THE THEORY OF NON-COMPLETE ALGEBRAIC SURFACES

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Abstract. We will describe the basic theory of non-complete algebraic surfaces developed by Japanese algebraic geometers. We will also mention some of the major results in this theory as well as several results about affine surfaces proved using this theory during the last thirty years.

§1. Introduction.

The Enriques-Kodaira classification of minimal algebraic surfaces can be briefly described as follows.

Let X be a smooth projective surface which does not contain any exceptional curve of the first kind (i.e. no smooth rational curve with self-intersection -1). We say that X is a relatively minimal surface.

(1) If $\kappa(X) = -\infty$ then either $X \cong \mathbf{P}^2$ or X admits a morphism $X \to B$ onto a smooth projective curve all whose fibers are isomorphic to $\mathbf{P}^1$. In this latter case X is called a relatively minimal ruled surface.

(2) If $\kappa(X) = 0$ then X is either an abelian surface, a $K-3$ surface, an Enriques surface, or a hyperelliptic surface (i.e. a quotient of a product of two elliptic curves by a finite group of automorphisms, acting freely).

(3) If $\kappa(X) = 1$ then X has an elliptic fibration $\varphi : X \to B$ onto a smooth projective curve. Kodaira gave a rich theory of possible singular fibers of φ, the monodromy action on the first integral homology group of a general fiber in a neighborhood of any singular fiber, a formula for the canonical bundle of X in terms of the canonical bundle of B and the singular fibers, study of variation of complex structures of the fibers of φ, etc. Elliptic fibrations continue to play a special and important role in surface theory.

(4) If $\kappa(X) = 2$ then X is called a surface of general type. Kodaira and Bombieri proved important results about the pluri-canonical maps given by the linear systems $|nK_X|$. A vanishing theorem (generalising the famous Kodaira vanishing theorem) proved by C.P. Ramanujam plays an important role in the study of these maps. Mumford proved that for large n this map is a birational morphism onto a projectively normal surface with at most rational double points. As expected, there are still many mysteries about the nature of possible invariants of a surface of general type like p_g (geometric genus), K_X^2, q_X (irregularity), fundamental group, etc.

It is a great surprise that most of these results have close analogues in the theory of non-complete surface theory. It was S. Iitaka's extraordinary intuition to introduce the notion of logarithmic Kodaira dimension, $\overline{\kappa}(V)$, of a smooth quasi-projective variety V. He proved many basic properties of this invariant. This was followed by a fundamental work of Y. Kawamata. Kawamata proved important structure theorems for smooth surfaces with $\overline{\kappa} \geq 1$. He also made an important study of the log pluricanonical map and the singularities of the image. In 1979, Miyanishi - Sugie - Fijita proved the Cancellation Theorem for $\mathbf{C^2}$. A crucial step in this proof was to prove a general result for varieites with $\overline{\kappa} = -\infty$. Fujita wrote an important paper about topology of non-complete algebraic surfaces which continues to be very useful.

One more important contribution in the theory of non-complete algebraic surfaces was a Bogomolov - Miyaoka - Yau type inequality proved by Kobayashi - Nakamura - Sakai. This inequality continues to play a crucial role in many results proved after 1990. The book ([33]) gives a detailed treatment of the theory of non-complete surface theory. It is safe to say that the Japanese algebraic geometers can take a patent on the theory of non-complete surfaces! Outside Japan this theory has been extensively used by H. Flenner, R.V. Gurjar, S. Kaliman, M. Koras, S. Kolte, S. Lu. A. Maharana, S. Orevkov, K. Palka, S. Paul, C.R. Pradeep, P. Russell, A.R. Shastri, M. Zaidenberg, D.-Q. Zhang and others to prove important results about affine surfaces.

Our aim is to survey these and other results in this area of algebraic geometry. We believe that commutative algebraists will benefit by studying this theory. Due to lack of time, we cannot include all the interesting results proved in this area and we will say only a few words about the proofs of the results. At the end we will state some unsolved problems.

§2. Some fundamental results about open algebraic surfaces

Let k be an algebraically closed field. Sometimes we my assume $k = \mathbb{C}$ when topological arguments are useful. For example, in the deep study of elliptic surfaces by K. Kodaira classical complex analysis is crucial.

All varieties will be defined over k. We begin with some basic surface theory.

Let X be a smooth projective (irreducible) surface, $p \in X$ a closed point and $f, g \in m_p$ elements of $\mathcal{O}_{X,p}$ without common factors. Then the ideal (f, g) is primary for the maximal ideal m_p. Therefore $\dim_k \mathcal{O}_{X,p}/(f, g)$ is finite. If

C, D are the scheme-theoretic curves defined by $\mathcal{O}_{X,p}/(f)$ and $\mathcal{O}_{X,p}/(g)$ respectively then the intersection multiplicity of C, D at p is defined to be this dimension and denoted by $(C \cdot D)_p$. If $C, D \subset X$ are scheme-theoretic curves without any common component then we define $C \cdot D = \sum_{p \in C \cap D} (C \cdot D)_p$. Let $\Gamma = \sum a_i C_i, \Delta = \sum b_j D_j$ be formal linear combinations of irreducible curve on X with integer (or rational) coefficients such that no C_i and D_j are same and only finitely many a_i, b_j are non-zero. Then we define

$$\Gamma \cdot \Delta = \sum_{i,j} a_i b_j C_i \cdot \Delta_j$$

We define $C \cdot C$ as follows.
Let $p \in C$ be any point. We can find a rational function $\varphi \in k(X)$ such that the divisor $(\varphi) + C = \sum b_j D_j$, where no D_j occurs in the support of C.
Define $C \cdot C = \sum b_j C \cdot D_j$. This is well defined. This uses the fact that the number of zeros and poles of a rational function on a smooth projective curves, counted properly, are equal.
It can be said that the whole of theory of algebraic surfaces depends on properties the intersection numbers $C \cdot D$.

