A geometric and topological view of classical optics

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Polarisation optics-two-level QM

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Every Ray of Light has therefore two opposite Sides, originally endued with a Property on which the[Pg 361] unusual Refraction depends, and the other two opposite Sides not endued with that Property.
(Newton, in Opticks, regarding double refraction)

$$E_x = \Re\left(a_1 \exp\left(i\phi_1\right)\exp\left(-i\omega t + kz\right)\right) \quad ; \quad E_y = \Re\left(a_2 \exp\left(i\phi_2\right)\exp\left(-i\omega t + kz\right)\right)$$

$$\boldsymbol{\psi} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} q_1 + ip_1 \\ q_2 + ip_2 \end{bmatrix}$$

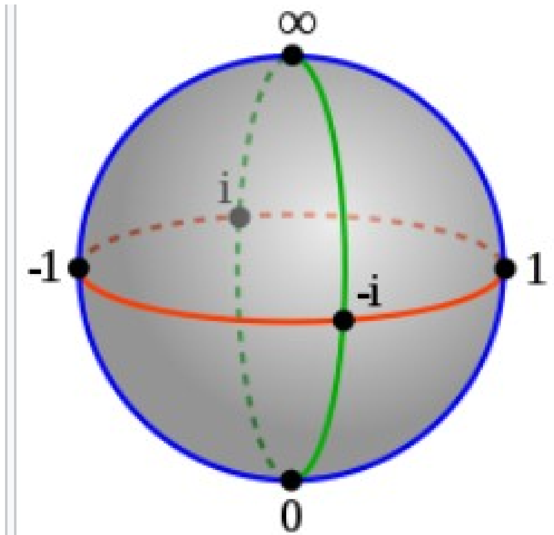
$$\bar{\boldsymbol{\psi}}\boldsymbol{\psi} = \bar{z}_1 z_1 + \bar{z}_2 z_2 = q_1^2 + p_1^2 + q_2^2 + p_2^2 = 1$$

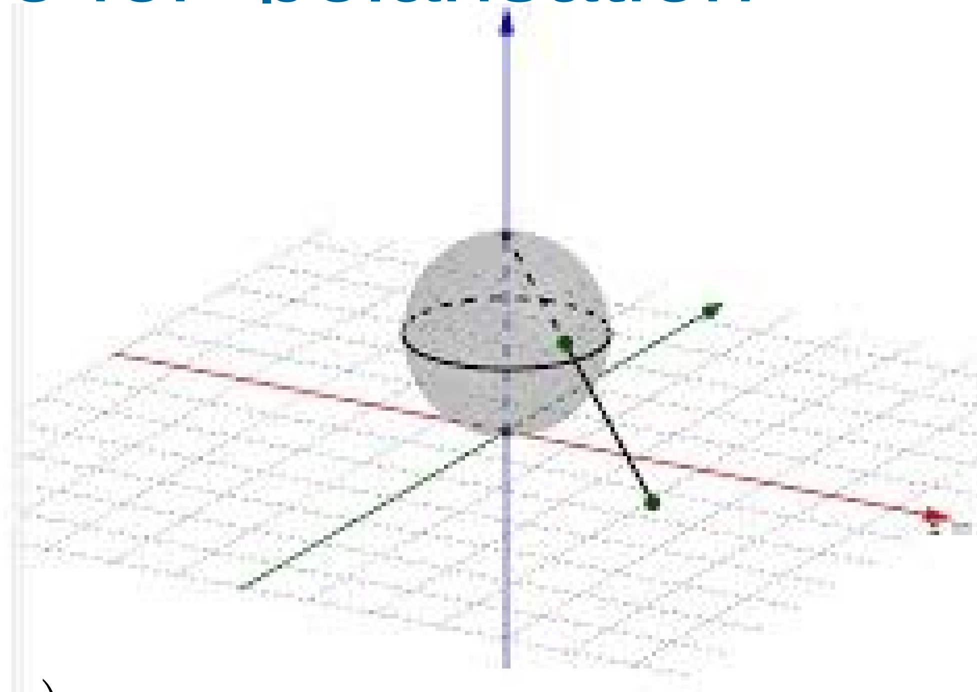
Polarisation state – ray space

- We are concerned with the shape and orientation of the (in general) elliptical orbit of the electric vector, not with its size, nor with phase, i.e position on the orbit., leaving us with two parameters, z_1, z_2 is equivalent to $\lambda z_1, \lambda z_2$
- In QM, we normalise the state anyway, and, the overall phase does not contribute to any physical property, the four real parameters in z_1, z_2 reduce to two
- Relative amplitude and relative phase of the two harmonic oscillators are encoded in the ratio $z = z_2/z_1$. So it would seem that we have a complex plane of z

BUT , $z_2 = 1$, $z_1 = 0$, $z = \infty$ *has to be treated as a single point….Hence we go to the Riemann sphere*

