Chern-Simons Theory and Knot Invariants

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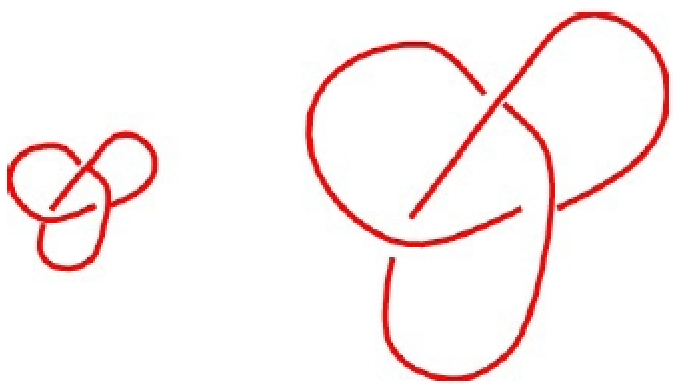
Meeting Ground for Mathematicians and Physicists

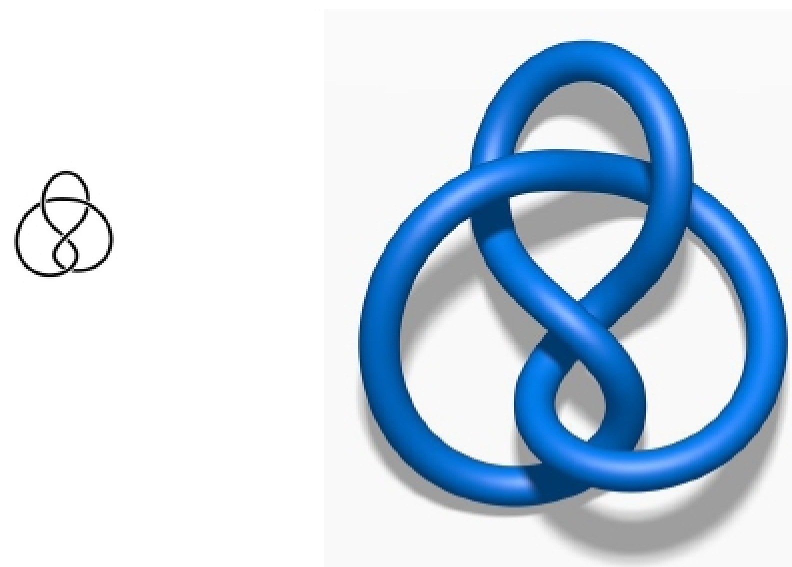
- Low dimensional topology and geometry are explained from **topological quantum field theories**
- **Chern-Simons theory** is one such theory which provides a natural framework for the study of **knots and links**

Plan

- Chern-Simons theory
- Direct Computation of knot invariants
- Main barrier to write the polynomial form
- Our conjecture- enables writing the polynomials for many knots
- Discussions and open problems

Chern-Simons Theory





Torus Knot :Trefoil (left) Hyperbolic Knot: Figure-Eight knot (right)

(small or big size does not matter)

Need a metric independent Theory(Chern-Simons)

Chern-Simons Theory

Chern-Simons action S is given by

$$S = \frac{k}{4\pi}\int_M \epsilon_{\mu\nu\lambda}\, d^3x\, Tr\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right)$$

where k is the coupling constant and gauge connection are matrix valued(gauge group G)

The knots carrying representation R are the Wilson loop observables whose expectation values gives **knot invariants**

Chern-Simons Theory

Wilson loop operators for knot *C* is $W_R(C) = Tr[P exp \oint A_\mu dx^\mu]$

whose expectation value gives **colored knot invariants** (color refers to representation R of group G)

$$V_R[C] = \langle W_R(C) \rangle = \frac{\int_M [\mathcal{D}A]\, W_R(C)\, exp(iS)}{\mathcal{Z}[M]}$$

where $\mathcal{Z}[M] = \int_M [\mathcal{D}A]\, exp(iS)$ (partition function)

