Optimal Control and Stabilization of Wake Flows

Bartosz Protas

Department of Mathematics & Statistics
McMaster University
Hamilton, ON, Canada
URL: `http://www.math.mcmaster.ca/bprotas`

Collaborators: A. Elcrat, D. Pelinovsky

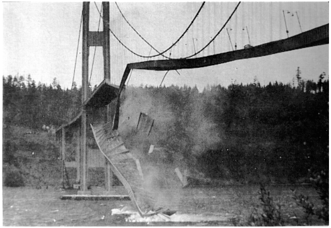
Funded by NSERC–Discovery (Canada) and CNRS (France)

December 12, 2012

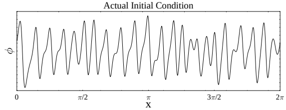
Motivation — Applications of Flow Control

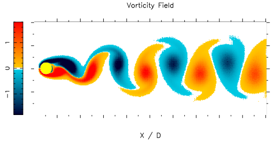
- Wake Hazard

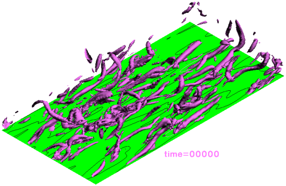
- Fluid–Structure Interaction



Model Problems







- Objectives:
  - Control fluid flow with the least amount of energy possible
  - Estimate flow based on incomplete and/or noisy measurements
- The Navier–Stokes system

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} + \boldsymbol{\nabla} p - \mu\Delta\mathbf{v} = \phi, & \text{in } \Omega\times(0,T) \\ \boldsymbol{\nabla}\cdot\mathbf{v} = 0, & \text{in } \Omega\times(0,T) \\ \text{Initial condition} & \text{on } \Gamma\times(0,T) \\ \text{Boundary condition} & \text{in } \Omega \text{ at } t=0 \end{cases}$$

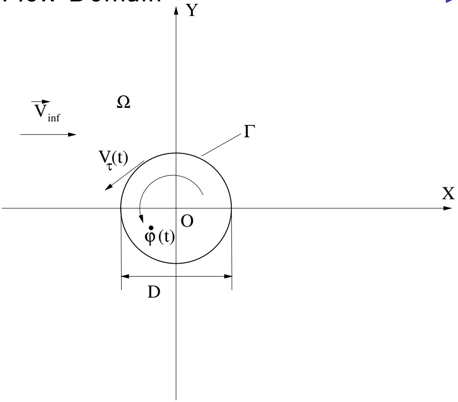
- Inverse problems

PART I

Optimal Open–Loop Control via Adjoint–Based Optimization

Statement of the Problem I

- Flow Domain



- Assumptions:
  - viscous, incompressible flow
  - plane, infinite domain
  - $Re = 150$

Statement of the Problem II

- Find $\dot{\varphi}_{opt} = \operatorname{argmin}_{\dot{\varphi}} \mathcal{J}(\dot{\varphi})$, where

$$\mathcal{J}(\dot{\varphi}) = \frac{1}{2}\int_0^T \left\{ \left[\begin{array}{c}\text{power related to}\\ \text{the drag force}\end{array} \right] + \left[\begin{array}{c}\text{power needed to}\\ \text{control the flow}\end{array} \right] \right\} dt$$
$$= \frac{1}{2}\int_0^T \oint_{\Gamma_0} \{[p(\dot{\varphi})\mathbf{n} - \mu\mathbf{n}\cdot\mathbf{D}(\mathbf{v}(\dot{\varphi}))]\cdot[\dot{\varphi}\,(\mathbf{e}_z \times \mathbf{r}) + \mathbf{v}_\infty]\}\, d\sigma\, dt$$

- Subject to:

$$\begin{cases} \left[\begin{array}{c} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mu\Delta\mathbf{v} + \nabla p \\ \nabla\cdot\mathbf{v} \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \end{array} \right] & \text{in } \Omega\times(0,T), \\ \mathbf{v} = 0 & \text{at } t = 0, \\ \mathbf{v} = \dot{\varphi}_{opt}\tau & \text{on } \Gamma \end{cases}$$

Abstract Framework I

- Constrained optimization problem

$$\begin{cases} \min_{(x,\varphi)} \tilde{\mathcal{J}}(x,\varphi) \\ S(x(\varphi),\varphi) = 0 \end{cases}$$

- Equivalent UNCONSTRAINED optimization problem (note that $x = x(\varphi)$)

$$\min_{\varphi} \tilde{\mathcal{J}}(x(\varphi),\varphi) = \min_{\varphi} \mathcal{J}(\varphi)$$

- First–Order OPTIMALITY CONDITIONS ($\mathcal{U}$ - Hilbert space of controls)

$$\forall_{\varphi' \in \mathcal{U}} \quad \mathcal{J}'(\varphi;\varphi') = \left(\boldsymbol{\nabla}\mathcal{J}, \varphi'\right)_{\mathcal{U}} = 0,$$

with the GÂTEAUX DIFFERENTIAL

$$\mathcal{J}'(\varphi;\varphi') = \lim_{\epsilon\to 0} \frac{1}{\epsilon}[\mathcal{J}(\varphi + \epsilon\varphi') - \mathcal{J}(\varphi)].$$

Abstract Framework II

- Minimization of $\mathcal{J}(\varphi)$ with a DESCENT ALGORITHM in $\mathcal{U}$
  $\Longrightarrow$ solution to a STEADY STATE of the ODE in $\mathcal{U}$

$$\begin{cases} \dfrac{d\varphi}{d\tau} = -\mathcal{Q}\nabla_{\varphi}\mathcal{J}(\varphi) & \text{on } \tau \in (0,\infty) \ \ \text{(pseudo–time)}, \\ \varphi = \varphi_0 & \text{at } \tau = 0. \end{cases}$$

- Typically well–behaved (quadratic) cost functionals
- Typically ill–behaved constraints: THE NAVIER–STOKES SYSTEM
  - nonlinear, nonlocal, multiscale, evolutionary PDE,
- Dimensions:
  - state: 10^6 — 10^7 DoF $\times$ 10^2 — 10^3 time levels
  - control: 10^4 — 10^5 DoF $\times$ 10^2 — 10^3 time levels
- No hope of using "matrix" formulation ...
- Formulation equivalent to Lagrange Multipliers

Differential of the Cost Functional

- The cost functional:

$$\mathcal{J}(\dot\varphi) = \frac{1}{2}\int_0^T \left\{ \left[\begin{array}{c} \text{power related to} \\ \text{the drag force} \end{array} \right] + \left[\begin{array}{c} \text{power needed to} \\ \text{control the flow} \end{array} \right] \right\} dt$$
$$= \frac{1}{2}\int_0^T \oint_{\Gamma_0} \{[p(\dot\varphi)\mathbf{n} - \mu\mathbf{n}\cdot\mathbf{D}(\mathbf{v}(\dot\varphi))]\cdot[\dot\varphi\,(\mathbf{e}_z\times\mathbf{r}) + \mathbf{v}_\infty]\}\, d\sigma\, dt,$$

