t, x, \cdot)$ be the associated transition kernel.

Step 1. We prove that each solution is concentrated on $D(A)$ at every time $t > 0$, for every initial condition x in H. With a computation similar to the one that yielded (1.8) we have

$$\mathbb{E}^{\mathbb{P}_x}\Big[\int_0^t \|A\xi_s\|^\delta \, ds\Big] < \infty,$$

for some $\delta > 0$. Thus $P(s, x, D(A)) = 1$ for almost every $s \in [0, t]$. Recall that, for $z \in D(A)$ and $r \geqslant 0$, $P(r, z, D(A)) = 1$. We deduce that $P(t - s, y, D(A)) = 1$, $P(s, x, \cdot)$–a. s. for almost every $s \in [0, t]$. Since the Chapman-Kolmogorov equation holds for almost every s, we have:

$$P(t, x, D(A)) = \frac{1}{t}\int_0^t \Big(\int P(t - s, y, D(A))\, P(s, x, dy)\Big)\, ds = 1.$$

Step 2. Given $x \in D(A)$, $s > 0$ and $B \subset D(A)$ measurable, we prove the following formula,

$$\big|P(s, x, B) - P_R(s, x, B)\big| \leqslant 2\mathbb{P}_x[\tau_R \leqslant s].$$

[3] A so–called *strong solutions*, where here "strong" has to be understood in PDE sense.

Indeed, by weak–strong uniqueness,

$$\begin{aligned} P(s,x,B) &= \mathbb{E}^{\mathbb{P}_x}[\xi_s \in B, \tau_R > s] + \mathbb{E}^{\mathbb{P}_x}[\xi_s \in B, \tau_R \leqslant s] \\ &= P_R(s,x,B) + \mathbb{E}^{\mathbb{P}_x}[\xi_s \in B, \tau_R \leqslant s] - \mathbb{E}^{\mathbb{P}^R_x}[\xi_s \in B, \tau_R \leqslant s]. \end{aligned}$$

Hence the first side of the inequality holds. The other side follows in the same way.

Step 3. We prove that the lemma holds if the initial condition is in $D(A)$. Let B be such that $\mathscr{L}_d(B) = 0$, hence $P_R(t,x,f^{-1}(B)) = 0$ for all $t > 0$, $x \in D(A)$ and $R \geqslant 1$, then

$$\begin{aligned} P(t+\epsilon,x,f^{-1}(B)) &= \int P(\epsilon,y,f^{-1}(B))\, P(t,x,dy) \\ &\leqslant 2\int_{\{\|Ay\|_H<R\}} \mathbb{P}_x[\tau_R \leqslant \epsilon]\, P(t,x,dy) + 2P(t,x,\{\|Ay\|_H \geqslant R\}), \end{aligned}$$

Since by Theorem 3.8 we have that $\mathbb{P}_x[\tau_R \leqslant s] \downarrow 0$ as $s \downarrow 0$ if $\|Ax\|_H < R$, by first taking the limit as $\epsilon \downarrow 0$ and then as $R \uparrow \infty$, we deduce, using also the first step, that $P(t+\epsilon,x,f^{-1}(B)) \to 0$ as $\epsilon \downarrow 0$. On the other hand by Lemma 3.7 (here we use the strong Feller property!) we have that $P(t,x,f^{-1}(B)) \leqslant \liminf_{\epsilon\to 0} P(t+\epsilon,x,f^{-1}(B)$, hence $P(t,x,f^{-1}(B)) = 0$.

Step 4. We finally prove that the lemma holds with initial conditions in H. We know that $P(t,x,f^{-1}(B)) = 0$ for all $t > 0$ and $x \in D(A)$ if $\mathscr{L}_d(B) = 0$. If $x \in H$ and $s > 0$ is a time such that the a. s. Markov property holds, then

$$P(t,x,f^{-1}(B)) = \int P(t-s,y,f^{-1}(B))\, P(s,x,dy) = 0,$$

since $P(s,x,D(A)) = 1$ by the first step. □

In the previous theorem the fact that we are dealing with image measures of a function does not play any essential role. Indeed, if there was a common reference measure to whom all $P_R(t,x,\cdot)$ are absolutely continuous, the same would be true for any Markov kernel $P(t,x,\cdot)$.

In particular, this idea has been used in **[FR07]** to prove that the Markov kernels of different Markov solutions are equivalent. The same remains true for the corresponding invariant measures (see Theorem 2.4).

3.4. Existence of densities for Markov solutions

The approach through Malliavin calculus allows to prove the most general result of this work in terms of *qualitative* existence of densities (at the price of stronger assumptions though), with respect to the results of the next chapters, namely Theorems 5.16, 6.3 and **??**. We start by introducing the finite dimensional functionals which are allowed by our results.

Let $f : D(A) \to \mathbf{R}^d$ be C^1 and define, for our purposes, a *singular point* x of f as a point where the range of $Df(x)$ is a proper subspace of $\mathbf{R}^d$. We are interested in C^1–functionals such that the set of singular points is not dense in $D(A)$. A few significant examples of such functionals are the following.

- Functions such as $f(u(t)) = \|u(t)\|_H^2$, as well as any other norm which is well defined on $D(A)$.
- In view of the results of the following sections, consider the case where $f \approx \pi_F$, where F is a finite dimensional subspace of $D(A)$, π_F is the projection onto F and f is given as $f(x) = (\langle\cdot, f_1\rangle, \dots, \langle\cdot, f_d\rangle)$, where $f_1, \dots, f_d$ is a basis of F.
- Given points $y_1, \dots, y_d \in \mathbb{T}_3$ (or in the corresponding bounded domain in the Dirichlet boundary condition case), the map $x \mapsto (x(y_1), \dots, x(y_d))$, defined on $D(A)$, is clearly differentiable and non–singular, since the elements of $D(A)$ are continuous functions by Sobolev's embeddings.

Our main result of existence of densities via Malliavin calculus works under the assumption 3.6, since we need Lemma 3.9 to work with smooth solutions. Hence $\mathcal{C}$ is a trace class operator, with $\operatorname{Tr}(A^{1+\epsilon}\mathcal{C}) < \infty$ and $\mathcal{C}^{-1/2}A^{-\delta} \in \mathscr{L}(H)$ for suitable ϵ and δ.

THEOREM 3.10. *Let Assumption 3.6 hold and let* $f : D(A) \to \mathbf{R}^d$ *be a map such that the set of singular points is not dense. Given an arbitrary Markov solution* $(\mathbb{P}_x)_{x\in H}$, *let* $u(\cdot, x)$ *be a random field with distribution* $\mathbb{P}_x$. *Then for every* $t > 0$ *and every* $x \in H$, *the law of the random variable* $f(u(t; x))$ *has a density with respect to the Lebesgue measure* $\mathscr{L}_d$ *on* $\mathbf{R}^d$.

REMARK 3.11. Under the Assumption (3.6) one can prove that the density we have obtained for the random variable $f(u(t; x))$ is almost surely positive. We postpone to Chapter 4 the discussion on this issue

By the results of [**DPD03, DO06, Oda07**] or [**Rom08**], each Markov solution converges to its unique invariant measure. The following result is a straightforward consequence of the theorem above.

COROLLARY 3.12. *Under the same assumptions of Theorem 3.10, given a Markov solution* $(\mathbb{P}_x)_{x\in H}$, *denote by* $\mu_\star$ *its invariant measure. Then the image measure* $f_\#\mu_\star$ *has a density with respect to the Lebesgue measure on* $\mathbf{R}^d$.

PROOF. If $(P(t, x, \cdot))_{t\geqslant 0, x\in H}$ is the corresponding Markov transition kernel and $E \subset \mathbf{R}^d$ has Lebesgue measure $\mathscr{L}_d(E) = 0$, then by Theorem 3.10 $P(t, x, f^{-1}E) = f_\# P(t, x, E) = 0$ for each $x \in H$ and $t > 0$. Then, by Chapman–Kolmogorov,

$$f_\#\mu_\star(E) = \int_{D(A)} P(t, x, f^{-1}E)\,\mu_\star(dx) = 0,$$

since $\mu_\star(D(A)) = 1$. $\square$

3.4.1. Absolute continuity for the smooth problem. In view of Lemma 3.9, we simply have to show that the law of $f(u_R(t;x))$ has a density with respect to the Lebesgue measure on $\mathbf{R}^d$ for every $R > 0$, $x \in D(A)$ and $t > 0$. We use Theorem 2.1.2 of [Nua06].

Let $x \in D(A)$, $t > 0$ and $R \geqslant 1$. Consider a finite–dimensional map $f : D(A) \to \mathbf{R}^d$ with non–dense singular points. We recall that to use Theorem 2.1.2 of [Nua06] we need to show the following two facts

- $u_R(t;x) \in \mathbb{D}^{1,2}(D(A))$, the space of square integrable random variables with a square–integrable Malliavin derivative defined in (3.1)
- the associated Malliavin matrix is a. s. invertible.

Given $x \in D(A)$, $s \geqslant 0$ and $k \in \mathbf{N}$, define $\eta_k(t;s,x)$ as the solution to

$$\begin{cases} d_t\eta_k + \nu A\eta_k + DB_R(u_R)\eta_k = 0, \\ \eta_k(s;s,x) = q_k, \end{cases} \qquad t \geqslant s \tag{3.8}$$

where $\eta_k(t) = \eta_k(t;s,x)$, B_R is defined in (3.5) and $(q_k, \sigma_k^2)_{k\in\mathbf{N}}$ is a system of eigenvectors and eigenvalues of the covariance operator $\mathcal{C}$ of the noise. A simple computation yields that

$$DB_R(v)\theta = \chi_R(\|Av\|_H^2)\big(B(\theta,v) + B(v,\theta)\big) + 2\chi_R'(\|Av\|_H^2)\langle Av, A\theta\rangle_H B(v,v).$$

It turns out that the Malliavin derivative $\mathcal{D}u_R(t)$ along the direction hq_k, where $h \in L^2_{loc}(0,\infty;R)$, is given by

$$\langle \mathcal{D}u_R(t;x), hq_k\rangle = \sigma_k \int_0^t \eta_k(t;s,x)h(s)\,ds \tag{3.9}$$

More generally, one has

$$\mathcal{D}_s u_R(t;x) = \mathbb{1}_{\{s\leqslant t\}}\langle \eta(t;s,x), \mathcal{C}^{\frac{1}{2}}\cdot\rangle,$$

where $\eta \in \mathcal{L}(D(A))$ solves

$$d_t\eta + A\eta + \langle DB_R(u_R), \eta\rangle = 0, \qquad t \geqslant s,$$

with initial condition $\eta(s;s,x) = \mathrm{Id}$ and the Malliavin derivative along a direction $h \in L^2_{loc}(0,\infty;D(A))$ is

$$\begin{aligned} \langle \mathcal{D}u_R(t;x), h\rangle &= \int_{\mathbf{R}} \langle \mathcal{D}_s u_R(t;x), h(s)\rangle\,ds = \int_0^t \langle \eta(t;s,x), \mathcal{C}^{\frac{1}{2}}h(s)\rangle\,ds \\ &= \sum_{k=0}^\infty \sigma_k \int_0^t \eta_k(t;s,x)h_k(s)\,ds, \end{aligned}$$

and $h_k = \langle h, e_k\rangle_H$.

LEMMA 3.13. *For every* $x \in D(A)$ *and* $t > 0$, $u_R(t,x) \in \mathbb{D}^{1,2}(D(A))$ *and its Malliavin derivative is given by the formulas above, where, as in* (3.1),

$$\mathbb{D}^{1,2}(D(A)) = \Big\{u : \mathbb{E}\big[\|Au\|_H^2\big] + \sum_{k=0}^{\infty} \mathbb{E}\int_0^t \|\mathcal{D}^s u \cdot q_k\|_{D(A)}^2 < \infty\Big\}.$$

Moreover, for every $k \in \mathbf{N}$ *and* $x \in D(A)$, *the function* $\eta_k(t; s, x)$ *is continuous in both variables* $s \in [0, \infty)$ *and* $t \in [s, \infty)$.

PROOF. Problem (3.6) admits a unique strong solution, so there is a measurable map $\Phi_{t,x}$ such that $u(t;x) = \Phi_{t,x}(W|_{[0,t]})$, where W is a cylindrical Wiener process on H. Given $h \in L^2_{loc}(0,\infty;\mathbf{R})$, we want to prove that

$$\langle \mathcal{D}u_R(t), hq_k\rangle = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(\Phi_{t,x}(W|_{[0,t]} + \epsilon K q_k) - \Phi_{t,x}(W|_{[0,t]})\big),$$

exists and has finite second moment. where $K(s) = \int_0^s h(r)\,dr$. Define $\theta_k = \int_0^t \eta_k(t; s, x) h(s)\,ds$, which is solution of

$$\frac{d}{dt}\theta_k + \nu A\theta_k + DB_R(u_R)\theta_k = h\,q_k, \qquad t \geqslant 0,$$

with initial condition $\theta_k(0) = 0$. Using estimates similar to those below, it is easy to prove that formula (3.9) holds and hence that $u \in \mathbb{D}^{1,2}(D(A))$.

Next, we prove that $\eta_k(t; s, x)$ is continuous in both s and t. By computing the derivative in time of $\|A\eta_k(t)\|_H^2$, using (1.4) and applying Gronwall's lemma, one gets

$$\sup_{[s,t]}\|A\eta_k\|_H^2 + \nu\int_s^t \|A^{\frac{3}{2}}\eta_k\|_H^2\,dr \leqslant \|Aq_k\|^2 e^{cR(t-s)}. \tag{3.10}$$

By multiplying the first line of (3.8) by $\dot\eta_k$, using again (1.4) and Gronwall's lemma, it also follows that

$$\int_s^t \|A^{\frac{1}{2}}\dot\eta_k\|_H^2\,dr \leqslant \|Aq_k\|_H^2\, e^{cR(t-s)}.$$

In particular the two estimates above imply that the map $t \mapsto \|A\eta_k\|_H^2$ has a derivative which is L^1 in time and in conclusion $t \mapsto \eta(t; s, x)$ is continuous in the t variable with values in $D(A)$.

We finally prove that $s \mapsto \eta_k(t; s, x)$ is continuous with values in $D(A)$. Fix s, it is sufficient to prove the two following statements

- if $s_n \uparrow s$, $\eta_k(t; s_n, x) \to \eta_k(t; s, x)$ in $D(A)$ for every $t \geqslant s$,
- if $s_n \downarrow s$, $\eta_k(t; s_n, x) \to \eta_k(t; s, x)$ in $D(A)$ for every $t > s$.

For the first statement, set $w_i(t) = \eta_k(t; s_n, x) - \eta_k(t; s, x)$, then w_i satisfies (3.8) with initial condition $w_i(s) = \eta_k(s; s_n, x) - q_k$ and, using the estimate corresponding to (3.10) for w_i, it follows that it is enough to prove that $\eta_k(s; s_n, x) \to$

q_k in $D(A)$. We have that

$$A\big(\eta_k(s;s_n,x)-q_k\big) = \big(e^{-\nu A(s-s_n)}-\mathrm{Id}\big)Aq_k + \\ -\int_{s_n}^{s} A^{\frac12} e^{-\nu A(s-r)} A^{\frac12} DB_R(u_R)\eta_k(r;s_n,x)\,dr,$$

the first term on the right hand side converges to 0 by standard facts on semigroups, while the second term converges to zero since the term inside the integral is bounded by $c(s-r)^{-1/2}$, by virtue of (1.4), (3.10) and semigroup estimates.