Blowing up

Recall that for any $p \in X$, we can construct another smooth projective surface $\widetilde{X}$ with a proper morphism $\pi : \widetilde{X} \to X$ such that $E := \pi^{-1}(p) \cong \mathbb{P}^1$ and $\pi : \widetilde{X} - E \xrightarrow{\sim} X - p$. A surprising result is that $E \cdot E = -1$. E is called the **exceptional curve** for the blowing up of X at p. A smooth projective irreducible curve $C \subset X$ such that $C \cdot C = -n$ is called a $(-n)-$curve if C is also rational. This terminology is mainly used when $n > 0$. These curves play an important role in the study of normal singular points of algebraic surfaces.

Recall that for any divisor $D = \sum b_j D_j$ on X (some b_j may be < 0), the linear system $H^0(X, \mathcal{O}(D)) = \{\varphi \in k(X) \mid (\varphi) + D \geq 0\}$ is finite dimensional. The set of effective divisors $(\varphi) + D$ as φ varies in $H^0(X, \mathcal{O}(D))$ is called the complete linear system of divisors which are effective and rationally equivalent to D. It is denoted by $|D|$. If $H^0(X, \mathcal{O}(D)) = (0)$, then $|D| = \phi$. We state two important results.

Riemann- Roch Theorem

Recall that Ω^1_X is the sheaf of regular 1-forms on X. This is a locally free sheaf on X of rank n if X is a smooth variety of dimension n. If $p \in X$, then a section of Ω^1_X at p has the form $f_1 dz_1 + f_2 z_2 + \cdots + f_n z_n$, where $f_i \in \mathcal{O}_{X,p}$ and $m_p = (z_1, \cdots, z_n)$.

The sheaf $\bigwedge^n \Omega^1_X$ is called the canonical line bundle of X and denoted by K_X. A section of K_X at p is of the form $f dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$, where $f \in \mathcal{O}_{X,p}$. Recall that $z_1, \cdots, z_n$ is a transcendence basis for $k(X)$. A rational n-form $w := \varphi \; dz_1 \wedge dz_2 \wedge \cdots \wedge \cdots \wedge dz_n$ with $\varphi \in k(X)$ has zeros and poles. We denote by (w) the divisor $\sum a_i D_i - \sum b_j \Gamma_j$ where $D_1, D_2, \cdots$ are zeros of w and $\Gamma_1, \Gamma_2, \cdots$ are poles of w (hence $a_i > 0, b_j > 0$). In case $n = 2, D_i, \Gamma_j$ are irreducible curves on X. If $n = 1$ and $\varphi \in k(X)$ then degree $(\varphi) = 0$. Here, for any divisor $D = \sum a_i p_i$ on X the integer $\sum a_i$ is called the degree of D, denoted by $\deg D$.

Riemann- Roch Theorem for curves

Let C be a smooth projective irreducible curve of genus g, i.e. $\dim_k H^1(C, \mathcal{O}_C) = g = \dim_k H^0(C, \Omega^1_C)$.

For any divisor D on C,

$\dim_k H^0(C, \mathcal{O}(D)) - \dim_k H^1(C, \mathcal{O}(D)) = \deg D + 1 - g$.

By Serre duality theorem

$H^1(C, \mathcal{O}(D)) \cong H^0(C, \mathcal{O}(K_C - D))$.

Riemann- Roch Theorem for surfaces

Let X be a smooth projective irreducible surfaces, D a divisor on X. Then $\dim_k H^0(X, \mathcal{O}(D)) - \dim_k H^1(X, \mathcal{O}(D)) + \dim_k H^2_k(X, \mathcal{O}(D)) = \frac{D^2 - D\cdot K_X}{2} + \chi(X, \mathcal{O})$

Here $\mathcal{O}$ is the structure sheaf of X and $\chi(X, \mathcal{O}) = \dim_k H^0(X, \mathcal{O}) - \dim_k H^1(X, \mathcal{O}) + \dim_k H^2(X, \mathcal{O})$. If C is an irreducible (not necessarily smooth) curve on X then the arithmetic genus of $C, p_a(C)$, is equal to $\frac{C^2 + C\cdot K_X}{2} + 1$. This formula is called the **adjunction formula**. If $\overline{C}$ is the desingularization (equivalently, normalization in its function field) of C, then $\dim_k H^1(\overline{C}, \mathcal{O}_{\overline{C}})$ is called the geometric genus of C, denoted by g_C. We have $p_a(C) - g_C = \sum \frac{e_i(e_i - 1)}{2}$.

Here the summation is over all singular points of C with multiplicity e_i (including infinitely near ones)

Finally, we state another deep result.