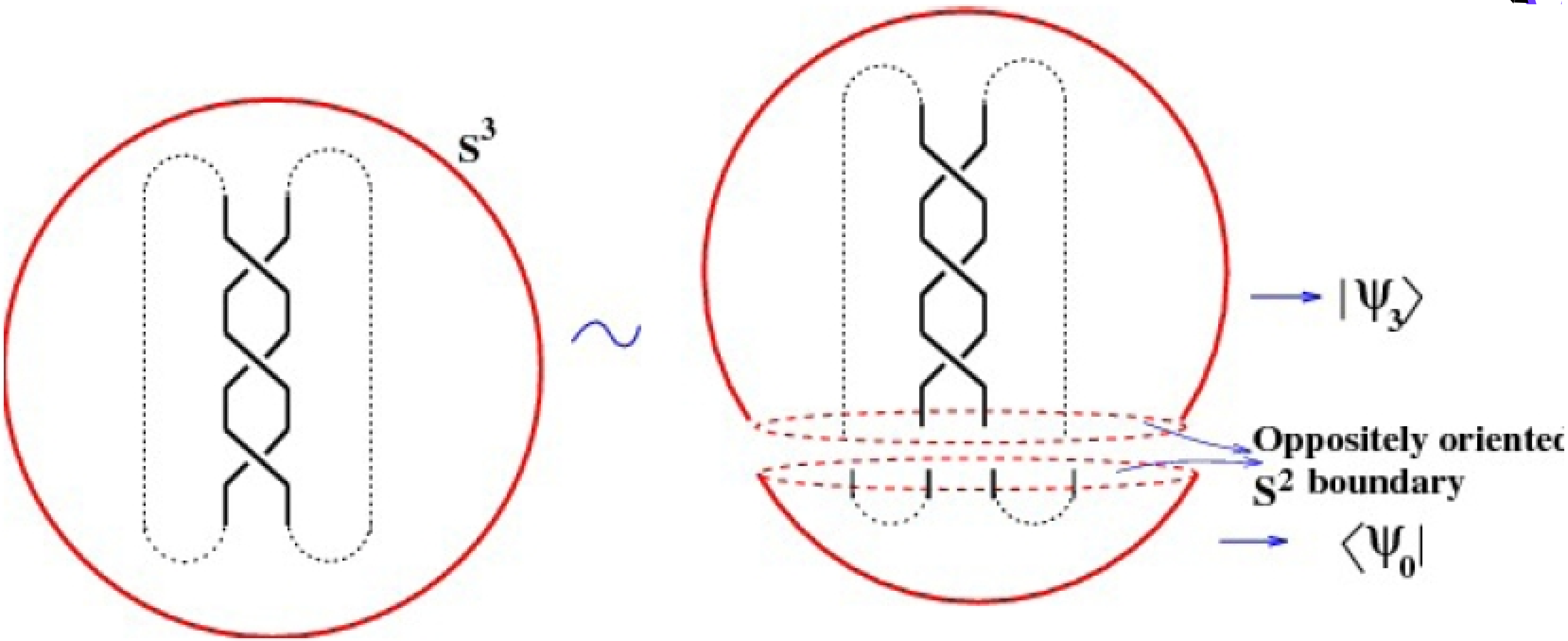
Computation of knot invariants

The colored knot invariants $V_R[C]$ are computed using the following two ingredients:

1) Chern-Simons on a three ball with boundary S^2 to states (WZNW conformal block space) –**Witten '89**
2) Redraw every knot or link as closure/plat of braid- **Birman, Alexander thm**

Knot invariant computation

- Take the trefoil :

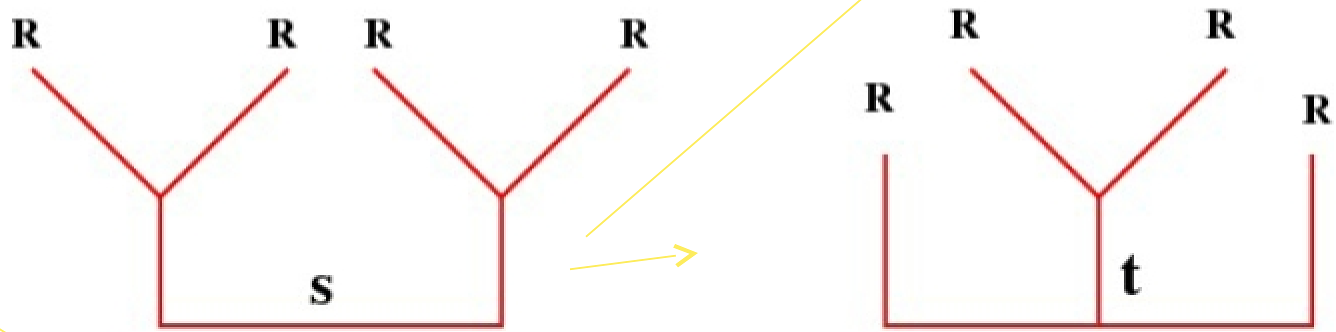


$$V_R[C] = \langle\psi_0|\psi_3\rangle = \langle\psi_0|\mathcal{B}^3|\psi_0\rangle$$