If on the other hand $s_n \downarrow s$, the same method shows that it is enough to prove that $\eta_k(s_n,s,x) \to q_k$ in $D(A)$, and this is immediate since we have already proved that $t \mapsto \eta_k(t;s,x)$ is continuous. □

We can proceed to the proof of our main result. Fix $x \in D(A)$, $t > 0$ and $R > 0$. By the chain rule for Malliavin derivatives, the Malliavin matrix $\mathcal{M}^f(t)$ of $f(u_R(t;x))$ is given by

$$\begin{aligned}\mathcal{M}^f_{ij}(t) &= \langle Df_i(u_R(t;x))\mathcal{D}u_R(t;x), Df_j(u_R(t;x))\mathcal{D}u_R(t;x)\rangle \\ &= \sum_{k=1}^{\infty}\int_0^t \big(Df_i(u_R(t;x))\mathcal{D}^s u_R(t;x)\cdot q_k\big)\big(Df_j(u_R(t;x))\mathcal{D}^s u_R(t;x)\cdot q_k\big)\,ds \\ &= \sum_{k=1}^{\infty}\sigma_k^2\int_0^t \big(Df_i(u_R(t;x))\eta_k(t;s,x)\big)\big(Df_j(u_R(t;x))\eta_k(t;s,x)\big)\,ds\end{aligned}$$

for $i,j = 1,\dots,d$, where $f = (f_1,\dots,f_d)$.

To show that $\mathcal{M}^f(t)$ is invertible a. s., it is sufficient to show that if $y \in \mathbf{R}^d$ and

$$\langle \mathcal{M}^f(t)y,y\rangle = \sum_{k=1}^{\infty}\sigma_k^2\int_0^t\Big|\sum_{i=1}^d Df_i(u_R(t;x))\eta_k(t;s,x)y_i\Big|^2\,ds$$

is zero, then $y = 0$. This is clearly true, since if $\langle \mathcal{M}^f(t)y,y\rangle = 0$, then

$$\sum_{i=1}^d y_i Df_i(u_R(t;x))\eta_k(t;s,x) = 0, \qquad \mathbb{P}-\text{a.s.},$$

for all $k \in \mathbf{N}$ and a. e. $s \leqslant t$. By Lemma 3.13 and continuity of u_R in $D(A)$, the above equality holds for all $s \leqslant t$. In particular for $s = t$ this yields

$$\sum_{i=1}^d y_i Df_i(u_R(t;x))q_k = 0, \quad \mathbb{P}-\text{a.s.},$$

for all $k \geqslant 1$. Under our assumptions on the covariance, the support of the law of $u_R(t;x)$ is the full space $D(A)$, see (3.7). Hence, $u_R(t;x)$ belongs to the set of non singular points of f with positive probability. We know that $(q_k)_{k\geqslant 1}$ is a basis of H, hence the family of vectors $(Df_1(u_R(t;x))q_k, \dots, Df_d(u_R(t;x))q_k)_{k\geqslant 1}$ spans all $\mathbf{R}^d$ with positive probability, and in conclusion $y = 0$.

CHAPTER 4

Intermezzo: positivity of the density

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In this chapter we take a short break from proving existence of densities and we focus first on the support of the laws of the solution at any time, and then, if such laws admit densities, we show that the densities are positive almost everywhere with respect to the Lebesgue measure.

The model result for the support is taken from [**Fla97**], although the idea that some information on the support can be obtained through an associated control problem dates back to [**SV72**]. The key idea in [**Fla97**] is that the equations (1.1) are well–posed around a strong solution, hence in particular one can get the continuity result in Section § 4.1.2. We additionally remark that other results of controllability, taking into account the regularity of the solution induced by the noise (see for instance [**Rom11a**]), are possible and the space V in the first section can be replaced (at most) by $V_{\frac{1}{2}}$, which is the largest space (among Hilbert–Sobolev spaces) where existence and uniqueness of local solutions of (1.1) can be proved [**FK62**].

Positivity of the density needs a bit more work, although it is based again on the idea that it is sufficient to work with regular solutions. The proof of positivity is based on a series of controllability results developed in [**AS05**, **AS08**] and it is taken from [**Shi07**]. Other relevant results along the same direction can be found in [**AKSS07**, **Shi06**].

4.1. A support theorem

We recall that the support of a Borel measure is the set of all points x such that every open neighbourhood of x has positive measure. It is then clear that the notion of support depends on the topology. In finite dimension this is not an issue, but in infinite dimension one should specify the underlying topology of the support.

As it regards weak solutions of the Navier–Stokes equations with noise, the natural topology for the support is the state space H. On the other hand we may find additional regularity, such as the one given by the energy inequality, namely V, or others such as (1.8), which tell us that $\mathbb{P}[u(t) \in V] = \mathbb{P}[u(t) \in D(A)] = 1$ for a. e. t (and all t for Markov solutions). With all this in mind, we shall say that a measure μ on H is *fully supported* on some space $\mathcal{V}$ smaller than H if $\mu(\mathcal{V}) = 1$ and $\mu(U) > 0$ for every open set U of $\mathcal{V}$.

We state the following theorem from [**Fla97**]. A more refined result which takes into account the regularity of the solutions (and hence shows smaller spaces on whom the laws are fully supported) as prescribed by the regularizing properties of the covariance can be found in [**FR08**, Proposition 6.1].

THEOREM 4.1. *Assume the covariance $\mathcal{C}$ is trace class and injective. Then for every $x \in V$, every $t > 0$, every ball B in in H and every* almost sure energy solution *(see Definition 3.4) u with initial condition x,*

$$\mathbb{P}[u(t) \in B] > 0.$$

From the theorem we immediately deduce that, given a solution u of (1.7) with initial condition in V, the support of the law of $u(t)$ is H for every $t > 0$. Something more can be said.

The first additional remark is that since $\mathbb{P}[u(t) \in V] = 1$ for a. e. $t > 0$, it follows from the above theorem that $u(t)$ is fully supported on V for a. e. t. The same holds as long as the law of $u(t)$ is concentrated in a smaller space than t. We notice also that for Markov solutions (see Chapter 3, solutions are fully supported on V for every t.

The second remark is that the theorem above extends to initial conditions in H. Indeed, this statement would be straightforward if u is a Markov process since, by Chapman–Kolmogorov, for any $s \in (0, t)$,

$$P(t, x, B) = \int P(t - s, y, B)\, P(s, x, dy),$$

where $P(t, x, \cdot)$ is the associated Markov kernel. Choose an s such that $P(s, x, V) = 1$ (there are infinitely many such s), then $y \in V$ and hence $P(t - s, y, B) > 0$ for $P(s, x, \cdot)$–a. e. y, and so $P(t, x, B) > 0$.

If on the other hand u is not Markov (or, more precisely, if we do not know if it is Markov), the conditional law of u given $\mathscr{F}_s$ is again an almost sure energy solution with initial condition $u(s)$ (this is a result from [**Rom09**]) and the theorem above still applies.

REMARK 4.2. In view of the results of the previous Chapter 3 and the following chapters 5, 6, 7, the above theorem implies in particular that for finite dimensional projections on a space F,

$$\mathbb{P}[\pi_F u(t) \in A] > 0$$

for every open set A of F, hence the support of the law of the random random variable $\pi_F u(t)$ is F for every $t > 0$ and every initial condition in H.

The rest of this section contains a sketch of the proof of the theorem.

4.1.1. The control problem. Denote by $\mathrm{Lip}([0,T];H)$ the space of H–valued Lipschitz continuous functions and let $\omega \in \mathrm{Lip}([0,T];H)$. Consider the following control problem

$$\frac{du}{dt} + \nu A u + B(u,u) = \frac{d\omega}{dt}, \tag{4.1}$$

with initial condition $u(0) = x$ and terminal condition $u(T) = x_1$.

PROPOSITION 4.3. *Let $T > 0$, $x \in V$ and $x_1 \in D(A)$. Then there exist $\bar{\omega} \in \mathrm{Lip}([0,T];H)$ and $\bar{u} \in C([0,T];V) \cap L^2(0,T;D(A))$ such that $\bar{u}$ and $\bar{\omega}$ solve the control problem (4.1) with initial condition $\bar{u}(0) = x$ and final condition $\bar{u}(T) = x_1$.*

PROOF. Consider (4.1) with $\omega = 0$. We know from Chapter 1 that for a short time there is a unique smooth solution u. Choose $T_1 < T$ so that $u(T_1) \in D(A)$ and set $\bar{\omega} = 0$ on $[0,T_1]$ and $\bar{u}$ the solution on $[0,T_1]$ corresponding to this choice of $\bar{\omega}$.

Define $\bar{u}$ on $[T_1,T]$ as linear interpolation between $u(T_1)$ and x_1, and $\bar{\omega}$ as

$$\frac{d\omega}{dt} = \frac{d\bar{u}}{dt} + \nu A\bar{u} + B(\bar{u},\bar{u}).$$

It is immediate to see that $\bar{\omega}$ and $\bar{u}$ have the required regularity. □

4.1.2. Continuity along the controllers. Fix $s \in (0,\frac{1}{2})$ and $p \in (1,\infty)$ such that $s - \frac{1}{p} > \frac{3}{8}$, and denote by Ω_0 the subspace of $W^{s,p}(0,T;H)$ such that $\omega(0) = 0$ if $\omega \in \Omega_0$. The parameters s,p have been chosen so that, under the standing assumptions on the covariance, $\mathcal{C}^{\frac{1}{2}}|_{[0,T]}W \in \Omega_0$. We recall that the fractional Sobolev space $W^{s,p}(0,T;H)$ is defined by the norm

$$\|\eta\|^p_{W^{s,p})0,T;H)} = \int_0^T \|\eta\|^p_H\,dt + \int_0^T\int_0^T \frac{\|\omega(t)-\omega(s)\|^p_H}{|t-s|^{1+sp}}\,ds\,dt$$

PROPOSITION 4.4. *Let $\bar{u}, \bar{\omega}$ be as in the statement of the previous proposition. Let $(\omega_n)_{n\in N}$ be a sequence in Ω_0 converging to $\bar{\omega}$ in Ω_0 and let,for each n, $u(\cdot;\omega_n)$ be the solution of (4.1) with control $\bar{\omega}_1$. If $\omega_n \to \bar{\omega}$ in Ω_0, then $\bar{u}(\cdot;\omega_n) \to u(\cdot;\bar{\omega})$ in H, uniformly in T.*

PROOF. Set $u_n = u(\cdot;\omega_n)$, let $\bar{z}$ be the solution of

$$\frac{d\bar{z}}{dt} + \nu A\bar{z} = \frac{d\bar{\omega}}{dt}$$

with initial condition $\bar{z}(0) = 0$, and define likewise z_n from ω_n. Set finally $\bar{v} = \bar{u} - \bar{z}$ and $v_n = u_n - z_n$. The idea we shall use in this proof, which also the key idea of the weak–strong uniqueness principle in Theorem 3.8, is that $\bar{u}$ is more regular than u_n. We use this fact a first time to deduce that, while each u_n satisfies the energy inequality (it is the deterministic analogue of $\mathcal{E}$ of Definition 3.4), $\bar{u}$ satisfies the energy equality. Therefore,

$$\frac{d}{dt}\|v_n - \bar{v}\|_H^2 + 2\nu\|v_n - \bar{v}\|_V^2 \leqslant \langle B(u_n, v_n - \bar{v}), z_n - \bar{z}\rangle + \langle B(u_n - \bar{u}, v_n - \bar{v}), \bar{u}\rangle$$
$$\leqslant c\|u_n\|_{L^4}\|v_n - \bar{v}\|_V^2\|z_n - \bar{z}\|_L^4 + c\|u\|_{L^4}\|v_n - \bar{v}\|_V^2\|z_n - \bar{z}\|_L^4.$$

Here we use again the additional regularity of $\bar{u}$ as $\bar{u} \in^\infty (0, T; L^4)$ to get rid of the second term. The application of Young's inequality and the Ladyzhenskaya inequality $\|\cdot\|_{L^4} \leqslant \|\cdot\|_H^{1/2}\|\cdot\|_V^{1/2}$ finally yield,

$$\|\bar{v}(t) - v_n(t)\|_H^2 \leqslant e^{cT}\Big(c_1 T + c_2\int_0^T \|v_n - z_n\|_{L^4\,ds}\Big)\sup_{[0,T]}\|\bar{z} - z_n\|_{L^4}^2.$$

We know that $\omega_n \to \overline{\omega}$ in Ω_0, hence $z_n \to \bar{z}$ converges in L^4 uniformly on $[0, T]$, and by these properties we deduce the statement of the proposition. □

4.1.3. The proof of the main theorem. We are ready to prove Theorem 4.1. Let $x \in V$, $t > 0$, $B \subset H$ a ball in H, and let u be a solution of (1.7) with initial condition x. Choose a point $x_1 \in A$ such that $x_1 \in B$ and let B' be a ball centred in x_1 such that $B' \subset B$. Then

$$\mathbb{P}[u(t) \in B] \geqslant \mathbb{P}[u(t) \in B']$$

By the result in Section §4.1.1 there are $\bar{\omega} \in Lip([0, T]; H)$ and $\bar{u} \in C([0, T]; V) \cap L^2(0, t; D(A))$ such that $\bar{u}$ solves the control problem (4.1) on $[0, t]$ with control $\bar{\omega}$, has initial position $\bar{u}(0) = x$ and final position $\bar{u}(t) = x_1$.

By the result in Section §4.1.2 there is a ball $\widehat{B} \subset \mathcal{W}$ centred in $\bar{\omega}$ such that $u(t, \omega) \in B'$ if $\omega \in \widehat{B}$. Therefore

$$\mathbb{P}[u(t) \in B] \geqslant \mathbb{P}[u(t) \in B'] \geqslant \mathbb{P}[\mathcal{C}^{\frac{1}{2}}W|_{[0,t]} \in \widehat{B}] > 0,$$

by the assumptions on the covariance.

4.2. Positivity of the density

Unfortunately the information given by the support theorem of the previous section is not enough to deduce positivity of the density. Indeed, let F be a finite–dimensional subspace of the state space H of the Navier–Stokes equations, and denote by f_F the density of the projection of the law of the solution at some time $t > 0$. The results of the previous section imply that for every ball B in F,

$$\int_B f_F(x)\,dx > 0.$$

If f_F is continuous, then f_F must be positive. On the other hand, in general the above condition does not imply that $f_F > 0$ $\mathbb{P}$–a.s., as one can see from the following example.

EXAMPLE 4.5. Let f be a density on $\mathbf{R}$ such that $\int_B f(x)\,dx > 0$ on every ball $B \subset \mathbf{R}$. Let $(q_n)_{n\in\mathbf{N}}$ be an enumeration of $\mathbf{Q}$ and $(\epsilon_n)_{n\in\mathbf{N}}$ be a sequence of real positive numbers such that $\sum_n \epsilon_n < \infty$. Let

$$A = \bigcup_{n\geqslant 1} B_{\epsilon_n}(q_n),$$

then the Lebesgue measure $\mathscr{L}_1(A)$ of A can be estimated as

$$\mathscr{L}_1(A) \leqslant \sum_n \mathscr{L}\big(B_{\epsilon_n}(q_n)\big) = 2\sum_n \epsilon_n < \infty.$$

Hence $\mathscr{L}_1(A^c) = \infty$ and $f(x) = \mathbb{1}_A(x)/\mathscr{L}_1(A)$ is a density such that the set $\{f = 0\}$ is of positive (infinite) Lebesgue measure. On the other hand, for every ball $B \subset \mathbf{R}$,

$$\int_B f(x)\,dx = \frac{\mathscr{L}_1(A\cap B)}{\mathscr{L}_1(A)} > 0,$$

because there is at least one $n \geqslant 1$ such that $B \cap B_{\epsilon_n}(q_n)$ is of positive measure.

The proof of positivity we give here is taken from [**Shi07**], using some controllability results of [**Shi06**, **AKSS07**]. We shall work for simplicity under the assumption of non–degeneracy of the covariance of the driving noise, as in different flavours is done in our Chapters 3, 5, 6 and 7. We remark that the original results of [**Shi07**] hold under much weaker assumptions on the covariance, similar to those we have used in Section § 2.3.

4.2.1. A few additional remarks on strong solutions. This short section is a supplement to what we have briefly pointed out at the end of Section § 1.1 on strong solutions. If one considers the non–random Navier–Stokes equations, it is well–known that given an initial condition and a forcing term, there is for a short time, depending on the two data, a strong solution. The main problem addressed in [**Fef06**] is to show that for smooth data this short time is indeed infinite.

