ICTS Lectures

Hilbert Space Techniques in Complex Analysis and Geometry

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Contents

I Complex Geometry **6**

1 Complex Manifolds and Holomorphic Vector Bundles **7**
1.1 Complex manifolds and holomorphic functions . . . 7
1.2 Examples of complex manifolds . . . 9
1.3 Holomorphic vector bundles . . . 15
1.3.1 The definitions . . . 15
1.3.2 Transition functions . . . 17
1.3.3 Holomorphic structure of the tangent bundle . . . 19
1.3.4 Some non-trivial examples of holomorphic line bundles . . . 21
1.4 Differential forms on complex manifolds . . . 23
1.4.1 Forms of bidgree (p, q) . . . 24
1.4.2 Exterior differential operators . . . 25
1.4.3 Twisted differential forms . . . 25

2 Hermitian metrics **27**
2.1 Hermitian metrics for complex vector bundles . . . 27
2.2 Hermitian Riemannian metrics for complex manifolds . . . 28
2.2.1 Structure of the metric . . . 28
2.2.2 Metric form . . . 29
2.2.3 Volume . . . 30

3 Connections and Curvature **32**
3.1 Connections . . . 32
3.1.1 Basic definition . . . 32
3.1.2 Induced connections . . . 33
3.1.3 Connections with additional symmetry . . . 34
3.1.4 The Kähler condition . . . 39
3.1.5 Induced connection on twisted forms . . . 41
3.2 Curvature . . . 44
3.2.1 Definition . . . 44
3.2.2 Curvature of the Chern connection . . . 44
3.2.3 Curvature of a line bundle . . . 45
3.2.4 Curvature of determinant bundles . . . 46

3.2.5 Symmetry of Kähler curvature . . . 47
3.2.6 Positivity . . . 50

II L^2 Estimates for $\bar{\partial}$ 55

4 L^2 Estimates for $\bar{\partial}$ in complex dimension 1 56
4.1 Domains in $\mathbb{C}$ . . . 56
4.2 L^2 estimates for $\bar{\partial}$ on Riemann surfaces . . . 65
4.2.1 Regularity of solutions . . . 68

5 The Bochner-Kodaira Identity 70
5.1 The Hilbert spaces . . . 70
5.2 Two $\bar{\partial}$-operators and their formal adjoints . . . 71
5.3 The formal identity . . . 73

6 L^2 Estimates on Compact Kähler manifolds 76
6.1 Hörmander's Theorem on compact Kähler manifolds . . . 76
6.2 A few useful applications . . . 78

7 L^2 Estimates on Complete Kähler manifolds 81
7.1 Extending the $\bar{\partial}$ operator to Hilbert spaces . . . 81
7.1.1 Closed, densely defined operators . . . 81
7.1.2 The maximal extension of $\bar{\partial}$ . . . 83
7.1.3 The Hilbert space adjoint $T^*_{p,q}$ . . . 84
7.2 Complete Kähler manifolds . . . 85
7.2.1 Complete Riemannian manifolds and exhaustion functions . . . 85
7.2.2 Approximation of twisted (p, q)-forms on complete Kähler manifolds . . . 85
7.2.3 The basic estimate for complete Kähler metrics . . . 87
7.3 Hörmander's Theorem . . . 87
7.3.1 Hörmander's Theorem for Complete Kähler metrics . . . 87
7.3.2 Comparison of L^2 norms from comparison of metrics . . . 88
7.3.3 Hörmander's Theorem for complete Kähler manifolds . . . 89
7.3.4 Weakly pseudoconvex Kähler manifolds . . . 90
7.3.5 Skoda's estimate . . . 91
7.4 Singular Hermitian metrics: an overview . . . 92

III Twisted Methods 96

8 Donnelly-Fefferman-Ohsawa Technique 97
8.1 The twisted basic estimate . . . 97
8.2 Donnelly-Fefferman-Ohsawa estimate for twisted $\bar{\partial}$ . . . 99
8.3 Ohsawa's $\bar{\partial}$ estimate . . . 99

8.4 Functions with self-bounded gradient . . . 100

9 Extension Theorems **104**
9.1 Extension without estimates . . . 104
9.1.1 Strongly pseudoconvex manifolds . . . 104
9.1.2 Stein manifolds . . . 108
9.1.3 Compact complex manifolds with positively curved line bundles . . . 110
9.1.4 The Kodaira Embedding Theorem . . . 112
9.2 $\mathbf{L^2}$ Extension . . . 115
9.2.1 Adjunction . . . 115
9.2.2 Statement of the $\mathbf{L^2}$ Extension Theorem . . . 116
9.2.3 Proof of Theorem 9.2.2 . . . 117

10 Some applications of the $\mathbf{L^2}$ Extension Theorem **123**
10.1 Beurling-Seip Theory of Interpolation . . . 123
10.1.1 Seip's Theorem . . . 124
10.1.2 A more general setting . . . 124
10.1.3 Weighted Bergman Inequalities . . . 127
10.2 Deformation Invariance of Plurigenera . . . 131
10.2.1 $\mathbf{L^2}$ extension of twisted canonical forms . . . 134
10.2.2 Positively twisted canonical sections . . . 135
10.2.3 Construction of the metric . . . 138
10.2.4 Examination of $e^{-\mu}$ on the central fiber: conclusion of the proof . . . 139
10.2.5 An $L^{2/m}$-estimate for m-canonical sections . . . 140

11 Berndtsson's Theorem on Plurisubharmonic Variation **142**
11.1 Berndtsson's Theorem . . . 142
11.1.1 Variations of Hilbert spaces . . . 142
11.1.2 Statement of Berndtsson's Theorem . . . 146
11.1.3 The Bergman projection . . . 147
11.1.4 Proof of Theorem 11.1.8 . . . 148
11.2 Application: The Suita Conjecture . . . 151
11.2.1 Formulation of the Suita Conjecture . . . 152
11.2.2 Alternate formulation of the Suita Conjecture . . . 153
11.2.3 Estimating the Bergman kernel . . . 155
11.3 Application: Sharp Estimates for L^2 extension . . . 158
11.3.1 Statement of the Sharp L^2 Extension Theorem . . . 158
11.3.2 Sharpness in Theorem 11.3.1 . . . 159
11.3.3 Beginning of the proof of Theorem 11.3.1 . . . 160
11.3.4 Dual formulation of the norm of the minimal extension . . . 160
11.3.5 Degeneration to the infinitesimal neighborhood . . . 163
11.3.6 The proof of Theorem 11.3.1 . . . 164
11.4 Equivalence of Sharp Extension and Positive Variation . . . 166

11.4.1 Log plurisubharmonicity of the Bergman kernel 166
11.4.2 Plurisubharmonic variation versus sharp L^2 extension 170

Preface

These notes assume some basic familiarity with real analysis, elementary complex analysis in one and several variables (such as the definition of holomorphic functions, Cauchy Integral Formula, and so on, up to the point of the Hartogs Phenomenon), and the basic definitions in the study of smooth manifolds, including differential forms.

Part I

Complex Geometry

Lecture 1

Complex Manifolds and Holomorphic Vector Bundles

1.1 Complex manifolds and holomorphic functions

1.1.1 DEFINITION. *A complex manifold of complex dimension n is a smooth manifold that has an atlas whose transition functions are holomorphic. That is to say, a complex manifold is a topological space X with an open cover $\{U_\alpha \,;\, \alpha \in J\}$ and homeomorphisms*

$$\varphi_\alpha : U_\alpha \to V_\alpha \overset{open}{\subset} \mathbb{C}^n, \qquad \alpha \in J,$$

such that for each pair $\alpha, \beta \in J$ the homeomorphism

$$\Phi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}|_{U\alpha\cap U\beta} : V_\beta \to V\alpha$$

is a holomorphic diffeomorphism. The open sets U_α are called coordinate charts, and the maps φ_α are called local coordinates. The maps $\Phi_{\alpha\beta}$ are called transition functions.

1.1.2 REMARK. By analogy with real manifolds, the collection $\{(U_\alpha, \varphi_\alpha) \,;\, \alpha \in J\}$ is called a *holomorphic atlas*. Two holomorphic atlases are said to be equivalent if their union is again a holomorphic atlas. A maximal holomorphic atlas is an equivalence class of holomorphic atlases. By using the inclusion of sets as a partial ordering, one sees that each maximal holomorphic atlas contains a unique maximal element, so we can identify the equivalence class with an actual holomorphic atlas. This maximal holomorphic atlas is huge, and in practice most of its charts are irrelevant in any given situation. However, it is convenient to have this notion because the maximal atlas is a unique object associated to the complex manifold, and hence characterizes its complex structure. Perhaps the most important thing to keep in mind is that once a single holomorphic atlas is found, this atlas uniquely determines a maximal holomorphic atlas. ◇

For a holomorphic map $F = (F^1, ..., F^n) : \Omega_1 \to \Omega_2$ between two open subsets of $\mathbb{C}^n$ one has the identity

$$\det(DF) = |\det \partial F|^2, \tag{1.1}$$

where D is the real derivative in $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and

$$\partial(F^1, ..., F^n) = \left(\frac{\partial F^i}{\partial z^j}\right)_{i,j=1}^n .$$

In particular, the holomorphic atlas with which the complex manifold is equipped has the property that the derivatives of its transition functions all have positive determinants, i.e., every complex manifold is orientable.

The definition of complex manifold is such that in the neighborhood of each of its points all of the concepts of complex analysis in one and several variables make sense, since they are invariant under holomorphic changes of coordinates. This feature facilitates the extension of the standard objects of complex analysis from domains in Euclidean space to complex manifolds. The rigorous definition is as follows.