$\mathcal{B}$ is a braiding operator

To see the polynomial form

- Expand the state Ψ_0 in a suitable basis in which $\mathcal{B}$ is diagonal
- Two such bases for four-punctured S^2 bdy will be



$$|\phi_s\rangle = \sum_t a_{st} \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} |\hat{\phi}_t\rangle$$

Wigner 6j

Polynomial for trefoil

Involves braiding in the middle strands and hence ϕ_t is suitable

$$|\Psi_0\rangle = \sum_t \mu_t \, |\hat{\Phi}_t\rangle$$

Here $\mu_t = \sqrt{dim_q t}$ (unknot normalisation)

$$V_R[C] = \langle \Psi_0 | \mathcal{B}^3 | \Psi_0 \rangle = \sum_t dim_q t \, (\lambda_t(R,R))^3$$

$$\lambda_t(R,R) = \epsilon_t q^{(2C_R - C_t/2)}$$

Hence polynomial in q

Polynomial for trefoil

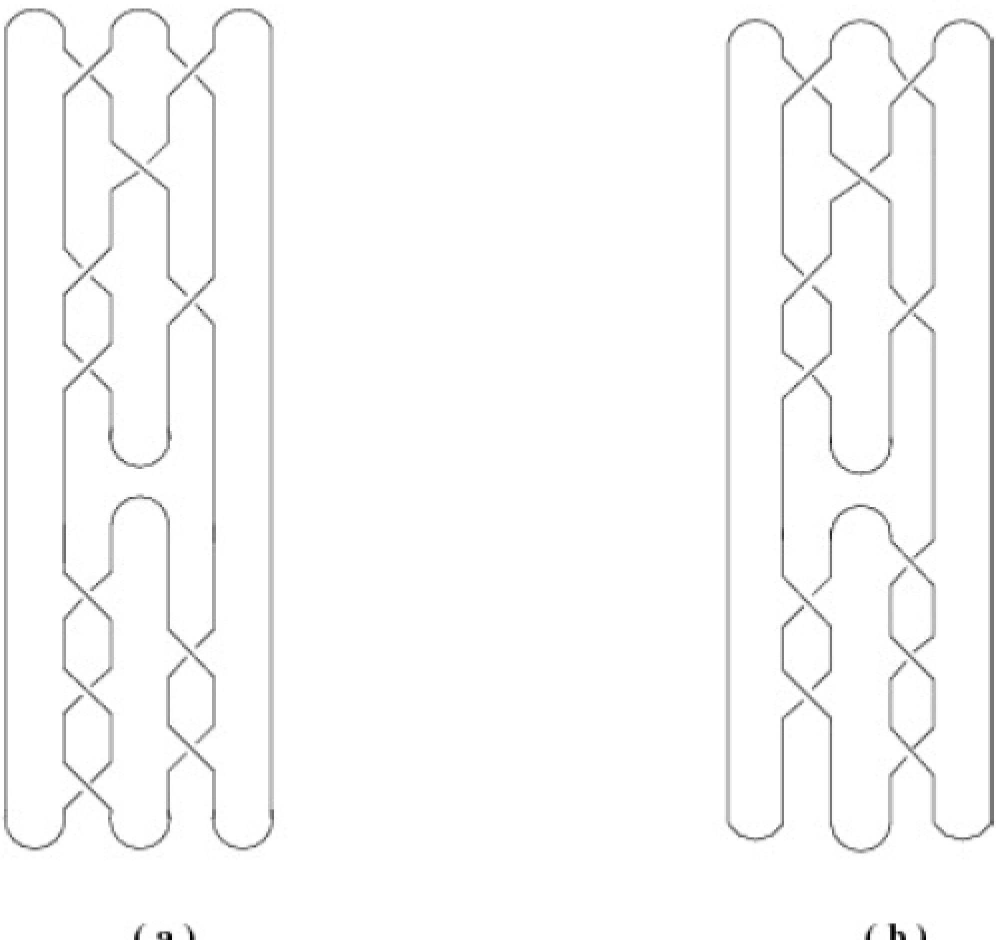
Chern-Simons knot invariants agree with well-known polynomials:

Special Cases: R=fundamental

Gauge Group	Polynomial
$SU(2)$	Jones'
$SU(N)$	Two-variable HOMFLY
$SO(N)$	Two-variable Kauffman