Given a (non–random) force η, consider the equation

$$\dot{u} + \nu A u + B(u) = \eta \tag{4.2}$$

and its linear counterpart,

$$\dot{z} + \nu A z = \eta, \tag{4.3}$$

with initial condition $z(0) = 0$. It is immediate to verify that $v = u - z$ solves

$$\dot{v} + \nu A v + B(v + z) = 0 \tag{4.4}$$

with the same initial condition of u.

Given an initial condition $u_0 \in V$ and a time $T > 0$, consider the following sets

$$\mathcal{S}_T(u_0) = \{z \in C([0,T];V) \cap L^2(0,T;V_2) : \\ \text{(4.4) has a unique solution in } C([0,T];V) \cap L^2(0,T;V_2)\}$$

and

$$\mathcal{T}_T(u_0) = \{\eta \in L^2(0,T;H) : \\ \text{(4.2) has a unique solution in } C([0,T];V) \cap L^2(0,T;V_2)\}$$

Define the operator M as

$$M_T\xi(t) = \int_0^t e^{-\nu(t-s)A}\eta(s)\,ds, \qquad t \in [0,T],$$

then it is easy to verify that, given $\eta \in L^2(0,T;H)$, $M_T\eta$ is solution of (4.3) and $\eta \mapsto M\eta$ maps $\mathcal{T}_T(u_0)$ into $\mathcal{S}_T(u_0)$. Likewise, if $\eta \in \mathcal{T}_T(u_0)$, then $R_T\eta = M_T\eta + \nu$ is the unique solution of (4.2) with initial condition u_0, where ν is the unique solution of (4.4).

It is easy to check that the operator R_T defined above on $\mathcal{T}_T(u_0)$ with values in $C([0,T];V) \cap L^2(0,T;V_2)$ is locally Lipschitz continuous and that $\mathcal{T}_T(u_0)$ is an open set in $L^2(0,T;H)$.

4.2.2. Solid controllability. For the purpose of proving positivity, approximate controllability is not enough and we need the stronger notion of solid controllability. We refer to [**AS08**] for a more detailed introduction to the idea and to its applications on equations of fluid dynamics.

DEFINITION 4.6. Equation (4.2) is *solidly controllable* at time $T > 0$ if for every $u_0 \in V$ and $r > 0$ there exist $\delta > 0$ and a compact set K in a subspace of a finite dimensional space F of $C^\infty([0,T];H)$ of admissible controls such that $K \subset \mathcal{T}_T(u_0)$ and for every continuous map $\phi : K \to F$,

$$\sup_{\eta\in K} \|\phi(\eta) - \pi_F R_T(\eta)\|_F \leqslant \delta \qquad \Longrightarrow B_r^F(0) \subset \phi(F),$$

where $B_r^F(0)$ is the ball centred at 0 with radius r in F.

We immediately notice that, if we choose $\phi = \pi_F R_T$ in the definition above, we conclude that for every $u_0 \in V$ and > 0 there is a suitable compact set K of controls in $\mathcal{T}_T(u_0)$ such that for every $x \in B_r^F(0)$ there is a control $\eta \in K$ with $\pi_F R_T(\eta) = x$. Hence solid controllability not only ensures that the elements of $B_r^F(0)$ are accessible through admissible controls, but it also states that this property is stable for small contninuous perturbations.

If, as in the first part of this chapter, we assume that the admissible controls are the image of $L(0,T;H)$ through the operator $\mathcal{C}^{\frac{1}{2}}$ such that $\mathcal{C}$ is injective and the trace of $A\mathcal{C}$ is finite, the proof of solid controllability is slightly easier. Indeed, the existence of a control can be granted as in Section §4.1.1 and the stability to perturbation follows from the fact that the map R_T is locally Lipschitz continuous on the open set $\mathcal{T}_T(u_0)$. We remark again that [**Shi06**] contain a proof of solid controllability for the three–dimensional Navier–Stokes for a more general set of controls, corresponding to a degenerate covariance.

4.2.3. Positivity. We are ready to prove positivity of the densities. Here we assume that a density exists. This statement has been proved under strong assumptions of regularization of the covariance in Chapter 3 and will be proved under weaker assumptions of regularization (although still strong assumptions of non–degeneracy). In the original work [**Shi07**] the author proves the inequality of the theorem below for measures, without assuming in any way existence of densities.

We finally notice that the fact that the lower bound ρ in the theorem below is a continuous function can be immediately *upgraded* to $\rho \in C^\infty(F)$ by elementary considerations. The main fact is indeed that the lower bound is continuous.

THEOREM 4.7. *Let* F *be a finite dimensional space, let* u *be the solution of* (1.1) *with covariance and assume that the random variable* $\pi_F u(1)$ *has a density* f_F *with respect to the Lebesgue measure of* F*. The there exists* $\rho \in C(F)$ *such that* $\rho > 0$ *a.e. and*

$$f_F \geqslant \rho, \qquad \text{a.e.}$$

PROOF. It is sufficient to show that for every $r > 0$ and almost every $y \in B^F_r(0)$ there are a ball B centred in y and a continuous function ρ_y such that $f_F \geqslant \rho_y$ and $\rho_y > 0$ in B. The statement of the theorem is then obtained by standard arguments of finite coverings and partitions of unit. In the proof we can also restrict all considerations on strong solutions, since by weak–strong uniqueness (see Theorem 3.8) for a Borel set A of F,

$$\int_A f_F = \mathbb{P}_{u_0}[\pi_F u(1) \in A] \geqslant \mathbb{P}_{u_0}[\pi_F u(1) \in A, \tau_\infty > 1],$$

where τ_∞ is the blow–up time, that is the first time the solution is infinite in V.

Let us then fix $r > 0$ and consider the function $f : \mathcal{S}_1(u_0) \to F$ defined as

$$f(z) = \pi_F\big(z(1) + v(1)\big), \qquad z \in \mathcal{S}_1(u_0),$$

where v solves equation (4.4) with initial condition u_0 and with the z chosen. In other words $z + v$ solves (4.2) with $\eta = \partial_t z + \nu A z$.

We wish to use the abstract implicit change of variables theorem 4.8 below. To this end we need to prove that $f \in C^1$ and that $Df(x)$ has full rank for a.e. x. The regularity of f is a lengthy albeit straightforward issue and is based on the fact that the equation (4.4) has a unique smooth solution. So we concentrate on the second issue.

From solid controllability we know that there is a compact set of a finite dimensional space X_0 of admissible controls such that $K \subset \mathcal{T}$ and $B^F_r(0) \subset \pi_F R_1(K)$.

Let Z_0 be the image of X_0 with respect to the linear operator M_1 definite above. The space Z_0 is finite dimensional and $M(K)$ is compact. Moreover $M(K) \subset \mathcal{S}_1(u_0)$, hence $B_r^F(0) \subset f(K)$. The set $\mathcal{S}_1(u_0)$ is open, hence there is an open subset O of $M(K)$ such that $B_r^F(O) \subset f(K)$.

The Sard lemma implies that a.e point of $B_r^F(O)$ is regular, that is given $y \in B_r^F(0)$, $Df(x)$ has full rank for a.e. x such that $f(x) = y$. Theorem 4.8 below implies that for a.e. $y \in B_r^F(0)$ there are a ball B in O, a ball B_y centred in y and a continuous function ρ_y, with $\rho_y > 0$ on B_y, such that the image through f of λ restricted to B satisfies

$$f_\# \lambda|_B \geqslant \rho_y \mathscr{L}_F,$$

where λ is the law of the solution of problem (4.3) with noise on the right–hand side (that is, problem (3.4)) at time $t = 1$. □

We conclude the section with the statement of the abstract theorem used in the proof above, which is taken from [**Shi07**, Theorem 2.4]. The theorem is essentially a change of variable obtained by an implicit function theorem. Other theorems of the same kind can be found in [**Bog98**]

THEOREM 4.8. *Let Z be a separable Banach space, λ a non–degenerate Gaussian measure with support equal to Z, and F a finite dimensional space. If $B = B_r(z_0) \subset Z$ is an open ball and $f : B \to Z$ is a C^1 function such that the image of $Df(z_0)$ is F, then there is a continuous function $\rho \in C(F)$ such that $\rho > 0$ in a neighbourhood of $f(z_0)$ and $f_\# \lambda|_B \geqslant \rho \mathscr{L}_F$.*

CHAPTER 5

Densities with the Girsanov theorem

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As a first method to show existence of densities, we use an *overkill*, namely the Girsanov theorem. Indeed, as we shall see, Girsanov's theorem ensures equivalence of the laws of two stochastic processes *at the level of paths*, rather than at the level of fixed times.

The Girsanov change of measure is used as a crucial step in the theory of stochastic differential equations and in mathematical finance. Here we are clearly more interested in former rather than in the latter.

In this chapter we give a thorough introduction to equivalence of measures. We turn then to equivalence of translates of (infinite dimensional) Gaussian measures and the Cameron–Martin theorem. Girsanov's theorem can be seen as a generalization of the Cameron–Martin theorem. We will see how to apply Girsanov's theorem to show weak uniqueness for stochastic differential equations and related in properties, both in finite and infinite dimension (as a complement to Section §2.2.2). We will finally prove existence of densities for finite dimensional projections of the Navier–Stokes equations with noise. The way we do it here does not allow to obtain any regularity information on the density. Regularity of the density will be the subject of the following chapters.

5.1. Generalities on equivalence of measures

The main purpose of this section is to convince the reader that equivalence in infinite dimension is a "big deal". Even if restricted to Gaussian measures, which is what we shall do in preparation to the Girsanov theorem, equivalence follows by strict conditions on mean and covariance of the measure.

We recall that, given two probability measures μ and ν, the measure μ is *absolutely continuous* with respect to ν, $\mu \ll \nu$, if every null–set of ν is a null set of μ. Radon–Nykodym's theorem ensures that if $\mu \ll \nu$, then there is a $L^1(\nu)$ function f such that $\mu = f\nu$. The two measures are *equivalent* if they are mutually absolutely continuous. Finally, the two measures are *mutually singular* if there is a measureable set such that $\mu(A) = 1$, $\nu(A) = 0$.

There are several distances that may show if measures are singular, for instance consider the *total variation* distance,

$$\|\mu - \nu\|_{TV} = \sup_{A\, meas.} |\mu(A) - \nu(A)|,$$

which is smaller or equal than one for probability measures, and equal to one if and only if the two measures are singular (total variation does not help in general to tell equivalent from non–equivalent measures apart if they are not singular).

Another quantity is the *Hellinger* integral, which is very effective for product measures. Given probability measures μ, ν let γ be a probability measure such that μ, ν are both absolutely continuous with respect to γ (for instance $\gamma = \frac{1}{2}(\mu + \nu)$). Write$\mu = f\gamma$ and $\nu = g\gamma$, and define

$$H(\mu, \nu) = \int \sqrt{f(x)g(x)}\,\gamma(dx)$$

PROPOSITION 5.1. *The following statement are equivalent,*

1. μ, ν *are singular,*
2. *there is a sequence* $(A_n)_{n\in\mathbf{N}}$ *such that* $\mu(A_n) \to 1$, $\nu(A_n) \to 0$,
3. $H(\mu, \nu) = 0$.

PROOF. The fact that (1) implies (2) is obvious (take $A_n = A$). We prove that (2) implies (3). Let $\gamma = \frac{1}{2}(\mu + \nu)$, $\mu = f\gamma$ and $\nu = g\gamma$, then

$$\begin{aligned} H(\mu, \nu) = \int \sqrt{fg}\, d\gamma = \int \sqrt{fg}\mathbb{1}_{A_n}\, d\gamma + \int \sqrt{fg}\mathbb{1}_{A_n^c}\, d\gamma \leqslant \\ \leqslant \Big(\int f\, d\gamma\Big)^{\frac{1}{2}} \Big(\int g\mathbb{1}_{A_n}\, d\gamma\Big)^{\frac{1}{2}} + \Big(\int g\, d\gamma\Big)^{\frac{1}{2}} \Big(\int f\mathbb{1}_{A_n^c}\, d\gamma\Big)^{\frac{1}{2}} \leqslant \\ \leqslant \sqrt{\nu(A_n)} + \sqrt{\mu(A_n^c)} \longrightarrow 0 \end{aligned}$$

Finally, (3) implies (1). Let again $\gamma = \frac{1}{2}(\mu + \nu)$, $\mu = f\gamma$ and $\nu = g\gamma$ and set $A = \{f > g\}$, then

$$H(\mu, \nu) = \int \sqrt{fg}\, d\gamma = \int \mathbb{1}_A \sqrt{fg}\, d\gamma + \int \mathbb{1}_{A^c} \sqrt{fg}\, d\gamma \geqslant$$
$$\geqslant \int g \mathbb{1}_A \, d\gamma + \int f \mathbb{1}_{A^c} \, d\gamma = \nu(A) + \mu(A^c),$$

that is $\nu(A) + \mu(A^c) \leqslant H(\mu, \nu)$. □

COROLLARY 5.2. *If there is a sequence of measurable maps* $(F^n)_{n\in\mathbf{N}}$ *such that* $H(F^n_\# \mu, F^n_\# \nu) = 0$*, then* μ, ν *are mutually singular.*

PROOF. We prove (2) of the previous theorem. Let f_n, g_n be the densities of $F^n_\# \mu$ and $F^n_\# \nu$ with respect to their arithmetic mean γ_n and set $B_n = \{f_n < g_n\}$. Then

$$F^n_\# \mu(B_n) + F^n_\# \nu(B^c_n) = \int f_n \mathbb{1}_{B_n} \, d\gamma_n + \int g_n \mathbb{1}_{B^c_n} \, d\gamma_n \leqslant$$
$$\leqslant \int \sqrt{f_n g_n} \mathbb{1}_{B_n} \, d\gamma_n + \int \sqrt{f_n g_n} \mathbb{1}_{B^c_n} \, d\gamma_n = \int \sqrt{f_n g_n} \, d\gamma_n = H(F^n_\# \mu, F^n_\# \nu).$$

Set $A_n = F_n^{-1}(B_n)$, then $\mu(A_n) \to 0$ and $\nu(A_n) \to 1$. □

We refer to [**DPZ92**] for further details on the subject.

5.1.1. The Cameron–Martin theorem. We wish to understand equivalence of a Gaussian measure with a translate of it. Let us start with the simplest example, namely one dimensional Gaussian measures. If μ is $\mathcal{N}(0, \sigma^2)$ and ν is $\mathcal{N}(m, \sigma^2)$, it is just a matter of elementary computations to see that the two measures are (clearly) equivalent with density $g(x) = \exp\big(-\frac{1}{2}(m/\sigma)^2 + mx/\sigma^2\big)$. The multi–variate case, namely when the two measures have a $d \times d$ covariance matrix $\mathcal{C}$, gives likewise $g(x) = \exp\big(-\frac{1}{2} m^T \cdot \mathcal{C}^{-1} \cdot m + m^T \cdot \mathcal{C}^{-1} \cdot x\big)$. Let us see what happens in infinite dimension.

Let μ be a probability measure on a Hilbert space H. The measure μ is Gaussian if $h_\# \mu$ is Gaussian on $\mathbf{R}$ for every $h \in H$ (here we see h as an element of the dual H', which is identified by the Hilbert space itslef, and $h_\# \mu$ is the image measure by the linear continuous map $h : H \to \mathbf{R}$). A Gaussian measure is uniquely identified by its *mean*

$$\langle m, h \rangle = \int_H \langle h, x \rangle \mu(dx)$$

and covariance matrix

$$\mathcal{C}_\mu(h, h') = \int_H \langle h, x \rangle \langle h', x \rangle \mu(dx)$$

with the associated covariance operator $\mathcal{C}_\mu : H \to H$, defined by

$$\mathcal{C}_\mu h = \int_H x\langle h, x\rangle \mu(dx).$$

The covariance operator of a Gaussian measure is trace class (see [**DPZ92**, Proposition 2.15]), hence there exists a complete orthonormal basis $(e_n)_{n\in\mathbf{N}}$ of eigenvectors with corresponding eigenvalues $(\lambda_n)_{n\in\mathbf{N}}$.