1.1.3 DEFINITION. *Let X and Y be complex manifolds.*

1. *A function $f : X \to \mathbb{C}$ on a complex manifold X said to be holomorphic (resp. harmonic, pluriharmonic, subharmonic, plurisubharmonic) if for each $pinX$ and each coordinate chart $\varphi : U \to V \subset \mathbb{C}^n$ the function $f \circ \varphi^{-1} : V \to \mathbb{C}$ is holomorphic (resp. harmonic, pluriharmonic, subharmonic, plurisubharmonic). The set of holomorphic functions on X is denoted $\mathcal{O}(X)$.*

2. *A map $F : X \to Y$ is said to be holomorphic if for each $f \in \mathcal{O}(Y)$, $f \circ F \in \mathcal{O}(X)$.*

3. *A holomorphic map $F : X \to Y$ is said to be a holomorphic diffeomorphism if it is bijective and the inverse map is holomorphic. In this case one also says that X and Y are biholomorphic.*

1.1.4 DEFINITION. *A subset Z of a complex manifold X is said to be a* complex submanifold *if for each $p \in X$ and every coordinate chart $\varphi_\alpha : U_\alpha \to V_\alpha$ such that $p \in U_\alpha$ there exist holomorphic functions $f_1, , ..., f_k \in \mathcal{O}(V_\alpha)$ such that*

$$Z \cap U_\alpha = \{x \in U_j \ ; \ f_1(\varphi_\alpha(x)) = \cdots = f_k(\varphi_\alpha(x)) = 0\}$$

and for each $p \in U_\alpha \cap Z$

$$df_1(\varphi_\alpha(p)) \wedge ... \wedge df_k(\varphi_\alpha(p)) \neq 0,$$

i.e., $df_1(\varphi_\alpha(p)), ..., df_k(\varphi_\alpha(p))$ are linearly independent.

For points $p \in Z$, the number k is called the complex codimension of Z at p, and if X has complex dimension n then the number $n - k$ is called the dimension of Z at p.

A complex submanifold of complex codimension 1 is called a smooth complex hypersurface.

The implicit function theorem shows that a complex submanifold of a complex manifold is itself a complex manifold. Our definition here is one of several standard definitions. In our definition a submanifold Z of a complex manifold X is always a closed subset of X (this would change if one replaced '$p \in X$' with '$p \in Z$' in our definition), and its topological structure as an abstract manifold is the relative topology it inherits as a subset of X. There are other standard definitions in which either or these properties fails.

1.1.5 PROPOSITION. *If X and Y are complex manifolds then a map $F : X \to Y$ is holomorphic if and only if its graph $\Gamma_F := \{(x, F(x)) \ ; \ x \in X\} \subset X \times Y$ is a complex submanifold.*

EXERCISES

1.1.1. Prove the formula (1.1).

1.1.2. Show that if $D \subset \mathbb{C}$ is an open set and $f : D \to \mathbb{C}$ is an injective holomorphic function then $f'(z) \neq 0$ for every $z \in D$.

In fact, the analogous fact holds for a holomorphic map $F : \Omega \to \mathbb{C}^n$ where $\Omega \subset \mathbb{C}^n$ is an open set: if F is injective then $\det DF$ is nowhere-zero. (The proof of this fact is elementary but not obvious; You are encouraged to try to prove it or to look it up, though don't spend to much time on it if you choose to do it yourself.) Most relevantly, it is crucial that the dimensions of the domain and of the range are equal. Show that the map

$$g : \mathbb{C} \ni t \mapsto (t^2, t^3) \in \mathbb{C}^2$$

is injective and that its derivative vanishes at the origin.

1.1.3. Show that if Z is a complex submanifold of a complex manifold X and $f : X \to Y$ is a holomorphic map then the restriction $f|_Z : Z \to Y$ is a holomorphic map.

1.1.4. Prove Proposition 1.1.5.

1.2 Examples of complex manifolds

To get a feeling for complex manifolds and how they work, it is useful to have a number of examples. There are many other interesting example besides those we present here.

1.2.1 EXAMPLE. Open subsets of $\mathbb{C}^n$ are complex manifolds. One can take a chart to be the entire space, and the coordinate map to be the identity. This atlas, which has only one coordinate chart, is automatically holomorphic by definition, but things become non-trivial when one obtains the maximal holomorphic atlas.

This set of examples is not particularly interesting; of course it contains the setting of almost all problems in classical complex analysis, but if this were the only example of a complex manifold, the theory would not exist. ◇

1.2.2 EXAMPLE (Graphs). Let $\Omega \subset \mathbb{C}^n$ be a domain and let $f^1, ..., f^k \in \mathscr{C}(\Omega)$. Then the set

$$\Gamma_f := \{(z, w^1, ..., w^k) \in \Omega \times \mathbb{C}^k \ ; \ f^i(z) = w^i, \ 1 \leq i \leq k\}$$

with its relative topology is a complex manifold. Indeed, a coordinate chart is given by the restriction to Γ_f of the projection $\pi : \Omega \times \mathbb{C}^k \to \Omega$ to the first factor.

Again this example is not so interesting, because the complex structure is determined by a single chart. Perhaps the one instructive aspect is that although Γ_f is a complex manifold in general, it is a complex submanifold of $\Omega \times \mathbb{C}^k$ if and only if $f_i \in \mathcal{O}(\Omega)$, $1 \leq i \leq k$. ◇

1.2.3 EXAMPLE (Zero sets of non-degenerate holomorphic mappings in $\mathbb{C}^n$). For a collection of functions $f_1, ..., f_k \in \mathcal{O}(\mathbb{C}^n)$ define

$$Z = \{x \in \mathbb{C}^n \ ; \ f_j(x) = 0 \text{ for all } 1 \le j \le k\}.$$

Assume moreover that the complex-valued differential 1-forms $df_1, ..., df_k$ are linearly independent at each point of Z. Then the set Z is a complex manifold of dimension $n - k$, as one can easily show from the implicit function theorem applied to the mapping $F = (f_1, ..., f_k) : \mathbb{C}^n \to \mathbb{C}^k$.

Closed submanifolds of $\mathbb{C}^n$ are called *Stein manifolds*. Those codimension-k closed submanifolds of $\mathbb{C}^n$ that are cut out by k holomorphic functions are called *complete intersections*. Not all closed submanifolds of $\mathbb{C}^n$ are complete intersections. ◇

1.2.4 EXAMPLE (The extended complex plane). The set $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, where $\{\infty\}$ is any singleton, is topologized by declaring that its open sets are

(i) open subset of $\mathbb{C}$, and

(ii) $\{\infty\} \cup (\mathbb{C} - K)$ where K is a compact subset of $\mathbb{C}$;

the latter are neighborhoods of ∞. With this topology, the space $\widehat{\mathbb{C}}$ is homeomorphic to the 2-sphere.

The open sets $U_1 = \mathbb{C}$ and $U_2 = \{\infty\} \cup (\mathbb{C} - \{0\})$ and homeomorphisms $\varphi_1 : U_1 \to \mathbb{C}$ and $\varphi_2 : U_2 \to \mathbb{C}$ defined by

$$\varphi_1(z) := z \quad \text{and} \quad \varphi_2(w) := \begin{cases} 1/w, & w \in \mathbb{C} - \{0\} \\ 0 \quad , & w = \infty \end{cases}$$

have transition functions

$$\Phi_{12}(w) = \frac{1}{w} \quad \text{and} \quad \Phi_{21}(z) = \frac{1}{z},$$

so the atlas defined by this open cover by two coordinate charts is holomorphic. The resulting complex manifold is called the *extended complex plane*, and also the *Riemann sphere*. ◇

1.2.5 EXAMPLE (Riemann surfaces). As we already pointed out, every complex manifold is orientable. One can ask if orientability and even-dimensionality guarantee the existence of a holomorphic atlas. In general this is not so, but remarkably in real dimension 2 every orientable surface admits at least one holomorphic atlas (and sometimes many incompatible atlases), as we now explain.

Let S be an orientable surface, i.e., 2-dimensional real manifold. Orientability means that there is an atlas of smooth coordinate charts such that the derivative matrices of all of the transition functions have positive determinant. There are exactly two incompatible maximal atlases with this property, and we shall fix one of them, referring to any coordinate chart in this atlas as *positively oriented*.

We can also equip S with a Riemannian metric g. Of course, there are always many different Riemannian metrics on any given manifold, but in two dimensions there is a remarkable class of local normal forms for such metrics: for any point $p \in S$ there is a coordinate chart U containing p and a coordinate map $\varphi : U \to V \subset \mathbb{R}^2$ such that

$$g = e^{\rho} \left((d\varphi^1)^2 + (d\varphi^2)^2 \right)$$

where ρ is a smooth function on U. In geometric language g is conformal to the Euclidean metric in this coordinate chart.

REMARK. Note that if S is oriented then we may assume that the coordinate chart is positively oriented: if a given coordinate map $\varphi = (\varphi^1, \varphi^2)$ is not positively oriented then $\psi := (\varphi^2, \varphi^1)$ is positively oriented, and clearly swapping the first and second components preserves the normal form.

Coordinate charts in which a metric g is conformal to the Euclidean metric are classically called *isothermal coordinates* (for the metric g). Their existence when the metric g is real analytic goes back to Gauss. For smooth metrics the existence theorem was proved by Korn and Lichtenstein, and is perhaps the first result in the theory of elliptic partial differential equations. In particular, it is non-trivial. There is a lot more to say about this theorem and results that extended it, but in the interest of preserving focus let us mention only that a proof is possible by combining standard Elliptic Regularity ideas with methods that we will develop later in the notes when we look at Hörmander's Theorem.

If we accept the theorem on existence of isothermal coordinates then the construction of a holomorphic atlas is simple: at each point of S choose an isothermal coordinate system with positive orientation, and let the atlas be composed of these chosen coordinate systems. Since these coordinate functions map the metric g to a multiple of the Euclidean metric, the transition functions preserve orientation and the derivatives of the transition functions, as maps of tangent spaces, map circles to circles. Elementary complex analysis shows that in dimension 2 a map that sends circles to circles and preserves orientation is holomorphic. Hence the atlas thus constructed is a holomorphic atlas.

A 1-dimensional complex manifold is called a Riemann surface. Example 1.2.4 of the RIemann sphere is a special case when the underlying manifold is S^2; its complex structure is determined by for example the round metric. It turns out that any two smooth metrics on the Riemann sphere are conformal, so that the complex structure of $\widehat{\mathbb{C}}$ is the only possible maximal holomorphic atlas one can put on $\widehat{\mathbb{C}}$ once the orientation is fixed.