- Expression for the Gâteaux differential:

$$\mathcal{J}'(\dot\varphi; h) = \frac{1}{2}\int_0^T \oint_{\Gamma_0} \Big\{ [p'(h)\mathbf{n} - \mu\mathbf{n}\cdot\mathbf{D}\,(\mathbf{v}'(h))]\cdot[\dot\varphi\,(\mathbf{e}_z\times\mathbf{r}) + \mathbf{v}_\infty] +$$
$$[p(\dot\varphi)\mathbf{n} - \mu\mathbf{n}\cdot\mathbf{D}(\mathbf{v}(\dot\varphi))]\cdot(\mathbf{e}_z\times\mathbf{r})\,h \Big\}\, d\sigma\, dt = B_1$$
$$= (\nabla\mathcal{J}(t), h)_{L_2([0,T])}$$

The fields $\{\mathbf{v}'(h), p'(h)\}$ solve the linearized perturbation system.

- How to calculate the GRADIENT $\nabla\mathcal{J}$?

Sensitivities and Adjoint States

- The linearized perturbation system

$$\begin{cases} \mathcal{N}\begin{bmatrix} \mathbf{v}' \\ p' \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}' + (\mathbf{v}'\cdot\boldsymbol{\nabla})\mathbf{v} - \mu\Delta\mathbf{v}' + \boldsymbol{\nabla} p' \\ -\boldsymbol{\nabla}\cdot\mathbf{v}' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{in } \Omega\times(0,T), \\ \mathbf{v}' = 0 & \text{at } t=0, \\ \mathbf{v}' = h\tau & \text{on } \Gamma\times(0,T) \end{cases}$$

- Duality pairing defining the adjoint operator

$$\left\langle \mathcal{N}\begin{bmatrix} \mathbf{v}' \\ p' \end{bmatrix}, \begin{bmatrix} \mathbf{v}^* \\ p^* \end{bmatrix} \right\rangle_{L_2(0,T;L_2(\Omega))} = \left\langle \begin{bmatrix} \mathbf{v}' \\ p' \end{bmatrix}, \mathcal{N}^*\begin{bmatrix} \mathbf{v}^* \\ p^* \end{bmatrix} \right\rangle_{L_2(0,T;L_2(\Omega))} + B_1 + B_2$$

- The adjoint system (TERMINAL VALUE PROBLEM !!)

$$\begin{cases} \mathcal{N}^*\begin{bmatrix} \mathbf{v}^* \\ p^* \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathbf{v}^*}{\partial t} - \mathbf{v}\cdot\left[\boldsymbol{\nabla}\mathbf{v}^* + (\boldsymbol{\nabla}\mathbf{v}^*)^T\right] - \mu\Delta\mathbf{v}^* + \boldsymbol{\nabla} p^* \\ -\boldsymbol{\nabla}\cdot\mathbf{v}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{in } \Omega\times(0,T), \\ \mathbf{v}^* = 0 & \text{at } t=T, \\ \mathbf{v}^* = \mathbf{r}\times(\dot{\varphi}\mathbf{e}_z) + \mathbf{v}_\infty & \text{on } \Gamma\times(0,T) \end{cases}$$

Cost Functional Gradient

- The ADJOINT STATE and DUALITY PAIRING can now be used to re–express the cost functional differential as:

$$\mathcal{J}'(\dot{\varphi}; h) = \frac{1}{2}\int_0^T \oint_\Gamma \{\mu R\mathbf{n}\cdot\mathbf{D}(\mathbf{v}^*)\cdot\tau + \mu\mathbf{n}\cdot\mathbf{D}(\mathbf{v}(\dot{\varphi}))\cdot(\mathbf{e}_z\times\mathbf{r})\}\; h\, d\sigma\, dt$$

- Identification of the COST FUNCTIONAL GRADIENT

$$\mathcal{J}'(\dot{\varphi}; h) = (\nabla\mathcal{J}(t), h)_{L_2([0,T])} = \int_0^T \nabla\mathcal{J}(t)\, h\, dt$$

$$\boxed{\nabla\mathcal{J}(t) = \frac{1}{2}\oint_\Gamma \{\mu R\mathbf{n}\cdot\mathbf{D}(\mathbf{v}^*)\cdot\tau + \mu\mathbf{n}\cdot\mathbf{D}(\mathbf{v}(\dot{\varphi}))\cdot(\mathbf{e}_z\times\mathbf{r})\}\; d\sigma}$$

Optimality (KKT) system

- Complete optimality system for $\dot\varphi_{opt}$, $[\mathbf{v}_{opt}, p_{opt}]$, and $[\mathbf{v}^*, p^*]$

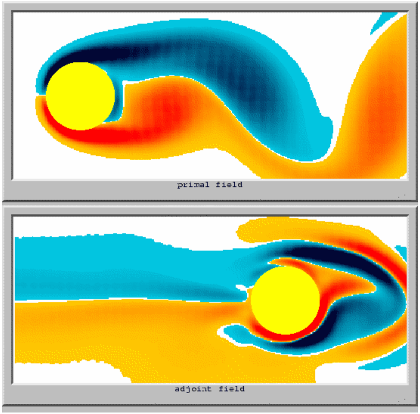
$$
\begin{cases}
\frac{1}{2}\oint_\Gamma \{\mu R\mathbf{n}\cdot\mathbf{D}(\mathbf{v}^*)\cdot\tau + \mu\mathbf{n}\cdot\mathbf{D}(\mathbf{v}(\dot\varphi_{opt}))\cdot(\mathbf{e}_z\times\mathbf{r})\}\,d\sigma = 0 \\
\begin{cases}
\begin{bmatrix} \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mu\Delta\mathbf{v} + \nabla p \\ \nabla\cdot\mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{in } \Omega\times(0,T), \\
\mathbf{v} = 0 & \text{at } t=0, \\
\mathbf{v} = \dot\varphi_{opt}\tau & \text{on } \Gamma
\end{cases} \\
\begin{cases}
\mathcal{N}^*\begin{bmatrix} \mathbf{v}^* \\ p^* \end{bmatrix} = \begin{bmatrix} -\frac{\partial\mathbf{v}^*}{\partial t} - \mathbf{v}\cdot\left[\nabla\mathbf{v}^* + (\nabla\mathbf{v}^*)^T\right] - \mu\Delta\mathbf{v}^* + \nabla p^* \\ -\nabla\cdot\mathbf{v}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{in } \Omega\times(0,T), \\
\mathbf{v}^* = 0 & \text{at } t=T, \\
\mathbf{v}^* = \mathbf{r}\times(\dot\varphi_{opt}\mathbf{e}_z) + \mathbf{v}_\infty & \text{on } \Gamma
\end{cases}
\end{cases}
$$

- A counterpart of the Euler–Lagrange equation
- Solved with an iterative Gradient Algorithm (e.g., Conjugate Gradients, quasi–Newton, etc.)