For other representations of different G we get colored knot invariants (**data for attempting classification problem**)

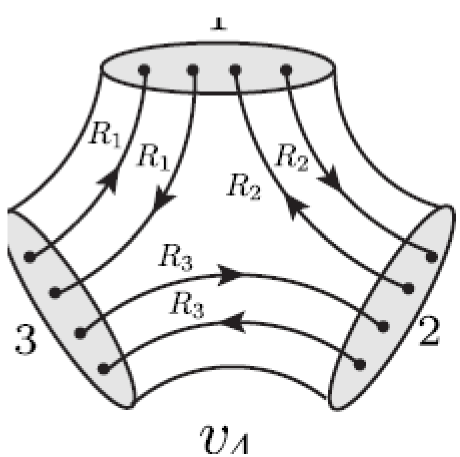
Kinoshita-Terasaka and Conway knot: **Mutants**

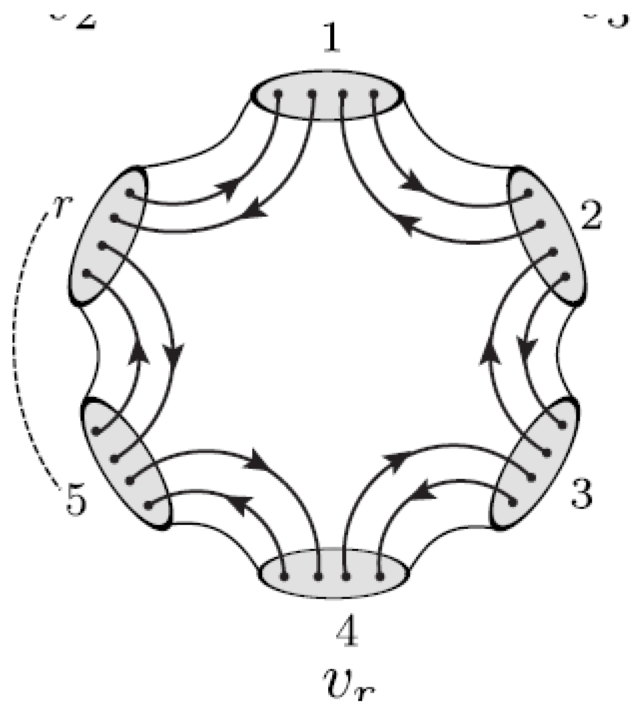


(a) (b)