M. Noether's formula

$$\chi(X, \mathcal{O}) = \frac{K_X^2 + \chi_{top}(X)}{12}$$

Here $\chi_{top}(X)$ is the topological Euler-characteristic X in case $k = \mathbb{C}$. These three results are very important for the theory of surfaces.

Using $C \cdot D$, we get an intersection form on the family of all divisors on X.

Hodge index theorem.

Let H be a divisor on X with $H \cdot H > 0$. If D is a divisor on X such that $H \cdot D = 0$ then either D is numerically equivalent to zero (i.e. $D \cdot C = 0$ for every curve C on X), or $D^2 < 0$.

A divisor H is ample if $H^2 > 0$ and $H \cdot C > 0$ for every irreducible curve C on X.

This is called the Nakai's criterion of ampleness.

If X is a smooth projective irreducible surface and $D \subset X$ a (possibly reducible) curve on X such that $X - D$ is affine then $\sum a_i D_i$ is ample for some effective divisor supported on D.

This is called the **Goodman's criterion of affineness**. It was also proved by Gizatullin.

Classification of surfaces

Let X_0 be a smooth projective surface. If $E \subset X_0$ is a $(-1)-$curve, then X_0 is obtained by blowing up a smooth point p_1 on a smooth projective surface $X_1, \pi : X_0 \to X_1$, such that $E = \pi^{-1}(p_1)$. This is Castelnuovo's criterion of contraction. We can see that the rank of the Neron-Severi group, or $\dim_k H^1(X_0, \Omega^1_{X_0})$, or $b_2(X_0)$ decreases by 1, i.e. $b_2(X_0) = b_2(X_1) + 1$. Since $b_2(X) > 0$ for any smooth projective surface X, if we repeat this procedure of contracting $(-1)-$curves then we reach a smooth projective surface X_n which has no $(-1)-$curve.

X_n is called a relatively minimal model of $k(X)$. It is unique except for rational or ruled surfaces, i.e. unless $k(X)$ has the form $k(C)(t)$, where C is a smooth projective curve and t an indeterminate over $k(C)$.

We will give some description of relatively minimal models using the invariant Kodaira dimension $\kappa(X)$.

Using the concept of logarithmic Kodaira dimension introduced by S. Iitaka, this has been extended to non-complete algebraic surfaces by S. Iitaka, Y. Kawamata, T. Fujita, M. Miyanishi, T. Sugie, with important contribution by F. Sakai, S. Tsunoda, R. Kobayashi.

Kodaira dimension

Let X be a smooth projective surface. If $H^0(X, \mathcal{O}(nK_X)) = (0)$ for all $n \geq 1$ then we say that $\kappa(X) = -\infty$.

Now assume that $|nK_X| \neq \phi$ for some $n \geq 1$. If $\dim H^0(X, \mathcal{O}(nK_X))$ is non-zero for some n but bounded for $n \geq 1$ then we write $\kappa(X) = 0$

If $\dim H^0(X, (nK_X))$ is unbounded as a function of n but grows linearly with n then $\kappa(X) = 1$.

Finally, if $\dim H^0(X, (nK_X))$ grows as a quadratic function of n then $\kappa(X) = 2$. We call $\kappa(X)$ the Kodaira dimension of X. If X, Y are birationally isomorphic smooth projective surfaces then $\kappa(X) = k(Y)$. The converse is, of course, false even when X is a curve.

Detailed structure of minimal models

If $\kappa(X) \geq 0$ then the relatively minimal model of $k(X)$ is unique upto an isomorphism.

Assume that X is a relatively minimal model.

The case $\overline{\kappa}(X) = -\infty$.

Then either $X \cong \mathbb{P}^2$ or there is a morphism $f : X \to C$ onto a smooth projective curve C such that every (scheme-theoretic) fiber of f is $\mathbb{P}^1$. There is a rank 2 vector bundle $V \to C$ such that the associated projective bundle is $X \to C$. Thus the theory of ruled surfaces is intimately tied to the theory of rank 2 vector bundles on smooth projective curves. There are still important unsolved questions about the geometry of $\mathbb{P}^2$ or ruled surfaces. X contains at most one irreducible curve C with $C^2 < 0$. If it exists then C is a cross-section of f. If l denotes a fiber of f then every divisor Δ on X is numerically $aC + bl$, where C is a fixed cross-section of f.

There are non-trivial restriction on a, b if Δ is irreducible, or ample, $\cdots$. If X is rational then either X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\exists C \subset X$ with C a $(-n)-$curve with $n > 0$. This is called a Hirzebruch surface $\sum_n$ (or $\mathbb{F}_n$).

We can show that $\mathbb{F}_n$ and $\mathbb{F}_m$ are diffeomorphic if $n - m$ is even and conversely.

The case $\kappa(X) = 0$.

Then $12K$ is a trivial line bundle. X is one of the following.

(i) K-3 surface (i.e. K_X is trivial and X is simply-connected).

(ii) An Enriques surface. Now $2K_X$ is trivial and $H^1(X, \mathcal{O}) = 0$.

(iii) A hyperelliptic surface with a morphism $f : X \to \mathbb{P}^1$ with every fiber an irreducible elliptic curve.

(iv) An abelian surface. (i.e. the tangent bundle T_X is trivial).