The Riemann sphere- CP^1 – is the Poincare (1888) sphere for polarisation

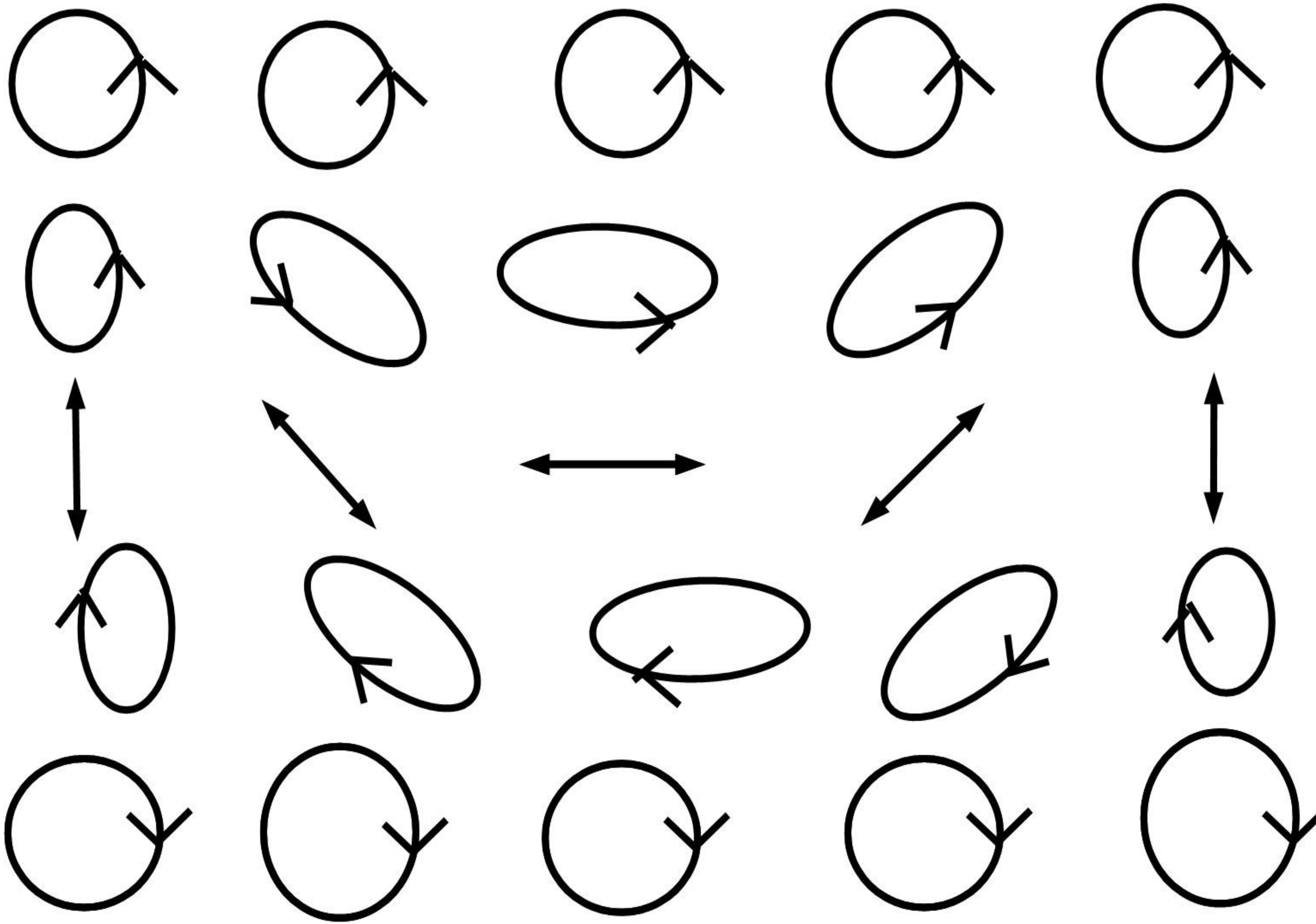




$$z_R = \frac{(z_1 - i\,z_2)}{\sqrt{2}} \quad ; \quad z_L = \frac{(z_1 + i\,z_2)}{\sqrt{2}}$$

Wikipedia article on the Riemann Sphere

geogebra.org



Stokes (1852) route to the Poincare sphere

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \begin{bmatrix} \bar{z}_1 & \bar{z}_2 \end{bmatrix} = \begin{bmatrix} z_1 \bar{z}_1 & z_1 \bar{z}_2 \\ z_2 \bar{z}_1 & z_2 \bar{z}_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} I+Q & U+iV \\ U-iV & I-Q \end{bmatrix}$$

$$\begin{bmatrix} \left(q_1^2+p_1^2\right) \quad \text{¿¿} & \left(q_1 q_2 + p_1 p_2\right) + i\left(p_1 q_2 - p_2 q_1\right) \\ \left(q_1 q_2 + p_1 p_2\right) - i\left(p_1 q_2 - p_2 q_1\right) & \left(q_2^2+p_2^2\right) \end{bmatrix}$$

$$\text{where } \left(q_1^2+p_1^2+q_2^2+p_2^2\right)^2 = I^2 = Q^2+U^2+V^2$$

The interference of polarised light

-Not a new subject –goes back to Fresnel and Arago (1817)

$$I = (\bar{\psi}_1 + \bar{\psi}_2)(\psi_1 + \psi_2) \text{ (intensity of two superposed beams)}$$
$$= \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 \text{ (individual intensities)}$$
$$+ \bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1 \text{ (interference terms)}$$

- By shifting the phase of any one beam by $\pi/2$, the imaginary part of the interference term is also measurable

Can interpret as interference between $\bar{\psi}_1$ and $\psi_1(\bar{\psi}_1\psi_2)$

Defining the phase difference for two arbitrary polarisation states

Extract from ‘Generalised Theory of Interference’ , Proc. Ind. Acad. Sci **44**, 247 (1956) by S. Pancharatnam.

the value of δ as defined above. Hence we will be guilty of no internal inconsistency if we make the following statement by way of a definition: *the phase advance of one polarised beam over another* (not necessarily in the same state of polarisation) *is the amount by which its phase must be retarded relative to the second, in order that the intensity resulting from their mutual interference may be a maximum.*

This phase advance is identically equal to δ, and the above definition holds only for non-orthogonal vibrations.

This definition has an unexpected property

Preston, 1897, Part II, Ch. 7, p. 50, Ex. 1). Since the Poincaré sphere has unit radius, we arrive at the following unexpected geometrical result.

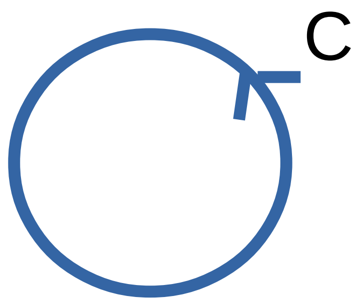
V. When a beam of polarisation C is decomposed into two beams in the states of polarisation A and B respectively, the phase difference δ between these beams is given by

$$|\delta| = \pi - \tfrac{1}{2}|E'| \tag{5 a}$$

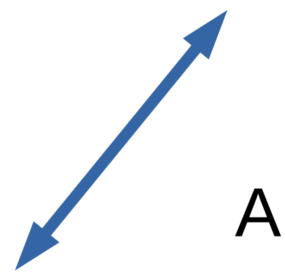
where the angle $|E'|$ is numerically equal to the area of the triangle C′BA colunar to ABC. (E′ is also the spherical excess of the triangle C′BA, *i.e.*, the excess of the sum of its three angles over π.)

The phase of $(\bar{\psi}_1\psi_2)(\bar{\psi}_2\psi_3)(\bar{\psi}_3\psi_1)$ is not zero - it equals half the solid angle made by the representatives of the three states on the Poincare sphere. This phase is independent of the phase choices for the three ψ 's