An Iterative Optimization Procedure

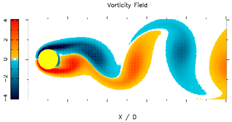
0. provide initial guess $\dot{\varphi}^0$
1. Solve for $\left\{\mathbf{v}(\dot{\varphi}^i); p(\dot{\varphi}^i)\right\}$ on $[0, T]$
2. Solve for $\left\{\mathbf{v}^*(\dot{\varphi}^i); p^*(\dot{\varphi}^i)\right\}$ on $[0, T]$
3. Use $\left\{\mathbf{v}(\dot{\varphi}^i); p(\dot{\varphi}^i)\right\}$ and $\left\{\mathbf{v}^*(\dot{\varphi}^i); p^*(\dot{\varphi}^i)\right\}$ to compute $\nabla \mathcal{J}^i(t)$ on $[0, T]$
4. update control according to $\dot{\varphi}^{i+1}(t) = \dot{\varphi}^i(t) - \alpha_i \gamma_i \left(\nabla \mathcal{J}(t)\right)$
5. iterate 1. through 4. until convergence, i.e. until $\boldsymbol{\nabla} J^i\,(t) \simeq 0$

Primal and Adjoint Simulations for Cylinder Rotation as Control

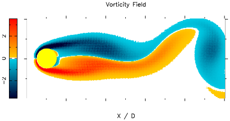


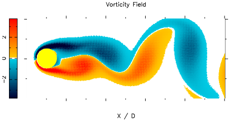
Results

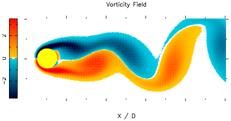
- No Control



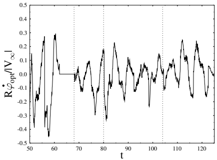
- Flow Pattern Modifications due to Control ($T = 6$)

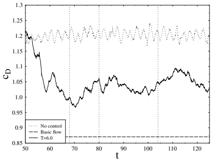


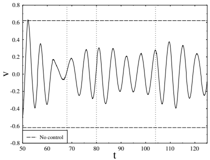




- Optimal Control $\dot{\varphi}_{opt}$, drag coefficient c_D, transverse velocity v



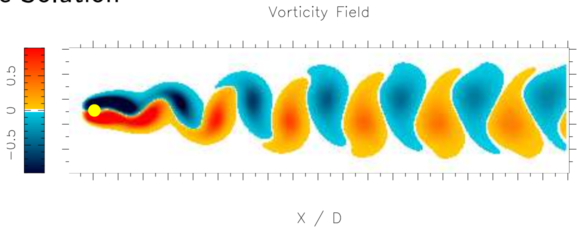




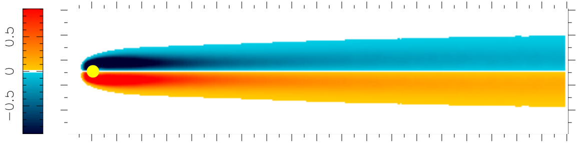
PART II
Feedback Stabilization of Laminar Wake Flows

Navier–Stokes Equation, $Re = 75$

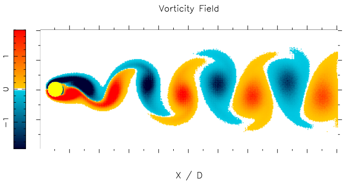
- Stable Solution

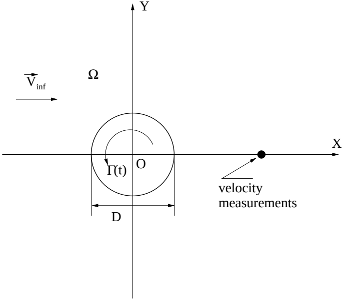


- Unstable Solution



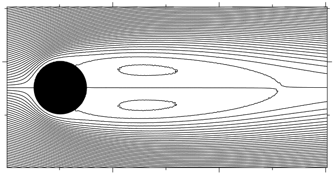
Wake Stabilization Problem





- Objectives — Given:
  - cylinder rotation $\Gamma(t)$ as FLOW ACTUATION ,
  - pointwise velocity measurements $[u(x_m), v(x_m)]$ as SYSTEM OUTPUT ,
- Determine the OPTIMAL STABILIZING FEEDBACK CONTROL LAW
- Assumptions:
  - incompressible flow
  - plane, infinite domain
  - $Re = 75$

- Steady symmetric solution as the target flow



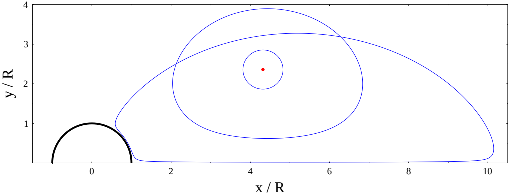
- For simplicity, let us focus on EULER FLOWS
- Steady state Euler equations in 2D:

$$\nabla \times [(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0] \quad \Rightarrow \quad \frac{d\omega}{dt} = 0 \quad \Rightarrow \quad \begin{cases} \Delta\psi = f(\psi) & \text{in} \;\; \Omega, \\ \psi = 0 & \text{on} \;\; \partial\Omega, \\ \psi \to U_\infty y & \text{for} \;\; |(x,y)| \to \infty \end{cases}$$

Arbitrariness of $f(\Psi)$ reflects NONUNIQUENESS of solutions

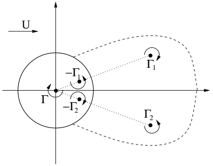
- Family of SADOVSKII FLOW solutions (Elcrat et al., JFM 409) Constant–vorticity vortex (parametrized by α) embedded in irrotational flow

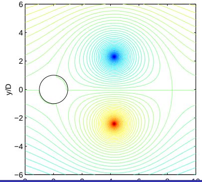
$$f(\psi) = \begin{cases} -\omega, & \psi \le \alpha, \\ 0, & \psi > \alpha, \end{cases}$$



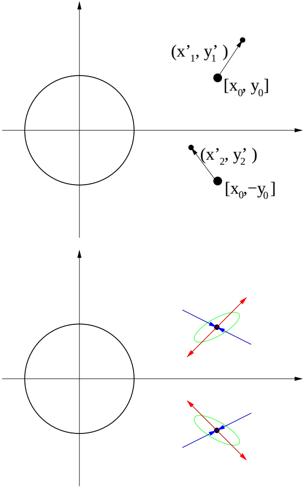
- CONTINUOUS families of solutions parametrized by $\Gamma = \iint_A \omega \, dA$
- For $\alpha \to -\infty$ (or, $|\omega| \to \infty$), FÖPPL'S (1913) POTENTIAL–FLOW SOLUTION

$$\begin{cases} (r^2 - R^2)^2 = 4r^2 y^2, \\ \Gamma = 2\pi \dfrac{(r^2 - R^2)^2 (r^2 + R^2)}{r^5} \end{cases}$$