Each of these classes has a rich geometry, with contributions from many mathematicians.
The surfaces in (ii) also admit an elliptic fibration $f : X \to \mathbb{P}^1$.

The case $\kappa(X) = 1$.

In this case there is a morphism $f : X \to C$ onto a smooth curve such that a general fiber of f is an elliptic curve, i.e. X is an elliptic surface.
For every singular fiber F_s, the arithmetic genus $p_a(F_s) = 1$ by semi-continuity theorem. This means $K \cdot F_s = 0$. Writing $F_s = \sum a_i A_i$, we know that $A_i^2 < 0$ for every i if F_s is reducible. Hence it is easy to deduce that each A_i is a $(-2)-$curve if F_s is reducible. Kodaira listed all possibilities that can occur for F_s. He also gave description of monodromy around each singular fiber.

The surfaces $f : X \to C$ which admit a cross-section are special, called **Basic Surface** by Kodaira. In general f can have multiple fibers. Kodaira discovered an important notion called a **logarithmic transform**. This enables to get rid of multiple fibers without changing the fibration outside this fiber, and conversely we can create multiple fibers with prescribed multiplicity from non-simplyconnected fibers.

Assuming that f has no multiple fiber, Kodaira showed how such a surface can be obtained from a Basic Surface by deformation. The knowledge of all the singular fibers enables us to calculate $\chi_{top}(X)$ and K_X.

Kodaira's work on elliptic surfaces uses deeply classical complex analysis.

The case $\kappa(X) = 2$.

Such a surface is called a **surface of general type**.

Here the deeper study of $|nK_X|$ was initiated by Kodaira. Using Kodaira Vanishing Theorem, and its variant Ramanujam Vanishing Theorem, we can show that $H^0(X, nK)$ has no base points if $n \geq 5$. For $K_X^2 >> 0$, this can be improved. The morphism given by $|nK|$ for $n \geq 5$ maps X birationally onto a surface with at most rational double points.
An inequality proved by Bogomolov-Yau-Miyaoka plays an important role.

$$K_X^2 \leq 3 \cdot \chi_{top}(X).$$

If equality occurs then the universal cover of X is the unit disc in $\mathbb{C}^2$. Yau's proof uses differential geometric method. This has been generalized to non-complete algebraic surfaces by Kobayashi- Nakamura-Sakai.

There are still many mysteries about surfaces of general type, e.g. which pairs (K^2, χ_{top}) of numbers can occur ? Similarly, $\pi_1(X)$ is still not completely understood though there are some important results in this connection.

Logarithmic Kodaira dimension

This theory was initiated by S. Iitaka. Later Y. Kawamata, T. Fujita, M. Miyanishi, F. Sakai, T. Sugie, R. Kobayashi, T. Tsunoda made important contributions to make the theory rich. It continues to find highly non-trivial applications, as we will mention at the end.

Let X be a smooth quasi-projective irreducible variety of dimension n. We can embed $X \subset V$, where V is a smooth projective variety such that $D := V - X$ is a divisor with simple normal crossings, i.e. for any $p \in D$, there exists local uniformising parameters $z_1, \cdots, z_n$ for $\mathcal{O}_{V,p}$ such that D is defined by $\{z_1 \cdot z_2 \cdots z_m = 0\}$ for some $m \leq n$. Such an embedding is guaranteed by H. Hironaka's work on resolution of singularities.

If $H^0(V, n(K_V + D)) = (0)$ for n for $n \geq 1$ then we write $\overline{\kappa}(X) = -\infty$. Otherwise $H^0(V, n(K + D)) \neq (0)$ for some $n \geq 1$. We can prove that $\dim_k H^0(V, n(K + D))$ is $O(n^r)$ for some r with $0 \leq r \leq n$. Then we write $\overline{\kappa}(X) = r$.

$\overline{\kappa}(X)$ is called the logarithmic Kodaira dimension of X.

Iitaka proved the following basic results.

1. If $f : V \to W$ is a dominating, generically finite map between smooth (irreducible) varieties then $\overline{\kappa}(V) \geq \overline{\kappa}(W)$.

2. If f is étale and proper then $\overline{\kappa}(V) = \overline{\kappa}(W)$.

3. If $\overline{\kappa}(V) \geq 0$ then any dominant morphism $f : V \to V$ is étale. (There are examples when such an f is not proper.)
   If $\overline{\kappa}(V) = \dim V$ then any dominant morphism $V \to V$ is an isomorphism. Further, Aut (V) is finite.

4. If $f : V \to W$ is a surjective morphism with connected general fiber then $\overline{\kappa}(V) \leq \dim W + \overline{\kappa}(F)$, where F is a "general" fiber of f. This is called the **"Easy" addition formula** of Iitaka.

Our main interest is the case of surfaces, where the theory has so far been most developed and effective.

Some results of Y. Kawamata ([23], [24], [25]).

(1) Let V be a smooth surface and $f : V \to B$ a morphism onto a smooth curve such that a general fiber F of f is irreducible. Then $\overline{\kappa}(V) \geq \overline{\kappa}(B) + \overline{\kappa}(F)$. This is a special case of **Iitaka's conjecture**. It is a very useful inequality. To state the next result and subsequent work, we explain the notion of Zariski - Fujita decomposition. Assume that $| n(K_X + D) | \neq \phi$ for some $n \geq 1$. Then there exists $Q-$divisors P, N on X satisfying the following properties.