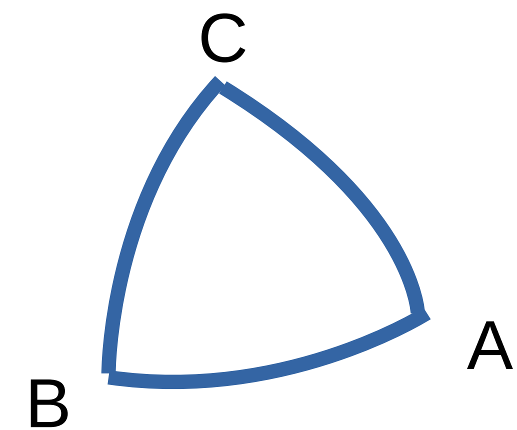
‘Pancharatnam phase’: an example



$$\psi_C = \begin{bmatrix} 1 \\ i \end{bmatrix}$$



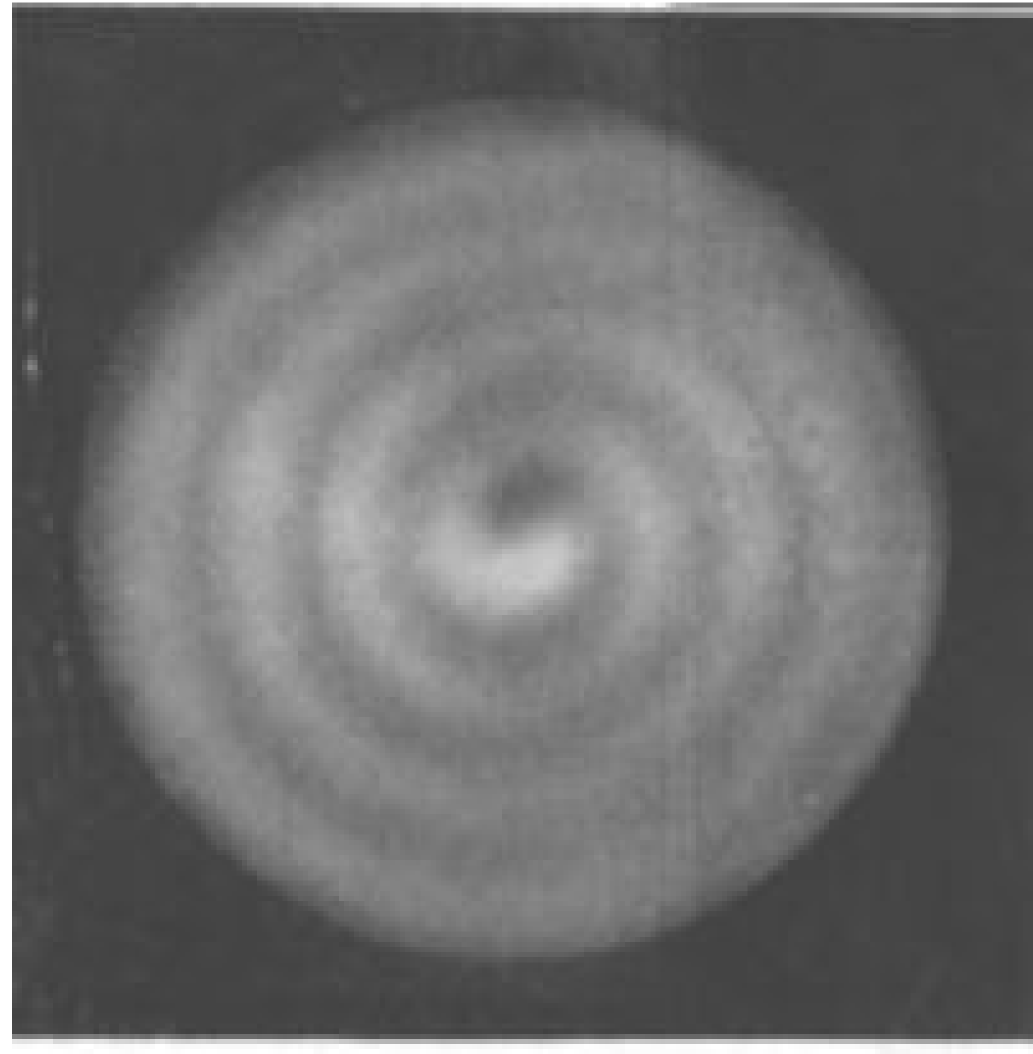
$$\psi_A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



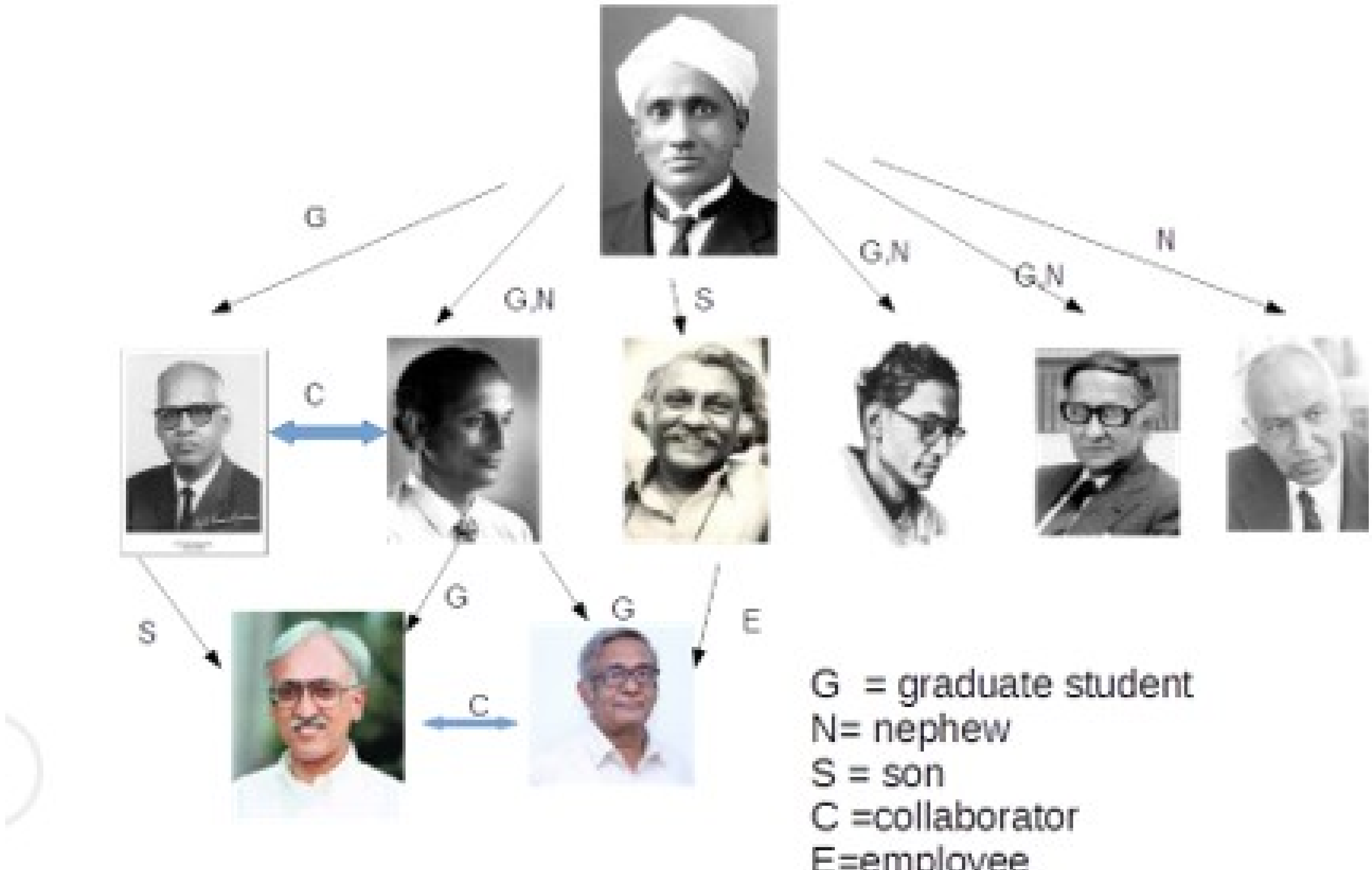
$$\psi_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Being ‘in phase with’ , considered as a relation, is reflexive and symmetric, but not transitive