We state a first result that shows how even dilations of Gaussian measures are mutually singular.

PROPOSITION 5.3. *Let μ be a centred Gaussian measure on an infinite dimensional Hilbert space H and $d^c : H \to H$ be the map $d_c(x) = cx$, then μ and $d^c_\#\mu$ are mutually singular for $c \neq \pm 1$.*

PROOF. The result follows immediately by the Feldman Hajek theorem (see Theorem 5.6 below), since if $\mathcal{C}_\mu$ is the covariance of μ, then the operator $\mathcal{A}$ therein is given by $\mathcal{A} = (c^2 - 1)I$, which is not Hilbert–Schmidt unless $c^2 = 1$.

A direct proof is given in [**Hai09**, Proposition 3.40]. Set $X_n = \lambda_n^{-1/2}\langle e_n, \cdot\rangle$. The X_n can be seen as random variables on the probability space (H, μ) and it turns out that they are independent standard Gaussian. On the other hand, on the probability space $(H, d^c_\#\mu)$ they are independent Gaussian with variance c^2. Hence by the strong law of large numbers, the event

$$\Big\{\lim_n \frac{1}{n}\sum_{k=1}^n X_k^2 = 1\Big\}$$

is a full set for μ and a null set for $d^c_\#\mu$, unless $c^2 = 1$. □

Denote by H_μ the closure of the range of the covariance $\mathcal{C}_\mu$ with respect to the nrom $\|x\|_\mu^2 = \langle \mathcal{C}_\mu^{-1}x, x\rangle$, or equivalently [**Hai09**, Exercise 3.35],

$$\|x\|_\mu = \sup\{\langle x, h\rangle : \langle \mathcal{C}_\mu h, h\rangle \leqslant 1\}.$$

In this case $H_\mu = \{x : \|x\|_\mu < \infty\}$. The space H_μ is the *Cameron–Martin space* and has a crucial role in understanding if translates of a Gaussian measure are equivalent to the original measure (this property is called *quasi–invariance*).

THEOREM 5.4 (Cameron–Martin). *Given $h \in H$, define the map $\tau_h : H \to H$ as $\tau^h x = x + h$. Then μ and $\tau^h_\#\mu$ are equivalent if and only if $h \in H_\mu$. Moreover, if $h \in H_\mu$, then the density is*

$$G_h(x) = e^{\langle \mathcal{C}_\mu^{-1}h, x\rangle - \frac{1}{2}\|h\|_\mu^2}.$$

PROOF. Assume $h \in H_\mu$, then $\int_H \langle \mathcal{C}_\mu^{-1}h, x\rangle^2 \mu(dx) = \|h\|_\mu^2$ and by Fernique's theorem $x \mapsto e^{\langle \mathcal{C}_\mu^{-1}h, x\rangle}$ is μ–integrable, hence the map G_h is strictly positive and $\int G_h(x)\mu(dx) = 1$. The measure $G_h\mu$ is clearly absolutely continuous with respect to μ, so we only need to prove that $G_h\mu = \tau^h\mu$. This can be done via characteristic functions, with $\widehat{\tau}^h\mu(\xi) = \exp(-\frac{1}{2}\langle\mathcal{C}_\mu\xi,\xi\rangle + i\langle\xi,h\rangle)$ (see details in Theorem 3.41 of [Hai09]).

Assume $h \notin H_\mu$, then for every $n \geqslant 1$ there is x with $\langle\mathcal{C}_\mu x, x\rangle = 1$ and $\langle x, h\rangle \geqslant n$. The image measure m_1 of μ with respect to the map $\langle x,\cdot\rangle$ is a standard Gaussian, while the image measure m_2 of $\tau^h_\#\mu$ is a Gaussian with mean $-\langle x,h\rangle$ and variance 1. It is elementary to check that $\|m_1-m_2\|_{TV} \geqslant 1-e^{-\frac{1}{8}\langle x,h\rangle^2}$, hence

$$\|\mu - \tau^h_\#\mu\|_{TV} \geqslant \|m_1 - m_2\|_{TV} \geqslant 1 - \exp -n^2/8,$$

and μ, $\tau^h_\#\mu$ are singular. □

REMARK 5.5. In finite dimension dilations and translates of Gaussian measures are clearly equivalent to the original measures and the Cameron–Martin space coincides with the support of the measure. The reader should compare with the infinite dimensional case.

It is interesting to notice that the Cameron–Martin space is a null set for the measure μ in infinite dimension, namely $\mu(H_\mu) = 0$. On the other hand the closure of the Cameron–Martin space coincides with the support of the measure. An even stronger hint that H_μ contains the core information of the measure is that if $(\xi_1,\dots,\xi_n\dots$ are independent standard Gaussian random variables, then

$$\sum_{n=1}^\infty \xi_n e_n$$

is a Gaussian random variable on H with law μ [DPZ92, Theorem 2.12].

For the sake of completeness, we mention the following theorem, which further analyse the equivalence of Gaussian measures in infinite dimension, whenever the covariance operators are different. A detailed and complete proof of this result can be found in [DPZ92, Theorem 2.23].

THEOREM 5.6 (Feldman–Hajek). *Let* $\mu_1 = \mathcal{N}(m_1,\mathcal{C}_1)$ *and* $\mu_2 = \mathcal{N}(m_2,\mathcal{C}_2)$ *Gaussian measures on a Hilbert space* H. *Then either* μ_1,μ_2 *are singular, or they are equivalent. Moreover, they are equivalent if and only if the following conditions hold,*

- $H_0 := \mathcal{C}_1^{\frac{1}{2}}H = \mathcal{C}_2^{\frac{1}{2}}H$,
- $m_1 - m_2 \in H_0$,
- *the operator* $\mathcal{A} := (\mathcal{C}_1^{-\frac{1}{2}}\mathcal{C}_2^{\frac{1}{2}})(\mathcal{C}_1^{-\frac{1}{2}}\mathcal{C}_2^{\frac{1}{2}})^\star - I$ *is Hilbert–Schmidt on* $\overline{H_0}$.

5.2. The Girsanov transformation

Let W be a cylindrical Wiener process on a Hilbert space H, let $\mathcal{C}$ be a trace class operator on H, and set μ_t the law of $\mathcal{C}^{1/2}W_t$. The Cameron–Martin space of μ_t is $\mathcal{C}^{1/2}H$ and $\|\cdot\|_{\mu_t} = \sqrt{t}\|\mathcal{C}^{-1/2}\cdot\|_H$. Let $x \in H$, then certainly $C^{1/2}x$ is in the Cameron–Martin space of μ_t and the law of $C^{1/2}x + C^{1/2}W_t$ is equivalent to μ_t, the law of $C^{1/2}W_t$, with density

$$G_x = e^{\langle \mathcal{C}^{-1/2}x, \mathcal{C}^{1/2}W_t\rangle - \frac{1}{2}\|\mathcal{C}^{-1/2}(\mathcal{C}^{1/2}x)\|_H^2}.$$

It turns out that by this density we have kept the same covariance and changed the mean (as long as the new mean is in the right space). We wish to do this at the level of the full path rather than for a fixed time. Since in Section 5.3 we will use the Girsanov transformation *only* at the level of finite dimensional approximations, we start with the finite dimensional version, following [**KS91**].

5.2.1. The Girsanov theorem. Let W be a d–dimensional standard Brownian motion and let X be a $\mathbf{R}^d$–valued adapted process with

$$\mathbb{P}\Big[\int_0^T |X_t|^2\,dt < \infty\Big] = 1.$$

Let

$$G_t = \exp\Big(\int_0^t X_s \cdot dW_s - \frac{1}{2}\int_0^t |X_s|^2\,ds\Big),$$

then $G_0 = 1$ and by the Itô formula,

$$G_t = 1 + \int_0^t G_s X_s \cdot dW_s,$$

hence G is a continuous local martingale. Assume that G is a martingale, then $\mathbb{E}[G_t] = 1$ and we can define a new probability $\mathbb{P}_t = G_t\mathbb{P}$. We wish to understand which is the law of W under the new probability space. By comparison with the Cameron–Martin example at the beginning of this section, we are changing the mean of W_t *for every* t and we are doing it in a cumulative way, so we can expect that in the new probability space W_t has the same law of

$$W_t + \int_0^t X_s\,ds$$

This is the content of Girsanov's theorem.

THEOREM 5.7. *Assume that* G *is a martingale. Define the process*

$$\widetilde{W}_t = W_t - \int_0^t X_s\,ds,$$

then for each fixed $T > 0$ *the process* $(\widetilde{W}_t)_{t\in[0,T]}$ *is a standard Brownian motion with respect to* $\mathbb{P}_T = G_T\mathbb{P}$.

The proof is based on the Levy characterization of Brownian motion and can be found, in full details, in [**KS91**].

REMARK 5.8. The theorem holds only on a *finite horizon*. It is easy to see that it cannot hold on the infinite time horizon. Let $\mu \neq 0$ and take X is constant, namely $X_t = \mu$ a.s.. Assume that the equivalence of Girsanov theorem extends to $[0,\infty)$ to a probability $\widetilde{\mathbb{P}}$. Then, by the law of large numbers,

$$\widetilde{\mathbb{P}}[\lim_{t\to\infty} \tfrac{1}{t}W_t = \mu] = \mathbb{P}[\lim_{t\to\infty} \tfrac{1}{t}\widetilde{W}_t = 0] = 1, \quad \text{and} \quad \mathbb{P}[\lim_{t\to\infty} \tfrac{1}{t}W_t = \mu] = 0.$$

The key point is then to understand when G is a martingale. Since G is a local martingale, $G_t \wedge T_n$ is a martingale for a sequence of stopping times $T_n \uparrow \infty$. By the Fatou lemma for the conditional expectation,

$$\mathbb{E}[G_t|\mathscr{F}_s] \leqslant \liminf_n \mathbb{E}[G_{t\wedge T_n}|\mathscr{F}_s] = \liminf_n G_{s\wedge T_n} = G_s,$$

for $s \leqslant t$, hence G is a super–martingale and it is a martingale if and only if $\mathbb{E}[G_t] = 1$ for all $t \geqslant 0$. A well–known sufficient condition for having G martingale is the following.

THEOREM 5.9 (Novikov condition). *With the positions set above, if*

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^T |X_s|^2\,ds\Big)\Big] < \infty,$$

then G *is a martingale.*

The above condition can be improved to the following *Kazamaki* condition, which states that if $\exp(\frac{1}{2}\int_0^t X_s \cdot dW_s)$ is a uniformly integrable sub–martingale, then G is a martingale.

5.2.2. Application to SDE. Girsanov theorem provides a simple way to prove existence and uniqueness of weak solution of stochastic differential equations (SDE) with additive noise and drift with minimal regularity. Consider

$$dX_t = b(t, X_t)\,dt + dW_t,$$

where W_t is a d–dimensional Brownian motion and $b : [0,\infty) \times \mathbf{R}^d \to \mathbf{R}^d$ is measurable and there is $c > 0$ such that $|b(t,x)| \leqslant c(1 + |x|)$. the first results ensures existence, the basic idea is to start with a Brownian motion and change probability, so that the same process turns out to be a solution.

PROPOSITION 5.10 (Existence). *For every $x \in \mathbf{R}^d$ there exists a solution to the above SDE.*

PROOF. To simplify the proof, assume that b is bounded. Let X be a d–dimensional standard Brownian motion started at x, then the process

$$G_t = \exp\Big(\int_0^t b(s, X_s)\cdot dX_s - \frac{1}{2}\int_0^t |b(s, X_s)|^2\,ds\Big)$$

is a martingale. By the Girsanov theorem, under the probability $G_T\mathbb{P}$, the process

$$W_t = X_t - x - \int_0^t b(s, X_s)\,ds$$

is a Brownian motion, hence (X, W) is a weak solution. □

Girsanov's theorem provides also a way to prove uniqueness. The basic idea is the same: a change of probability maps back the solution to a Brownian motion.

PROPOSITION 5.11 (Uniqueness). *Let* $(\Omega^i, \mathscr{F}^i, \mathbb{P}^i, (\mathscr{F}^i_t)_{t\geqslant 0}, X^i, W^i)$, $i = 1, 2$ *be two weak solutions of the SDE above with* $X^1_0 = X^2_0$. *If*

$$\mathbb{P}^i\Big[\int_0^T |b(s, X^i_s)|^2\,ds < \infty\Big] = 1, \qquad i = 1, 2,$$

then (X^1, W^1) *and* (X^2, W^2) *have the same distribution.*

PROOF. Assume again for simplicity that b is bounded. For $i = 1, 2$, the processes

$$G^i_t = \exp\Big(-\int_0^t b(s, X^i_s)\cdot\,dW^i_s - \frac{1}{2}\int_0^t |b(s, X^i_s)|^2\,ds\Big)$$

are martingale on their respective filtered probability space. Given $T > 0$ we can define on $\mathscr{F}^i_T$ the probability $\widetilde{\mathbb{P}}^i = G^i_T\mathbb{P}^i$. Under $\widetilde{\mathbb{P}}^i$, the process

$$X^i_t = X_0 + \int_0^t b(s, X^i_s)\,ds + W^i$$

is a d–dimensional Brownian motion, hence under $\widetilde{\mathbb{P}}^i$ also W^1, W^2 have the same distribution, as well as G^1, G^2. This implies easily that (X^1, W^1) and (X^2, W^2) have the same distribution. □

5.2.3. A slight modification. In this final preliminary section we state a criterion that ensures solutions to SDEs with different drift have equivalent laws, under a much weaker assumption than Novikov. The result is very much in the spirit of the uniqueness proposition proved above and is contained in Theorem 7.19 and Theorem 7.20 (in the formulation of Section 7.6.4) of [LS01]. Here we give a slightly simplified formulation, closer to our needs.

Consider the two SDEs

$$d\xi = B(\xi)\,dt + \mathcal{C}dW,$$
$$d\eta = b(\eta)\,dt + \mathcal{C}dW,$$

with a common initial condition, where $B : \mathbf{R}^d \to \mathbf{R}^d$ and $b : \mathbf{R}^d \to \mathbf{R}^d$, $\mathcal{C}$ is a $m \times d$ positive symmetric matrix and W is m–dimensional standard Wiener process. Assume that *both problems have a unique strong solution*.

THEOREM 5.12. *With the above assumptions and positions, assume that for* $T > 0$

$$\mathbb{P}\Big[\int_0^T \Big(B(\xi_t)^\star \mathcal{C}^{-1} B_t(\xi_t) + b(\xi_t)^\star \mathcal{C}^{-1} b_t(\xi_t)\Big)\,dt < \infty\Big] = 1,$$

then the law of ξ *is absolutely continuous with respect to the law of* η *on* $[0, T]$*. Moreover, the density is given by*

$$\exp\Big(\int_0^t (B(\eta) - b(\eta))^\star \mathcal{C}^{-1}\,d\eta_s - \frac{1}{2}\int_0^t (B(\eta) - b(\eta))^\star \mathcal{C}^{-1}(B(\eta) - b(\eta))\,ds\Big).$$

The idea of the proof here is to use localization, by means of a sequence of stopping times which make the integral (which is almost surely finite) bounded. With this at hands, one can apply Girsanov's theorem on the stopped process. the final result is recovered in the limit.

A straightforward extension of the above theorem ensures equivalence of laws.

COROLLARY 5.13. *Assume additionally that*

$$\mathbb{P}\Big[\int_0^T \Big(B(\eta_t)^\star \mathcal{C}^{-1} B_t(\eta_t) + b(\eta_t)^\star \mathcal{C}^{-1} b_t(\eta_t)\Big)\,dt < \infty\Big] = 1,$$

then the laws of ξ *and* η *on* $[0, T]$ *are equivalent measures.*

5.2.4. The infinite dimensional version. The infinite dimensional version of the Girsanov theorem clearly has to take into account that the directions for the change of variables should be compatible with the Cameron–Martin theorem. This is the only modification to extend Girsanov's theorem to infinite dimension. The version given here is [DPZ92, Theorem 10.14].

