Non Hermitian description of the Quasi-Zeno dynamics of a quantum particle

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J. Phys. A **48**, 115304 (2015),
PRA **91**, 062115 (2015),
arXiv:1708.06496 (2017).

WE-Heraeus Seminar: Search and problem solving by random walks

Outline

- Introduction: Time of arrival, first passage time in quantum mechanics. First detection time.
- Quantum evolution for a system subjected to repeated measurements.
- Connection to non-Hermitian Hamiltonians.
- Example: lattice model of a free quantum particle.
- Conclusions

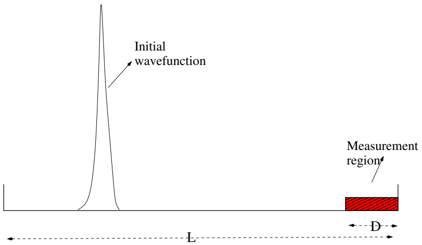
Quantum Quasi-Zeno Dynamics: Transitions mediated by frequent projective measurements near the Zeno regime, Elliott, Vedral, Phys. Rev. A (2016).

RENEWAL APPROACH:
Quantum Renewal Equation for the first detection time of a quantum walk, Friedman, Kessler, Barkai, J.Phys.A (2016).

Quantum walks: the first detected passage time problem, Friedman, Kessler, Barkai, Phys. Rev. E (2017).

First detection of a quantum walker on an infinite line, Thiel, Barkai, Kessler, Phys. Rev. Lett. (2018).

Introduction - The time of arrival of a quantum particle



A quantum particle is released from a confined region at time $t = 0$ and it is allowed to move within a larger box. Its initial wavefunction $\Psi(x, t = 0)$ is localized in space.
A particle detector is placed somewhere inside the bigger box over the region D.

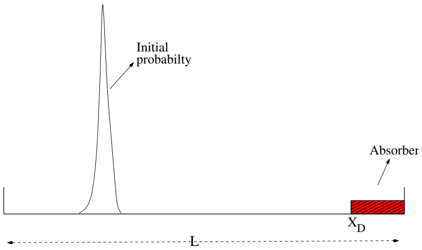
What is the probability $p_t dt$ that the detector clicks, and thus that the particle is detected (for the first time), during the time interval $(t, t + dt)$.

A related question: What is the probability S_t that the particle survives being detected till time t. Clearly $p_t = -dS_t/dt$.

This is the main question that we address. From the point of view of experiments, seems to be a reasonable question to ask.

How do we calculate p_t, S_t using quantum theory ?

Recall: First passage problem for a Brownian particle



A Brownian particle is released from a localized region inside a box at time $t = 0$.
Initial probability distribution $P_0(x)$.
What is the probability it arrives at the point X_D (for the first time) in the time interval $(t, t + dt)$.

Solution: Put an absorbing boundary at X_D.

- Solve diffusion equation $\partial_t P(x,t) = D\partial_x^2 P(x,t)$ with:

  Boundary conditions — $P(X_D, t) = 0$ (absorber at X_D)
  $D\partial_x P(x,t)|_{x=a} = 0$ (box impermeable at $x = a$)

  Initial condition — $P(x, t = 0) = P_0(x)$.

- Survival probabilty: $S(t) = \int_a^{X_D} dx P(x,t) \sim 1/t^{1/2}$ (for $a \to -\infty$).

- First passage probability distribution: $p(t) = -dS/dt = -D\partial_x P(x,t)|_{x=X_D} \sim 1/t^{3/2}$.

Computing first passage in quantum system

For quantum case, we cannot proceed in a similar fashion.

- $P(x,t) = |\Psi(x,t)|^2$. Not clear how to set absorbing boundary condition for $\Psi(x,t)$.
- Need to talk about measurements.
  — Quantum measurements change the state of the system.

For our purpose, we need the following rules from quantum mechanics.

- States described by wave-functions $\Psi(x,t)$.
- Unitary time evolution through the Schrodinger equation $i\partial_t\Psi(x,t) = H\Psi(x,t)$.
- Observables are described by Hermitian operators. Observed values correspond to eigenvalues of operators.
- Measurement postulate — talks about the outcome of instantaneous measurements. Given any observable O, a measurement on a state $\Psi(x,t)$ gives us one of the eigenvalues of O. The measurement postulate gives us
  1. the probability of each outcome
  2. the state of the system after the experiment, for any given outcome — consider SELECTIVE MEASUREMENTS.