Complex conjugation, which reverses the orientation, gives a distinct (in the sense that its maximal atlas is not compatible with the one defined on $\widehat{\mathbb{C}}$ in Example 1.2.4) but biholomorphic complex structure on $\widehat{\mathbb{C}}$. The biholomorphic map is the map given by complex conjugation in each coordinate chart. ◇

1.2.6 EXAMPLE (Projective space). Let V be a complex vector space of complex dimension $n+1$. We say that $x \in V - \{0\}$ is equivalent to $y \in V - \{0\}$, and write $[x] = [y]$, if there is a complex number λ such that $y = \lambda x$. The relation [ ] is clearly an equivalence relation. We define the projective space $\mathbb{P}(V)$ to be the set of all equivalence classes of [ ]. The map

$$[\,] : V - \{0\} \to \mathbb{P}(V)$$

is continuous if we topologize $\mathbb{P}(V)$ by declaring that $U \subset \mathbb{P}(V)$ is open if and only if $[\,]^{-1}(U)$ is open (in $V - \{0\}$, of course).

A holomorphic function $F : V \to \mathbb{C}$ is said to be *homogeneous* if $F(\lambda v) = \lambda^k F(v)$ for some $k \in \mathbb{N}$ and all $\lambda \in \mathbb{C}$ and $v \in V$. The zero set of such a homogeneous holomorphic function on $V - \{0\}$ gives rise to a well defined (and closed) subset of $\mathbb{P}(V)$.

If we fix $n+1$ linearly independent functions $z^0, ..., z^n \in V^*$ (which are often called linear coordinates) then we identify V with $\mathbb{C}^{n+1}$ via the map

$$V \ni v \mapsto (z^0(v), ..., z^n(v)) \in \mathbb{C}^{n+1}.$$

Any homogeneous function of degree k is given by a homogeneous polynomial of degree k in the linear coordinates $z^0, ..., z^n$. With such a choice of coordinates, we thus define the open subsets

$$U_i := \{[v] \in \mathbb{P}(V) \; ; \; z^i(v) \neq 0\}, \quad i = 0, ..., n.$$

The map

$$\varphi_i : U_i \ni [v] \mapsto \frac{1}{z^i(v)}(z^0(v), ..., z^{i-1}(v), z^{i+1}(v), ..., z^n(v)) \in \mathbb{C}^n$$

is clearly one-to-one and onto. In other words

$$\zeta_i^\mu = \begin{cases} \frac{z^{\mu-1}}{z^i} & 1 \leq \mu \leq i \\ \frac{z^\mu}{z^i} & \mu > i \end{cases}$$

are coordinates on U_i, and $\varphi_i = (\zeta_i^1, \ldots, \zeta_i^n)$.

Now suppose we want to pass from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$. Assume without loss of generality that $i < j$. Then one easily computes that

$$\zeta_i^\mu = \begin{cases} 1/\zeta_j^{i+1} & \mu = j \\ \zeta_j^\mu/\zeta_j^{i+1} & 1 \leq \mu \leq i \quad \text{and} \quad \mu > j \\ \zeta_j^{\mu+1}/\zeta_j^{i+1} & i < \mu < j \end{cases},$$

i.e.,

$$\zeta_i = \varphi_i \circ \varphi_j^{-1}(\zeta_j) = \frac{1}{\zeta_j^{i+1}} \left(\zeta_j^1, ...\zeta_j^i, \zeta_j^{i+2}, ..., \zeta_j^j, \underbrace{1}_{j^{\text{th}} \text{ spot}}, \zeta_j^{j+1}, ..., \zeta_j^n \right)$$

Thus $\varphi_i \circ \varphi_j^{-1}$ is a holomorphic diffeomorphism on $\varphi_j(U_i \cap U_j)$, and so $\mathbb{P}(V)$ is a complex manifold of dimension n.

When $V = \mathbb{C}^{n+1}$ we write $\mathbb{P}_n := \mathbb{P}(\mathbb{C}^{n+1})$. ◇

1.2.7 EXAMPLE (Zero locus of homogeneous polynomials). We use the notation of Example 1.2.6. Suppose $F_1, ..., F_k : \mathbb{C}^{n+1} \to \mathbb{C}$ are homogeneous holomorphic polynomials, possibly with different degrees of homogeneity, and let

$$Z := \{z \in \mathbb{C}^{n+1} \; ; \; F_j(z) = 0, 1 \leq j \leq k\}.$$

On the set $[\ ]^{-1}(U_j)$, we may write

$$F_\ell = (z^j)^{d_\ell} f_{\ell,j}(\zeta_j), \quad 1 \le \ell \le k.$$

It follows that on $\mathbb{C}^{n+1} \cap [\ ]^{-1}(U_j) \cap \{F_\ell = 0\}$,

$$dF_\ell = (z^j)^{d_\ell} df_{j,\ell}(\zeta_j) \supset Z.$$

Thus by the Implicit Function Theorem we see that $[Z]$ is a smooth submanifold of $\mathbb{P}_n$ if and only if Z is a smooth submanifold of $\mathbb{C}^{n+1} - \{0\}$. Both manifolds have codimension k. Thus the common zero loci of homogeneous polynomials on $\mathbb{C}^{n+1}$ whose differentials are independent on the part of this zero locus that lies away from the origin, define a submanifold of $\mathbb{P}_n$.

Note that we must remove the origin. For example, in $\mathbb{C}^2$ the common zero set of $z^0 = z^1 = 0$ is a pair of transverse lines through the origin, which is not a submanifold. On the other hand, the induced set in $\mathbb{P}_1$ is a pair of distinct points.

Not all codimension k submanifolds of $\mathbb{P}_n$ are cut out by exactly k homogeneous polynomials. Nevertheless, it is a fact, which we shall prove later on, that all submanifolds of $\mathbb{P}_n$ are cut out by homogeneous polynomials. $\diamond$

1.2.8 EXAMPLE (The blowup of a point). Let $\mathbb{B} \subset \mathbb{C}^{n+1}$ denote the unit ball and let

$$\tilde{\mathbb{B}}_o := \{(z,\ell) \in \mathbb{B} \times \mathbb{P}_n \ ; \ z \in \ell\}.$$

Since the projection map $\mathbb{B} \times \mathbb{P}_n \to \mathbb{B}$ is clearly proper[1], so is its restriction $\pi : \tilde{\mathbb{B}}_o \to \mathbb{B}$ to the closed subset $\tilde{\mathbb{B}}_o$.

We claim that $\tilde{\mathbb{B}}_o$ is a complex manifold. Indeed, away from the origin in $\mathbb{B}$, every point lies on exactly one complex line through the origin, and thus π is a biholomorphic map away from $\pi^{-1}(0)$. On the other hand, if we choose projective coordinates $[x^0, ..., x^n]$ on $\mathbb{P}_n$ then $\pi^{-1}(\mathbb{B})$ is given by the zero sets of the holomorphic functions

$$f^j(z,[x]) := z^j - \frac{x^j}{x^i} z^i, \quad j \in \{0,...,n\} - \{i\}$$

in the coordinate chart $V_i := \{(z,[x]) \in \mathbb{B} \times \mathbb{P}_n \ ; \ |z| < 1 \text{ and } x^i \neq 0\}$ in $\mathbb{B} \times \mathbb{P}_{n-1}$. We compute, for the case $i = 0$, where $z^0, ..., z^n, \zeta^1, ..., \zeta^n$, $\zeta^j = x^j/x^i$, are coordinates on V_0, that

$$df^j(z,\zeta) = dz^j - \zeta^j dz^o - z^o d\zeta^j, \quad , 1 \le j \le n,$$

and these are clearly independent. Thus $\tilde{\mathbb{B}}_o$ is a complex manifold. Moreover, $\pi(z,[1,\zeta]) = z$, so $E := \pi^{-1}(0) = \{0\} \times \mathbb{P}_n$ is also a submanifold of $\mathbb{B} \times \mathbb{P}_n$, clearly biholomorphic to $\mathbb{P}_n$. But in fact, E is a submanifold of $\tilde{\mathbb{B}}_o$. Indeed, if we let $\phi : \tilde{\mathbb{B}}_o \to \mathbb{P}_n$ denote the restriction to $\tilde{\mathbb{B}}_o$ of the projection $\mathbb{B} \times \mathbb{P}_n \to \mathbb{P}_n$. Then, over the open set $U_i = \{[x] \in \mathbb{P}_n \ ; \ x^i = 1\}$, the map

$$\phi^{-1}(U_i) = \{((\lambda x^o, ..., \underbrace{\lambda}_{i^{\text{th}} \text{ position}}, ..., \lambda x^n), [x]) \ ; \ \lambda \in \mathbb{C}, \ [x] \in U_i\},$$

[1] Recall that a map between topological spaces is said to be *proper* if the inverse image of compact sets is compact.

and the map

$$\phi^{-1}(U_i) \ni \lambda x, [x] \mapsto (\lambda, \lambda x^1, ..., \lambda x^{i-1}, \lambda x^{i+1}, ..., \lambda^n) \in \mathbb{C} \times \varphi_i(U_i) = \mathbb{C} \times \mathbb{C}^{n-1} \cong \mathbb{C}^n$$

is a coordinate chart, and $E \cap \tilde{\mathbb{B}}_o$ is mapped to $\{0\} \times \mathbb{C}^{n-1}$ by this chart map.