Linearized Föppl Model — Open–Loop Stability



- Linearization with respect to the perturbation variables $\mathbf{X}'$

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_{10} \\ y_{10} \\ x_{20} \\ y_{20} \end{bmatrix} + \epsilon \begin{bmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \end{bmatrix}$$

- The linear system:

$$\frac{d}{dt}\mathbf{X}' = \mathcal{A}\mathbf{X}', \text{ where } \mathcal{A} = \left.\frac{D\mathcal{F}}{D\mathbf{X}}\right|_0$$

- Eigenvalues:

$$\lambda_1 = \lambda > 0 \qquad \lambda_3 = i\,\lambda_{Im}$$
$$\lambda_2 = -\lambda < 0 \qquad \lambda_4 = -i\,\lambda_{Im}$$

Linearized Föppl Model & Vortex Shedding

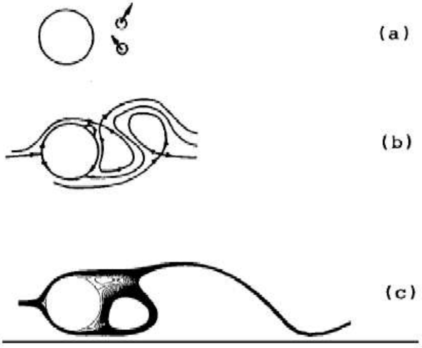


FIG. 13. Similarity between (a) the unstable eigenmode $\mathbf{V}_1$ of the low-dimensional model, (b) the sketch plotted by Perry *et al.*,[10] and (c) the instantaneous streamlines obtained in this paper by direct numerical simulation at Reynolds number Re=100 at time $t=255$.

S. Tang & N. Aubry, *Physics of Fluids* **9**, 2550-2561, (1997)

Controllability & Observability

- CONTROLLABILITY — starting from an arbitrary initial state, can the control drive the state to zero?

$$\text{rank}\left[\mathcal{B}\ \ \mathcal{A}\mathcal{B}\ \ \mathcal{A}^2\mathcal{B}\ \ \mathcal{A}^3\mathcal{B}\right] = 2 \neq 4 \text{ (not controllable!)}$$

- OBSERVABILITY — starting from an arbitrary initial guess, can one reconstruct the state of the system based on available measurements?

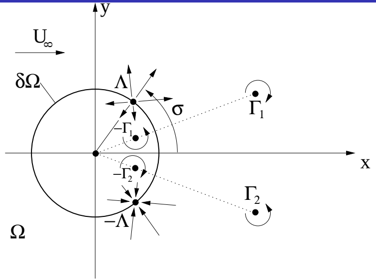
$$\text{rank}\left[\mathcal{C}^T\ \ \mathcal{A}^T\mathcal{C}^T\ \ (\mathcal{A}^T)^2\mathcal{C}^T\ \ (\mathcal{A}^T)^3\mathcal{C}^T\right] = 4 \text{ (fully observable)}$$

- MINIMAL REPRESENTATION — the smallest subsystem that is both CONTROLLABLE and OBSERVABLE $\left(x_{c/o} = \frac{x_1 - x_2}{2},\ y_{c/o} = \frac{y_1 + y_2}{2}\right)$

$$\frac{d}{dt}\underbrace{\mathcal{T}_{c/o}\mathbf{X}'}_{\begin{bmatrix}\mathbf{X}'_{c/o}\\ \mathbf{X}'_2\end{bmatrix}} = \underbrace{\mathcal{T}_{c/o}\mathcal{A}\mathcal{T}_{c/o}^T}_{\begin{bmatrix}\mathcal{A}_{c/o} & 0\\ 0 & \mathcal{A}_{22}\end{bmatrix}}\ \mathbf{X}' + \underbrace{\mathcal{T}_{c/o}\mathcal{B}}_{\begin{bmatrix}\mathcal{B}'_{c/o}\\ 0\end{bmatrix}}\ u + \underbrace{\mathcal{T}_{c/o}\mathcal{G}}_{\begin{bmatrix}\mathcal{G}'_{c/o}\\ \mathcal{G}_2\end{bmatrix}}\ w, \text{ where } \mathcal{T}_{c/o} = \begin{bmatrix}1/2 & 0 & -1/2 & 0\\ 0 & 1/2 & 0 & 1/2\\ 1/2 & 0 & 1/2 & 0\\ 0 & 1/2 & 0 & -1/2\end{bmatrix}$$

Singular Solutions — a Paradigm for Studying Controllability & Observability

- Wall Blowing & Suction



- Actuation modeled as a Sink & Source pair with the induction

$$V_\Lambda(z) = \frac{\Lambda}{2\pi}\left(\frac{1}{z - e^{i\sigma}} - \frac{1}{z - e^{-i\sigma}}\right),$$

- The linearized systems is completely controllable

The Control (Stabilization) Problem

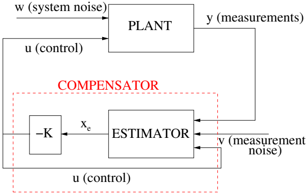
- Find a stabilizing feedback control $u = -\mathcal{K}\,\mathbf{X}'$, such that

$$\mathcal{J}(u) = E\left[\int_0^\infty (\mathbf{y}^*\mathbf{Q}\mathbf{y} + u\mathcal{R}u)\,dt\right] = \min, \quad \text{with} \quad \begin{cases} \dfrac{d}{dt}\mathbf{X}' = \mathcal{A}\,\mathbf{X}' + \mathcal{B}\,u + \mathcal{G}\,w \\ \mathbf{y} = \mathcal{C}\,\mathbf{X}' + \mathcal{D}\,u + \mathcal{H}\,w + \mathbf{v} \end{cases}$$

$$\mathbf{Q} = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}, \ \mathcal{R} > 0, \ w \text{ — system noise (uncertainty)}, \ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ — measurement noise}$$

- System noise w may reflect the effect of (small) nonlinearities
- Linear–Quadratic–Gaussian (LQG) compensation approach

SEPARATION PRINCIPLE — independent solution of the control and estimation problems



Controller — Linear–Quadratic Regulator (LQR)

A STABILIZING FEEDBACK CONTROL which minimizes the cost functional

$$\mathcal{J}(u) = E\left[\int_0^\infty (\mathbf{y}^*\mathbf{Q}\mathbf{y} + u\mathcal{R}u)\,dt\right]$$

is given by

$$u = \mathcal{K}\,\mathbf{X}' = \mathcal{R}^{-1}\mathcal{B}^*\mathcal{S}\,\mathbf{X}',$$

where $\mathcal{S}$ is a symmetric and positive–definite solution to the ALGEBRAIC RICCATI EQUATION

$$\mathcal{S}\,\mathcal{A} + \mathcal{A}^T\mathcal{S} - \mathcal{S}\,\mathcal{B}\,\mathcal{R}^{-1}\,\mathcal{B}^T\,\mathcal{S} + \mathbf{Q} = 0$$