Can we compute, using these rules, the probability distribution of the "time of first detection" of a quantum particle ?

First detection under repeated measurements

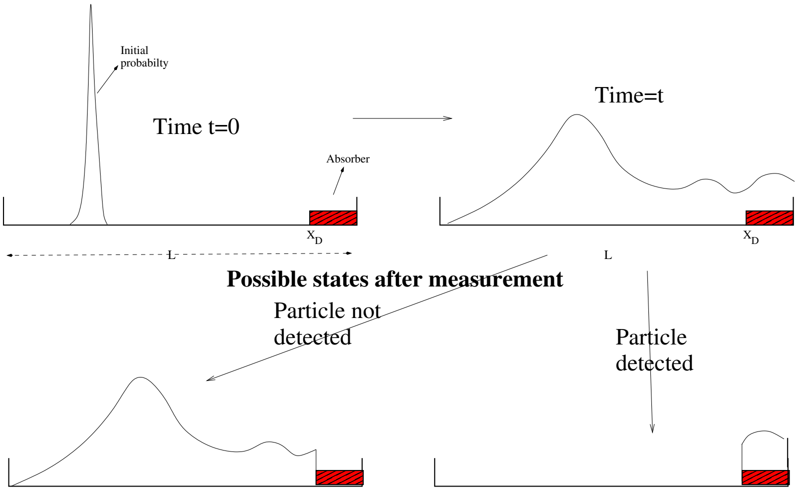
S.Dhar, S.Dasgupta, D. Sen, A. Dhar
[J. Phys. A **48**, 115304 (2015), PRA **91**, 062115 (2015)].

- We consider a simple lattice model for a free quantum particle in a box, which is subjected to regular instantaneous measurements, made to probe whether the particle is at a prescribed site.

- Time intervals between measurements is τ. We ask for the probability that the particle is detected, for the first time, on the n-th measurement, i.e at time $t = n\tau$.

- Main results:
  - Effective dynamics by a non-Hermitian Hamiltonian.
  - For a $1D$ lattice with N sites, there is a time regime $N \lesssim t \lesssim N^2$, where survival probability (for any finite τ) decays as a power law $P(t) \sim 1/t^{\alpha}$.
  - New results: for the time regime $t \lesssim N$.
  - In the limit $\tau \to 0$, the particle is never detected — Zeno's paradox.
    [B. Misra and E. C. G. Sudarshan, J. Math. Phys. 18, 756 (1977).]

Schematic of unitary evolution and projective measurments



Model and method

Lattice with position states labeled as $|r\rangle$.

$$H = \sum_{r,s=1}^{N} H_{r,s} \, |r\rangle\langle s| \; , \quad |\psi(t)\rangle = U_t|\psi(0)\rangle, \quad \text{where} \quad U_t = e^{-iHt} \; .$$

Projection operator $A = \sum_{r \in D} |r\rangle\langle r|$ corresponds to measurements to detect if particle is in domain D. If state is $|\psi\rangle$, then probability of detection is

$$p = \sum_{r \in D} |\langle r|\psi\rangle|^2 = \langle\psi|A|\psi\rangle$$

$B = 1 - A \rightarrow$ coresponds to non-detection of particle in D.

Note: $A + B = 1$ and $AB = 0, \; A^2 = A, B^2 = B$.

Probability of non-detection is $\quad P = \langle\psi|B|\psi\rangle = 1 - p$

Wavefunction immediately after measurement:

$|\psi^+\rangle = A|\psi\rangle$ if particle detected.

$|\psi^+\rangle = B|\psi\rangle$ if particle not detected.
(with appropriate normalizations)

After first measurement, unitarily evolve the state $|\psi^+\rangle = B|\psi\rangle$ until the next measurement.

Repeated measurements

Consider sequence of measurements $n = 1, 2 \ldots$ at intervals of time τ which continue until a particle is detected. *Thus time evolution* $=$ *sequence of unitary evolutions followed by projections onto the subspace corresponding to* B.

$|\psi_n^-\rangle$ — wavefunction immediately befor n^{th} measurement.