The Pancharatnam Connection

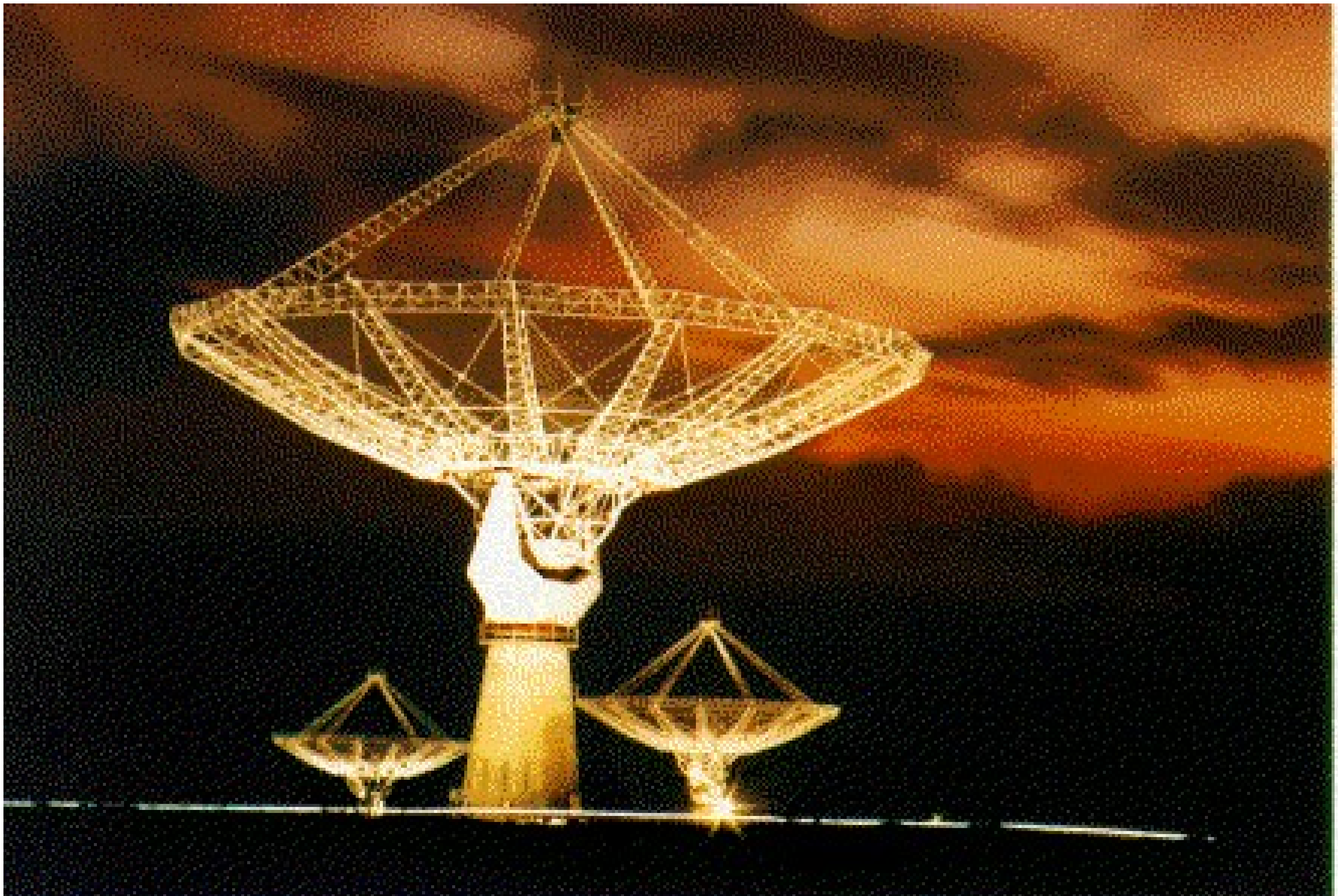


Current Science: August 1994
Resonance: April 2013



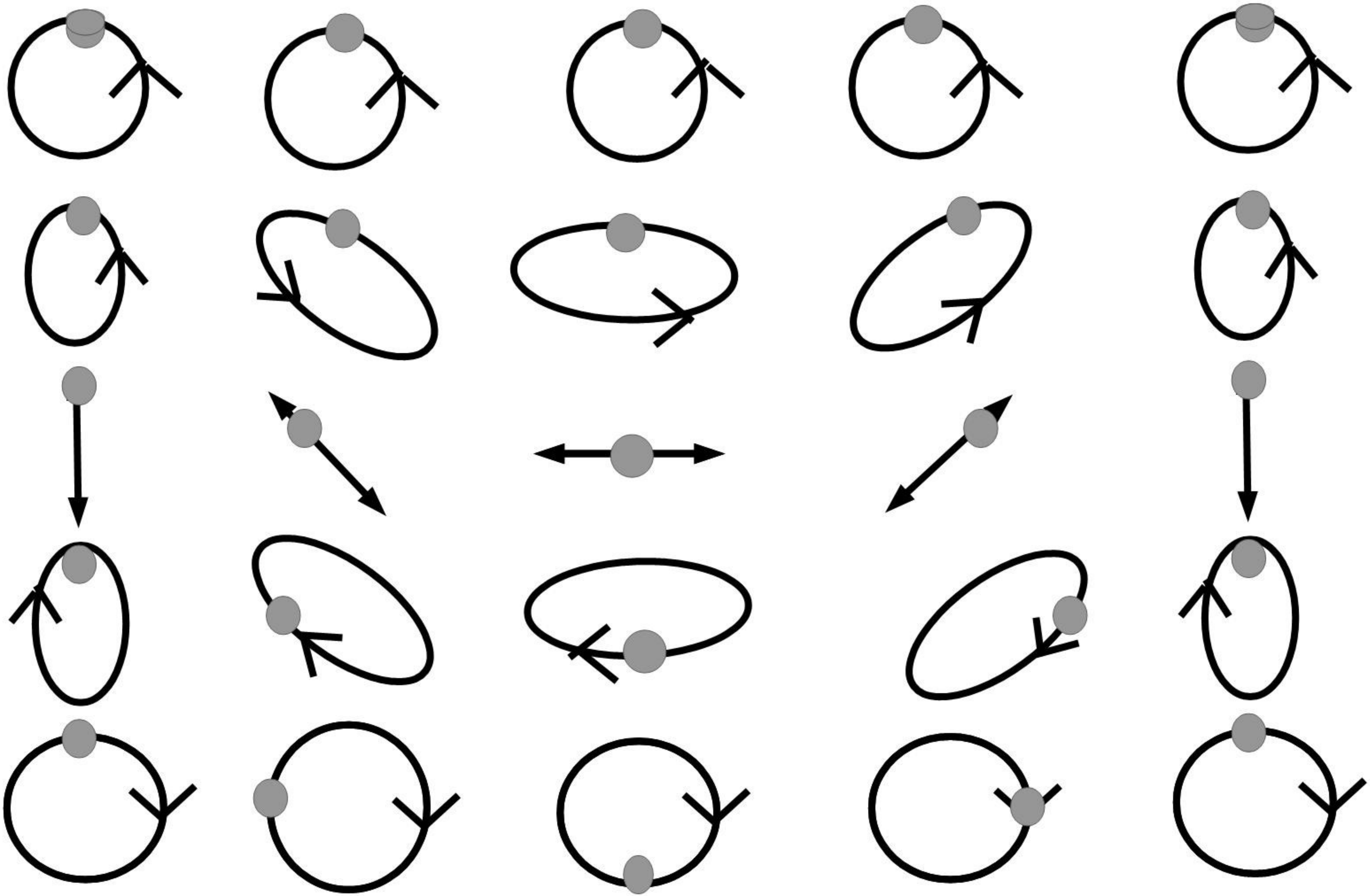
G
G,N
S
G,N
G,N
N
C
S
G
G
E
C
G = graduate student
N= nephew
S = son
C =collaborator
E=employee