THEOREM 5.14. *Let* $\mathcal{C}$ *be a trace class operator on* H*,* W *a cylindrical Wiener process on* H*. Let* $\psi : [0, \infty) \to \mathcal{C}^{1/2}H$ *be a predictable process such that*

$$G_t = \exp\Big(\int_0^T \langle \mathcal{C}^{-1}\psi(s), dW_s\rangle - \frac{1}{2}\int_0^T \|\mathcal{C}^{-\frac{1}{2}}\psi(s)\|_H^2\,ds\Big)$$

is a martingale. Then the process

$$\widetilde{W}_t = W_t - \int_0^t \mathcal{C}^{-\frac{1}{2}}\psi(s)\,ds, \qquad t \in [0, T]$$

is a cylindrical Wiener process with respect to the probability measure $G_T\mathbb{P}$*.*

Novikov's condition extends to this setting, once its formulation takes the topology suggested by the Cameron–Martin theorem into account.

REMARK 5.15 (An odd example). the reader may have the feeling that in the presentation we have given there is an implicit suggestion that only the

Cameron–Martin directions may give equivalent laws. This is somewhat incorrect, at least with random directions, as shown in the following example due to [**Fer02**].

On $\mathbf{R}^N$ let $\mu = \mathcal{N}(0, I)$ be the centred Gaussian measure with covariance the identity. Its Cameron–Martin space is $\ell^2(\mathbf{R})$. Let G be a random variable with law μ and consider the random variable Y, independent of G, whose law is given by

$$\mathbb{P}[y_n = 0] = 1 - \frac{1}{n}, \qquad \mathbb{P}[y_n = \sqrt{2\log n}] = \frac{1}{n}.$$

Clearly Y is not $\ell^2(\mathbf{R})$–valued (it is not even ℓ^∞–valued), but if one considers $X_\lambda = G + \lambda Y$ and denotes by μ_λ its law, then there are values of λ such that μ and μ_λ are equivalent measures (this happens actually if and only if $\lambda \in [0, 1)$).

Actually the main theorem of Fernique's paper is in this case to state the existence of a random variable Y' (not necessarily independent from G!) with values in $\ell^2(\mathbf{R})$ such that, for $\lambda < 1$, $G + \lambda Y'$ has law μ_λ. So in some sense our "implicit suggestion" still keeps some value.

As an immediate application, let us consider equivalence of laws for stochastic PDEs. Consider the two problems

$$dz + Az\,dt = \mathcal{C}^{\frac{1}{2}}\,dW,$$
$$du + Au\,dt + B(u)\,dt = \mathcal{C}^{\frac{1}{2}}\,dW,$$

with the same initial condition, where A is for instance $-\Delta$ with suitable boundary conditions. The Novikov condition in this setting reads

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^T \|\mathcal{C}^{-\frac{1}{2}}B(z_t)\|_H^2\Big)\Big] < \infty.$$

Under this assumption we know that the process $\widetilde{W}_t = W_t + \int_0^t \mathcal{C}^{-\frac{1}{2}}B(z_s)\,ds$ is a cylindrical Wiener process, hence in the new probability space the process z solves the equation

$$dz + Az\,dt = \mathcal{C}^{\frac{1}{2}}\,dW = \mathcal{C}^{\frac{1}{2}}(d\widetilde{W} - \mathcal{C}^{-\frac{1}{2}}B(z)\,dt) = -B(z)\,dt + \mathcal{C}^{\frac{1}{2}}\,d\widetilde{W},$$

the equation solved by u. In conclusion, the law of u on $[0, T]$ is absolutely continuous with respect to the law of z.

See [**Fer10**] for an application of the results of [**LS01**] to the infinite dimensional setting.

5.3. Existence of densities for Navier–Stokes with the Girsanov theorem

We turn finally to the proof of existence of densities for finite dimensional projections of the Navier–Stokes equatons. With respect to Chapter 3, we have some restrictions, namely we can only consider a special class of finite dimensional spaces, as well as improvement, as the covariance is more general. As with Malliavin calculus, we consider a special class of solutions.

Assume that the covariance $\mathcal{C} \in \mathscr{L}(H)$ is of trace–class and that $\ker(\mathcal{C}) = \{0\}$ (less restrictive but essentially identically assumptions may work, see the remarks in Section 5.3.4 below). The solution we consider here is the limit of Galerkin approximations (as explained in Section 1.3), so we fix a process u and a sequence $(u^{N_k})_{k\in\mathbf{N}}$ such that $u^{N_k} \rightharpoonup u$ in law and for each k, u^{N_k} solves the stochastic ODE (1.9), with initial condition the projection onto H_{N_k} of $u(0)$. We fix also a finite dimensional space $F = \operatorname{span}[e_{n_1}, \dots, e_{n_d}]$ and a time $t > 0$, which without loss of generality is taken equal to $t = 1$.

For $N \geqslant n_d$, let $\mathbb{P}^N$ be the distribution of the solution of the Galerkin system at level N with initial condition $\pi_N u(0)$ and denote by f_N the density of the random variable $\pi_F u^N(1)$. We already know from Section § 2.3 that $u^N(1)$ admits a smooth density, hence f_N is smooth as well.

5.3.1. The approximated Girsanov density. We use the version of Girsanov's theorem detailed in Section 5.2.3. Let v^N be the solution of

$$dv^N + \big(\nu A v^N + B^N(v^N) - \pi_F B^N(v^N)\big)\,dt = \pi_N \mathcal{C}^{\frac{1}{2}}\,dW,$$

with the same initial condition as u^N. The idea is that the drift has been changed so that the projection of v^N on F solves a simple linear equation, namely $\pi_F v^N$ is solution of

$$dz^F + \nu A z^F = \pi_F \mathcal{C}^{\frac{1}{2}}\,dW, \tag{5.1}$$

with initial condition $\pi_F u(0)$, which in particular is decoupled from $v^N - \pi_F v^N$. Moreover, it is easy to prove, with essentially the same methods that yield (1.10), that

$$\mathbb{E}\Big[\sup_{[0,T]} \|v^N\|_H^p\Big] < \infty. \tag{5.2}$$

The next idea is to exploit the fact that we are working in finite dimension. We recall that the main difficulty of the three–dimensional Navier–Stokes equations stem from the non–linear term, or more precisely from its lack of regularity. The key point here is that when the dimension is finite, all topologies are equivalent and we by–pass the lack of regularity. More quantitatively, $\sup_{\|w\|_{W^{1,\infty}}=1}\langle v, w\rangle$ is a norm on $\pi_N H$, which is therefore equivalent to the norm of H on $\pi_N H$. We can then write:

$$\langle \pi_N B(v), w\rangle = -\langle B(v, \pi_N w), v\rangle \leqslant c\|w\|_{W^{1,\infty}}\|v\|_H^2,$$

therefore

$$\|\pi_N B(v)\|_H \leqslant c_N \|v\|_H^2, \qquad v \in H, \tag{5.3}$$

with a constant c_N that clearly explodes as $N \uparrow \infty$.

Let us verify the assumption of Theorem 5.12. Since the covariance $\mathcal{C}^{\frac{1}{2}}$ is invertible on $\pi_N H$, we deduce from (1.10), (5.2) that

$$\int_0^t \|\mathcal{C}^{-\frac{1}{2}}\pi_N B(u^N)\|_H^2\,ds < \infty, \qquad \int_0^t \|\mathcal{C}^{-\frac{1}{2}}\pi_N B(v^N)\|_H^2\,ds < \infty. \qquad \mathbb{P}-a.s.$$

hence the process

$$G_t^N = \exp\Big(\int_0^t \langle \mathcal{C}^{-\frac{1}{2}}\pi_F B(u^N), dW_s\rangle - \frac{1}{2}\int_0^t \|\mathcal{C}^{-\frac{1}{2}}\pi_F B(u^N)\|_H^2\,ds\Big), \tag{5.4}$$

is positive, finite $\mathbb{P}$–a. s. and a martingale. Moreover, under the probability measure $\widetilde{\mathbb{P}}_N(d\omega) = G_t^N \mathbb{P}_N(d\omega)$ the process

$$\widetilde{W}_t = W_t - \int_0^t \mathcal{C}^{-\frac{1}{2}}\pi_F B(u^N)\,ds$$

is a cylindrical Wiener process on H and $\pi_F u^N(t)$ has the same distribution of $\pi_F v^N(t)$, hence of the solution z^F of (5.1), which is independent of N.

This fact allows us to give an alternative proof of existence of the density of the random variable $\pi_F u^N(t)$. Indeed, for every measurable $E \subset F$,

$$\mathbb{P}_N[z^F(t) \in E] = \widetilde{\mathbb{P}}_N[\pi_F u^N(t) \in E] = \mathbb{E}^{\mathbb{P}_N}\big[G_t^N \mathbb{1}_E(\pi_F u^N(t))\big]$$

and if $\mathscr{L}_F(E) = 0$, where $\mathscr{L}_F$ is the Lebesgue measure on F, then $\mathbb{1}_E(z^F(t)) = 0$. Since $G_t^N > 0$, $\mathbb{P}_N$–a. s., we have that $\mathbb{1}_E\big(\pi_F u^N(t)\big) = 0$, $\mathbb{P}_N$–a. s., that is $\mathbb{P}_N[\pi_F u^N(t) \in E] = 0$. In conclusion $\pi_F u^N(t)$ has a density with respect to the Lebesgue measure on F.

5.3.2. The limit Girsanov density. Consider now the weak martingale solution u of the infinite dimensional problem, which is limit in law of a sequence $(u^{N_k})_{k\in\mathbf{N}}$ of Galerkin approximation. In order to show that $G_t^{N_k}$ is convergent, we use Skorokhod's theorem: there exist a probability space, a Wiener process, a sequence of random variables $(U_k)_{k\in\mathbf{N}}$ and a random variable U such that for every k U_k and u^{N_k} have the same law, as well as u and U, and U_k converges almost surely to U in $C([0,T];H_w)$ — where we recall that H_w is the space H with the weak topology — and in $L^2(0,T;H)$ for every $T > 0$. In particular the sequence $(U_k)_{k\geqslant 1}$ is a. s. bounded in $L^\infty(0,T;H)$ and thus a. s. strongly convergent in $L^p(0,T;H)$ for every $T > 0$ and every $p < \infty$.

Denote by $\widetilde{G}_t^k$ the density G^{N_k} in (5.4) where u^{N_k} is replaced by U_k, and $\widetilde{G}$ the density in (5.4) where u^N is replaced by U. The density $\widetilde{G}_t$ is strictly positive and finite almost surely, since by (5.3) and (1.10),

$$\frac{1}{2}\int_0^t \|\mathcal{C}^{-\frac{1}{2}}\pi_F B(U)\|_H^2\,ds < \infty, \qquad \text{a. s.}$$

on the limit solution.

The problem here is that, although $\widetilde{G}^k$ converges almost surely to G, we do not have good enough exponential moments for U (that is, u) to ensure L^1 convergence of $\widetilde{G}_k$ to $\widetilde{G}$. Indeed, $B(U)$ is quadratic in U, hence $\|\mathcal{C}^{-\frac{1}{2}}\pi_F B(U)\|_H^2$ is quartic in U. Such an exponential moment would give very light tails to u, lighter than Gaussian.

5.3.3. Existence of the density. Nevertheless something can be still done, as long as we are only interested in showing *existence* of a density (quantitative estimates will be considered in the next chapters).

We show that $\pi_F u(t)$ has a density with respect to the Lebesgue measure on F. Let $E \subset F$ with $\mathscr{L}_F(E) = 0$, then for every open set J such that $E \subset J$ we have by Fatou's lemma (notice that $\mathbb{1}_J$ is lower semi–continuous with respect to the weak convergence in H since J is finite dimensional),

$$\begin{aligned}\mathbb{E}[G_t \mathbb{1}_E(\pi_F u(t))] = \mathbb{E}[\widetilde{G}_t \mathbb{1}_E(\pi_F U(t))] \leqslant \\ \leqslant \mathbb{E}[\widetilde{G}_t \mathbb{1}_J(\pi_F U(t))] \leqslant \liminf_k \mathbb{E}[\widetilde{G}^k_t \mathbb{1}_J(\pi_F U^k(t))] = \\ = \liminf_k \mathbb{E}[G^{N_k}_t \mathbb{1}_J(\pi_F u^{N_k}(t))] = \mathbb{P}[z^F(t) \in J],\end{aligned}$$

hence $\mathbb{E}[G_t \mathbb{1}_E(\pi_F u(t))] = 0$ since J can have arbitrarily small measure and $z^F(t)$ has Gaussian density. As already pointed out, $G_t > 0$ almost surely, hence we can deduce that $\mathbb{P}[\pi_F u(t) \in E] = 0$. In conclusion, we have proved the following result.

THEOREM 5.16. *Let F be a finite dimensional subspace of $D(A)$ generated by the eigenvalues of A, namely $F = \operatorname{span}[e_{n_1}, \dots, e_{n_d}]$ for some arbitrary indexes $n_1, \dots, n_d$. For every $t > 0$ the projection $\pi_F u(t)$ has a density with respect to the Lebesgue measure on F, where u is* any *solution of Navier–Stokes with initial condition in H whose law is a limit point of the spectral Galerkin approximations.*

REMARK 5.17. Under the standing assumptions on the covariance taken in Chapter 4, which are stronger than those of this chapter, we can deduce by Theorem 4.7 that the density obtained in the theorem above is positive almost everywhere on F.

5.3.4. Additional remarks and extensions. We conclude the section with a few comments on the above theorem on existence of densities with the Girsanov theorem.

Identification of the law. One of the ways we have found out Girsanov's theorem may be useful, is in proving uniqueness, as in Proposition 5.11 of Section 5.2.2. We recall that finite dimensional marginals *do* identify an infinite dimensional measure, and for a Markov process (whose existence has been ensured in Theorem 3.5) the one dimensional time marginals are enough to identify the whole process. In view of these facts, one may be tempted to use the finite dimensional densities to make up a proof of uniqueness in law.

As already mentioned, the bounds on the sequence $(\widetilde{G}^N_t)_{N\geqslant 1}$ are not strong enough to deduce a stronger convergence to $\widetilde{G}_t$ and hence to deduce, for instance, the representation

$$\mathbb{E}[\phi(z^F(t))] = \mathbb{E}\big[G_t\phi(\pi_F u(t))\big] \tag{5.5}$$

in the limit, for smooth functions $\phi : F \to \mathbf{R}$. Although this formula would provide a representation for the (unknown) density of $\pi_F u(t)$ in terms of the (known) density of z^F, solution of (5.1), this would not characterise the law of $\pi_F u(t)$ by any means, since the factor G_t which appears in the formula depends on the sub–sequence $(N_k)_{k\in\mathbf{N}}$ which ensures that $\mathbb{P}_{N_k} \rightharpoonup \mathbb{P}$.

Vice versa, one could use the inverse density

$$\overline{G}^N_t = \exp\Big(-\int_0^t \langle \mathcal{C}^{-\frac{1}{2}}\pi_F B(v^N), dW_s\rangle - \frac{1}{2}\int_0^t \|\mathcal{C}^{-\frac{1}{2}}\pi_F B(v^N)\|^2_H\, ds\Big),$$

which is also a martingale (here we are invoking Corollary 5.13 rather than Theorem 5.12) to get in the limit

$$\mathbb{E}[\phi(\pi_F u(t))] = \mathbb{E}\big[\overline{G}_t\phi(z^F(t))\big] \tag{5.6}$$

but the bound (5.2) for v^N is not uniform in N.

Multi–dimensional time–marginals. An additional advantage of the Girsanov theorem is that it ensures equivalence at the level of paths rather than at the level of the state space. We can use this fact to give a slight improvement to the above theorem. This should be compared with the results of the next chapters, that improve Theorem 5.16 from a quantitative point of view, but that cannot manage multi–dimensional time–marginals.