$|\psi_n^+\rangle$ — wave function immediately after n^{th} measurement.

$$\text{Clearly} \qquad |\psi_n^-\rangle = U_\tau |\psi_{n-1}^+\rangle, \quad |\psi_n^+\rangle = B|\psi_n^-\rangle, \quad \text{where } \; U_\tau = e^{-iH\tau}.$$

Iterating and defining $\widetilde{U} = BU_\tau$, we get

$$|\psi_n^-\rangle = U_\tau \widetilde{U}^{n-1}|\psi(0)\rangle \quad \text{and} \quad |\psi_n^+\rangle = \widetilde{U}^n|\psi(0)\rangle \; .$$

Survival probability (probability of no detection) after n measurements is

$$S_n = \langle \psi_n^+ | \psi_n^+ \rangle.$$

Proof $\rightarrow$

Repeated measurements

Let S_n be probability of survival after n measurements. Then clearly

$$S_1 = \langle \psi_1^- | B | \psi_1^- \rangle = \langle \psi(0) | U_\tau^\dagger B\, B U_\tau | \psi(0) \rangle = \langle \psi(0) | \widetilde{U}^\dagger \widetilde{U} | \psi(0) \rangle = \langle \psi_1^+ | \psi_1^+ \rangle \ .$$

S_1 is normalizing factor for $|\psi_1^+\rangle$ and also for $|\psi_2^-\rangle$.

Survival probability after second measurement
$=$ (probability of non-detection at $n = 1$) $\times$ (probability of non-detection at $n = 2$) .

$$\text{Hence} \ \ S_2 = S_1 \times \frac{\langle \psi_2^- |}{\sqrt{S_1}} B \frac{|\psi_2^- \rangle}{\sqrt{S_1}} = \langle \psi(0) | \widetilde{U}^{\dagger 2} \widetilde{U}^2 | \psi(0) \rangle = \langle \psi_2^+ | \psi_2^+ \rangle \ .$$

Proceeding iteratively in this way, we get

$$S_n = \langle \psi(0) | \widetilde{U}^{\dagger n} \widetilde{U}^n | \psi(0) \rangle = \langle \psi_n^+ | \psi_n^+ \rangle \ .$$

Need to understand the evolution $|\Psi_n^+\rangle = \widetilde{U}^n |\Psi_0^+\rangle$, where $\widetilde{U} = BU_\tau \equiv Be^{-iH\tau}B \ \rightarrow$

Perturbation theory

Diagonalizing $\widetilde{U} \equiv Be^{-iH\tau}B$ is difficult in general.

For τ small ($\tau << 1/\gamma$), expect small change of wavefunction — so try perturbation theory.

$$H = H_S + H_M + V \ , \quad \text{system + measuring device + interaction}$$

$$\text{where } H_S = \sum_{l,m} H_{l,m}|l\rangle\langle m|, \quad H_M = \sum_{\alpha,\beta} H_{\alpha,\beta}|\alpha\rangle\langle\beta| \qquad V = \sum_{l,\alpha} V_{l,\alpha}|l\rangle\langle\alpha| + V_{\alpha,l}|\alpha\rangle\langle l| \ .$$

l, m - system index. α, β - measuring device index.

Expanding the effective evolution operator $\widetilde{U} = Be^{-iHt}B$ to second order in τ gives

$$\widetilde{U} = B\left[I - iH\tau - \frac{\tau^2}{2}H^2 + \dots \right] B \qquad [\text{Note} : B = \sum_l |l\rangle\langle l|]$$

$$= I - iH_S\tau - \frac{\tau^2}{2}H_S^2 - \frac{\tau^2}{2}\sum_{l,m}\sum_{\alpha} V_{l,\alpha}V_{\alpha,m}|l\rangle\langle m| + \dots$$

$$= e^{-iH_{eff}\tau} + \mathcal{O}(\tau^3),$$

H_{eff} is the effective Hamiltonian controlling the time-evolution.

Effective non-Hermitian Hamiltonian for system

Thus we see that, for small τ, the time evolution of a wave-packet in the box is described by the following non-Hermitian Hamiltonian.

$$\text{where } H_{eff} = H_S + V_{eff}, \quad \text{and} \quad V_{eff} = -\frac{i\tau}{2} \sum_{l,m} \sum_{\alpha} V_{l,\alpha} V_{\alpha,m} |l\rangle\langle m| \ .$$

$V_{l,\alpha}$- Hopping matrix elements between system and device sites.