The CLOSED–LOOP system is thus:

$$\begin{cases} \dfrac{d}{dt}\mathbf{X}' = (\mathcal{A} - \mathcal{B}\mathcal{K})\,\mathbf{X}' + \mathcal{G}\,w \\ \mathbf{y} = (\mathcal{C} - \mathcal{D}\mathcal{K})\,\mathbf{X}' + \mathcal{H}\,w + \mathbf{v} \end{cases}$$

Estimator — Kalman Filter

The OPTIMAL estimate $\mathbf{X}'_e$ of the state $\mathbf{X}'$ which minimizes THE ERROR COVARIANCE

$$E\left[\int_0^\infty (\mathbf{X}' - \mathbf{X}'_e)^T(\mathbf{X}' - \mathbf{X}'_e)\,dt\right]$$

is given by solutions of the following ESTIMATOR SYSTEM :

$$\begin{cases} \frac{d}{dt}\mathbf{X}'_e = (\mathcal{A} - \mathcal{L}\mathcal{C})\,\mathbf{X}'_e + (\mathcal{B} - \mathcal{L}\mathcal{D})u - \mathcal{L}\mathbf{y} \\ \mathbf{y}_e = \mathcal{C}\,\mathbf{X}'_e + \mathcal{D}\,u, \end{cases}$$

where $\mathcal{L}$ is a symmetric and positive–definite solution to the ALGEBRAIC RICCATI EQUATION

$$\mathcal{A}\,\mathcal{L} + \mathcal{L}\,\mathcal{A}^T - \mathcal{L}\,\mathcal{C}^T\,\mathcal{R}^{-1}\,\mathcal{C}\,\mathcal{L} + \mathcal{G}\,\mathbf{Q}\,\mathcal{G}^T = 0$$

Center Manifold in the Closed–Loop Nonlinear System

- The nonlinear Föppl system with closed–loop control

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{bmatrix} = \begin{bmatrix}\mathcal{A}_{11} & 0\\ 0 & \mathcal{A}_{22} - \mathcal{B}\mathcal{K}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{bmatrix} + \begin{bmatrix}\mathbf{g}_1(\boldsymbol{\xi},\boldsymbol{\eta})\\ \mathbf{g}_2(\boldsymbol{\xi},\boldsymbol{\eta})\end{bmatrix}$$

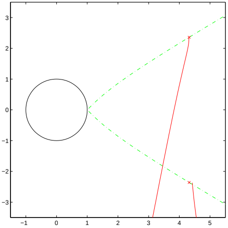
where $\boldsymbol{\eta}$ are the MINIMAL REPRESENTATION (STABLE MANIFOLD) coordinates and $\boldsymbol{\xi}$ are the CENTER MANIFOLD coordinates

- A_{11} has PURELY IMAGINARY EIGENVALUES
$\Longrightarrow$ neutral linear stability INCONCLUSIVE for the nonlinear system
- THEOREM: There exists an invariant manifold given by $\boldsymbol{\eta} = \boldsymbol{\Phi}(\xi) = 0$
Proof — via a direct calculation of an invariant manifold reduction
- THEOREM: Periodic solutions of the REDUCED SYSTEM

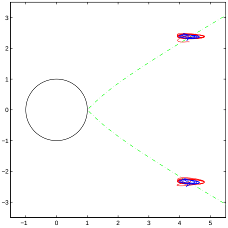
$$\frac{d}{dt}\boldsymbol{\xi} = A_{11}\boldsymbol{\xi} + \mathbf{g}_1(\boldsymbol{\xi}, 0)$$

are STABLE for small initial data
Proof — by examining the Hamiltonian reduced to the center manifold

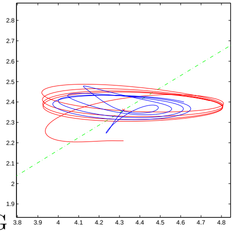
- Linear Feedback Control (LQG)



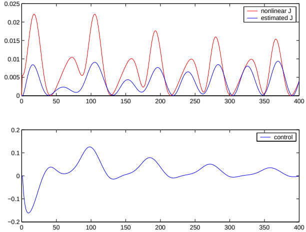
NO CONTROL



LQG CONTROL



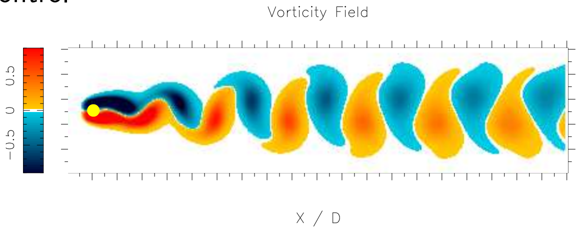
LQ… CONTROL (ZOOM)



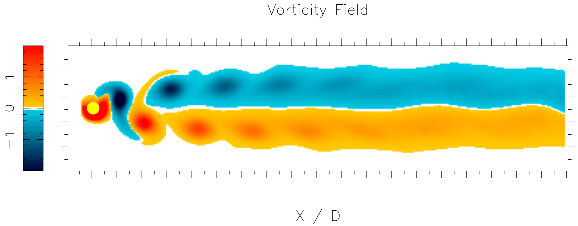
LQG CONTROL PARAMETERS

LQG stabilization of vortex shedding at $Re = 75$

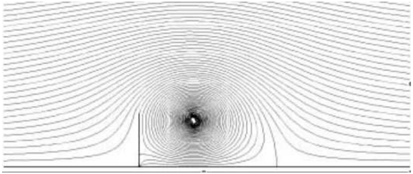
- No Control

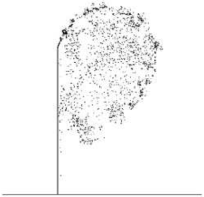


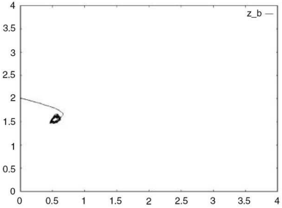
- LQG Control



Stabilization of Trapped Vortices

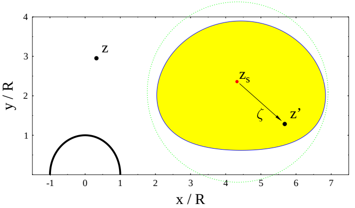






Wall suction stabilization of trapped vortices
Zannetti & Iollo, *Theor. Comp. Fluid Dyn.* **16**, (2003)