Fix $0 < t_1 < t_2 < \cdots < t_m \leqslant T$, if E is a null–set in F^m, then for every open set J such that $E \subset J$, we have, as before

$$\begin{aligned}\mathbb{E}[G_T \mathbb{1}_E(\pi_F u(t_1),\dots,\pi_F u(t_m))] &\leqslant \liminf_k \mathbb{E}[G^{N_k}_T \mathbb{1}_J(\pi_F u^{N_k}(t_1),\dots,\pi_F u^{N_k}(t_m))]\\ &= \mathbb{P}[(z^F(t_1),\dots,z^F(t_m)) \in J],\end{aligned}$$

and we conclude as above since $(z^F(t_1),\dots,z^F(t_m))$ is jointly Gaussian. These considerations can be summarised in the following result.

COROLLARY 5.18. *Under the assumptions and positions of Theorem 5.16, for every positive* $t_1, t_2, \dots, t_m$ *the random variable* $(\pi_F u(t_1),\dots,\pi_F u(t_m))$ *has a joint density with respect to the Lebesgue measure on* $F^m \approx \mathbf{R}^{md}$.

Extension to degenerate covariances. By running through the proof of Theorem 5.16, one can notice that we have really used the fact that the covariance is invertible *only* on F. Indeed, existence of a smooth density at the level of Galerkin approximations holds under much weaker assumptions on the covariance, as we have seen in Section § 2.3. In the end our assumption on the covariance is restrictive and rather than requiring $\ker(\mathcal{C}) = \{0\}$, we can impose $\mathsf{F} \cap \ker(\mathcal{C}) = \{0\}$, which allows for a degenerate covariance.

On the other hand, if one wants to analyse the densities *at any* frequency, there is no substantial difference between the two assumptions. The extension to a problem with a *real* degenerate covariance seems to be tough and is the subject of a work in progress.

CHAPTER 6

Besov regularity of the density

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Malliavin calculus is one of the most powerful methods to prove existence and regularity of densities of random variables, and it has proved itself extremely versatile in being effective, even beyond the Gaussian realm. We have indeed used Malliavin calculus in Chapter 3 for our purposes, but it is clear that so far we can consider the Navier–Stokes equations in three dimensions a problem which is *non Malliavin differentiable*.

We wish to give a hint which should convince the reader about this statement. Consider the equations in abstract form (1.7), and deduce the equation solved by the Malliavin derivative $\mathcal{D}_v u$ along a direction v,

$$\frac{d}{dt}\mathcal{D}_v u + \nu A \mathcal{D}_v u + B(u, \mathcal{D}_v u) + B(\mathcal{D}_v u, u) = \mathcal{C}^{\frac{1}{2}} v.$$

It turns out that $\mathcal{D}_v u$ satisfies the linearization of (1.7), which is the same equation, up to the term on the right–hand side, solved by the difference of two solutions. In other words, whatever decent estimate we may get out of the equation above might prove much more useful for the issue of uniqueness. There is no hope at this stage to be able to prove anything useful in this direction.

We turn then to methods for existence of densities for non Malliavin differentiable problems. The one we present here is taken from [**DR12**], is based on a simple idea of [**FP10**] (see also [**Fou08**]) and has been later used also in [**DF12**, **Fou12**]. The idea, in one form or another, is simply to approximate the random variable under examination, obtain regularity estimates for the densities of the approximating sequence and try to balance the approximation rate (small) and the size of the densities (large). The same idea has been used in several recent papers, such as [**BC11a**, **BC11b**, **BF11**, **DM11**], and recently furtherly developed in [**BC12**]. A different approach for non Malliavin differentiable problems based on Girsanov's formula has been introduced in [**KHT12**, **HKHY12**].

In this chapter we show that the density found in the previous chapter with the Girsanov change of probability has a little bit more regularity than the baisc one provided by the Radon–Nykodym theorem. We work again under the assumptions on the covariance of Section §5.3, although again the remarks of Section §5.3.4 still apply.

6.1. A primer on Besov spaces

Besov spaces, together with the Triebel–Lizorkin spaces, are a scale of function spaces introduced to capture the fine properties of regularity of functions, beyond on the one hand the Sobolev spaces, and on the other hand the spaces of continuous functions. Indeed, Besov spaces contain both. The main references we shall use on this subject are [**Tri83**, **Tri92**].

6.1.1. The Littlewood–Paley decomposition. Let $\varphi \in C_c^\infty(\mathbf{R}^d)$ such that $\varphi \equiv 1$ if $|x| \leqslant 1$ and $\operatorname{supp}\varphi \subset \{y \in \mathbf{R}^d : |y| \leqslant 2\}$, and set

$$\varphi_0(x) = \varphi(x), \qquad \varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x), \quad j \geqslant 1, x \in \mathbf{R}^d,$$

then

- $\operatorname{supp}\varphi_j \subset \{y \in \mathbf{R}^d : 2^{j-1} \leqslant |y| \leqslant 2^{j+1}\}$, for $j \geqslant 1$,
- $\sum_{j=0}^\infty \varphi_j(x) = 1$, for all $x \in \mathbf{R}^d$.

If now $f \in \mathcal{S}'(\mathbf{R}^d)$, for each j the function $f_j = \mathcal{F}^{-1}(\varphi_j \widehat{f})$ is real analytic and $f = \sum_j f_j$. Given $p, q \in [1, \infty]$ and $s \in \mathbf{R}$, set

$$\|f\|_{B^s_{p,q}} = \|(2^{js}\|f_j\|_{L^p(\mathbf{R}^d)})\|_{\ell^q}$$

The Besov space $B^s_{p,q}(\mathbf{R}^d)$ is the space of all tempered distributions such that the above norm is finite. The space does not depend on the function φ, although the norm does depend, and different choices of φ give raise to equivalent norms.

6.1.2. Besov spaces via the difference operator. The definition given above via the Littlewood–Paley decomposition, although holds for all the possible values of the parameters (even for $p, q \in (0, 1)$), it is not the best suited for our purposes. At least when $s > 0$, there is an alternative equivalent definition (see

[Tri83, Theorem 2.5.12] or [Tri92, Theorem 2.6.1]) in terms of differences. Define

$$(\Delta_h^1 f)(x) = f(x+h) - f(x),$$
$$(\Delta_h^n f)(x) = \Delta_h^1(\Delta_h^{n-1} f)(x) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(x+jh)$$

then the following norms, for $s > 0$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$,

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \Big(\int_{\{|h|\leqslant 1\}} \frac{\|\Delta_h^n f\|_{L^p}^q}{|h|^{sq}} \frac{dh}{|h|^d}\Big)^{\frac{1}{q}}$$

and for $q = \infty$,

$$\|f\|_{B^s_{p,\infty}} = \|f\|_{L^p} + \sup_{|h|\leqslant 1} \frac{\|\Delta_h^n f\|_{L^p}}{|h|^\alpha},$$

where n is any integer such that $s < n$, are equivalent norms of $B^s_{p,q}(\mathbf{R}^d)$ for the given range of parameters.

We remark that alternative definitions of these spaces can be given in terms of real or complex interpolation spaces.

6.1.3. A few properties. There are countless properties of Besov and related spaces which can be found, for instance, in [Tri83, Tri92] and similar books. Here we list a few of them that will be used in the sequel.

The following inclusions are understood as continuous embeddings of Banach spaces,

- $B^s_{p,q}(\mathbf{R}^d) \subset L^p(\mathbf{R}^d)$, for all $s > 0$, $p, q \in [1, \infty]$,
- $B^s_{p,q}(\mathbf{R}^d) \subset B^s_{p,q'}(\mathbf{R}^d)$, if $q \leqslant q'$,
- $B^s_{p,q}(\mathbf{R}^d) \subset B^{s'}_{p,q'}(\mathbf{R}^d)$, if $s' < s$, $p, q, q' \in [1, \infty]$,
- $B^s_{p,q}(\mathbf{R}^d) \subset B^{s'}_{p',q}$ if $1 \leqslant p, q < \infty$, $s \in (0, d/p)$ and $\frac{s}{d} - \frac{1}{p} = \frac{s'}{d} - \frac{1}{p'}$.

For some values of the parameters, we find known spaces. Indeed,

- $B^s_{p,p}(\mathbf{R}^d) = W^{s,p}(\mathbf{R}^d)$, for $p \in [1, \infty)$ and $s > 0$ non–integer,
- $B^s_{\infty,\infty}(\mathbf{R}^d) = C^s(\mathbf{R}^d)$ for all non–integers $s > 0$.

The topological dual of a Besov space has a fairly simple expression [Tri83, Theorem 2.11.2], which justifies the effort of defining them for negative "derivative" indices, namely for $s \in \mathbf{R}$, $p, q \in [1, \infty)$,

$$(B^s_{p,q}(\mathbf{R}^d))' = B^{-s}_{p',q'}(\mathbf{R}^d), \tag{6.1}$$

where p' qnd q' are the conjugate Hölder exponents of p, and q, in other words $1/p + 1/p' = 1$, likewise for q.

Finally, we are interested in the mapping properties of Besov spaces with respect to derivatives. From [Tri83, Theorem 2.3.8] we see that

$$(I_d - \Delta_d)^{\frac{\sigma}{2}} : B^s_{p,q}(\mathbf{R}^d) \longrightarrow B^{s-\sigma}_{p,q}(\mathbf{R}^d) \tag{6.2}$$

is an isomorphism for $s, \sigma \in \mathbf{R}$ and $p, q \in [1, \infty]$, where I_d is the identity function on $\mathbf{R}^d$.

6.2. Besov regularity of the densities

When we have applied the Malliavin calculus to study densities, we have seen that the key point is the integration by parts formula, so that the density turns out to be a smoothing functional. Here we proceed in a similar way, by doing an integration by parts (by using the difference operator introduced above) which "transfers" only a bit of derivatives from the test function to the density.

The main problem is to handle the non–linearity and to do it we introduce an auxiliary process which coincides with the original one up to the very last moment, in which the non–linearity is killed and only the noise runs the process. We gain regularity using the smoothing effect of the noise and we pay the discrepancy between the original and the auxiliary process. The trade–off allows us to get the final result.

6.2.1. The assumptions. The assumptions are exactly the same of the previous chapter, namely we assume that the covariance $\mathcal{C} \in \mathscr{L}(H)$ is of trace–class and that $\ker(\mathcal{C}) = \{0\}$ (again less restrictive but "morally identical" assumptions may work, see the remarks in Section 5.3.4. The solution we consider here is the limit of Galerkin approximations (as explained in Section 1.3), so we fix a process u and a sequence $(u_{N_k})_{k\in N}$ such that $u_{N_k} \rightharpoonup u$ in law and for each k, u_{N_k} solves the stochastic ODE (1.9), with initial condition the projection onto H_{N_k} of $u(0)$. We fix also a finite dimensional space $F = \operatorname{span}[e_{n_1}, \dots, e_{n_d}]$ and a time $t > 0$, which without loss of generality is taken equal to $t = 1$.

For $N \geqslant n_d$, let f_N be the density of the random variable $\pi_F u^N(1)$, where u^N is the solution of (1.9). We already know from Section § 2.3 that $u^N(1)$ admits a smooth density, hence f_N is smooth as well.

6.2.2. The auxiliary process. For every $\epsilon < 1$, denote by $\eta_\epsilon = \mathbb{1}_{[0,1-\epsilon]}$ the indicator function of the interval $[0, 1-\epsilon]$. Denote by $u^{N,\epsilon}$ the solution of

$$du^{N,\epsilon} + \big(\nu A u^{N,\epsilon} + B(u^{N,\epsilon}) - (1-\eta_\epsilon)\pi_F B(u^{N,\epsilon})\big)\,dt = \pi_N \mathcal{C}^{\frac{1}{2}}\,dW,$$

where π_N is the projection onto H_N, and notice that $u^{N,\epsilon}(s) = u^N(s)$ for $s \leqslant t - \epsilon$. Moreover for $t \in [1-\epsilon, 1]$, $v = \pi_F u^{N,\epsilon}$ satisfies

$$\begin{cases} dv + \nu\pi_F A v\,dt = \pi_F \mathcal{C}^{\frac{1}{2}}\,dW, \\ v(1-\epsilon) = \pi_F u^{N,\epsilon}(1-\epsilon). \end{cases} \tag{6.3}$$

Therefore, conditional to $\mathscr{F}_{1-\epsilon}$, $\pi_F u^{N,\epsilon}(1)$ is a Gaussian random variable with mean $\pi_F u^{N,\epsilon}(1-\epsilon)$ and covariance

$$\mathcal{C}_F = \int_0^\epsilon \pi_F\, e^{-\nu\pi_F A s}\,\mathcal{C}\,e^{-\nu\pi_F A s}\,\pi_F\,ds.$$

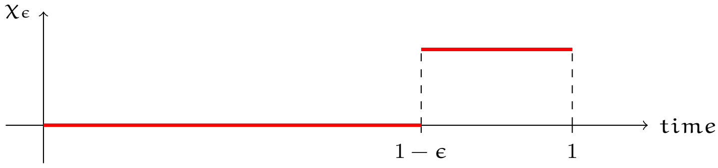

FIGURE 1. The timeline given by the function χ_ϵ.

The process $u^{N,\epsilon}$ can be re–interpreted as the *one–step explicit Euler approximation* of $\pi_F u^N(1)$, starting at $\pi_F u^N(1-\epsilon)$ and with time–step ϵ.

We denote by $g_{\epsilon,N}$ the density of $\pi_F u^{N,\epsilon}(1)$ conditional to $\mathscr{F}_{1-\epsilon}$ with respect to the Lebesgue measure. Since $\mathcal{C}_F$ is bounded and invertible on F and its eigenvalues are all of order ϵ, it is easy to see, by a simple change of variable (to get rid of the random mean and to extract the behaviour in ϵ) and the smoothness of the Gaussian density, that

$$\|g_{\epsilon,N}\|_{B^n_{1,1}} \leqslant c\epsilon^{-\frac{n}{2}},$$

holds almost surely with a deterministic constant $c > 0$. We give a probabilistic proof of this statement, to practice with the Besov spaces.