Connection to another non-Hermitian Hamiltonian

Consider the dynamics of a particle evolving with the following Hamiltonian:

$$H_{NH} = H + \Gamma H' \quad \text{where} \quad H' = -i\gamma \sum_{\alpha=1}^{N_D} |\alpha\rangle\langle\alpha| \ ,$$

and H is the tight-binding Hamiltonian defined earlier.

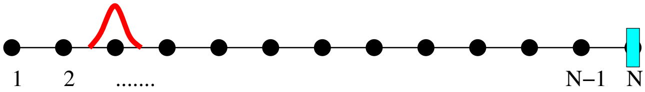
For large Γ one can do a perturbation theory in $1/\Gamma$. One then sees that energy levels of the "system" states are described by the effective Hamiltonian

$$H_{eff} = H_S - \frac{i}{\gamma\Gamma} \sum_{l,m} \sum_{\alpha} V_{l,\alpha} V_{\alpha,m} |l\rangle\langle m| \ .$$

This is identical to our effective Hamiltonian if we make the identification:

$$\frac{\tau}{2} = \frac{1}{\gamma\Gamma} \ .$$

Let us now look at a simple example $\rightarrow$



$$H = -\gamma \sum_{l=1}^{N-1} \left(|l+1\rangle\langle l| + |l\rangle\langle l+1| \right) .$$

$A = |N\rangle\langle N|$ and $B = \sum_{l=1}^{N-1} |l\rangle\langle l|$.

Effective Hamiltonian for the $N-1$ sites system is given by

$$H_{eff} = H_S + V_{eff},$$

$$\text{where } \; H_S = -\sum_{l=1}^{N-2} \left(|l+1\rangle\langle l| + |l\rangle\langle l+1| \right) \; \text{ and } \; V_{eff} = -\frac{i\tau}{2}|N-1\rangle\langle N-1| .$$

Eigenvalues and eigenvectors of H_S (with $N-1$ sites) are given by

$$\epsilon_q = -2\cos\left(\frac{q\pi}{N}\right), \; \phi_q(l) = \sqrt{\frac{2}{N}} \sin\left(\frac{ql\pi}{N}\right) \;, \; \text{q=1,2,...,N-1}.$$

Treat V_{eff} as a perturbation to find the eigenstates, eigenvalues of H_{eff}.

Free particle in a box

First order perturbation theory gives for the eigenvalues of H_{eff}

$$\mu_q = \epsilon_q + \langle \phi_q | V_{eff} | \phi_q \rangle = \epsilon_q - \frac{i}{2}\alpha_q, \quad \text{with} \;\; \alpha_q = \frac{2\tau}{N} \sin^2\left(\frac{q\pi}{N}\right) \; .$$

Hence, eigenstates of H_S decay exponentially with time.

After time $t = n\tau$, the state of the system is given by

$$|\phi_q(t)\rangle = e^{-iH_{eff}t}|\phi_q\rangle = e^{-\alpha_q t/2} e^{-i\epsilon_q t}|\phi_q\rangle.$$

Survival probability $S_q(t)$ of initial energy eigenstates is

$$S_q(t) = \langle \phi_q(t) | \phi_q(t) \rangle = e^{-\alpha_q t} \; ,$$

The decay rate α_q depends on τ and vanishes in the limit $\tau \to 0$.
Thus, if we make too frequent measurements, the particle is not able to evolve into the domain D and so — is never detected!!

This is the *quantum Zeno effect*.