Giant Metrewave Radio Telescope



Adding phase to the Poincare sphere

- The unit intensity / normalisation condition puts us on the three-sphere S^3
- Each orbit is, at least topologically, a circle S^1
- ‘Collapse’ each orbit to a point, we get a two-sphere S^2
- *Can we introduce the usual spherical co-ordinates for polarisation state and one more for phase? How does this S^2 worth of circles sit inside the S^3*



Filling the three sphere with circles and tori
Clifford (1873) parallels

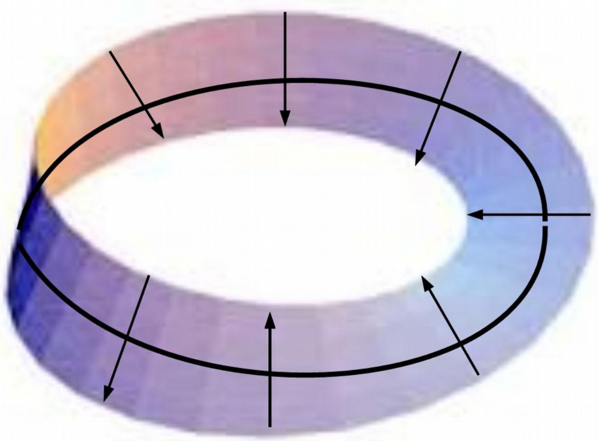
- The three sphere can be represented in 3-d euclidean space using two solid spheres,representing the ‘northern’ and southern’ hemispheres, just as we could represent a two sphere by using two discs

- Each closed orbit of the 2-d oscillator is a clrcle, they have to be non-intersecting, and between them they fill the 3-sphere. Incidentally, any two are linked, and also have a constant separation – hence the ‘parallels’

- Each line of latitude of the Poincare sphere corresponds to a torus (incidentally ‘flat’!)

Beyond Cartesian duality

- No difficulty in assigning *x* and *y* to points on a plane, or an angle θ and a height *z* to a point on a cylinder
- Not so easy to assign an angle θ and a height *z* to points on a Mobius band. We need two 'patches' to do the job
- This is only locally a 'Cartesian product, but globally 'twisted'
- This kind of situation led a few mathematicians to define ? invent ? discover? fibre bundles, around (1930-1940) . The geometric object that captures how polarisation and phase are intertwined, is now known as the 'Hopf fibration' after Heinz Hopf (1931)



Not all fibre bundles are twisted! The cylinder and the plane are examples.

The Hopf fibration (1931)

§ 5.

Eine Abbildung der S^3 auf die S^2 mit $\gamma = 1$.

Der euklidische R^4 mit den Koordinaten x_1, x_2, x_3, x_4 sei auf den euklidischen R^3 mit den Koordinaten ξ_1, ξ_2, ξ_3 folgendermaßen abgebildet:

$$\xi_1 = 2(x_1 x_3 + x_2 x_4), \quad \xi_2 = 2(x_2 x_3 - x_1 x_4), \tag{1}$$
$$\xi_3 = x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