Chern-Simons does not detect mutation

Generalisations involving many boundaries





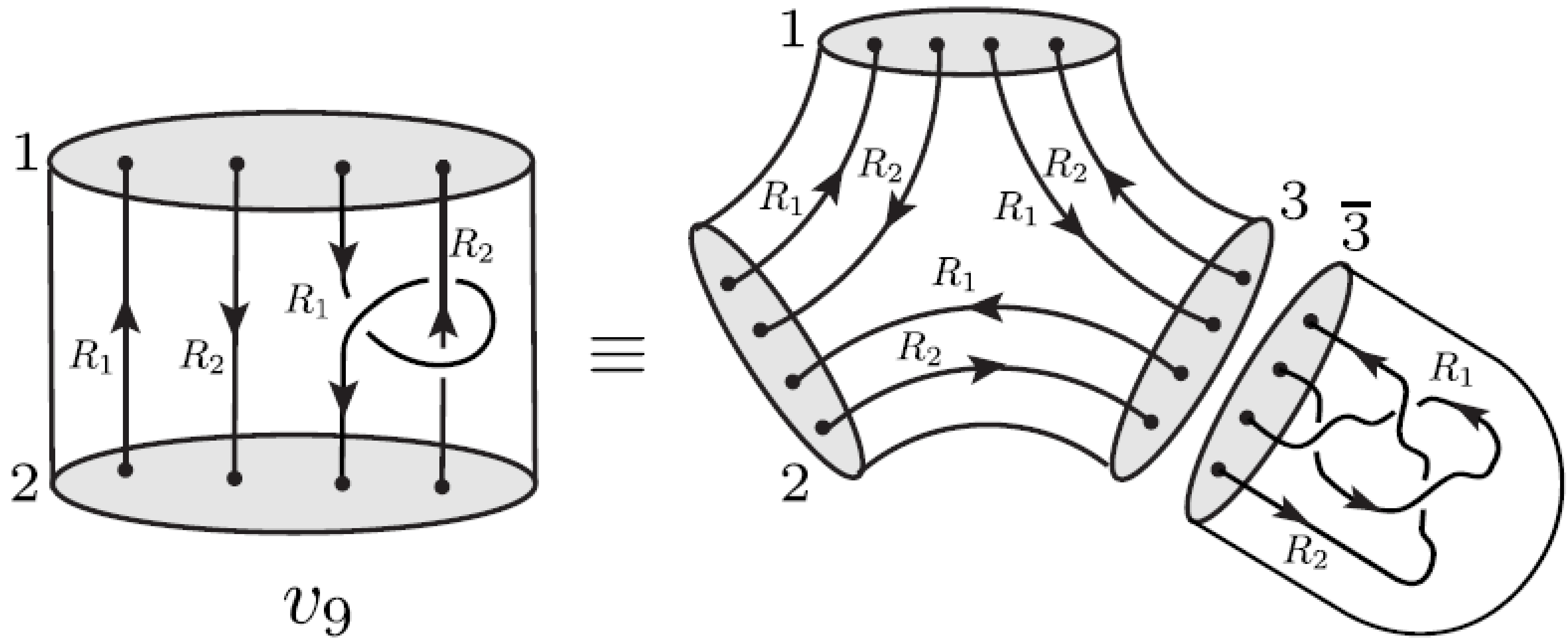
Generalisations to r boundaries

$$\nu_r = \sum_{R_s} \frac{|\phi^{(1)side}_{R_s}\rangle |\phi^{(2)side}_{R_s}\rangle \cdots |\phi^{(r)side}_{R_s}\rangle}{(dim_q R_s)^{\frac{r-2}{2}}}$$

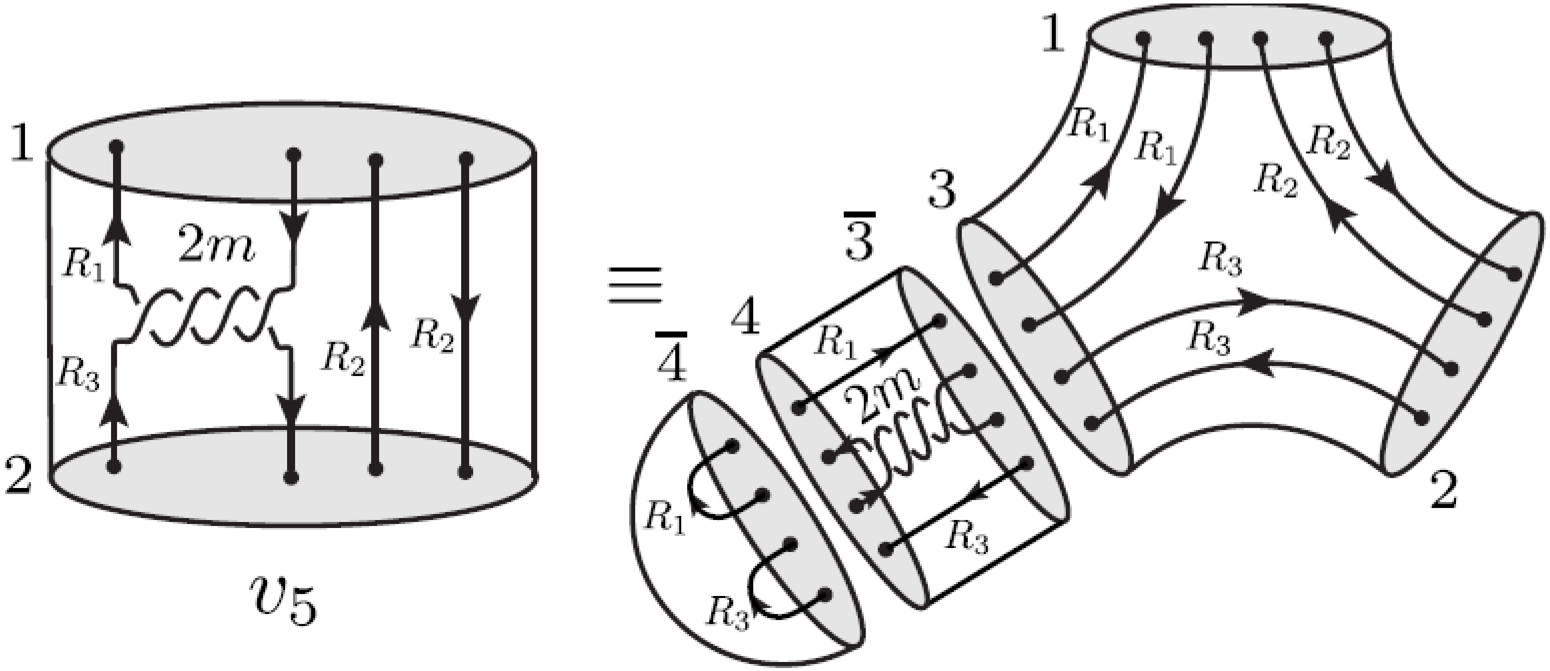
- Many knots can be redrawn as gluing of such three-balls with suitable braiding