Particle in a box: Survival probability of initially localized state

Now Consider case when initial state is a position eigenstate $|\psi(t=0)\rangle = |\ell\rangle$.
Time evolution is given by

$$|\psi(t)\rangle = e^{-iH_{eff}t}|\ell\rangle = \sum_q \phi_q(\ell) e^{-\alpha_q t/2} e^{-i\epsilon_q t}|\phi_q\rangle \ ,$$

so that the survival probability becomes

$$S_\ell(t) = \langle\psi(t)|\psi(t)\rangle = \sum_{q=1}^{N} \frac{2}{N} \sin^2(\frac{q\pi\ell}{N}) e^{-\frac{2\tau t}{N}\sin^2(\frac{q\pi}{N})} \ .$$

For large N, in the time window where $t\tau/N$ is large but $t\tau/N^3$ is small, it can be shown that the survival probability is given by

$$S_\ell(t) = \frac{1}{\sqrt{8\pi x}}\left[1 - e^{-\ell^2/2x}\right], \quad \text{where} \quad x = \frac{t\tau}{N}.$$

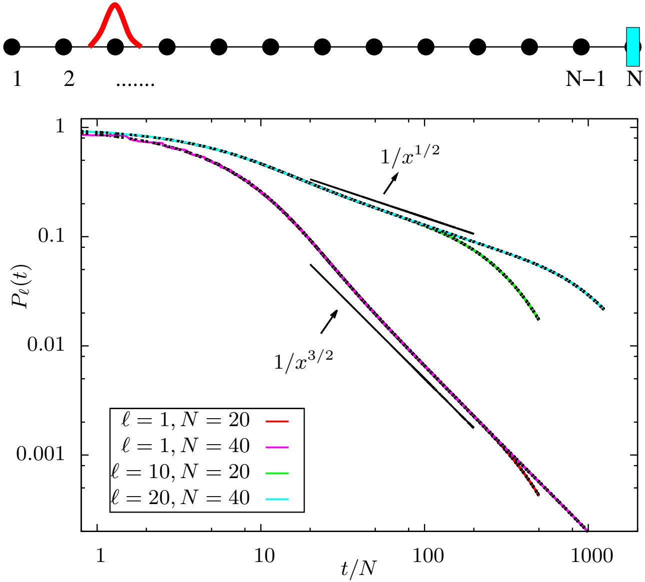
For initial condition close to boundary — $S_t \sim 1/t^{3/2}$ at large t.

For initial condition within the bulk — $S_t \sim 1/t^{1/2}$.

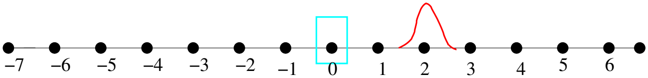
After times $t \gtrsim N^3$ there is an exponential decay with time.

Comparision between exact results and those obtained from perturbation theory →

Comparision between analytic and exact numerical results



Detector at origin of an infinite lattice



- Studied by H. Friedman, D. Kessler, F. Thiel, E. Barkai — Exact results from renewal-type approach.

  (a)Particle has finite probability of survival at infinite times (non-recurrenunlike 1D random walk).

  (b) Given that there is detection, the probability of detection decays as

  $$p_n \sim \frac{4\tau}{\pi n^3} \cos^2\left(2\gamma\tau n + \frac{\pi}{4}\right). \qquad \text{Random Walk : } p_n \sim \frac{1}{n^{3/2}}$$

  for initial condition $a = 0$.

Some questions we ask (Lahiri and Dhar [arXiv:1708.06496]):

- Can these results be obtained from the non-Hermitian Hamiltonian models?
- Why is the decay $p_n \sim 1/n^3$ of detection probability different from the $\sim 1/n^{5/2}$ form seen in our earlier set-up?
  — This can be understood as a finite-size effect.

Detector at centre of a finite lattice - the corresponding Non-Hermitian Hamiltonian models

Mapping-I

$$H_{eff}^{(1)} = H_S^{(1)} + V_{eff}^{(1)};$$

$$H_S^{(1)} = -\sum_{x=-L}^{-2} (|x\rangle\langle x+1| + |x+1\rangle\langle x|) - \sum_{x=1}^{L-1} (|x\rangle\langle x+1| + |x+1\rangle\langle x|)\,;$$

$$V_{eff}^{(1)} = -\frac{i\tau}{2}\,(|1\rangle\langle 1| + |-1\rangle\langle -1| + |1\rangle\langle -1| + |-1\rangle\langle 1|)\,.$$