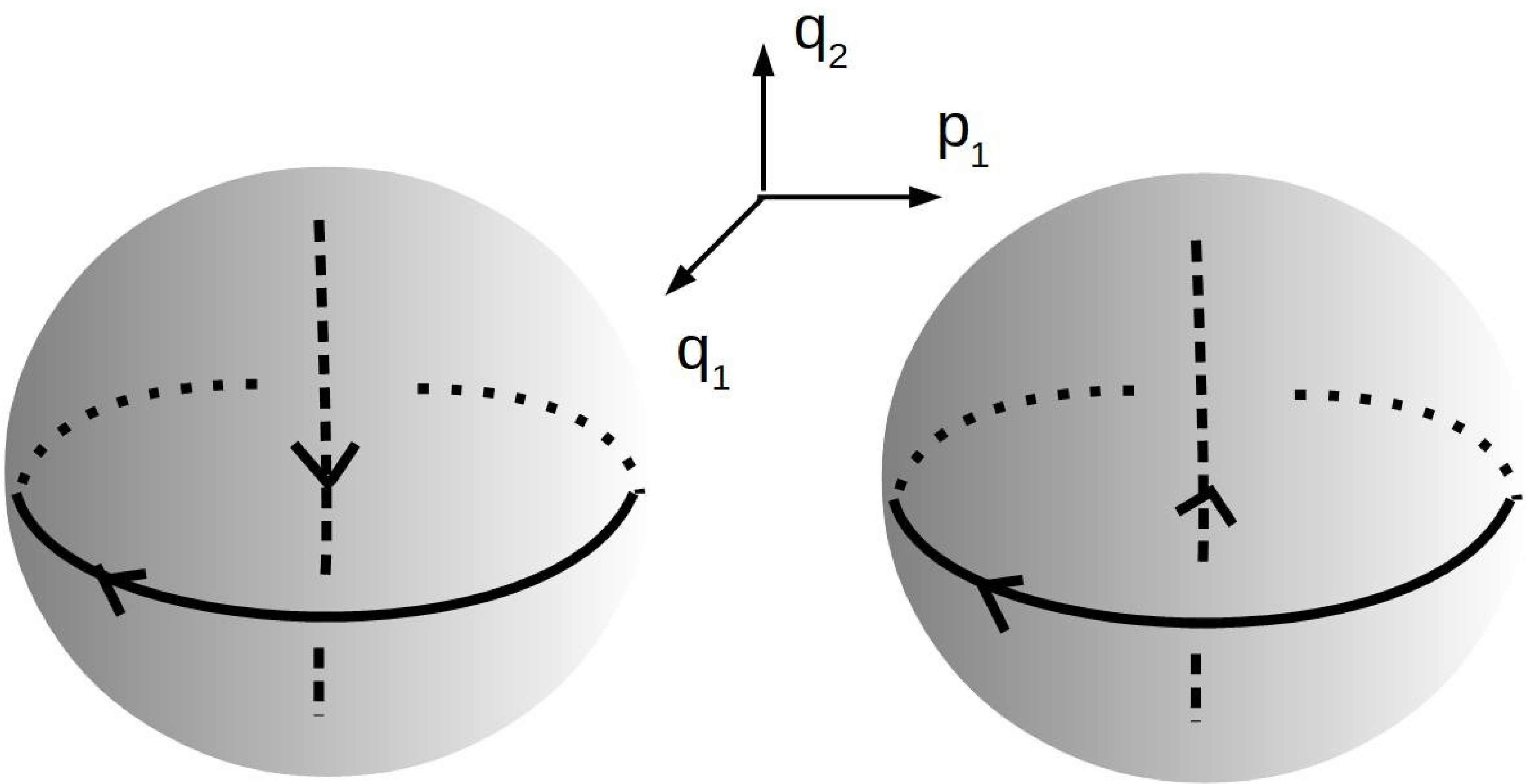
Dann ist

$$\xi_1^2 + \xi_2^2 = 4(x_1^2 + x_2^2) \cdot (x_3^2 + x_4^2),$$

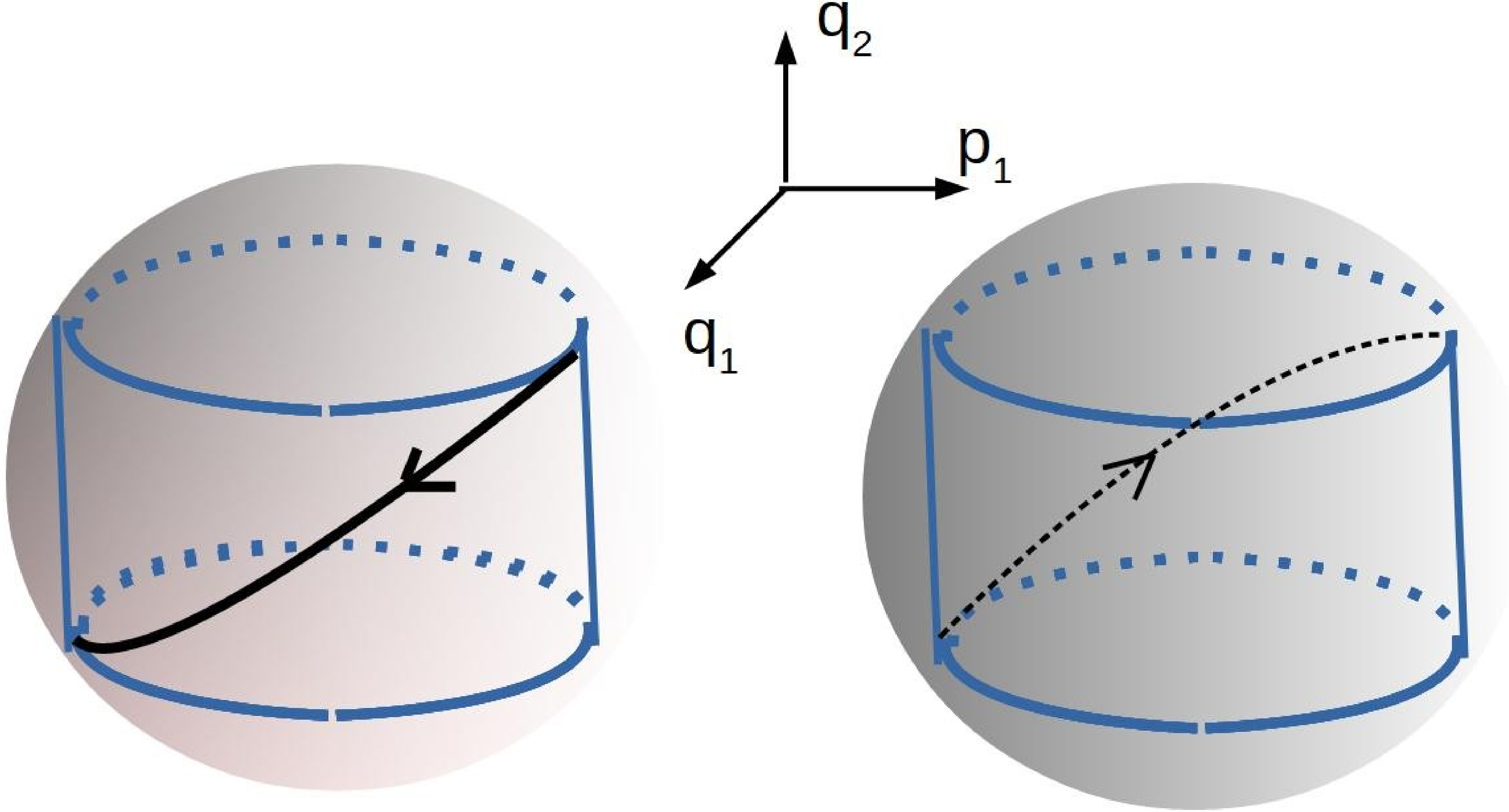
also

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^2; \tag{2}$$

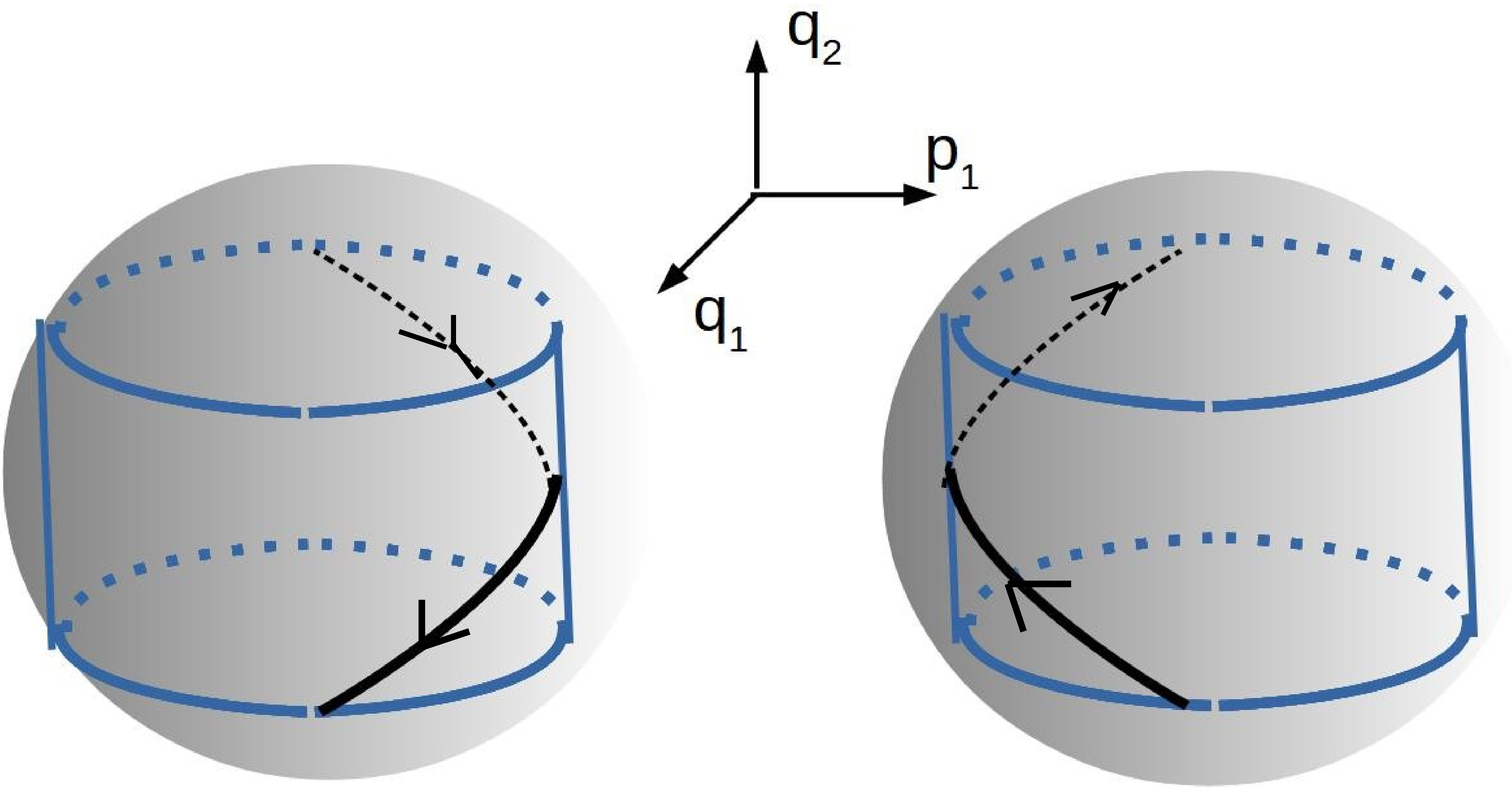
die dreidimensionale Kugel mit dem Radius r um den Nullpunkt R^4 als Mittelpunkt wird somit auf die zweidimensionale Kugel mit dem Radius r^2 um den Nullpunkt des R^3 als Mittelpunkt abgebildet; insbesondere ist die Einheitskugel S^2 des R^3 das Bild der Einheitskugel S^3 des R^4. Diese Abbildung f der S^3 auf die S^2 wollen wir betrachten[12]).