- OBJECTIVE — Construct a singular (potential flow) solution approximating for large $|\mathbf{x}|$ the finite–vortex solution of Euler equations



- Potential induced by a VORTEX PATCH

$$\tilde{W}_P(z) = (\varphi + i\psi)(z) = \frac{1}{2\pi i}\int_P \ln(z - z')\omega(z')\, dA(z')$$

$$= \frac{\Gamma_0}{2\pi i}\ln(z - z_s) + \frac{1}{2\pi i}\int_P \ln\left(1 - \frac{\zeta}{z - z_s}\right)\omega(z_s + \zeta)\, dA(\zeta)$$

$$= \frac{\Gamma_0}{2\pi i}\ln(z - z_s) - \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{c_n}{n}(z - z_s)^{-n}, \qquad |z - z_s| > \zeta_m,$$

where

$$c_n(z_s) = \int_P \omega(z_s + \zeta)\zeta^n\, dA(\zeta)$$

- Use the CIRCLE THEOREM to enforce the boundary conditions & truncate

$$W_N(z) = W_C(z) + W_{F,N}(z) = W_C(z) + \tilde{W}_{P,N}(z) + \tilde{W}_{Q,N}(z) + \overline{\tilde{W}}_{P,N}\left(\frac{R^2}{z}\right) + \overline{\tilde{W}}_{Q,N}\left(\frac{R^2}{z}\right)$$

$$= U_\infty\left(z + \frac{R^2}{z}\right) - \frac{\Gamma_0}{2\pi i}\left[\ln(z - z_s) - \ln\left(z - \frac{R^2}{\overline{z}_s}\right) - \ln(z - \overline{z}_s) + \ln\left(z - \frac{R^2}{z_s}\right)\right] -$$

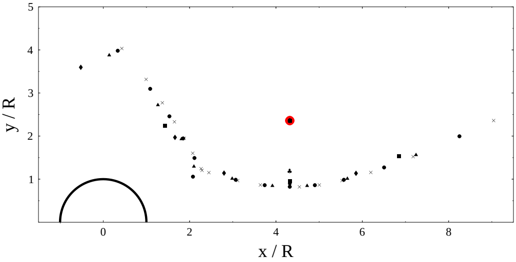
$$\frac{1}{2\pi i}\sum_{n=1}^{N}\frac{1}{n}\left[\frac{c_n}{(z - z_s)^n} - (-1)^n\frac{\overline{c}_n}{\left(z - \frac{R^2}{\overline{z}_s}\right)^n}\left(\frac{z}{\overline{z}_s}\right)^n - \frac{\overline{c}_n}{(z - \overline{z}_s)^n} + (-1)^n\frac{c_n}{\left(z - \frac{R^2}{z_s}\right)^n}\left(\frac{z}{z_s}\right)^n\right],$$

- Nonlinear dynamical system — looking for fixed points

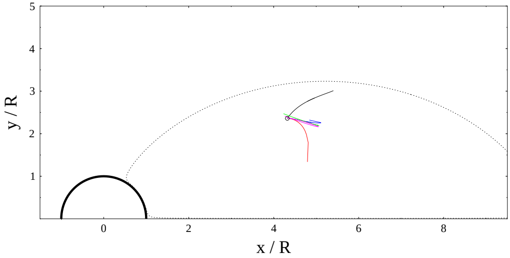
$$\hat{V}_N(z_N) = U_\infty\left(1 - \frac{R^2}{z_N^2}\right) - \frac{\Gamma_0}{2\pi i}\left[-\frac{1}{\left(z_N - \frac{R^2}{\overline{z}_N}\right)} - \frac{1}{(z_N - \overline{z}_N)} + \frac{1}{\left(z_N - \frac{R^2}{z_N}\right)}\right] +$$

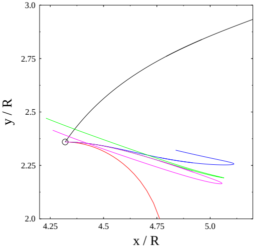
$$\frac{1}{2\pi i}\sum_{n=1}^{N}\left[(-1)^{n+1}\frac{R^2\overline{c}_n}{\left(z_N - \frac{R^2}{\overline{z}_N}\right)^{n+1}}\frac{z_N^{n-1}}{\overline{z}_N^{n+1}} - \frac{\overline{c}_n}{(z_N - \overline{z}_N)^{n+1}} - (-1)^{n+1}\frac{R^2 c_n}{\left(z_N - \frac{R^2}{z_N}\right)^{n+1}}\frac{1}{z_N^2}\right] = 0$$

- Higher–Order Föppl systems form a TWO–PARAMETER family depending on:
  1. the truncation order N
  2. the area A of the vortex region desingularizing the classical Föppl solution
- THEOREM: for $A = 0$, the classical Föppl equilibrium z_0 is also a solution of higher–order systems of arbitrary order N
- Additional SPURIOUS ROOTS appearing for higher truncation orders N (their number can be estimated from Bézout's theorem)

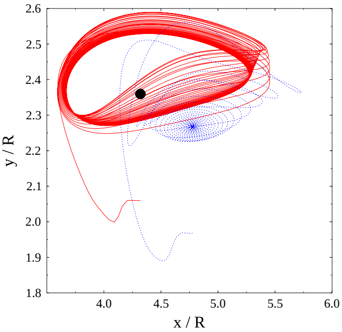


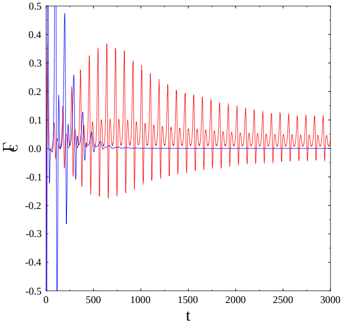
- THEOREM: For a vortex region with a "small" area A, the N–th order Föppl system admits an equilibrium z_N that is "close" to z_0 , i.e., the classical Föppl equilibrium
  Proof — via an application of the Fundamental Continuity Theorem to intersections of algebraic curves
- The LOCUS of the HIGHER–ORDER FÖPPL EQUILIBRIA corresponding to A increasing from 0 to A_{max} for different N





- Presence of the Center Manifold is structurally unstable
  - The Center Manifold disappears upon perturbation of the operator $\tilde{\mathcal{A}} = \mathcal{A} + \sum_k^N \mathcal{A}_k$ with terms corresponding to higher–order Föppl equilibria (unless $\mathcal{A}_k$ have special structure which is not the case)
  - the uncontrollable modes in the higher–order Föppl system are now exponentially stable

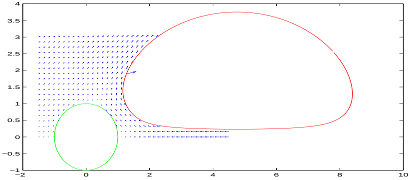




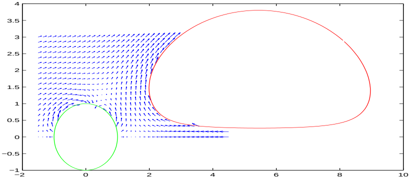
PART III
Inverse Problem of Vortex Design