Mapping-II

$$H_{eff}^{(2)} = H_S^{(2)} + V_{eff}^{(2)};$$

$$H_S^{(2)} = -\sum_{x=-L}^{L-1} (|x\rangle\langle x+1| + |x+1\rangle\langle x|);$$

$$V_{eff}^{(2)} = -\frac{2i}{\tau}|0\rangle\langle 0|.$$

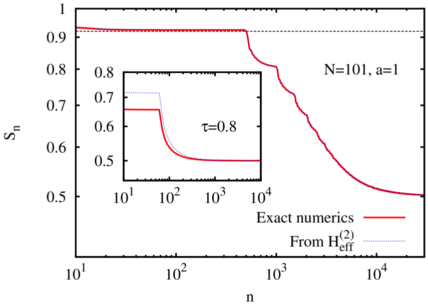
Schrödinger equation: $i\partial\psi_x t = -\psi_{x+1}(t) - \psi_{x-1}(t) - (2i/\tau)\,\delta_{x0}\,\psi_0(t)$.
—Studied by Luck, Krapivsky, Mallick (JSP, 2014).

Accuracy of the non-Hermitian Hamiltonian descriptions.

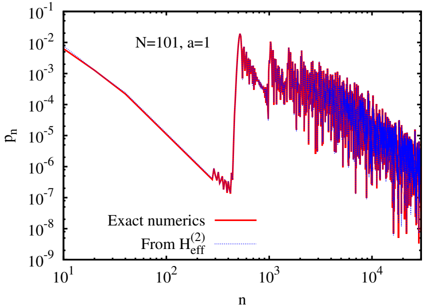
- Here I discuss only the second mapping, for which exact results can be obtained.

- Compare predictions of $H^{(2)}_{eff}$ with exact numerics ($\gamma\tau = 0.1$).

Survival probability



First detection probability

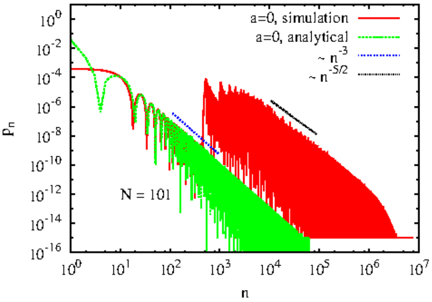


- The agreement is very good — at all times
- Three time scales
- Saturation of $S(\infty)$ on a finite lattice (eigenstates with vanishing amplitude at the detector site do not decay— continue to be eigenstates of H_{eff}).

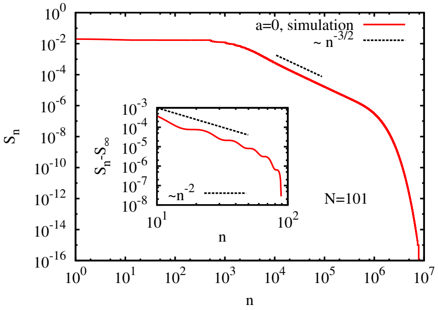
First passage on infinite lattice

Comparison with infinite lattice result ($a = 0$ case).

First detection probability



Survival probability



- Three time scales —
  - $t \lesssim N$ — ballistic time scale.
  - $N \lesssim t \lesssim N^3$ — Smallest Im[eigenvalue]$\sim N^{-3}$.
  - $N^3 \lesssim t$ — Exponential decay regime.

Other results

- From Krapivsky, Luck, Mallick results, we can get first detection probability

$$p^{(a)}(t) = 2\frac{a^2}{\Gamma}\frac{J_a^2(2t)}{t^2} \sim \left(\frac{\tau a^2}{\pi}\right)\frac{\cos^2(2t - a\pi/2 - \pi/4)}{t^3}$$

  Exact result for survival probability $S(t \to \infty)$.

  — Very good agreement with exact numerics and exact results of Barkai et al

- First return in the Aubry-Andre-Harper model — ask what happens in a system where the free unitary evolution is non-ballistic.

Summary

- An attempt to find the time of arrival of a quantum particle into a specified region. Make repeated instantaneous measurements to detect presence of particle in specified region.
- Non-unitary evolutions can be effectively described by two different non-Hermitian Hamiltonians.
- Study of particle moving on a 1D lattice with one detector site. Surprising degree of agreement between perturbation theory, exact numerics and exact results from renewal approach (Barkai et al).
- Interesting finite size effects.
- Zeno effect for continuous measurements: particle never detected. For any finite measurement time interval, survival probability has interesting features — such as power-law tails.

Other things

- Experiments: Cold atoms released from a trap.
- Weak measurements — talk by S. Dasgupta