Now let X be any complex manifold, and let $p \in X$. Then there is a neighborhood of p in X that is biholomorphic to $\mathbb{B}$ via a biholomorphism that sends p to the origin. If we replace $\mathbb{B}$ by $\tilde{\mathbb{B}}_o$ in X (via the chart map) then we obtain a new complex manifold $\tilde{X}_p$ that is biholomorphic to X away from p, and in place of p has a smooth submanifold biholomorphic to a projective space of dimension $\dim_{\mathbb{C}}(X) - 1$, i.e., a hypersurface. We also have a proper holomorphic map $\pi : \tilde{X}_p \to X$ such that

$$\pi : \tilde{X}_p - \pi^{-1}(\{p\}) \to X - \{p\}$$

is biholomorphic. ◇

1.2.9 DEFINITION. *The complex manifold $\tilde{X}_p$ is called the* blowup of X at p*, and the holomorphic map $\pi : \tilde{X}_p \to X$ is called the blowdown map. The smooth hypersurface $\pi^{-1}(\{p\})$ is called the* exceptional hypersurface *for π.*

1.2.10 EXAMPLE (Complex Tori). Let Λ be a lattice in a complex vector space V of dimension n. (Recall that a lattice in a real vector space is a collection of vectors that is closed under addition and whose convex hull is the whole vector space.) We say that $x \in V$ is equivalent to $y \in V$, and write $x \sim y$, if there exists $\lambda \in \Lambda$ such that $y = x + \lambda$. The set of equivalence classes is denoted V/Λ or $\mathbb{T}_\Lambda(V)$, and we have a map

$$\pi : V \to \mathbb{T}_\Lambda(V)$$

sending x to its equivalence class. As with $\mathbb{P}(V)$, we endow $\mathbb{T}_\Lambda(V)$ with the coarsest topology that makes π continuous, i.e., $U \subset \mathbb{T}_\Lambda(V)$ is open if and only if $\pi^{-1}(U)$ is open in V.

We define coordinate charts on $\mathbb{T}_\Lambda(V)$ as follows. Let $x \in V$ and let U be a neighborhood of x such that

$$U \cap (U + \lambda) = \emptyset \text{ for all } \lambda \in \Lambda - \{0\}.$$

Then $\pi|_U$ is a homeomorphism. We let $\varphi_U := (\pi|_U)^{-1} : \pi U \to U$ be our coordinate neighborhood. Clearly the set of all such coordinate charts covers $\mathbb{T}_\Lambda(V)$, and if any two such charts intersect on $\mathbb{T}_\Lambda(V)$, then their coordinate chart images in V intersect after a translation. Thus the transition functions are holomorphic.

1.2.11 DEFINITION. *The manifold $\mathbb{T}_\Lambda(V)$ is called a complex torus.*

It is easy to see that complex tori are all diffeomorphic to the Cartesian product of $2n$ circles. However, among all of these mutually diffeomorphic manifolds there are many different complex structures. For example, it turns out that some complex tori have no proper submanifolds of positive dimension. On the other hand, some complex tori can be realized as submanifolds of projective space, and therefore have many submanifolds.

Tori that can be embedded in some projective space are called *Abelian varieties*, and have a special role in algebraic geometry. ◇

EXERCISES

1.2.1. Prove the last statement at the end of Example 1.2.2, i.e., that Γ_f is a complex submanifold of $\Omega \times \mathbb{C}^k$ if and only if $f_i \in \mathcal{O}(\Omega)$, $1 \le i \le k$.

1.2.2. Show that if $L : V \to W$ is an injective linear map then the map

$$\mathbb{P}(L) : \mathbb{P}(V) \ni [v] \mapsto [Lv] \in \mathbb{P}(W)$$

is well-defined and injective.

1.2.3. Show that

(a) for every holomorphic map $F : \mathbb{P}(V) \to \mathbb{P}(W)$ there exists a homogeneous holomorphic map $\Phi : V \to W$ such that $F([v]) = [\Phi(v)]$, and that

(b) every holomorphic diffeomorphism of $\mathbb{P}(V)$ is of the form $\mathbb{P}(L)$ (c.f. Exercise 1.2.2) for some $L \in GL(V)$.

1.2.4. Show that $\widehat{\mathbb{C}}$ and $\mathbb{P}_1$ are biholomorphic.

1.2.5. Show that the sets

$$R := \{[x,y,z] \in \mathbb{P}_2 \ ; \ xy = z^2\} \quad \text{and} \quad E := \{[x,y,z] \in \mathbb{P}_2 \ ; \ y^2 z = x^3 + xz^2 + z^3\}$$

are Riemann surfaces, that R is biholomorphic to $\mathbb{P}_1$, and that E is not biholomorphic to $\mathbb{P}_1$.
HINT: *Show that the meromorphic* 1*-form*

$$\omega := \frac{d\xi}{\eta}$$

on $U_2 = \{z \neq 0\}$ *of* $\mathbb{P}_2$ *with coordinates* $\xi = x/z$ *and* $\eta = y/z$, *is holomorphic on* $E \cap U_2$, *and extends to a well-defined holomorphic* 1*-form on* E.

1.3 Holomorphic vector bundles

1.3.1 The definitions

- A holomorphic vector bundle of rank r is a triple $(V, X, \pi : V \to X)$ such that

  (i) V and X are complex manifolds,

  (ii) π is a holomorphic map, and

  (iii) each $p \in X$ is contained in an open set U on which there are holomorphic maps $e_1, ..., e_r : U \to V$ such that

  $$\pi e_i = \mathrm{Id}_U \quad \text{and} \quad \mathrm{span}_{\mathbb{C}}\{e_1(x), ..., e_r(x)\} = V_x \text{ for all } x \in U,$$

  where $V_x := \pi^{-1}(x)$ denotes the fiber of π over $x \in X$. Such a collection of maps $\{e_1, ..., e_r\}$ is called a *frame for* V *over* U.

Observe that if $\{e_i\}$ and $\{\tilde{e}_i\}$ are two frames defined over the same open set U, then there are holomorphic functions $g_i^j \in \mathcal{O}(U)$ such that $g_i^j(p) \in GL(r, \mathbb{C})$ for all $p \in U$ and

$$\tilde{e}_i = g_i^j e_j.$$

1.3.1 REMARK. If "holomorphic" and "complex manifold" are replaced by "smooth (or continuous)" and "smooth manifold (or topological space)" one obtains the definition of complex vector bundle. ◇

- A holomorphic vector bundle of rank 1 is called a *holomorphic line bundle*. Holomorphic line bundles play an especially important role in complex analytic and algebraic geometry.

- A map of holomorphic vector bundles is a holomorphic map $F : V \to W$ such that

  (i) the diagram

$$\begin{array}{ccc} V & \xrightarrow{F} & W \\ \downarrow & & \downarrow \\ X & \xrightarrow{\mathrm{Id}} & X \end{array}$$

  commutes, and

  (ii) for each $x \in X$ the map
$$F_x := F|_{V_x} : V_x \to W_x$$
  is linear.

  Two vector bundles are isomorphic if there are holomorphic vector bundle maps $F : V \to W$ and $G : W \to V$ such that $FG = \mathrm{Id}_V$ and $GF = \mathrm{Id}_W$.

- A section s of a holomorphic vector bundle $\pi : V \to X$, i.e., a right inverse for π, is said to be holomorphic (resp. smooth, measurable, etc.) if it is holomorphic (resp. smooth, measurable, etc.) as a map $X \to V$.

1.3.2 EXAMPLE (Trivial bundles). The simplest example of a holomorphic vector bundle is the trivial bundle $\pi : X \times \mathbb{C}^r \to X$, where π denotes the projection to the first factor. If a vector bundle $V \to X$ is isomorphic to the trivial bundle, then for any basis $\mathbf{e}_1, ..., \mathbf{e}_r$ of $\mathbb{C}^r$ the isomorphism $F : X \times \mathbb{C}^r \to V$ defines a frame

$$e_i(x) := F(x, \mathbf{e}_i), \quad 1 \leq i \leq r$$

over the whole of X. Conversely, a global frame for a vector bundle $V \to X$ defines an isomorphism F^{-1}, where F is given by the same formula and then extended fiberwise-linearly. That is to say, if we fix a basis $\mathbf{e}_1, ..., \mathbf{e}_r$ of $\mathbb{C}^n$, we define the isomorphism $F : X \times \mathbb{C}^r \to V$ by

$$F^{-1}(f^i(x)e_i(x)) := (x, f^i(x)\mathbf{e}_i).$$

Thus a vector bundle is isomorphic to the trivial bundle if and only if the vector bundle has a global frame. In particular, every (holomorphic) vector bundle is locally trivial. ◇

1.3.3 EXAMPLE (New bundles from old, pullbacks, etc).

(i) Observe that if $V \to X$ and $W \to X$ are holomorphic vector bundles the so are $V^* \to X$, $V \otimes W \to X$ and $V \oplus W \to X$. Thus $\mathrm{Sym}^k(V) \to X$ and $\Lambda^k V \to X$ are holomorphic vector bundles, as are all vector bundles obtained from holomorphic vector bundles from multi-$\mathbb{C}$-linear operations. On the other hand, the complex conjugate bundle $\overline{V} \to X$, which we will meet often in the text, is in general *not* a holomorphic vector bundle.

(ii) In topology (or even category theory) one has the so-called *fiber product*: Given morphisms $f : X \to Z$ and $g : Y \to Z$ one defines

$$X \times_Z Y := \{(x, y) \in X \times Y \ ; \ f(x) = g(y)\}.$$

There are projection maps $X \times_Z Y \to X$ and $X \times_Z Y \to Y$ given by the restriction to $X \times_Z Y$ of the Cartesian projections $X \times Y \to X$ and $X \times Y \to Y$.

If $\pi : V \to Y$ is a holomorphic vector bundle and $f : X \to Y$ is a holomorphic map then

$$f^* V = V \times_Y X \to X$$

is a holomorphic vector bundle, called the pullback of V by f.

(iii) Given holomorphic vector bundles $V \to X$ and $W \to Y$, one defines the holomorphic vector bundle

$$V \boxtimes W = p_X^* V \otimes p_Y^* W \to X \times Y,$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the Cartesian projections.

1.3.4 EXAMPLE. A vector bundle map $F : V \to W$ can be identified with a holomorphic section of the bundle $W \otimes V^*$. ◇

1.3.2 Transition functions

Let $V \to X$ be a holomorphic vector bundle. As already pointed out, the choice of a frame over an open set $U \subset X$ yields a vector bundle isomorphism ψ to the trivial bundle $U \times \mathbb{C}^r \to U$, defined by

$$\psi(t^i e_i(x)) = (x, t).$$

With two such frames, and the corresponding isomorphisms ψ and ψ', one can form a map

$$\psi' \circ \psi^{-1} : U \cap U' \times \mathbb{C}^r \to U \cap U' \times \mathbb{C}^r.$$

It is clear from the definitions that

$$\psi' \circ \psi^{-1}(x, t) = (x, g_{U'U}(x)t)$$

for some holomorphic function $g_{U'U} : U \cap U' \to GL(r, \mathbb{C})$. The functions $g_{U'U}$ are called *transition functions*, and they contain all of the information of (the isomorphism class of) the vector bundle.