(i) $(K_X + D) \approx P + N (\approx$ means numerical equivalence).

(ii) P is nef, i.e. $P.C \geq 0$ for every curve C on X

(iii) $N = \sum_{i=1}^{\ell} a_i C_i$ where all a_i are non-negative rational numbers and C_i are irreducible curves.

(iv) The intersection form on $\cup_i^{\ell} C_i$ is negative definite.

(v) $P.C_i = 0 \;\; \forall i$.

Kawamata, Fujita, Miyanishi, Tsunoda have given a clear recipe for finding this decomposition. For details, see [3].

(2) Kawamata proved that if $\overline{\kappa}(V) = 0$ then $nP \approx 0$ for some n (the converse is also true and easy). In this case Kawamata considered the quasi-Albanese map $f : V \to A$ where A is a quasi-Albenese variety (this is an algebraic group which is an extension of the algebraic torus by an abelian variety). Kawamata proved important properties of this map in [25].

(3) If $\overline{\kappa}(V) = 1$ then there is a morphism $f : V \to B$ such that a general fiber of f is either an elliptic curve or $\mathbf{C}^*$. In this case $P^2 = 0$. The map f is natural. This is quite useful in some considerations. In particular,

if V is affine then f is a $\mathbf{C}^*$-fibration. In [32] Miyanishi has described all the possible singular fibers of a $\mathbf{C}^*$-fibration on a smooth affine surface.

(4) If $\overline{k}(V) = 2$ then the ring $R = \oplus_{n\geq 0} H^0(X, n(K_X + D))$ is finitely generated. Define $V_c = Proj\ R$. V_c is called the **quasi-canonical model** of V.

Now assume that $\overline{\kappa}(V) = 2$.
There is a birational morphism $V \rightarrow V_c$ given by the linear system $|nP|$ for large n.
If Γ_c is the image of D on V_c then the pair (V_c, Γ_c) has so called **log canonical singularities**. This means the following.

(i) $n(K_{V_c} + \Gamma_c)$ is a Cartier divisor for some integer n.

(ii) If $f : Y \to V_c$ is a minimal resolution of singularities then $K_Y + \triangle = f^*(V_c + \Gamma_c) + \sum_{j=1}^{n} a_j E_j$ where $a_j \in Q$ and $-1 \leq a_j \leq 0$.

Here $\triangle$ is the proper transform of Γ_c in Y and $E_1, \ldots, E_n$ are the irreducible exceptional curves for f.

A complete classification of dual graphs of log canonical singularities is possible by Kawamata's work. ([31]).

The Kobayashi - Nakamura -Sakai inequality.

With the above notation, let $LCS(V_c, \Gamma_c)$ be the set of all log canonical singularities of V_c which are not quotient singular points and not contained in *Supp* (Γ_c).

Define $V_{c,0} = V_c - \Gamma_c - LCS(V_c, \Gamma_c)$.
If p is a quotient singularity of V_c let G_p be the local fundamental group of a germ $V_{c,p}$ and $| G_p |$ its order. Then we have

$$0 < (K_{V_c} + \Gamma_c)^2 \leq 3\{\chi_{\text{top}}(V_{c,0}) + \sum_p (\frac{1}{| G_p |} - 1)\},$$

where p ranges over all quotient singular points of $V_{c,0}$. The proof of this is differential geometric. ([26]). A very important consequence of this for our purpose is the following result.

Corollary. Let V be a smooth affine surface with $\overline{k}(V) = 2$. Then $\chi_{\text{top}}(V) > 0$.

Remarks. (1) In the projective case, the analogous result is a famous result of Castelnuovo. Its proof is much simpler than the above theorem.
(2) A.J. Parameswaran and R.V. Gurjar have generalized this to arbitrary smooth surfaces which are connected at infinity, e.g. for all smooth affine surfaces.

The case $\overline{\kappa} = -\infty$.

Here the fundamental result proved by Miyanishi - Sugie - Fujita and by P. Russell is as follows. (see [18]).

Theorem. Let X be a smooth projective surface and D a connected normal crossing divisor on X. Then $\overline{\kappa}(X - D) = -\infty$ iff $X - D$ contains a Zariski - open set isomorphic to $B \times \mathbf{C}$, where B is a curve.

When D is not connected and $\overline{k}(X - D) = -\infty$, we have the following important result due to Miyanishi - Tsunoda ([38]) and Keel - Mckernan ([27]).

Theorem. There is a smooth algebraic surface $\widetilde{V}$ with a dominant morphism $\widetilde{V} \to V$ such that $\widetilde{V}$ contains a cylinder-like open set $B \times \mathbf{C}$, where B is a smooth curve.

The proof of this is quite involved. It should be remarked that in [38] and [27] ideas from Mori theory have been used.

Fujita's work.

In an influential paper [3], T. Fujita studied the topology of non-complete surfaces. Especially important in this paper is his classification of **NC-minimal** affine surfaces with $\overline{\kappa} = 0$ which are **Q**-homology planes (defined below). Fujita listed all possible boundary divisors of a smooth NC-minimal affine surface with $\overline{\kappa} = 0$. A striking consequence of this work is the result that there are no **Z**-homology planes V with $\overline{\kappa}(V) = 0$. A direct proof of this result was given by R.V. Gurjar.