- So colored knot invariants involves braiding eigenvalues and Wigner 6j (unknown for SU(N))

Few more building blocks

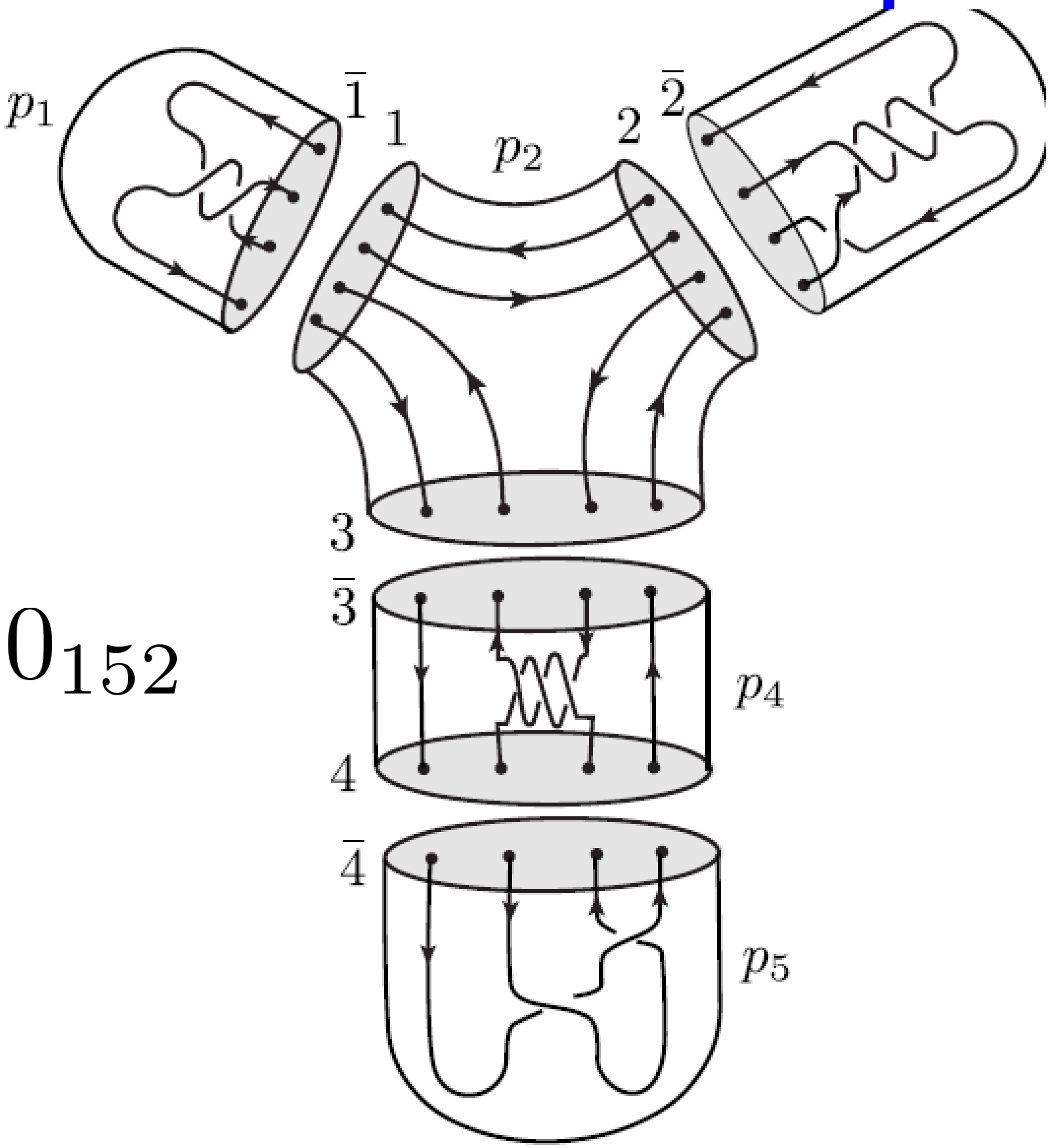


Few more building blocks

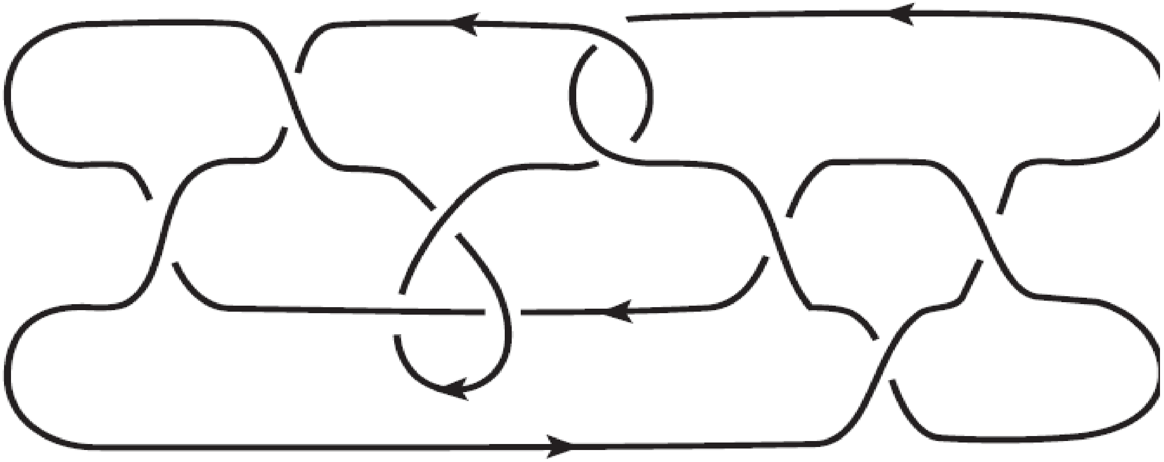


Few examples

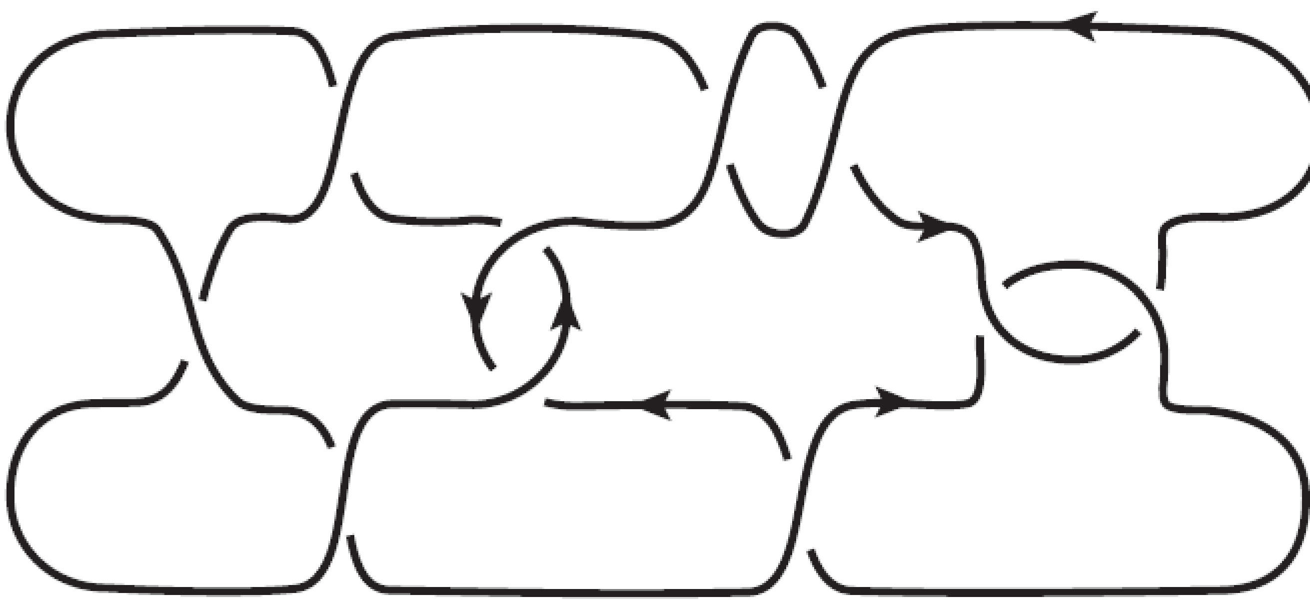
10_{152}



Few examples



10_{154}



10_{132}

Colored Knot invariant for 10_{152}

$$
= \sum_{l,l_1,r,u,v,x,y,z} \frac{1}{\epsilon_l^{R,\overline{R}} \sqrt{\dim_q l}\ \epsilon_{l_1}^{R,\overline{R}} \sqrt{\dim_q l_1}}\ \epsilon_z^{R,\overline{R}} \sqrt{\dim_q z}
$$