To be more precise, suppose we cover X by a collection of open sets U_α, $\alpha \in A$, over each of which one has a frame for V. Then we get a collection of holomorphic transition functions

$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C}) \; ; \; \alpha, \beta \in A\}.$$

It is easy to see that the conditions

$$g_{\alpha\alpha} = \mathrm{Id} \quad \text{and} \quad g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathrm{Id}, \qquad \alpha, \beta, \gamma \in A$$

on their domains of definition.

Conversely, if one is given an open cover $\{U_\alpha \; ; \; \alpha \in A\}$ of X and a collection of holomorphic functions

$$\mathscr{T} := \{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C}) \; ; \; \alpha, \beta \in A\}$$

that satisfy the so-called *cocycle condition*

$$g_{\alpha\alpha} = \mathrm{Id} \quad \text{and} \quad g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathrm{Id}, \qquad \alpha, \beta, \gamma \in A$$

on their domains of definition. (By taking $\gamma = \alpha$ one sees that $g_{\alpha\beta}^{-1} = g_{\beta\alpha}$.) Then one can define the total space

$$V(\mathscr{T}) := \left(\coprod_{\alpha \in A} U_\alpha \times \mathbb{C}^r \right) / \sim,$$

where

$$U_\alpha \times \mathbb{C}^r \ni (x, t) \sim (y, s) \in U_\beta \times \mathbb{C}^r \quad \iff \quad x = y \quad \text{and} \quad s = g_{\alpha\beta}(x) t,$$

(the cocycle condition implies that $\sim$ is an equivalence relation) and the map $\pi(\mathscr{T}) : V(\mathscr{T}) \to X$ by $\pi(\mathscr{T})([x,t]) = x$. It is easy to see that $V(\mathscr{T})$ is a holomorphic vector bundle whose transition functions are $\mathscr{T}$. Moreover, if $\mathscr{T}$ is a collection of holomorphic transition functions for $V \to X$ the $V(\mathscr{T}) \to X$ is isomorphic to $V \to X$. Indeed, if we denote by $e_{1\alpha}, ...e_{r\alpha}$ the frame over U_α then the map

$$F([x,t]) := t^1 e_{1\alpha}(x) + \cdots + t^r e_{r\alpha}, \quad x \in U_\alpha(x)$$

yields a well-defined holomorphic vector bundle isomorphism.

On the other hand, if $V \to X$ and $W \to X$ are isomorphic, they need not have the same transition functions.

1.3.5 PROPOSITION. *If*

$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C}) \; ; \; \alpha, \beta \in A\} \quad \textit{and} \quad \{\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C}) \; ; \; \alpha, \beta \in A\}$$

are transition functions for $V \to X$ and $W \to X$ over the same open cover (if the open covers are locally finite then by taking finite intersections this can always be arranged) then $V \to X$ and $W \to X$ are isomorphic if and only if there exist maps

$$\{h_\alpha : U_\alpha \to GL(r, \mathbb{C}) \; ; \; \alpha \in A\}$$

such that $\tilde{g}_{\alpha\beta} = h_\alpha^{-1} g_{\alpha\beta} h_\beta$.

The proof is left as an exercise.

1.3.3 Holomorphic structure of the tangent bundle

On a complex manifold X one has the real tangent bundle $T_X \to X$; this is a smooth real vector bundle, and one can then define the complex vector bundle $T_X \otimes_{\mathbb{R}} \mathbb{C} \to X$. The points of the total space of $T_X \otimes_{\mathbb{R}} \mathbb{C}$ are called *complex tangent vectors*.

Now, for a holomorphic coordinate system $z = (z^1, ..., z^n)$ on an open set $U \subset X$ (i.e., an element of the maximal holomorphic atlas of X) one can define the complex tangent vectors

$$\frac{\partial}{\partial z^i} := \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right), \quad 1 \le i \le n, \tag{1.2}$$

where $z^i = x^i + \sqrt{-1}y^i$. These vectors depend on the local coordinate system, but their span does not (Exercise 1.3.2). For each $x \in U$ we define

$$T^{1,0}_{X,x} := \mathrm{Span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}\right\}.$$

The elements of $T^{1,0}_{X,x}$ are called $(1,0)$-*vectors* at x.

For each $\xi \in T^{1,0}_{X,x}$ the vector $\bar{\xi} \in T_X \otimes_{\mathbb{R}} \mathbb{C}$ (the complex conjugate of ξ) does not lie in $T^{1,0}_{X,x}$ (Exercise 1.3.3). Defining

$$T^{0,1}_{X,x} := \overline{T^{1,0}_{X,x}},$$

we obtain the decomposition

$$T_X \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}_{X,x} \oplus T^{0,1}_{X,x}. \tag{1.3}$$

Define the vector bundle $\pi : T^{1,0}_X \to X$ by

$$T^{1,0}_X := \coprod_{x\in X} T^{1,0}_{X,x} \quad \text{and} \quad \pi^{-1}(x) := T^{1,0}_{X,x},$$

with the vector bundle structure given by the frames (1.2). The chain rule shows that $T^{1,0}_X \to X$ is a holomorphic vector bundle.

From basic complex analysis one sees that if U is an open set in X, $x \in U$, $f \in \mathcal{O}(U)$ and $\xi \in T^{1,0}_{X,x}$ then

$$\xi f = (2\mathrm{Re}\,\xi)\, f.$$

Moreover, if $\iota : T_{X,x} \hookrightarrow T_{X,x} \otimes_{\mathbb{R}} \mathbb{C}$ denotes the natural inclusion and $\pi^{1,0} : T_{X,x} \otimes_{\mathbb{R}} \mathbb{C} \to T^{1,0}_{X,x}$ denotes the projection to the first factor in the decomposition (1.3) then the composite map

$$s^{1,0} := \pi^{1,0} \circ \iota : T_X \to T^{1,0}_X$$

is a real isomorphism of vector bundles whose inverse is the map $\xi \mapsto 2\mathrm{Re}\,\xi$.

From these observations one can give $T_X \to X$ the structure of a holomorphic vector bundle in two separate but equivalent ways. The first and simplest way is to map T_X isomorphically to $T^{1,0}_X$; since the latter is a holomorphic vector bundle, so is the former.

1.3.6 DEFINITION. *The vector bundle $T_X^{1,0} \to X$ is called the holomorphic tangent bundle. The dual vector bundle $T_X^{*1,0} \to X$ is called the holomorphic cotangent bundle of X.*

Note that in a local coordinate system a frame for the holomorphic cotangent bundle is given by the complex 1-forms $dz^1, ..., dz^n$.

The second way is to determine the complex structure on T_X directly. To do so, it is first necessary to make explicit the real operator (of the underlying real vector space) associated to multiplication by $\sqrt{-1}$ in a complex vector space. To cut to the chase, the reader should verify (Exercise 1.3.4) that, the operator $J_o : T_X \to T_X$ defined pointwise in a holomorphic coordinate system $z^i = x^i + \sqrt{-1}y^i$ by

$$J_o \frac{\partial}{\partial x^i} := \frac{\partial}{\partial y^i} \quad \text{and} \quad J_o \frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}, \qquad 1 \le i \le n \tag{1.4}$$

is (i) well-defined, independent of the holomorphic cooridnate system, and (ii) satisfies

$$2\text{Re}\,(\sqrt{-1}\xi) = J_o 2\text{Re}\,\xi, \qquad \xi \in T_X^{1,0}.$$

1.3.7 REMARK (Almost complex structures and almost complex manifolds). The operator J_o is a vector bundle map $J_o : T_X \to T_X$ that satisfies

$$J_o^2 = -\text{Id}.$$

Any vector bundle endomorphism $J \in \text{Hom}(T_X, T_X)$ such that $J^2 = -\text{Id}$ has eigenvalues $\pm\sqrt{-1}$, and is therefore only diagonalizable in $T_X \otimes_{\mathbb{R}} \mathbb{C}$. In the latter bundle every eigenvector has a complex conjugate eigenvector, so that the eigenspaces

$$E_+(J) := \{v \in T_X \otimes_{\mathbb{R}} \mathbb{C}\ ;\ Jv = \sqrt{-1}v\} \quad \text{and} \quad E_-(J) := \{v \in T_X \otimes_{\mathbb{R}} \mathbb{C}\ ;\ Jv = -\sqrt{-1}v\}$$

are isomorphic under the complex conjugation map, and together span $T_X \otimes_{\mathbb{R}} \mathbb{C}$. The endomorphism J is therefore in 1-1 correspondence with such a splitting, and either object is called an *almost complex strucutre*. A real manifold with an almost complex structure is called an *almost complex manifold.* and as a matter of notation, one also writes

$$E_+(J) = T_X^{1,0}(J) \quad \text{and} \quad E_-(J) = T_X^{0,1}(J)$$

for the J-eigenspaces.

The almost complex structure J_o has an additional feature: one can find an atlas for X so that (1.4) holds on every coordinate chart. An almost complex structure J is said to be *integrable* if there is a holomorphic atlas with respect to which J is given by (1.4).

If the manifold X is a surface, i.e., of real dimension equal to 2, then every almost complex structure is integrable (if we accept the theorem on existence of isothermal coordinates). Indeed, if we choose any metric g_o on X and let $g := g_o + J^* g_o$ then J is an isometry of g whose square is $-\text{Id}$. If we then choose an atlas of isothermal coordinates for g then J will be an isometry of the Euclidean metric in a coordinate chart, and hence either J or $-J$ will satisfy (1.4).

In higher dimension, integrability is more subtle, as there are non-integrable almost complex structures. The coordinate charts that make (1.4) hold, if they exist, are again solutions of a partial differential equation. Fortunately the existence of solutions is equivalent to an algebraic condition called *involutivity* on the almost complex structure, which is analogous to the Fröbenius involutivity condition:

$$[\Gamma(X, T_X^{1,0}(J)), \Gamma(X, T_X^{1,0}(J))] \subset \Gamma(X, T_X^{1,0}(J)),$$

where $[\cdot,\cdot]$ is the complex linear extension of the Lie bracket of vector fields on X.