C.P. Ramanujam's characterization of $\mathbf{C^2}$.

Around 1970, C.P. Ramanujam gave a beautiful topological characterization of $\mathbf{C^2}$ as an affine variety. Recall the definition of the fundamental group at infinity, $\pi_1^\infty(V)$, for a normal affine surface V. Let $V \subset X$ be a projective embedding such that X is smooth outside V and $D := X \setminus V$ is a divisor with simple normal crossings. For a suitable neighborhood U of D in X the fundamental group of ∂U is well-defined. It is called the **fundamental group at infinity** of V.

The characterization of $\mathbf{C^2}$ due to C.P. Ramanujam is the following:

Theorem. *A smooth affine surface which is simply-connected at infinity is isomorphic to* $\mathbf{C^2}$.

The proof of this used Mumford's method in his famous paper on topology of normal singular surfaces. This result can be considered as the first major result in the theory of open algebraic surfaces. Ramanujam's method was in turn used by R.V. Gurjar, A.R. Shastri, tammo Tom Dieck, Ted Petrie, and others. Ramanujam also constructed the first example of a smooth contractible surface which is not isomorphic to $\mathbf{C^2}$. This example has $\overline{\kappa} = 2$. Later, all possible $\mathbf{Q}$-homology planes with $\overline{\kappa} = 1$ were constructed as $\mathbf{C}^*$-fibrations over $\mathbf{P}^1$. ([10]). Some of these surfaces were shown to be hypersurfaces in $\mathbf{C^3}$ by tammo Tom Dieck and Ted Petrie. They also constructed some $\mathbf{Q}$-homology planes starting from suitable line arrangements in $\mathbf{P}^2$.

A formula of Suzuki.

M. Suzuki proved an important formula for the Euler-Poincaré characteristic of a smooth affine surface fibered over a curve in ([41]). There is a useful improvement of this by Zaidenberg in ([42]).

Theorem. Let $f : V \to B$ be a surjective morphism from a smooth affine surface onto a smooth curve B such that a general fiber F of f is irreducible. Let $p_1, p_2, \ldots, p_m$ be all the points in B such that f is not C^∞-locally trivial in a neighborhood of p_i. Then

$$\chi_{top}(V) = \chi_{top}(B) \cdot \chi_{top}(F) + \Sigma_1^m(\chi_{top}(F_i) - \chi_{top}(F))$$

where $F_i = f^{-1}(p_i)$. Further, every difference in any bracket on the right hand side of this equality is non-negative. If equality holds for some i then F is isomorphic to either $\mathbf{C}$ or $\mathbf{C}^*$ and $F_{i_{red}}$ is isomorphic to F.

The proof of this uses plurisubharmonic functions. An algebraic geometric proof was given by R.V. Gurjar in [5].

§3. Recent results.

Most of the results we are going to describe deal with Q-homology planes, log del Pezzo surfaces and actions of the additive group $(\mathbf{C}, +)$ on affine varieties. For a more exhaustive list of papers in Affine Algebraic Geometry and an overview of the subject, see ([34]).

A smooth affine surface V is a *Q-homology plane* if $H_i(V; Q) = (0)$ for $i > 0$. A normal projective surface X is called a *log del Pezzo surface* if X has at most quotient singularities and $-K_X$ is ample. We begin with some new results about $Q-$homology planes.

Theorem 1. ([39], [42]).

Let V be a $Q-$homology plane with $\overline{\kappa}(V) = 2$. Then V does not contain any contractible curve.

Roughly speaking, if such a curve C exists then $\overline{\kappa}(V - C) = 2, \chi_{\text{top}}(V - C) = 0$. This contradicts the corollary of Kobayashi-Nakamura-Sakai inequality mentioned earlier.

Theorem 2. Let V be a $Q-$homology plane. If $\overline{\kappa}(V) = 1$ then V contains at least one and at most two contractible curves. ([10], [36]). If $\overline{\kappa}(V) = 0$ then V contains at most two contractible curves. (The actual result in this case is more precise. ([13]).

Theorem 3. Let V be a $\mathbf{Z}$-homology plane with $\overline{\kappa}(V) = 2$. If V has a $\mathbf{C}^{**}-$ fibration then $P^2 < 2$, where P is the nef part in the Zariski - Fujita decomposition of $K_X + D$. ([37]).

Theorem 4. New proofs of Abhyankar - Moh - Suzuki and Lin - Zaidenberg theorems, viz. any irreducible contractible curve $C \subset \mathbf{C^2}$ has the equation $\{Z_1^P = Z_2^q\}$ in some coordinates Z_1, Z_2 on $\mathbf{C^2}$. ([11]).

In the new proof of the Lin-Zaidenberg theorem one uses the observation that $\chi_{top}(\mathbf{C}^2 - C) = 0$ and hence $\overline{\kappa}(\mathbf{C}^2 - C) \leq 1$.

Theorem 5. New Topological proof of Cancellation Theorem for $\mathbf{C^2}$, viz. $V \times \mathbf{C} \approx \mathbf{C^3} \Rightarrow \mathbf{V} \approx \mathbf{C^2}$. ([4])

This proof uses the method of D. Mumford and C.P. Ramanujam of studying the tubular neighborhood of a normal crossing divisor on a smooth surface.