LEMMA 6.1. *Let $g_{\epsilon,N}$ be the density with respect to the Lebesgue measure of the random variable $\pi_F u^{N,\epsilon}(1)$ conditional to $\mathscr{F}_{1-\epsilon}$. Then for every $\alpha > 0$ there is $c_\alpha > 0$ such that almost surely,*

$$\|g_{\epsilon,N}\|_{B^\alpha_{1,1}} \leqslant c\epsilon^{-\frac{\alpha}{2}}.$$

PROOF. Fix an integer $n > \alpha$, and let $\phi \in \mathcal{S}(\mathbf{R}^d)$ and $h \in \mathbf{R}^d$ with $|h| \leqslant 1$, then,

$$\begin{aligned}\int_{\mathbf{R}^d} (\Delta^n_h g_{\epsilon,N})(x)\phi(x)\,dx &= \sum_{j=0}^n (-1)^{n-j}\binom{n}{j}\int_{\mathbf{R}^d} g_{\epsilon,N}(x+jh)\phi(x)\,dx\\ &= \sum_{j=0}^n (-1)^{n-j}\binom{n}{j}\int_{\mathbf{R}^d} g_{\epsilon,N}(x)\phi(x-hj)\,dx\\ &= \sum_{j=0}^n (-1)^{n-j}\binom{n}{j}\mathbb{E}\big[\phi(\pi_F u^{N,\epsilon}(1)-jh)\,|\,\mathscr{F}_{1-\epsilon}\big].\end{aligned}$$

We are going to re–write each term in the right hand side of the above formula by means of Girsanov theorem. Let

$$\widetilde{h} = \nu\mathcal{C}^{-\frac{1}{2}}A_N(I-e^{-\nu A_N\epsilon})^{-1}h, \tag{6.4}$$

so that

$$\int_{1-\epsilon}^1 e^{-\nu A_N(1-s)}\,\pi_F\mathcal{C}^{\frac{1}{2}}\widetilde{h}\,ds = h$$

and define the Girsanov density

$$G_s(h) = \exp\Big(\int_{1-\epsilon}^{s} \langle \widetilde{h}, dW_s\rangle - \frac{1}{2}\int_{1-\epsilon}^{s} \|\widetilde{h}\|_H^2\, dr\Big), \qquad s \geqslant 1-\epsilon. \tag{6.5}$$

It turns out that, conditional to $\mathscr{F}_{1-\epsilon}$, $\pi_F u^{N,\epsilon}(1) - jh$ has the same distribution of $\pi_F u^{N,\epsilon}(1)$ under the new probability space (which will be denoted by $\widetilde{\ }$), hence

$$\begin{aligned}\mathbb{E}[\phi(\pi_F u^{N,\epsilon}(1) - jh)|\mathscr{F}_{1-\epsilon}] &= \widetilde{\mathbb{E}}[\phi(\pi_F u^{N,\epsilon}(1))|\mathscr{F}_{1-\epsilon}]\\ &= \mathbb{E}[G_1(jh)\phi(\pi_F u^{N,\epsilon}(1))|\mathscr{F}_{1-\epsilon}].\end{aligned}$$

By Lemma (6.2) this implies that

$$\begin{aligned}\int_{\mathbf{R}^d} (\Delta_h^n g_{\epsilon,N})(x)\phi(x)\,dx &= \Big|\mathbb{E}\Big[\phi\big(\pi_F u^{N,\epsilon}(1)\big)\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}G_1(-jh)\Big|\mathscr{F}_{1-\epsilon}\Big]\Big|\\ &\leqslant \|\phi\|_\infty\Big|\mathbb{E}\Big[\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}G_1(-jh)\Big]\Big|\\ &\leqslant c\epsilon^{-\frac{n}{2}}|h|^n\|\phi\|_\infty\end{aligned}$$

since $\|\widetilde{h}\|_H \leqslant \frac{c}{\epsilon}|h|$ and since the Girsanov density is independent of $\mathscr{F}_{1-\epsilon}$. By duality, this immediately yields

$$\|\Delta_h^n g_{\epsilon,N}\|_{L^1(\mathbf{R}^d)} \leqslant c\big(1\wedge(\epsilon^{-\frac{n}{2}}|h|^n)\big)$$

and hence the conclusion. □

LEMMA 6.2. *Let* $h \in \mathbf{R}^d$ *and let* G *be as defined in* (6.5). *Then there is a number* $c > 0$ *(depending on* d*) such that*

$$\Big|\mathbb{E}\Big[\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}G_1(jh)\Big]\Big| \leqslant c\epsilon^{\frac{n}{2}}|\widetilde{h}|^n,$$

where $\widetilde{h}$ *is given in* (6.4).

PROOF. By the Cauchy–Schwarz inequality,

$$\begin{aligned}\Big|\mathbb{E}\Big[\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}G_1(jh)\Big]\Big|^2 &\leqslant \mathbb{E}\Big[\Big(\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}G_1(jh)\Big)^2\Big]\\ &= \sum_{a,b=0}^{n}(-1)^{a+b}\binom{n}{a}\binom{n}{b}\mathbb{E}[G_1(ah)G_1(bh)]\\ &= \sum_{a,b=0}^{n}(-1)^{a+b}\binom{n}{a}\binom{n}{b}e^{ab\epsilon|\widetilde{h}|^2},\end{aligned}$$

since it is elementary to verify that

$$\mathbb{E}[G_1(ah)G_1(bh)] = \mathbb{E}\big[G_1\big((a+b)h\big)\, e^{ab\epsilon|\widetilde{h}|^2}\big] = e^{ab\epsilon|\widetilde{h}|^2}.$$

Consider the function

$$\varphi(y) = \sum_{a,b=0}^{n} (-1)^{a+b}\binom{n}{a}\binom{n}{b} e^{aby},$$

since

$$\varphi^{(k)}(0) = \frac{d^k\varphi}{dy^k}(0) = \Big(\sum_{a=0}^{n}(-1)^a\binom{n}{a}a^k\Big)^2$$

it is elementary[1] to prove that $\varphi(0) = \varphi'(0) = \cdots = \varphi^{(n-1)}(0) = 0$ and $\varphi^{(n)}(0) = (n!)^2$. In particular, $|\varphi(y)| \leqslant cy^n$ for y in a bounded set. □

6.2.3. The splitting. We build up the splitting which will provide the two terms whose trade–off will provide the final result. Fix $\phi \in \mathcal{S}(\mathbf{R}^d)$ and $h \in \mathbf{R}^d$ with $|h| < 1$, then

$$\begin{aligned} \mathbb{E}[(\Delta_h\phi)(\pi_F u^N(1))] &= \mathbb{E}[(\Delta_h\phi)(\pi_F u^N(1)) - (\Delta_h\phi)(\pi_F u^{N,\epsilon}(1))] \\ &\qquad + \mathbb{E}[(\Delta_h\phi)(\pi_F u^{N,\epsilon}(1))] \\ &= \boxed{\textbf{NE}} + \boxed{\textbf{PE}} \end{aligned} \tag{6.6}$$

and the estimate of the density f_N of $\pi_F u^N(1)$ in Besov spaces is split in two terms. The first term, which we call *numerical error* using the interpretation of the auxiliary process as numerical approximation, takes into account the error we have by replacing $\pi_F u^N(1)$ with $\pi_F u^{N,\epsilon}(1)$. The second term is the *probabilistic error*, since we will estimate it using Lemma 6.1.

6.2.4. Estimate of the errors. With Lemma 6.1 in hands, the estimate of the probabilistic error $\boxed{\textbf{PE}}$ is easy, indeed, by a discrete integration by parts

$$\begin{aligned} \boxed{\textbf{PE}} &= \mathbb{E}\big[\mathbb{E}[(\Delta_h\phi)(\pi_F u^{N,\epsilon}(1))|\mathscr{F}_{1-\epsilon}]\big] = \\ &= \mathbb{E}\Big[\int_{\mathbf{R}^d}\Delta_h\phi(x)g_{\epsilon,N}(x)\,dx\Big] = \mathbb{E}\Big[\int_{\mathbf{R}^d}\phi(x)\Delta_{-h}g_{\epsilon,N}(x)\,dx\Big] = \\ &\leqslant \|\phi\|_{L^\infty}\|h\|\mathbb{E}[\|g_{\epsilon,N}\|_{B^n_{1,1}}] \leqslant c\|\phi\|_{L^\infty}\epsilon^{-\frac{1}{2}}\|h\|. \end{aligned}$$

The numerical error $\boxed{\textbf{NE}}$ can be estimated as follows

$$\begin{aligned} \big|\mathbb{E}[(\Delta_h\phi)(\pi_F u^N(1)) - (\Delta_h\phi)(\pi_F u^{N,\epsilon}(1))]\big| &\leqslant \\ \leqslant c[\phi]_\alpha \mathbb{E}\big[\|\pi_F\big(u^N(1) - u^{N,\epsilon}(1)\big)\|^\alpha\big], \end{aligned}$$

[1]Just consider the expansion of $(1+x)^n$, differentiate k times at $x = 1$ and take linear combinations.

where $[\phi]_\alpha$ is the Hölder semi-norm of $C^\alpha(\mathbf{R}^d)$, and $\alpha \in (0,1)$ will be suitably chosen later. Since

$$\pi_F\big(u^N(1) - u^{N,\epsilon}(1)\big) = -\int_{1-\epsilon}^1 e^{-\nu A(1-s)}\,\pi_F B(u^N(s), u^N(s))\,ds,$$

the two estimates (5.3) and (1.10) together yield

$$\mathbb{E}\big[\|\pi_F\big(u^N(1) - u^{N,\epsilon}(1)\big)\|\big] \leqslant c_F \int_{1-\epsilon}^1 \mathbb{E}[\|u^N(s)\|_H^2]\,ds \leqslant c_F(\|x\|_H^2 + 1)\epsilon.$$

By gathering together the estimates of the two error terms and by choosing $\epsilon = \|h\|^{\frac{2}{2\alpha+1}}$, we have

$$\big|\mathbb{E}[(\Delta_h\phi)(\pi_F u^N(1))]\big| \leqslant c[\phi]_\alpha\epsilon^\alpha + c\epsilon^{-\frac{1}{2}}\|h\|\|\phi\|_\infty = c\|\phi\|_{C^\alpha}\|h\|^{\frac{2\alpha}{2\alpha+1}}$$

where the number c is independent of N. By a discrete integration by parts,

$$\int_{\mathbf{R}^d}(\Delta_h f_N)(x)\phi(x)\,dx = \int_{\mathbf{R}^d}(\Delta_{-h}\phi)(x)f_N(x)\,dx = \big|\mathbb{E}[(\Delta_{-h}\phi)(\pi_F u^N(1))]\big|,$$

hence we have proved that for every $h \in \mathbf{R}^d$ with $|h| \leqslant 1$ and $\alpha_1 = \frac{2\alpha}{2\alpha+1}$,

$$\Big|\int_{\mathbf{R}^d}\phi(y)\frac{(\Delta_h f_N)(x)}{|h|^{\alpha_1}}\,dy\Big| \leqslant c\|\phi\|_{C^\alpha}. \tag{6.7}$$

6.2.5. Conclusion. We wish to deduce from the above inequality the following claim:

The sequence $(f_N)_{N\geqslant n_d}$ is bounded in $B^s_{1,\infty}(\mathbf{R}^d)$ for some $s \in (0,1)$.

Before proving the claim, we show how it immediately implies the statements of the theorem. Indeed, since for every $s' < s$ by the Sobolev embeddings,

$$B^s_{1,\infty}(\mathbf{R}^d) \subset B^{s'}_{1,1}(\mathbf{R}^d) = W^{s',1}(\mathbf{R}^d) \subset L^p(\mathbf{R}^d)$$

for every $p \in [1, \frac{d}{d-s'}]$, hence for every $p \in [1, \frac{d}{d-s})$ by choosing s' arbitrarily close to s, it follows that the sequence $(f_{N_k})_{k\geqslant 1}$ is uniformly integrable and hence convergent to a positive function $f \in L^p(\mathbf{R}^d)$ which is the density of $\pi_F u(1)$ (we recall here that $\mathbb{P}_{N_k} \rightharpoonup \mathbb{P}$, where $\mathbb{P}$ is the law of u, hence the limit is unique along the subsequence $(N_k)_{k\geqslant 1}$). Moreover, since the bound in the claim is independent of N, it follows that $f \in B^s_{1,\infty}(\mathbf{R}^d)$ and hence, using again the embedding of Besov spaces into Lebesgue spaces we have obtained above, $f \in L^p(\mathbf{R}^d)$ for every $p \in [1, \frac{d}{d-s})$.

It remains to show the above claim. Let $\psi \in \mathcal{S}(\mathbf{R}^d)$ and set $\phi = (I - \Delta_d)^{-\beta/2}\psi$, where $\beta > \alpha$ will be suitably chosen later. Notice that $C^\alpha(\mathbf{R}^d) = B^\alpha_{\infty,\infty}(\mathbf{R}^d)$, and $(I - \Delta_d)^{-\beta/2}$ maps continuously $B^{\alpha-\beta}_{\infty,\infty}(\mathbf{R}^d)$ into $B^\alpha_{\infty,\infty}(\mathbf{R}^d)$, hence

$$\|\phi\|_{C^\alpha} \leqslant c\|\phi\|_{B^\alpha_{\infty,\infty}} \leqslant c\|\psi\|_{B^{\alpha-\beta}_{\infty,\infty}} \leqslant c_0\|\psi\|_{L^\infty},$$

where the last inequality follows from the fact that $L^\infty(\mathbf{R}^d) \hookrightarrow B^{\alpha-\beta}_{\infty,\infty}(\mathbf{R}^d)$, since $B^{\alpha-\beta}_{\infty,\infty}(\mathbf{R}^d)$ is the dual of $B^{\beta-\alpha}_{1,1}(\mathbf{R}^d)$ and $B^{\beta-\alpha}_{1,1}(\mathbf{R}^d) \hookrightarrow L^1(\mathbf{R}^d)$, since $\beta > \alpha$.

Let $g_N = (I - \Delta_d)^{-\beta/2} f_N$, then (6.7) yields

$$\Big|\int_{\mathbf{R}^d} \psi(y)(\Delta_h g_N)(y)\, dy\Big| \leqslant c_0 |h|^{\alpha_1} \|\psi\|_{L^\infty},$$

hence $\Delta_h g_N \in L^1(\mathbf{R}^d)$ and

$$\|\Delta_h g_N\|_{L^1} \leqslant c_0 |h|^{\alpha_1}.$$

Moreover, by [AS61, Theorem 10.1], $\|g_N\|_{L^1} \leqslant c\|f_N\|_{L^1} = c$, hence $(g_N)_{N \geqslant n_d}$ is a bounded sequence in $B^{\alpha_1}_{1,\infty}(\mathbf{R}^d)$ and, since $(I - \Delta_d)^{\beta/2}$ maps $B^{\alpha}_{1,\infty}(\mathbf{R}^d)$ continuously onto $B^{\alpha-\beta}_{1,\infty}(\mathbf{R}^d)$, it follows that $(f_N)_{N \geqslant n_d}$ is a bounded sequence in $B^{\alpha_1-\beta}_{1,\infty}(\mathbf{R}^d)$ for every $\beta > \alpha$.

The key point here is if we have gained some kind of smoothing effect. The only possibility is if there is any $\alpha \in (0, 1)$ such that $\alpha_1 > \alpha$, and hence $\alpha_1 > \beta$ for β small enough. An elementary computation shows that $\alpha_1 > \alpha$ if $\alpha < \frac{1}{2}$ and the optimal choice of α (the one that maximizes $\alpha_1 - \alpha$) gives boundedness in $B^s_{1,\infty}$ for every $s < \frac{1}{2}(3 - 2\sqrt{2})$.

There is some space for improvements in the trade–off that yields (6.7) and we have two possibilities: either we improve the probabilistic error **PE**, or we improve the numerical error **NE** (well, or both!).

Improving **NE** turns out to be not feasible in the general case and we will do it in Section 6.3 for a special class of solutions. So we head for the improvement of **PE**, as already suggested by Lemma 6.1.

6.2.6. Further Besov regularity. Here we try to improve **PE**. Replace (6.6) with

$$\mathbb{E}[(\Delta^n_h \phi)(\pi_F u^N(1))] = \mathbb{E}[(\Delta^n_h \phi)(\pi_F u^N(1)) - (\Delta^n_h \phi)(\pi_F u^{N,\epsilon}(1))] + \mathbb{E}[(\Delta^n_h \phi)(\pi_F u^{N,\epsilon}(1))], \tag{6.8}$$

for some integer $n \geqslant 1$. The error term **NE** is of the same order in terms of ϵ (only the pre–factor changes). On the other hand, again by Lemma 6.1,

$$\begin{aligned}\boxed{\mathsf{PE}} &= \mathbb{E}\big[\mathbb{E}[(\Delta^n_h \phi)(\pi_F u^{N,\epsilon}(1))|\mathscr{F}_{1-\epsilon}]\big] = \\ &= \mathbb{E}\Big[\int_{\mathbf{R}^d} \phi(x)\Delta^n_{-h} g_{\epsilon,N}(x)\, dx\Big] \leqslant \|\phi\|_{L^\infty}\|h\|^n \mathbb{E}[\|g_{\epsilon,N}\|_{B^n_{1,1}}] \leqslant \\ &\qquad\qquad \leqslant c\|\phi\|_{L^\infty}\epsilon^{-\frac{n}{2}}\|h\|^n,\end{aligned}$$

and the choice $\epsilon = \|h\|^{\frac{2n}{2\alpha+n}}$ yields

$$\big|\mathbb{E}[(\Delta^n_h \phi)(\pi_F u^N(1))]\big| \leqslant c[\phi]_\alpha \epsilon^\alpha + c\epsilon^{-\frac{n}{2}}\|h\|^n \|\phi\|_\infty = c\|\phi\|_{C^\alpha}\|h\|^{\alpha_n}$$

where $\alpha_n = \frac{2\alpha n}{2\alpha+n}$, hence

$$\Big|\int_{\mathbf{R}^d} \phi(y)\frac{(\Delta_h^n f_N)(x)}{|h|^{\alpha_n}}\,dy\Big| \leqslant c\|\phi\|_{C^\alpha}.$$

As above, this yields that $(f_N)_{N\geqslant n_d}$ is bounded in $B^{\alpha_n-\beta}_{1,\infty}(\mathbf{R}^d)$ for every $\beta > \alpha$. By suitably choosing $n \geqslant 1$, $\alpha \in (0,1)$ and $\beta > \alpha$, the number $\alpha_n - \beta$ runs over all reals in $(0,1)$: this can be easily seen by noticing that $\alpha_n \to 2\alpha$ as $n \to \infty$. In conclusion $(f_N)_{N\geqslant n_d}$ is bounded in $B^s_{1,\infty}(\mathbf{R}^d)$ for every $s < 1$. We have proved the following result.