More precisely, the algebraic condition of involutivity measures the obstruction to solving the aforementioned partial differential equation (in fact, it's a system of PDE). The result that proves the existence of a holomorphic atlas when the manifold has an involutive almost complex structure is called the Newlander-Nirenberg Theorem. As was already indicated, it is a generalization of the theorem on the existence of isothermal coordinates.

With the methods developed in these notes it is almost possible to prove the Newlander-Nirenberg Theorem; such a proof is due to J. J. Kohn, who wrote in his article on the subject that the idea for the proof came from D. C. Spencer. Unfortunately we will not have sufficient time to discuss this result. ◇

1.3.4 Some non-trivial examples of holomorphic line bundles

1.3.8 EXAMPLE (The canonical bundle). The canonical bundle $K_X \to X$ is the determinant of the holomorphic cotangent bundle:

$$K_X := \det T_X^{*1,0} := \underbrace{T_X^{*1,0} \wedge \dots \wedge T_X^{*1,0}}_{n \text{ copies}}.$$

In a holomorphic local coordinate chart a frame of K_X is given by the n-form $dz^1 \wedge \dots \wedge dz^n$. Thus the transition functions for K_X are the determinants of the transition functions for $T_X^{*1,0}$.(Note that if X is a Riemann surface then of course $K_X = T_X^{*1,0}$.)

The name 'canonical' refers to the fact that K_X is essentially the only natural (often) non-trivial line bundle defined on every complex manifold X. ◇

1.3.9 EXAMPLE (The Tautological bundle and its dual, the hyperplane bundle). Consider the space

$$\mathbb{U} := \{(z, \ell) \in \mathbb{C}^{n+1} \times \mathbb{P}_n \; ; \; z \in \ell\}.$$

This space was first introduced as the blowup of the origin in $\mathbb{C}^{n+1}$ (where the notation $\tilde{\mathbb{C}}_o^{n+1}$ was used), with the blowdown map being the restriction to $\mathbb{U}$ of the projection to the first factor. Let $\pi : \mathbb{U} \to \mathbb{P}_n$ be the restriction to $\mathbb{U}$ of the projection to the second factor. Then π is holomorphic, and the preimage of a point $\ell \in \mathbb{P}_n$, thought of as a line through the origin, is the set of points comprising the line ℓ. In the chart

$$U_o = \{[1, z] \; ; \; z \in \mathbb{C}^n\}$$

we have the holomorphic section

$$\mathbf{e}_o([1, z]) = ((1, z), [1, z])$$

which defines a frame for $\pi : \mathbb{U} \to \mathbb{P}_n$ over U_o. More generally, in the chart

$$U_j = \{[z^1, ..., z^j, 1, z^{j+1}, ..., z^n] \, ; \, z \in \mathbb{C}^n\}$$

we have the section

$$\mathbf{e}_j([z^1, ..., z^j, 1, z^{j+1}, ..., z^n]) = ((z^1, ..., z^j, 1, z^{j+1}, ..., z^n), [z^1, ..., z^j, 1, z^{j+1}, ..., z^n]).$$

It is easy to see that the change of frame over $U_i \cap U_j$ is holomorphic. Thus $\mathbb{U} \to \mathbb{P}_n$ is a holomorphic line bundle, called the *tautological line bundle* (and sometimes also the universal line bundle).

The line bundle $\mathbb{U} \to \mathbb{P}_n$ is not trivial. We shall see this non-triviality by showing that the dual bundle is not trivial.

The dual line bundle to $\mathbb{U} \to \mathbb{P}_n$ is called the *hyperplane line bundle*, and denoted $\mathbb{H} \to \mathbb{P}_n$. Its fiber $\mathbb{H}_\ell$ consists of the set of linear functionals on the 1-dimensional subspace ℓ of $\mathbb{C}^{n+1}$. Thus a linear function $\lambda : \mathbb{C}^{n+1} \to \mathbb{C}$ defines a section $s_\lambda : \mathbb{P}_n \to \mathbb{H}$ of $\mathbb{H} \to \mathbb{P}_{n+1}$ via the formula

$$\langle s_\lambda(\ell), (z, \ell) \rangle := \lambda(z).$$

It is easy to check that this section is holomorphic, and that $s_\lambda(\ell) = 0$ if and only if $\ell \subset \mathrm{Kernel}(\lambda)$. (As we shall see later on, every holomorphic section of $\mathbb{H} \to \mathbb{P}_n$ is of the form s_λ for some $\lambda \in (\mathbb{C}^{n+1})^*$.)

By contrast, $\mathbb{U} \to \mathbb{P}_n$ has no global holomorphic sections other than the zero section. Indeed, suppose $\mathbb{U} \to \mathbb{P}_n$ has a holomorphic section $\theta : \mathbb{P}_n \to \mathbb{U}$. Given any linear functional $\lambda \in (\mathbb{C}^{n+1})^* - \{0\}$, we have the section s_λ. The duality pairing

$$g(\ell) := \langle s_\lambda(\ell), \theta(\ell) \rangle$$

is thus a well-defined holomorphic function on $\mathbb{P}_n$. Since $\mathbb{P}_n$ is compact, g must be constant. Since every non-identically zero linear functional on $\mathbb{C}^{n+1}$ has a non-trivial kernel for $n \geq 1$, $g = 0$, and therefore θ vanishes on the complement of the zero locus of s_λ. But by the identity principle, $\theta = 0$. In particular, $\mathbb{U} \to \mathbb{P}_n$ *is not trivial*, and therefore neither is $\mathbb{H} \to \mathbb{P}_n$.

We shall see the line bundles $\mathbb{U} \to \mathbb{P}_n$ and $\mathbb{H} \to \mathbb{P}_n$ very often in the sequel. ◇

1.3.10 EXAMPLE (The line bundle of a hypersurface). Let X be a complex manifold and let Z be a smooth complex hypersurface. By definition there is an open cover $\{U_\alpha \, ; \, \alpha \in J\}$ and holomorphic functions $f_\alpha \in \mathcal{O}(U_\alpha)$ such that

$$Z \cap U_\alpha = \{f_\alpha = 0\} \qquad \text{and} \qquad df_\alpha(z) \neq 0 \text{ for every } z \in Z \cap U_\alpha.$$

It follows that the functions

$$g_{\alpha\beta} := \frac{f_\alpha}{f_\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$$

have no zeros or poles. The collection $\mathscr{T}$ of all of these functions obviously satisfies the cocycle condition, and hence there is a holomorphic line bundle

$$L_Z \to X$$

whose transition functions are $\mathscr{T}$.

1.3.11 DEFINITION. *The line bundle L_Z is called the line bundle associated to the smooth hypersurface Z.*

We will meet the line bundle L_Z when we study L^2 extension. $\diamond$

EXERCISES

1.3.1. Prove Proposition 1.3.5.

1.3.2. Show that the span $T^{1,0}_{X,x}$ of the vectors (1.2) is independent of the choice of local coordinates.

1.3.3. Show that if $\xi \in T^{1,0}_{X,x}$ then $\bar{\xi} \notin T^{1,0}_{X,x}$.

1.3.4. Show that the operator J defined by (1.4) is independent of the holomorphic coordinate system and that it is intertwined with $\sqrt{-1}$ via the map $s^{1,0}$, i.e., $s^{1,0}J = \sqrt{-1}s^{1,0}$.

1.3.5. Show that every holomorphic section of the hyperplane line bundle $\mathbb{H} \to \mathbb{P}_n$ of Example 1.3.9 is of the form s_λ for some $\lambda \in (\mathbb{C}^{n+1})^*$.

1.3.6. Find all the holomorphic sections the line bundle $T^{1,0}_{\mathbb{P}_1} \to \mathbb{P}_1$.

1.3.7. Show that the only holomorphic section of the line bundle $(T^{1,0}_{\mathbb{P}_1})^* \to \mathbb{P}_1$ dual to the tangent bundle is the zero section.

1.3.8. Show that the line bundle $K_{\mathbb{P}_n} \to \mathbb{P}_n$ is isomorphic to $\mathbb{U}^{\otimes(n+1)} \to \mathbb{P}_n$.

1.4 Differential forms on complex manifolds

Not surprisingly, on a complex manifold one wants to consider complex-valued differential forms, i.e., sections of the bundle

$$\mathscr{E}_X := \bigoplus_{r=1}^{2n} \Lambda^r_X \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{r=1}^{2n} \bigwedge^r (T_X \otimes_{\mathbb{R}} \mathbb{C}).$$

The sections of this bundle form an algebra with respect to the wedge product (or more precisely, the complex linear extension of the a priori real wedge product), and the differential d (again extended $\mathbb{C}$-linearly) acts on the sections of $\mathscr{E}_X \to X$, mapping sections of $\Lambda^r_X \to X$ to sections of $\Lambda^{r+1}_X \to X$:

$$d(\Gamma(X, \Lambda^r_X)) \subset \Gamma(X, \Lambda^{r+1}_X).$$

Forms in the kernel of d are called *closed*, and forms in the image of d are called *exact*.

1.4.1 Forms of bidgree (p, q)

The splitting $T_X \otimes_{\mathbb{R}} \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1}$ induces a splitting

$$\Lambda_X^r \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=r} \bigwedge^p T_X^{1,0} \wedge \bigwedge^q T_X^{0,1},$$

and, with the notation

$$\Lambda_X^{p,q} := \bigwedge^p T_X^{1,0} \wedge \bigwedge^q T_X^{0,1},$$

one has projections

$$\pi^{p,q} : \Lambda_X^r \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda_X^{p,q}.$$

The wedge product sends an element of $\Lambda_{X,x}^{p,q}$ and an element of $\Lambda_{X,x}^{p',q'}$ to an element of $\Lambda_{X,x}^{p+p',q+q'}$.

The smooth vector bundle $\Lambda_X^{p,q} \to X$ is a holomorphic vector bundle if and only if $q = 0$.