Theorem 6. Let V be a $Q-$homology plane with $\overline{\kappa}(V) = 1$. If $f : V \to V$ is an étale map then f is an isomorphism. ([12]).

In this paper a counter example is found for an analogous result when $\overline{\kappa}(V) = 0$. When $\overline{\kappa}(V) = -\infty$ then except possibly for $V = \mathbf{C^2}$ and two more cases same result is true.

Theorem 7. A $Q-$homology plane is rational. ([40]), ([13]), ([16]).
The proof of this is very long and involved. The proof uses the Kobayashi-Nakamura-Sakai inequality in a very useful way.

Theorem 8. Let $\mathbf{C}^*$ act on $\mathbf{C^3}$ algebraically such that dim $\mathbf{C^3}//\mathbf{C}^* = \mathbf{2}$. Then $\mathbf{C^3}//_{\mathbf{C}^*} \approx \mathbf{C^2}/\mathbf{G}, \mathbf{G}$ a finite group of automorphisms of $\mathbf{C^2}$. ([28]).
The proof of this is quite long and technical and uses the inequality of Kobayashi-Nakamura-Sakai. It is an important step in the proof by Koras - Russell- Kaliman- Makar Limanov of the following important result.
'Any algebraic action of $\mathbf{C}^*$ on $\mathbf{C^3}$ is linearizable'.

Using similar ideas as in the proof of Theorem 8, Koras-Russell proved the following important result about contractible surfaces. ([29]).

Theorem 9. Let V be a normal contractible affine surface such that V has at most quotient singularities. Assume that a resolution of singularities $\widetilde{V}$ of V has $\overline{\kappa}(V) = -\infty$. Then $\overline{\kappa}(V - Sing\ V) = -\infty$.

Using this result R.V. Gurjar proved the following result about 2-dimensional quotients of $\mathbf{C^n}$ modulo reductive algebraic groups. ([6])

Theorem 10. Let G be a reducible algebraic group acting regularly on $\mathbf{C^n}$ such that $\mathbf{C^n//G}$ is 2-dimensional. Then $\mathbf{C^n//G}$ is isomorphic to $\mathbf{C^2/\Gamma}$, where Γ is a finite group of automorphisms of $\mathbf{C^n}$.

Now we turn to log del Pezzo surfaces.

Theorem 11. The fundamental group of the smooth locus of a log del Pezzo surface is finite. ([18]), ([2]).
The proof in ([18]) is long and technical. The proof in ([2]) uses differential geometric methods and is short.

Additive group actions It is a standard result that a regular action of $(\mathbf{C}, +)$ on an affine variety V is equivalent to the existence of a locally nilpotent derivation δ on the coordinate ring of V.
One of the big successes of the use of such an action is the result due to L. Makar-Limanov that the Russell 3-fold $\{x + x^2y + z^2 + t^3 = 0\}$ is not isomorphic to $\mathbf{C^3}$. ([35]). The notion of Makar-Limanov invariant of a normal affine variety arose out of this work. This measures essentially how many different $(\mathbf{C}, +)$ actions exist on a normal affine variety. Of particular interest are normal affine surfaces V which admit two independent $(\mathbf{C}, +)$-actions. Such a surface is called an ML_0-surface. There are several papers dealing with

the structure of these surfaces. For example, it is proved in ([7]) that the fundamental group at infinity for V is finite cyclic. This is equivalent to the statement that V has a normal projective completion X such that the curve $D : X - V$ is a linear chain of smooth rational curves and the intersection form on D has non-zero determinant.
It is proved in ([1]) that if a smooth ML_0 surface V is either a UFD or a complete intersection then V is isomorphic to the hypersurface $\{xy = p(z)\}$, where $p(z)$ has no repeated roots.
There are still some mysteries about the structure of ML_0 surfaces.

Another important result about $(\mathbf{C}, +)$ actions is the following result due to S. Kaliman. ([22]).

Theorem 12. Any $(\mathbf{C}, +)$ action on $\mathbf{C^3}$ without fixed points is conjugate to a translation.

The proof of this result uses algebraic topology in a nice way.

Miscellaneous results.

Theorem 13. Let V be a smooth affine surface with $\chi(V) < 0$. Then there is a morphism $f : V \to B$ such that B is a smooth curve with $\overline{k}(B) = 1$ and a general fiber of f is $\mathbf{C}$. ([14]).
This result generalizes Castelnuovo's characterization of ruled surfaces. We can classify all smooth affine surfaces with $\chi(V) \leq 0$.

Recently, S. Kaliman has proved an important result in connection with a generalization of the Abhyankar-Moh-Suzuki theorem in higher dimension. ([21]).

Theorem 14. Let $f(X, Y, Z)$ be an irreducible polynomial in three variables over $\mathbf{C}$. If $\{f = \lambda\}$ is isomorphic to $\mathbf{C}^2$ for all but finitely many constants $\lambda \in \mathbf{C}$ then there exist polynomials $g(X, Y, Z), h(X, Y, Z)$ such that $\mathbf{C}[X, Y, Z] = \mathbf{C}[f, g, h]$.