THEOREM 6.3. *Given an initial condition* $x \in H$, *a finite dimensional subspace* F *of* $D(A)$ *generated by the eigenvectors of* A, *namely* $F = \mathrm{span}[e_{n_1},\dots,e_{n_d}]$ *for some arbitrary indices* $n_1,\dots,n_d$, *and a time* $t > 0$, *the projection* $\pi_F u(t)$ *has a density* $f_{F,t}$ *with respect to the Lebesgue measure on* F, *where* u *is any solution of the Navier–Stokes equations which is limit point of the spectral Galerkin approximations.*

Moreover $f_{F,t} \in B^s_{1,\infty}(\mathbf{R}^d)$, *hence* $f_{F,t} \in W^{s,1}(\mathbf{R}^d)$, *for every* $s \in (0,1)$, *and* $f_{F,t} \in L^p(\mathbf{R}^d)$ *for any* $p \in [1, \frac{d}{d-1})$.

REMARK 6.4. We remark also in this chapter that under the standing assumptions on the covariance taken in Chapter 4, which are stronger than those of this chapter, we can deduce by Theorem 4.7 that the density obtained above is positive almost everywhere on F.

6.3. Additional regularity for stationary solutions

In this section we try to improve the numerical error. The name itself suggests that a way to do it is to use a better numerical approximation (first order explicit Euler is indeed the most basic approximation). It turns out that by doing so we end up with the requirement that some moments of the infinite dimensional solution should be finite, and, as far as we know, this is not true for an ordinary solution.

If we turn to a special class of solutions, the *stationary solutions*, the requirements are met and we can improve the numerical error. Stationary solutions are those solutions whose (infinite dimensional) distribution is invariant by time–shifts and the idea that stationary solutions may have better regularity properties has been already exploited, see for instance [**FR02**, **Oda06**].

To be more precise, here we consider a sub–class of stationary solutions, namely those which are limit of Galerkin approximations. Consider again problem (1.9), under fairly general assumptions (which are met in this section) it admits a unique invariant measure. This fact can be proved as in [**Fla08**], or one can look at [**Rom04**], but the main argument that implies uniqueness is the same of Section §2.3. Denote by $\mathbb{P}_N$ the law of the process started at the invariant measure of the finite dimensional problem. Every limit point of the sequence $(\mathbb{P}_N)_{N\geqslant 1}$ is a stationary solution of the Navier–Stokes equations with noise, that is a probability measure which is invariant with respect to the forward time–shift (other methods can be used to show existence of stationary solutions, see for instance [**FG95**]).

We aim to prove the following result, by slightly modifying the arguments we have previously detailed. The assumptions we use are the same of Section 6.2.1.

THEOREM 6.5. *Let F be a finite dimensional subspace of $D(A)$ generated by the eigenvalues of A, namely $F = \operatorname{span}[e_{n_1}, \dots, e_{n_d}]$ for some arbitrary indices $n_1, \dots, n_d$. Let u be a stationary solution of the Navier–Stokes equations with noise which is a limit point of a sequence of stationary solutions of the spectral Galerkin approximation. Then the projection $\pi_F u(1)$ has a density f_F with respect to the Lebesgue measure on F, such that $f_F \in B^s_{1,\infty}(\mathbf{R}^d)$ for every $s \in (0, 2)$.*

The proof of this result is the same of Theorem 6.3, but we introduce a new auxiliary process which provides a better estimate, in terms of ϵ of the numerical error, while the probabilistic error is of the same order.

Let $u^{N,\epsilon}$ be the solution of the following equation,

$$du^{N,\epsilon} + \big(\nu A u^{N,\epsilon} + B(u^{N,\epsilon}) - (1-\eta_\epsilon)\pi_F B(u^{N,\epsilon}) + \\ + (1-\eta_\epsilon)\pi_F B(e^{-A(s-1+\epsilon)} u^{N,\epsilon}(1-\epsilon))\big)\,ds = \pi_N \mathcal{C}^{\frac12}\,dW_s$$

so that again $u^N(t) = u^{N,\epsilon}(t)$ for $t \leqslant 1-\epsilon$, and for $t \geqslant 1-\epsilon$ the process $\pi_F u^{N,\epsilon}$ satisfies

$$dv + \big(\nu \pi_F A v + \pi_F B(e^{-A(s-1+\epsilon)} u^{N,\epsilon}(1-\epsilon))\big)\,ds = \pi_F \mathcal{C}^{\frac12}\,dW_s,$$

which is the same equation as in (6.3) with an additional adapted external forcing. We can interpret this new auxiliary process as a second order approximation.

Conditional to $\mathscr{F}_{1-\epsilon}$, $\pi_F u^{N,\epsilon}(1)$ is Gaussian with covariance $\mathcal{C}_F$. Thus, if we write formula (6.8) in this setting, the probabilistic error satisfies, as before,

$$\boxed{\mathsf{PE}} \leqslant c\epsilon^{-\frac{n}{2}} |h|^n \|\phi\|_\infty.$$

We claim that

$$\mathbb{E}\big[\|\pi_F u^N(1) - \pi_F u^{N,\epsilon}(1)\|_H\big] \leqslant c\epsilon^{\frac32}, \tag{6.9}$$

hence

$$\boxed{\mathsf{NE}} \leqslant c[\phi]_\alpha \epsilon^{\frac32\alpha},$$

and so the choice $\epsilon = |h|^{2n/(3\alpha+n)}$ and the position $\alpha_n = \frac{3\alpha n}{3\alpha+n}$ yield

$$\Big|\int_{R^d} \phi(x) \frac{(\Delta_h^n f_N)(x)}{|h|^{\alpha_n}}\,dx\Big| \leqslant c\|\phi\|_{C^\alpha}.$$

The above estimate implies that $(f_N)_{N\geqslant n_d}$ is a bounded sequence in $B^s_{1,\infty}$, for every $s < \alpha_n - \alpha$. Since $\alpha_n - \alpha \to 2\alpha$ as $n \to \infty$ and $\alpha \in (0,1)$ can be arbitrarily chosen, we conclude that $(f_N)_{N\geqslant n_d}$ is bounded in $B^s_{1,\infty}$ for every $s < 2$.

The proof is complete once we verify the claim in formula (6.9). This is done in the following lemma.

LEMMA 6.6. *Let* u^N *be a stationary solution of the Galerkin problem* (1.9) *and* $u^{N,\epsilon}$ *be defined as above. Then there is a number* $c > 0$ *independent of* N *and* ϵ *such that*

$$\mathbb{E}\big[\|\pi_F u^N(1) - \pi_F u^{N,\epsilon}(1)\|_H\big] \leqslant c\epsilon^{\frac{3}{2}}.$$

PROOF. We have that

$$\pi_F\big(u^N(1) - u^{N,\epsilon}\big) = \int_{1-\epsilon}^1 e^{-\nu A(1-s)}\,\pi_F\big(B(e^{-\nu A(s-1+\epsilon)}\,u^N(1-\epsilon)) - B(u^N(s))\big)\,ds,$$

hence by (5.3) and Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\big[\|\pi_F\big(u^N(1) - u^{N,\epsilon}(1)\big)\|\big] \leqslant\\
&\quad\leqslant c\int_{1-\epsilon}^1 \mathbb{E}\big[\big(\|\,e^{-\nu A(s-1+\epsilon)}\,u^N(1-\epsilon)\|_H + \|u^N(s)\|_H\big)\cdot\\
&\qquad\qquad \cdot\|\,e^{-\nu A(s-1+\epsilon)}\,u^N(1-\epsilon) - u^N(s)\|_H\big]\,ds\\
&\quad\leqslant c\int_{1-\epsilon}^1 \mathbb{E}\big[\big(\|u^N(1-\epsilon)\|_H + \|u^N(s)\|_H\big)^4\big]^{\frac{1}{4}}\cdot\\
&\qquad\qquad \cdot\mathbb{E}\big[\|\,e^{-\nu A(s-1+\epsilon)}\,u^N(1-\epsilon) - u^N(s)\|_H^{\frac{4}{3}}\big]^{\frac{3}{4}}\,ds\\
&\quad\leqslant c\int_{1-\epsilon}^1 \mathbb{E}\big[\|\,e^{-\nu A(s-1+\epsilon)}\,u^N(1-\epsilon) - u^N(s)\|_H^{\frac{4}{3}}\big]^{\frac{3}{4}}\,ds,
\end{aligned}$$

since $\mathbb{E}[\|u^N(s)\|_H^4]$ is finite, constant in s and uniformly bounded in N. Now, for $s \in (1-\epsilon, 1)$,

$$\begin{aligned}
e^{-\nu A(s-1+\epsilon)}\,u^N(1-\epsilon) - u^N(s) &= \int_{1-\epsilon}^s e^{-\nu A(s-r)}\,B(u^N(r))\,dr\\
&\quad - \int_{1-\epsilon}^s e^{-\nu A(s-r)}\,\mathcal{C}^{\frac{1}{2}}\,dW_r
\end{aligned}$$

and so

$$\begin{aligned}
&\mathbb{E}\big[\|\,e^{-\nu A(s-1+\epsilon)}\,u^N(1-\epsilon) - u^N(s)\|_H^{\frac{4}{3}}\big] \leqslant\\
&\quad\leqslant \mathbb{E}\Big[\Big(\int_{1-\epsilon}^s \big\|e^{-\nu A(s-r)}\,B(u^N(r))\big\|_H\,dr\Big)^{\frac{4}{3}}\Big] + \mathbb{E}\Big[\Big\|\int_{1-\epsilon}^s e^{-\nu A(s-r)}\,\mathcal{C}^{\frac{1}{2}}\,dW_r\Big\|_H^{\frac{4}{3}}\Big]\\
&\quad = \boxed{1} + \boxed{2}.
\end{aligned}$$

To estimate $\boxed{1}$ we use the inequality

$$\|A^{-\frac{1}{2}}B(v)\|_H \leqslant c\|v\|_{L^4}^2 \leqslant c\|v\|_H^{\frac{1}{2}}\|v\|_V^{\frac{3}{2}},$$

standard estimates on analytic semigroups and we exploit the fact that u^N is stationary,

$$\begin{aligned}
\boxed{1} &\leqslant \mathbb{E}\Big[\Big(\int_{1-\epsilon}^{s} \frac{c}{\sqrt{s-r}}\|u^N(r)\|_H^{\frac{1}{2}}\|u^N(r)\|_V^{\frac{3}{2}}\,dr\Big)^{\frac{4}{3}}\Big] \\
&\leqslant c\epsilon^{\frac{1}{3}}\mathbb{E}\Big[\int_{1-\epsilon}^{s} \frac{1}{(s-r)^{\frac{2}{3}}}\|u^N(r)\|_H^{\frac{2}{3}}\|u^N(r)\|_V^{2}\,dr\Big] \\
&= c\epsilon^{\frac{2}{3}}\mathbb{E}\big[\|u^N\|_H^{\frac{2}{3}}\|u^N\|_V^{2}\big] \\
&= c\epsilon^{\frac{2}{3}}.
\end{aligned}$$

The second term is standard,

$$\boxed{2} \leqslant \mathbb{E}\Big[\Big\|\int_{1-\epsilon}^{s} e^{-\nu A(s-r)}\,\mathcal{C}^{\frac{1}{2}}dW_r\Big\|_H^2\Big]^{\frac{2}{3}} \leqslant \Big(\frac{1}{2}\epsilon\,\mathrm{Tr}(\mathcal{C})\Big)^{\frac{2}{3}} = c\epsilon^{\frac{2}{3}},$$

hence

$$\mathbb{E}\big[\|\, e^{-\nu A(s-1+\epsilon)}\, u^N(1-\epsilon) - u^N(s)\|_H^{\frac{4}{3}}\big] \leqslant c\epsilon^{\frac{2}{3}},$$

and in conclusion

$$\mathbb{E}\big[\|\pi_F\big(u^N(1) - u^{N,\epsilon}\big)\|\big] \leqslant c\epsilon^{\frac{3}{2}},$$

which completes the proof. $\square$

CHAPTER 7

The Fokker–Planck equation

In this last chapter we discuss a classical approach to existence and regularity of densities, namely the Fokker–Planck equation (also known as Kolmogorov *forward* equation). The Fokker–Planck equation describes the evolution of the density of the Itô process solution of a stochastic equation. The Fokker–Planck equation, as well as the Kolmogorov equation, were the first method to approach diffusions, before Itô [**Itô44**] (and Doeblin [**Yor00, Doe00**]) introduced the stochastic integral.

We give a new proof of regularity of densities of finite dimensional projections, which is completely analytic and does not rely on probabilistic ideas. The regularity is again in the class of Besov spaces, as in the previous chapter. The subject and the methods of this chapter are part of a work in progress and we present only the simplest result, concerning the density of a projection at fixed time. **[This part has not been completed yet]**

List of symbols

$\mathscr{L}_d$	the Lebesgue measure on $\mathbf{R}^d$.
$\mathcal{B}(E)$	the space of bounded measurable functions on a space E.
$C(E)$	the space of real valued continuous functions on E.
$C_b(E)$	the space of bounded continuous functions on E.
$C^m(\mathbf{R}^d)$	the space of m–times differentiable functions with continuous m^{th} derivative.
$C^s(\mathbf{R}^d)$	for non–integer $s > 0$ is the space of functions in $C^{[s]}$ whose derivative of order $[s]$ is Hölder continuous of exponent $s - [s]$.
$C_c^\infty(\mathbf{R}^d)$	the space of real valued infinitely differentiable functions with compact support.
$C_p^\infty(\mathbf{R}^d)$	the space of real valued infinitely differentiable functions with polynomial growth at infinity.
$\mathcal{S}(\mathbf{R}^d)$	the Schwartz space of rapidly decreasing infinitely differentiable functions.
$\mathcal{S}'(\mathbf{R}^d)$	the space of tempered distributions, that is the dual of $\mathcal{S}(\mathbf{R}^d)$.
$\widehat{f}$, $\mathcal{F}f$	the Fourier transform of $f : \mathbf{R}^d \to \mathbf{R}$, $$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} f(x)\, e^{-ix\cdot\xi}\, dx.$$.
$\widehat{f}$, $\mathcal{F}^{-1}f$	the inverse Fourier transform of $f : \mathbf{R}^d \to \mathbf{R}$, $$\widehat{f}(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} f(\xi)\, e^{ix\cdot\xi}\, d\xi.$$.
$\|\cdot\|_E$	norm of the Banach space E.
$\langle\cdot,\cdot\rangle_E$	scalar product of the Hilbert space E.
$\ell^p(\mathbf{R}^d)$	the space of p–summable sequences in $\mathbf{R}^d$ with norm $$\|(x_n)_{n\in\mathbf{N}}\|_{\ell^p} = \Big(\sum_{n=0}^{\infty} \|x_n\|^p\Big)^{\frac{1}{p}}$$ if $p < \infty$. If $p = \infty$, $\ell^\infty(\mathbf{R}^d)$ is the set bounded sequences with norm $\|(x_n)_{n\in\mathbf{N}}\|_{\ell^\infty} = \sup_n \|x_n\|$.
$\mathrm{span}[a_1, a_2 \dots]$	the linear space generated by elements $a_1, a_2, \dots$.

$\mathscr{L}(\mathsf{E})$ space of linear bounded operators on the Banach space E.

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