1.4.1 DEFINITION. *The sections of $\Lambda_X^{p,q} \to X$ are called forms of bidegree (p, q), or (p, q)-forms.*

Locally every (p, q)-form is of the form

$$\alpha = f_{I\bar{J}} dz^I \wedge d\bar{z}^J, \tag{1.5}$$

where $dz^I \wedge d\bar{z}^J := dz^{i_1} \wedge ... \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge ... \wedge d\bar{z}^{j_q}$.

The expression (1.5) is not unique, i.e., different choices of coefficient functions $f_{I\bar{J}}$ can result in the same α. However the coefficient functions are uniquely determined by the form if one imposes the assumption of skew symmetry on the coefficient functions $f_{I\bar{J}}$ of the (p, q)-form α: if $\sigma \in \mathbb{S}_p$ and $\tau \in \mathbb{S}_q$ are permutations then one can impose the condition

$$f_{I_\sigma \overline{J_\tau}} = (\mathrm{sgn}\sigma)(\mathrm{sgn}\tau) f_{I\bar{J}}$$

on the coefficients of α, where a permutation $\nu \in \mathbb{S}_r$ acts on an r-tuple $K = (k_1, ..., k_r)$ by the formula

$$K_\nu = (k_{\nu(1)}, ..., k_{\nu(r)}).$$

Some authors use increasing indices I and J, meaning $1 \le i_1 < i_2 < ... < i_p \le n$ and similarly for J. However, because the wedge product and exterior derivative are naturally skew-symmetric object, representing a form using skew symmetric coefficients often results in much simpler computations.

1.4.2 REMARK. Here and in the rest of the text we are using summation convention: one sums over all the possible values of the (multi)indices of the same letter, and these appear in pairs with one a subscript and the other a superscript; moreover, the conjugation-parity is important. This summation convention is a bit confusing at first, but in fact it respects the various dualities that arise in complex geometry, and after a while it passes from the status of 'confusing' to the status of 'assisting', helping one to correctly think of the geometric nature of the quantities at hand.

The last sentence in the previous paragraph may seem cryptic, but it is traditional, and the author is confident that anyone that sticks to this convention will eventually see its benefits, and will find the remark about duality eventually illuminating. ◇

1.4.2 Exterior differential operators

The exterior algebra of a complex manifold is equipped with two additional differential operators: these are the operators ∂ and $\bar{\partial}$, and they are defined on sections of $\Lambda^{p,q}_X \to X$ by

$$\partial \alpha := \pi^{p+1,q} d\alpha \quad \text{and} \quad \bar{\partial}\alpha := \pi^{p,q+1} d\alpha, \qquad \alpha \in \Gamma(X, \Lambda^{p,q}_X).$$

One has the following proposition.

1.4.3 PROPOSITION. *$d = \partial + \bar{\partial}$, and consequently $\bar{\partial}^2 = 0$.*

The proof is left as an exercise.

1.4.4 REMARK. Since the splitting of $T_X \otimes_{\mathbb{R}} \mathbb{C}$ makes sense on an almost complex manifold, one also has the splitting of differential forms into components of bidegree (p, q) in such manifolds. Proposition 1.4.3 fails in a general almost complex manifold, and in fact it holds if and only if the almost complex structure is involutive, i.e., if and only if

$$[\Gamma(X, T^{1,0}_X(J)), \Gamma(X, T^{1,0}_X(J))] \subset \Gamma(X, T^{1,0}_X(J))$$

(c.f. Remark 1.3.7). In these notes all of our almost complex structures are integrable, so we will not discuss the issue of integrability any further. red

1.4.3 Twisted differential forms

1.4.5 DEFINITION. *Let X be a complex manifold and let $E \to X$ be a holomorphic vector bundle. An E-valued $(p.q)$-form is a section of the vector bundle*

$$\Lambda^{p,q}_X \otimes E \to X.$$

If the vector bundle E is not explicitly referred to then one says that such a section is a twisted differential form.

After one tensors with a holomorphic vector bundle, the exterior operator is no longer well defined. More precisely, if we choose a frame $\xi_1, ..., \xi_r$ for E and write

$$u = u^{\nu}_{I\bar{J}} dz^I \wedge d\bar{z}^J \otimes \xi_\nu$$

then the form

$$\frac{\partial u^{\nu}_{I\bar{J}}}{dz^k} dz^k \wedge dz^I \wedge d\bar{z}^J \otimes \xi_\nu$$

does depend on the holomorphic coordinate system, and therefore does not define a global section of the bundle $E \otimes \Lambda^{p+1,q}_X \to X$. However, remarkably, the form

$$\frac{\partial u^{\nu}_{I\bar{J}}}{d\bar{z}^k} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J \otimes \xi_\nu$$

is indeed globally defined. This fact, which is left to the exercises, immediately implies the following proposition.

1.4.6 PROPOSITION–DEFINITION. *Let X be a complex manifold and let $E \to X$ be a holomorphic vector bundle, and let $p, q \in \{1, ..., n\}$. Then there is a local operator*

$$\bar{\partial} : \Gamma(X, \Lambda^{p,q}_X \otimes E) \to \Gamma(X, \Lambda^{p,q+1}_X \otimes E)$$

defined as follows. If $u = u^{\nu}_{I\bar{J}} \xi_{\nu} \otimes dz^I \wedge d\bar{z}^J$ in some local coordinates and a local frame then

$$\bar{\partial} u := \frac{\partial u^{\nu}_{I\bar{J}}}{d\bar{z}^k} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J \otimes \xi_{\nu}$$

1.4.7 REMARK. It is worth making one more observation; an observation that will greatly simplify many computations. Since

$$\Lambda^{p,q}_X \cong \Lambda^{p,0}_X \otimes \Lambda^{0,q}_X \cong \Lambda^{p,0}_X \otimes K^*_X \otimes \Lambda^{n,q}_X,$$

one can write

$$\Lambda^{p,q}_X \otimes E \cong \Lambda^{n,q}_X \otimes F$$

where $F = \Lambda^{p,0}_X \otimes K^*_X \otimes E$. The crucial point here is that F is a holomorphic vector bundle, and so the $\bar{\partial}$ operator of Proposition-Definition 1.4.6 is computed locally in exactly the same way for E-valued (p, q)-forms as for F-valued (n, q)-forms. Consequently when working with twisted forms, especially when focus on the $\bar{\partial}$-equation, it often suffices to consider only twisted (n, q)-forms. $\diamond$

1.4.8 REMARK. Note that while the exterior differential operator $\bar{\partial}$ extends naturally to holomorphically twisted forms, the exterior operator ∂ does not have such a natural extension. The reason, as one can see after doing Exercise 1.4.2, is that the transition functions for a holomorphic vector bundle are annihilated by $\bar{\partial}$, but generally not by ∂. $\diamond$

EXERCISES

1.4.1. Prove Proposition 1.4.3.

1.4.2. Prove Proposition-Definition 1.4.6. (For the sake of developing intuition, it may be useful to start with the case $p = q = 0$.)

1.4.3. Show that there are no non-zero $\bar{\partial}$-closed $(n, 0)$-forms on $\mathbb{P}_n$.

Lecture 2

Hermitian metrics

The terminology 'Hermitian metrics' is ambiguous in complex geometry: on the one hand, it can refer to metrics for a complex vector bundle, and on the other, for certain Riemannian metrics on almost complex manifolds. In the case where the underlying manifold is a complex manifold, there is a link between the two notions.

To avoid the standard ambiguity, we shall refer to the first type of metric as a Hermitian metric, and to the second type of metric as a Hermitian Riemannian metric.

2.1 Hermitian metrics for complex vector bundles

Let M be a manifold and let $E \to M$ be a complex vector bundle.

2.1.1 DEFINITION. *A Hermitian metric for $E \to M$ is a section $\mathfrak{H}$ of the bundle $E^* \otimes \overline{E^*} \to M$ such that for all $x \in M$ and $v, w \in E_x$,*

(i) $\langle \mathfrak{H}, v \otimes \bar{w} \rangle = \overline{\langle \mathfrak{H}, w \otimes \bar{v} \rangle}$ (i.e., $\mathfrak{H}$ is Hermitian symmetric), and

(ii) $\langle \mathfrak{H}, v \otimes \bar{v} \rangle > 0$ for all $v \neq 0$ (i.e., $\mathfrak{H}$ is positive-definite).

In other words, $\mathfrak{H}$ defines a sesquilinear, positive definite Hermitian form on each fiber E_x of the vector bundle $E \to M$. ◇

If $\alpha^1, ..., \alpha^r$ is a frame for E^* over an open set U then one can write

$$\mathfrak{H} = \mathfrak{H}_{i\bar{j}} \alpha^i \cdot \bar{\alpha}^j$$

for functions $\mathfrak{H}_{i\bar{j}}$ over U satisfying

$$\overline{\mathfrak{H}_{i\bar{j}}} = \mathfrak{H}_{j\bar{i}} \quad \text{and} \quad \mathfrak{H}_{i\bar{j}} a^i \bar{a}^j \geq \varepsilon ||a||^2$$

for some positive function ε on U and all $a \in \mathbb{C}^r$. (Here $\alpha^i \cdot \bar{\alpha}^j = \alpha^i \otimes \bar{\alpha}^j + \bar{\alpha}^j \otimes \alpha^i$ is the symmetric product.) That is to say, at each $x \in U$ the matrix $\left(\mathfrak{H}_{i\bar{j}}(x)\right)_{i,j=1}^r$ is Hermitian and positive-definite. The regularity of the functions $\mathfrak{H}_{i\bar{j}}$ is declared to be the regularity of $\mathfrak{H}$.

Although we will think of the Hermitian metric $\mathfrak{H}$ for E as a section of $E^* \otimes \overline{E^*}$, we will often treat it as a Hermitian inner product on the fibers of E. As such, we will abusively write

$$\mathfrak{H}(v, w) := \langle \mathfrak{H}, v \otimes \bar{w} \rangle .$$

Exercises

2.1.1. Let $\mathbb{U} \to \mathbb{P}_n$ be the tautological line bundle of Example 1.3.9. Use the Euclidean metric on $\mathbb{C}^{n+1}$ to define a Hermitian metric on $\mathbb{U} \to \mathbb{P}_n$.