Vortex Design as an Inverse Problem

- Euler Flows with different Boundary Conditions



Homogeneous BCs

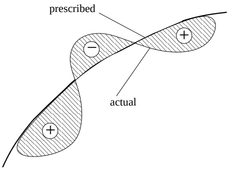


Nonhomogeneous BCs

- Statement of the Inverse Problem — Determine the boundary streamfunction ψ_b (equivalently, normal velocity $\frac{\partial \psi_b}{\partial s} = \mathbf{v} \cdot \mathbf{n}\big|_{\partial\Omega}$) to obtain a flow with a prescribed shape of the vortex region

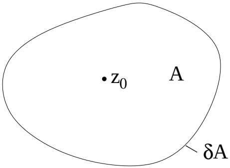
- Possible formulations

  - Minimize the area of the region $\Delta A(\psi_b)$ between the prescribed and actual patch boundaries, i.e.,



$$\mathcal{J}(\psi_b) = \frac{1}{2}\iint_{\Delta A(\psi_b)} dxdy$$

  - Characterization of the domain shape via MOMENTS $n = 1, \dots$



$$M_n = \int_A (z-z_0)^n \, dA = \frac{i}{2(n+1)} \oint_{\partial A} (z-z_0)^{n+1} \, d\bar{z}$$

(M_0 — area, M_1 — centroid, M_2 — ellipticity, ...)

- Cost Functional

$$\mathcal{J}(\psi_b) = \frac{1}{2}\sum_{n=1}^{N} \alpha_n \left[M_n(\psi_b) - \tilde{M}_n \right]^2$$

$\{\tilde{M}_n\}_{n=1}^{N}$ — moments of prescribed vortex boundary (given)

- OPTIMAL BOUNDARY CONDITION $\hat{\psi}_b$ determined by

$$\nabla \mathcal{J}(\hat{\psi}_b) = 0$$

- 2D steady Euler equation is a FREE–BOUNDARY PROBLEM

$$\begin{cases} |A(\psi_b)|\Delta\psi_{in} = \Gamma & \text{in } A(\psi_b), \\ \Delta\psi_{out} = 0 & \text{in } \Omega\backslash A(\psi_b), \\ \psi_{in} = \psi_{out} = \alpha & \text{on } \partial A(\psi_b), \\ \dfrac{\partial\psi_{in}}{\partial n} = \dfrac{\partial\psi_{out}}{\partial n} & \text{on } \partial A(\psi_b), \\ \psi_{out} = \psi_b & \text{on } \partial\Omega \end{cases}$$

- Differentiation of free–boundary problems requires tools of SHAPE–DIFFERENTIAL CALCULUS (Sokolowski & J.-P. Zolésio, 1992)
- Linear perturbation system obtained for $\psi_b + \epsilon\psi_b'$ (weak form)

$$
\begin{cases}
\mathcal{L}\psi' \triangleq |A(\psi_b)|\Delta\psi' - \left.\dfrac{\Gamma}{\frac{\partial\psi}{\partial n}}\right|_{\partial A(\psi_b)} \delta\left(\mathbf{x} - \mathbf{x}|_{\partial A(\psi_b)}\right)\psi' \\
\qquad + \dfrac{\Gamma}{|A(\psi_b)|}\left(\oint_{\partial A(\psi_b)} \left.\dfrac{\psi'}{\frac{\partial\psi}{\partial n}}\right|_{\partial A(\psi_b)} d\sigma\right) H(\psi-\alpha) = 0, & \text{in } \Omega \\
\psi' = \psi_b' & \text{on } \partial\Omega
\end{cases}
$$

- ADJOINT SYSTEM derived using the identity

$$\langle \mathcal{L}\psi', \psi^* \rangle = \langle \psi', \mathcal{L}^*\psi^* \rangle + b$$

$$\begin{cases} \mathcal{L}^*\psi^* \triangleq |A(\psi_b)|\Delta\psi^* + \\ \qquad + \dfrac{\Gamma}{\left.\frac{\partial\psi}{\partial n}\right|_{\partial A(\psi_b)}} \left(\dfrac{\int_{A(\psi_b)} \psi^* \, d\Omega}{|A(\psi_b)|} + \psi^* \right) \delta\left(\mathbf{x} - \mathbf{x}|_{\partial A(\psi_b)}\right) \\ \qquad = \dfrac{|A(\psi_b)|}{\left.\frac{\partial\psi}{\partial n}\right|_{\partial A(\psi_b)}} \delta\left(\mathbf{x} - \mathbf{x}|_{\partial A(\psi_b)}\right), & \text{in } \Omega \\ \psi^* = 0 & \text{on } \partial\Omega \end{cases}$$

- GRADIENT

$$\boxed{\nabla \mathcal{J}(\psi_b) = \left.\frac{\partial\psi^*}{\partial n}\right|_{\partial\Omega}}$$

Numerical Solution

- Noting that $\Delta\psi^* = 0$ in A and $\Omega\backslash A$, we have

$$\psi^*(z) = \frac{1}{2\pi}\oint_{\partial A(0)} \gamma^*(\zeta) \ln|z_0 - \zeta|\, ds_\zeta + \text{Image Terms}$$

  γ^* — density of the single–layer potential (defined on ∂A)

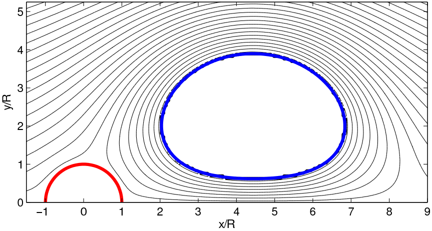
- Adjoint systems reduces to Boundary Integral Equation

$$\left.\frac{\partial\psi}{\partial n}\right|_{\partial A(0)} \gamma^*(z_0) + \omega \oint_{\partial A(0)} \gamma^*(\zeta) s_1(z_0,\zeta)\, ds_\zeta + \omega \sum_{k=2}^{4} \oint_{\partial A(0)} \gamma^*(\zeta) s_k(z_0,\zeta)\, ds_\zeta =$$

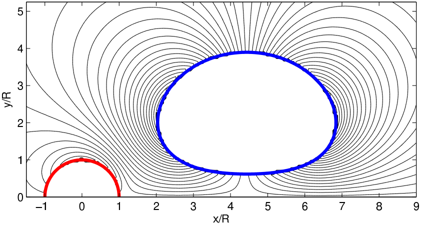
$$= \sum_{n=1}^{N} \alpha_n \left\{ \Re[M_n(\psi_b) - \tilde{M}_n]\Re[(z-z_0)^n] + \Im[M_n(\psi_b) - \tilde{M}_n]\Im[(z-z_0)^n] \right\}$$