Theorem 15. An Affine Mumford Theorem. Let V be a normal affine contractible surface such that $V - Sing\ V$ is simply-connected. Then V is smooth.

The proof of this result, and the next one, depend essentially on the theory of open algebraic surfaces (and some topology) described earlier. ([8]).

Theorem 16. Let V be a normal affine $\mathbf{Z}$-homology plane such that $\overline{\kappa}(V - Sing\ V) = 2$. Then V has at most one singular point and it is a cyclic quotient singularity.

The proof of this result is highly technical and long. It uses almost the whole theory of non-complete algebraic surfaces described earlier. ([9]).

Theorem 17. Recently Alok Maharana classified all smooth $\mathbf{Q}$-homology planes of the form $\{z^n = f(x, y)\}$.
Gurjar-Maharana classified smooth surfaces $S := \{z^n - f(x, y) = 0\}$ such that $\overline{\kappa}(S) \leq 1$.
Gurjar-Shameek Paul classified all factorial affine surfaces V such that $\overline{\kappa}(V - Sing\ V) \leq 1$.
These three results make essential use of the theory of open algebraic surfaces.

Log del Pezzo surfaces.

The study of normal projective surfaces with at most quotient singulaities arises naturally. Considering the log Kodaira dimension of the smooth locus of such a surface and using the classification due to Kawamata mentioned earlier one can get valuable information about these surfaces. Among them, log del Pezzo surfaces play a special role. If X is a log del Pezzo surface then it is easy to see that X is rational. Using Kawamata-Mori theory one can often reduce the study of these surfaces to log del Pezzo surfaces of rank 1. There are important papers about log del Pezzo surfaces by Demazure, Gurjar-Pradeep-Zhang, Hidaka-Watanabe, Miyanishi-Zhang and Zhang. We will not go into any details of the results proved in these papers.

Complements of plane curves.

Let $C \subset \mathbf{P}^2$ be an irreducible curve of degree $d > 1$. It is clear that if C is smooth and $d > 3$ the $\overline{\kappa}(\mathbf{P}^2 \setminus C) = 2$. Hence the classification of C such that $\overline{\kappa}(\mathbf{P}^2 \setminus C) \leq 1$ is a significant problem. There are many papers dealing with this by Iitaka, Kishimoto, Kojima, Sakai-Matsuoka, Tsunoda, Wakabayashi, Yoshihara, and others.

It is easy to construct plane curves of arbitrary genus with maximum (permissible by the genus formula) number of ordinary double points but finding rational curves which have at most unibranch singular points is not an easy task. Here the geometry of $\mathbf{P}^2$ plays an important role. The inequality of Kobayashi-Nakamura-Sakai puts a strong restriction about the number and types of such singular points but the final word on this problem has not been said.

§4. Some Open Problems

(1) Let X be a smooth projective rational surface and $C \subset X$ a smooth irreducible curve. Is $\pi_1(X - C)$ finite?
This is true if $\overline{k}(X - C) \leq 1$. ([19]). If this result has an affirmative answer then we get a striking consequence.
'Let X be as above. If $\varphi : X \to \mathbf{P}^1$ is a morphism with connected fibers, then φ has at most one multiple fiber'.

(2) Let X be a normal projective, rational surface with a unique singular point p. If p is a quotient singular point, is $\pi_1(X - p)$ finite?
This is true if $\overline{k}(X - p) \leq 1$. ([19]).

(3) Let $V := \{X^2 + Y^3 + Z^5 = 0\}$. Is every étale map $V \to V$ an isomorphism?

Miyanishi - Masuda have proved this for all surfaces $\{X^a + Y^b + Z^c = 0\}$ with a, b, c pairwise coprime integers > 1, except for $\{a, b, c\} = \{2, 3, 5\}$. ([30]).

More generally, Miyanishi has raised the question whether any étale self-map of a quotient $\mathbf{C}^2/G$ (G a finite group of automorphisms) is an isomorphism. This problem can be reduced to the case when G is either $\mathbf{Z}/(2)$ or the binary icosahedral group of order 120. The answer is not known for either of these groups.

A tantalising open problem is the following:
Question. Is there a smooth affine surface V with an étale self map whose image misses a non-empty finite subset?

(4) Classify all smooth affine surfaces V with a proper self-map of degree > 1.
Gurjar and Zhang have found the answer if either $\overline{\kappa}(V) \geq 0$, or if $\pi_1(V)$ is infinite.
It is not known if the affine surface $\{x^2 + y^2 + z^2 = 1\}$ has a proper self-morphism of degree > 1.

(5) Let W be a smooth affine 3-fold with a $G_a := (\mathbf{C}, +)$ action, $V := W//G_a$ and $\pi : W \to V$ the quotient morphism. It is not known if every irreducible component of any fiber of π is isomorphic to $\mathbf{A^1}$. This is not known even for $W = \mathbf{C^3}$.
Similarly, are the singularities of V quotient singularities?

(6) Let W be a smooth affine contractible 3-fold which admits three independent locally nilpotent derivations. Is W isomorphic to $\mathbf{C^3}$?

We have listed only few of the interesting results due to lack of time and space. It is clear that this theory will continue to play an important role for a long time.

Similarly, the study of normal singular points of surfaces (particularly rational singularities) is very useful for surface theory. Because of lack of time we have not touched upon this theory.

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