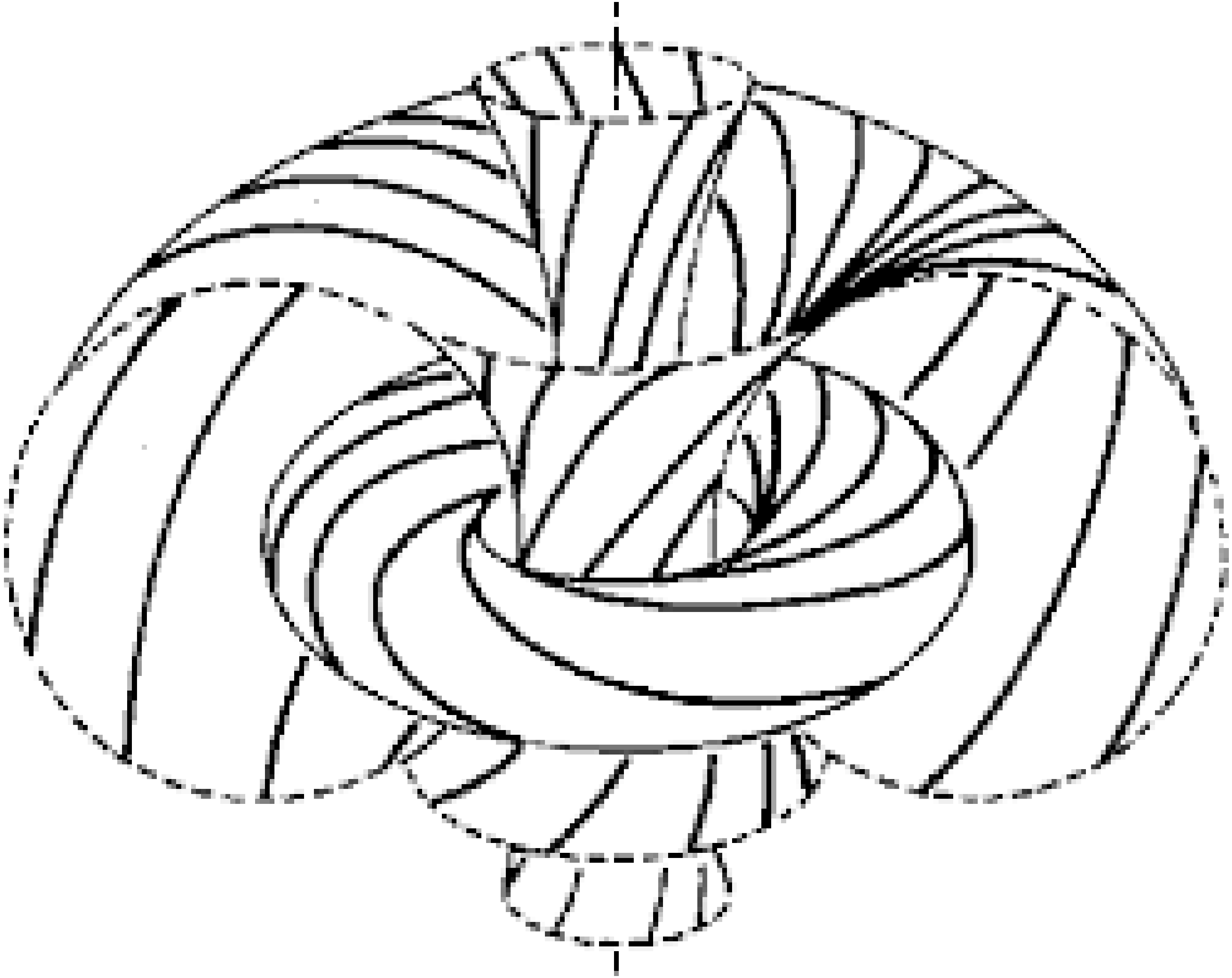


$$p_2 = -\sqrt{\left(1 - q_1^2 - p_1^2 - q_2^2\right)} \qquad p_2 = +\sqrt{\left(1 - q_1^2 - p_1^2 - q_2^2\right)}$$



q_2
p_1
q_1


q2
p1
q1

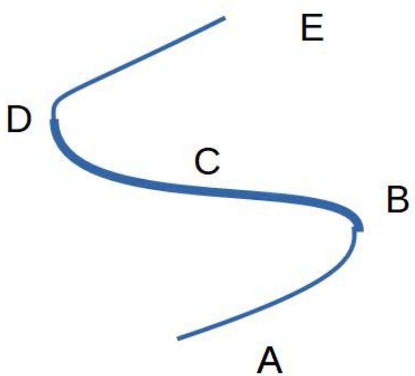


Hopf fibration in stereographic projection- from Penrose and Rindler, “Spinors and Spacetime” volume 1

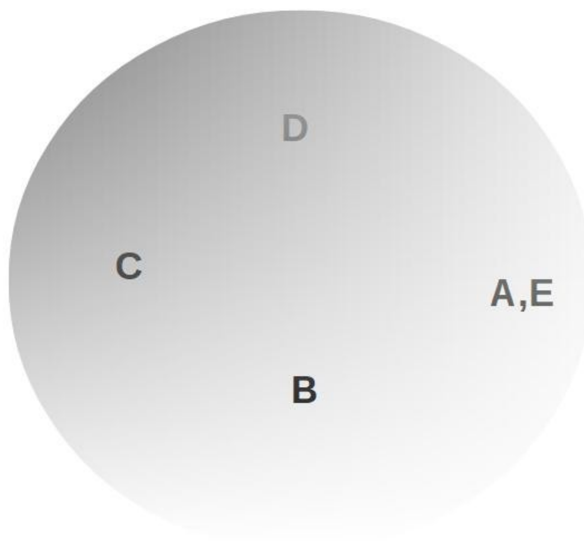
One more geometric phase associated with polarised light

- In a medium with refractive index dependent on space, light rays can describe a curve and indeed, a space curve like a helix
- This can also be achieved by sending light along a fibre and bending the fibre into a space curve.
- In a WKB kind of limit, one can ask what happens to the polarisation – its enough to know what happens to linear polarisations since we can then use them as a basis.
- With respect to a local plane approximating the curve (by taking three nearby points), the polarisation does not change – but this plane changes as we move along the curve!

The geometry of ‘coiled light’



One turn of a helix



tangent vectors on a sphere

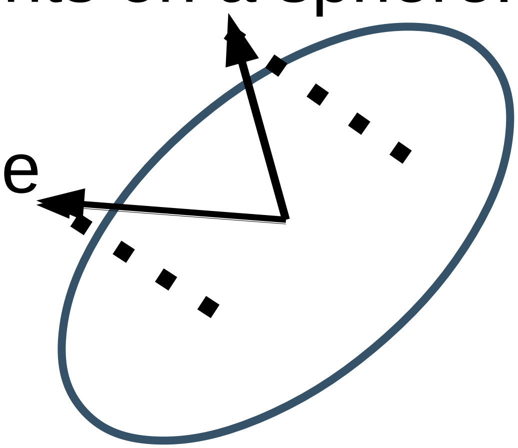
The electric vector lives in the plane transverse to the direction of propagation, that is, in the tangent plane to the sphere. As the direction changes, vectors in this plane undergo ‘parallel displacement’

Combining the two geometric phases: polarisation of beams going in all directions

- First guess: The polarisation can now be in any plane, so why not a 3-d harmonic oscillator, described by three complex amplitudes z_1, z_2, z_3
- The QM analogue is a three level system, and we are interested in its ray space, i.e. ignoring normalisation and overall phase, so (z_1, z_2, z_3) same as $\lambda(z_1, z_2, z_3)$
- Four real parameters,
- The trick of taking ratios would give $(1, z_2/z_1, z_3/z_1)$, hence two Riemann spheres – but has a problem!

Three equivalent ways to get the ray space of a three level system

- Use two complex numbers, the roots of the equation
  $z_1\alpha^2 + z_2\alpha + z_3 = 0$ Unordered pair of complex numbers, each of which can be put on a sphere
- Think of this as a spin one object, made up of two spin half states, symmetrised, (dropping the spin zero part which is antisymmetric) – due to Majorana, works for any spin *n/2* – unordered *n*-tuple of points on a sphere.
- Two unit vectors, order immaterial bisector giving normal to the plane and sense of the orbit and the the foci of the ellipse given by the in plane components

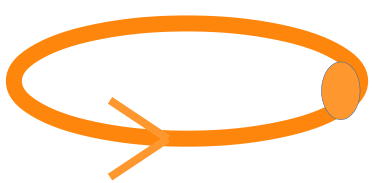


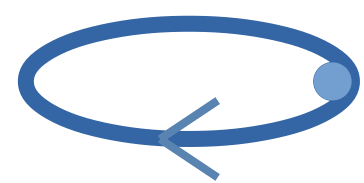
Topology of the full space of all directions and all polarisations: twisted or not?

- The geometric phase can be expressed in terms of the path in the auxiliary spin 1 / three level ray space ***CP^2***
- However, the full space we are after is not ***CP^2*** Each elliptical orbit corresponds to two oppsite ***k***'s
- Mere doubling of ***CP^2*** will not do – because the boundary between the left and right rotating ellipses is the set of linear vibrations, each of which corresponds to an entire circle worth of ***k***'s
- The solution – if we can find a basis for each ***k***, which is smooth on the sphere of directions, then we clearly can associate two complex numbers -so one Poincare sphere, smoothly, for all ***k***'s , so we are back to ***S^2 x S^2***

A smooth basis choice for all directions..

- Cannot use linear basis pair - ‘hairy ball theorem’
- Could work with general (complex) basis pair
- Can be manufactured by suitable sources
- Geometric phases can be calculated, agreeing with / generalising / unifying earlier results





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