- Solved using spectral interpolation with analytic treatment of the singularity (Kreiss, 1999)

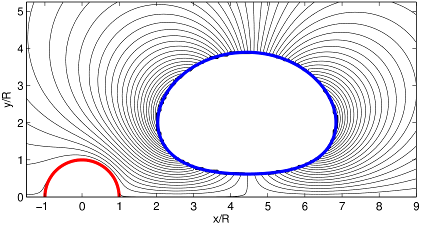
- direct problem (ψ)



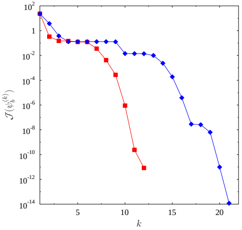
- perturbation problem (ψ')

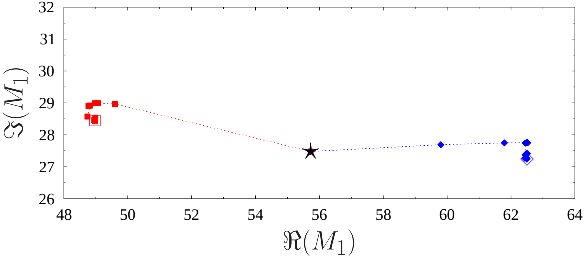


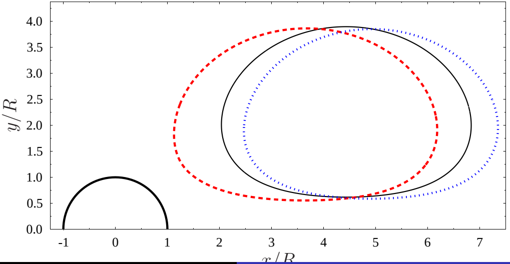
- adjoint problem (ψ^*)



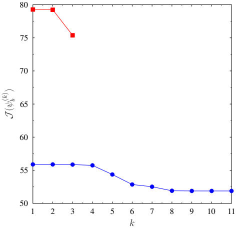
- STEP 1: $\mathcal{J}(\psi_b) = \frac{1}{2}[M_1(\psi_b) - \tilde{M}_1]^2$
- Results

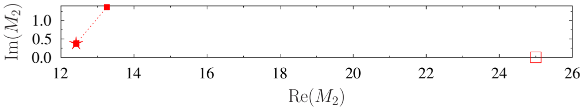


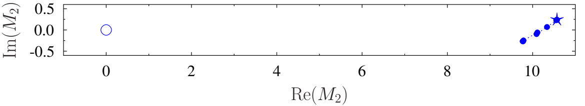


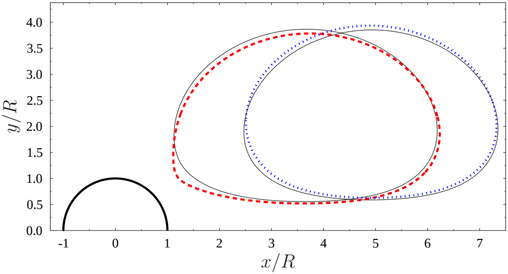


- STEP 2: $\mathcal{J}(\psi_b) = \frac{\alpha_1}{2}[M_1(\psi_b) - \tilde{M}_1]^2 + \frac{\alpha_2}{2}[M_2(\psi_b) - \tilde{M}_2]^2$
- Results

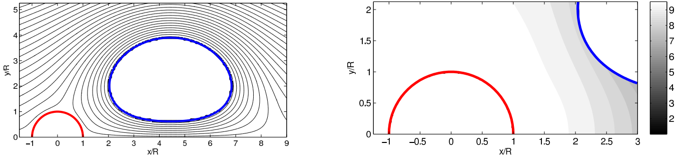




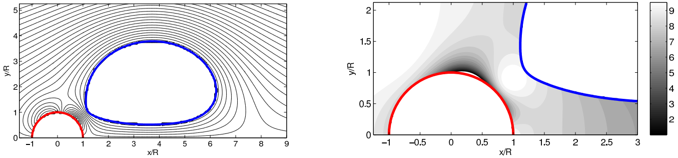




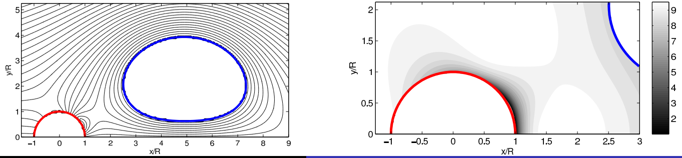
- No Control ($\psi_b \equiv 0$)



- Step 2, Case A



- Step 2, Case B



Conclusions

- Formulation of PDE control and estimation problems as constrained optimization
  - PDE–constrained gradients via Adjoint Equations
  - Vorticity form of the adjoint equations
  - Optimization of free boundary problems via shape–differential calculus
- Closed–Loop (Feedback) Control:
  - Control of singular solutions of Euler equations
  - Success of the linear (LQG) stabilization
  - Insights from the Nonlinear (Center Manifold) Analysis
- Vortex Design Problem:
  - Formulated an optimal control (design) problem for Euler flows with distributed vorticity
  - Key Enabler: shape–differentiation

- References Part I — Open–Loop Control:
  - B. Protas and W. Liao, "Adjoint–Based Optimization of PDEs in Moving Domains", *Journal of Computational Physics* **227** 2707–2723, 2008.
  - B. Protas, T. R. Bewley and G. Hagen, "A comprehensive framework for the regularization of adjoint analysis in multiscale PDE systems", *Journal of Computational Physics* **195**(1), 49-89, 2004.
  - B. Protas, "On the "Vorticity" Formulation of the Adjoint Equations and its Solution Using Vortex Method", *Journal of Turbulence* **3**, 048, 2002.
  - B. Protas and A. Styczek, "Optimal Rotary Control of the Cylinder Wake in the Laminar Regime", *Physics of Fluids* **14**(7), 2073–2087, 2002.
  - B. Protas, "Vortex Design Problem", *Journal of Computational and Applied Mathematics* **236**, 1926–1946, 2012.
- References Part II — Closed–Loop (Feedback) Control:
  - B. Protas, "Vortex Dynamics Models in Flow Control Problems", *Nonlinearity* **21**, R203–R250 (invited paper), 2008.
  - B. Protas, "Center Manifold Analysis of a Point–Vortex Model of Vortex Shedding with Control", *Physica D* **228** (2), 179–187, 2007.
  - B. Protas, "Higher–order Föppl models of steady wake flows", *Physics of Fluids* **18**(11), 117109, 2006.
  - B. Protas, "Linear Feedback Stabilization of Laminar Vortex Shedding Based on a Point Vortex Model", *Physics of Fluids* **16**(12), 4473-4488, 2004.