$$
\times\ \epsilon_u^{R,\overline{R}} \sqrt{\dim_q u}\ \epsilon_r^{R,R} \sqrt{\dim_q r}\ \epsilon_v^{R,R} \sqrt{\dim_q v}
$$

$$
\times\ a_{rl_1}\begin{bmatrix} R & R \\ \overline{R} & \overline{R} \end{bmatrix}\ a_{ll_1}\begin{bmatrix} R & \overline{R} \\ R & \overline{R} \end{bmatrix}\ a_{yl_1}\begin{bmatrix} R & R \\ \overline{R} & \overline{R} \end{bmatrix}\ a_{yz}\begin{bmatrix} R & R \\ \overline{R} & \overline{R} \end{bmatrix}\ a_{lu}\begin{bmatrix} R & \overline{R} \\ R & \overline{R} \end{bmatrix}
$$

$$
\times\ a_{vl}\begin{bmatrix} R & R \\ \overline{R} & \overline{R} \end{bmatrix}\ (\lambda_r^{(+)}(R,R))^3\ \lambda_y^{(+)}(R,R)\ \lambda_z^{(-)}(R,\overline{R})
$$

$$
\times\ (\lambda_u^{(-)}(R,\overline{R}))^2\ \lambda_l^{(-)}(R,\overline{R})\ (\lambda_v^{(+)}(R,R))^3 .
$$

We need SU(N) Wigner 6j to see the polynomial form

Quantum SU(N) Wigner6j

SU(2):Kirillov-Reshitikin

$$a_{j_{12}\, j_{23}} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix} = (-1)^{j_1+j_2+j_3+j_4} \sqrt{[2j_{12}+1][2j_{23}+1]} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix}$$

SU(N): Satoshi Nawata, Zodinmawia, PR (2013)

$$a_{\lambda_{12}\lambda_{23}} \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix} = \epsilon_{\{\lambda_i\}} \sqrt{\dim_q V_{\lambda_{12}} \dim_q V_{\lambda_{23}}} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_4 & \lambda_{23} \end{Bmatrix}$$

For a class of symmetric and their conjugate representation

Discussions and Open Problems

- Attempts to understand homological invariants from refined Chern-Simons theory
- Quantum SU(N) Wigner 6j for arbitrary representations to work out mutant knot invariants
- SO(N) quantum Wigner 6j required to obtain colored Kauffman

- **Knots,Links,Three-Manifold Invariants from Chern-Simons Field Theory**

Work done with

(a) Romesh Kaul and T.R. Govindarajan (IMSc):

1) Nucl. Phys. ***B402****(1993) 548*

2) Nucl. Phys. ***B422****(1994) 291*

*3) Mod. Phys. Lett.****A9****(1994)3205*

*4) Mod. Phys. Lett.****A10****(1995)1635*

(b) I.P. Ennes, A.V. Ramallo, J.M. Sanchez de Santos (Univ. of Santiago,Spain)

*5) Int. J. Mod. Phys.****A13****(1998) 2931*

(c) Swatee Naik (Math. Dept, Univ. of Nevada,USA)

*6) Commun. Math. Phys.****209****(2000)29*

(d) Romesh Kaul (IMSc, Chennai)

*7)Commun. Math. Phys.****217****(2001)295-314*

- **With Satoshi Nawata(NIKEF) and Zodinmawia(IITB)}:**

(1) `` Super-A-polynomials for Twist Knots,"JHEP 1211 (2012) 157.

(2))``Multiplicity-free quantum 6j-symbols for $U_q(sl_N)$,'' Lett.Math.Phys. 103 (2013) 1389-1398

(3)``Colored HOMFLY polynomials from Chern-Simons theory,"e-Print: arXiv:1302.5144 [hep-th]({\it to appear in Journal of Knot theory and Ramifications})

(4)``Colored Kauffman Homology and Super-A-polynomials,"e-print:arXiv:1310.2240

With Zodinmawia:

``SU(N) Racah and non-torus links",

Nucl. Phys B870(2013)205

Thank You