2.1.2. Let X be a complex manifold and let $L \to X$ be a holomorphic line bundle. Let $s_1, ..., s_m$ be sections of L whose common zero locus is empty. Show that $L \to X$ has a Hermitian metric h such that

$$h(s_i, s_i) \le 1$$

for all $1 \le i \le m$.

2.2 Hermitian Riemannian metrics for complex manifolds

Let X be an almost complex manifold and let J be the almost complex structure defined by (1.4).

2.2.1 DEFINITION. *A Riemannian metric g on X is said to be Hermitian if $J^*g = g$, i.e.,*

$$g(J\xi, J\eta) = g(\xi, \eta)$$

for all $x \in X$ and all $\xi, \eta \in T_{X,x}$. ◇

2.2.1 Structure of the metric

The condition of J-invariance puts some strong restrictions on the form of the metric; conditions that are best seen in the decomposition of the complexified tangent space according to the eigenspaces of J. In terms of the splitting $T_X \otimes_{\mathbb{R}} \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1}$, let us write (the complexification of) our metric as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

i.e.,

$$g(v_1 + \bar{w}_1, v_2 + \bar{w}_2) = v_1 \cdot Av_2 + v_1 \cdot B\bar{w}_2 + \bar{w}_1 \cdot Cv_2 + \bar{w}_1 \cdot D\bar{w}_2.$$

Let us analyze the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$.

First, if $v_1, v_2 \in T_X^{1,0}$, then

$$g(v_1, v_2) = g(Jv_1, Jv_2) = g(\sqrt{-1}v_1, \sqrt{-1}v_2) = -g(v_1, v_2),$$

and thus $A = 0$. Since the metric g is the complexification of a real metric, $g(\bar{v}, \bar{w}) = \overline{g(v, w)}$, and therefore $D = 0$.

Again by conjugation we have $\overline{g(v, \overline{w})} = g(\overline{v}, w)$, and thus

$$C = \overline{B}.$$

Finally, since g is symmetric, $C = B^{\text{trans}}$, and therefore

$$B^\dagger = B,$$

where † means transpose conjugate. Thus we have

$$g = \begin{pmatrix} 0 & h \\ \overline{h} & 0 \end{pmatrix}$$

for some Hermitian metric h for $T_X^{1,0}$. We therefore see that Riemannian Hermitian metrics for X are in one-to-one correspondence with Hermitian metrics for $T_X^{1,0}$. This fact partially explains the ambiguity in the name 'Hermitian metric'.

We compute that for real vectors v and w, one has

$$\begin{aligned} 2h(s^{1,0}v, \overline{s^{1,0}w}) &= 2g(\tfrac{1}{2}(v - \sqrt{-1}Jv), \tfrac{1}{2}(w + \sqrt{-1}Jw)) \\ &= \frac{1}{2}\big(g(v,w) + g(Jv,Jw) + \sqrt{-1}(g(v,Jw) - g(Jv,w))\big) \\ &= g(v,w) - \sqrt{-1}g(Jv,w), \end{aligned}$$

where in the last equality we used the Hermitian symmetry of g. Therefore

$$g(v,w) = 2\mathrm{Re}\, h(s^{1,0}v, \overline{s^{1,0}w}),$$

and hence one can recover the metric g on X from a metric h on $T_X^{1,0}$.

2.2.2 Metric form

The negative of the imaginary part of $2h(s^{1,0}v, \overline{s^{1,0}w})$, i.e.,

$$\omega_g(v,w) := g(Jv,w),$$

is called the metric form of the Hermitian Riemannian metric g. The form ω_g is skew symmetric, since

$$g(Jv,w) = g(w,Jv) = g(Jw,J^2v) = -g(Jw,v).$$

The formula $g(v,w) = \omega_g(v,Jw)$ shows that there is a 1-1 correspondence between g, ω_g and h.

In terms of frames, if $\alpha^1, ..., \alpha^n$ is a local frame for $T_X^{*1,0}$ then a Hermitian metric for $T_X^{1,0}$ may be written

$$h = h_{i\bar{j}}\alpha^i \otimes \bar{\alpha}^j.$$

In terms of this frame, if $g(v,w) := 2\mathrm{Re}\, h(s^{1,0}v, \overline{s^{1,0}w})$ is the associated Hermitian Riemannian metric then

$$\omega_g = \sqrt{-1}h_{i\bar{j}}\alpha^i \wedge \bar{\alpha}^j.$$

Indeed, the right hand is well-defined because the vector bundles $T_X^{*1,0} \otimes T_X^{*0,1}$ and $T_X^{*1,0} \wedge T_X^{*0,1}$ have the same transition functions, and we compute that

$$\begin{aligned}\sqrt{-1}h_{i\bar{j}}\alpha^i \wedge \bar{\alpha}^j(v,w) &= \sqrt{-1}h_{i\bar{j}}\alpha^i \wedge \bar{\alpha}^j\left(s^{1,0}v + \overline{s^{1,0}v}, s^{1,0}w + \overline{s^{1,0}w}\right)\\ &= \sqrt{-1}h_{i\bar{j}}\left(\left\langle \alpha^i, s^{1,0}v\right\rangle \overline{\langle \alpha^j, s^{1,0}w\rangle} - \left\langle \alpha^i, s^{1,0}w\right\rangle \overline{\langle \alpha^j, s^{1,0}v\rangle}\right)\\ &= -2\mathrm{Im}\ h_{i\bar{j}}\left\langle \alpha^i, s^{1,0}v\right\rangle \overline{\langle \alpha^j, s^{1,0}w\rangle} = -\mathrm{Im}\ 2h(s^{1,0}v, s^{1,0}w),\end{aligned}$$

where the second-to-last equality follows from the Hermitian symmetry of $h_{i\bar{j}}$.

2.2.3 Volume

Given a Riemannian metric g on an m-dimensional real manifold, in a local coordinate system $x = (x^1, ..., x^m)$ one can define the m-form

$$\sqrt{\det\left(g\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right)\right)}dx^1 \wedge \cdots \wedge dx^m.$$

If one changes to another coordinate system $y = (y^1, ..., y^m)$ then

$$\sqrt{\det\left(g\left(\tfrac{\partial}{\partial y^i}, \tfrac{\partial}{\partial y^j}\right)\right)}dy^1 \wedge \cdots \wedge dy^m = \frac{\left|\det\left(\frac{\partial y^i}{\partial x^j}\right)\right|}{\det\left(\frac{\partial y^i}{\partial x^j}\right)}\sqrt{\det\left(g\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right)\right)}dx^1 \wedge \cdots \wedge dx^m.$$

If one can choose an atlas whose transition functions all have positive determinant then one gets a globally defined form

$$dV_g := \sqrt{\det\left(g\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right)\right)}dx^1 \wedge \cdots \wedge dx^m$$

of top degree with no zeros, i.e., a volume form. The existence of such an atlas is the definition of orientability of a manifold. In particular, if the manifold is complex then the transition functions, being holomorphic, have this property due to (1.1). (In this case the manifold is not only orientable, but in fact oriented, i.e., there is a preferred atlas whose transition functions have derivatives with positive determinant.)

If the manifold is complex and the Riemannian metric is furthermore Hermitian then the situation for the volume form is even better: In particular, for a Riemannian Hermitian metric g on a complex manifold X

$$dV_g = \frac{1}{n!}\omega_g^n,$$

where ω_g is the metric form of g. Indeed, if in local complex coordinates $z = (z^1, ..., z^n)$ and corresponding real coordinates $\xi = (\xi^1, ..., \xi^{2n})$ where $\xi^{2i-1} = \mathrm{Re}\, z^i$, $\xi^{2i} = \mathrm{Im}\, z^i$, we write $g = g_{ij}d\xi^i d\xi^j$ and $h = h_{\alpha\bar{\beta}}dz^\alpha \cdot d\bar{z}^\beta$ then

$$\det\left((g_{ij})_{ij=1}^{2n}\right) = (-1)^n(\det(h_{\alpha\bar{\beta}}))^2,$$

and thus

$$
\begin{aligned}
\omega^n &= \sqrt{-1}^n h_{\alpha_1\bar\beta_1}...h_{\alpha_n\bar\beta_n} dz^{\alpha_1}\wedge d\bar z^{\beta_1}\wedge ...\wedge dz^{\alpha_1}\wedge d\bar z^{\beta_1}\\
&= \frac{(-1)^{n(n+1)/2}}{(2\sqrt{-1})^n} h_{\alpha_1\bar\beta_1}...h_{\alpha_n\bar\beta_n} dz^{\alpha_1}\wedge ...\wedge dz^{\alpha_n}\wedge d\bar z^{\beta_1}\wedge ...\wedge d\bar z^{\beta_n}\\
&= \left(\sum_{\alpha_1,...,\bar\beta_n} \operatorname{sgn}\begin{pmatrix} 1 & ... & n & \bar 1 & ... & \bar n\\ \alpha_1 & ... & \alpha_n & \bar\beta_1 & ... & \bar\beta_n\end{pmatrix} h_{\alpha_1\bar\beta_1}...h_{\alpha_n\bar\beta_n}\right)\\
&\qquad\qquad \cdot \frac{(-1)^{n(n+1)/2}}{(2\sqrt{-1})^n} dz^1\wedge ...\wedge dz^n\wedge d\bar z^1\wedge ...\wedge d\bar z^n\\
&= n!\det(h_{\alpha\bar\beta})\frac{dz^1\wedge d\bar z^1}{2\sqrt{-1}}\wedge ...\wedge \frac{dz^n\wedge d\bar z^n}{2\sqrt{-1}}.
\end{aligned}
$$

Let us consider a complex submanifold Y of our Hermitian manifold X. We can endow Y with a Hermitian metric simply by restricting the metric from the ambient space.

Now choose local coordinates $z^1, ..., z^n$ on X in such a way that the functions $z^1, ..., z^k$ are coordinates on the submanifold Y. The tangent spaces of Y are then defined by the vanishing of the differentials $dz^{k+1}, ..., dz^n$. In other words, in these coordinates, the Hermitian metric for Y is given by

$$h|_Y = h_{\alpha\bar\beta} dz^\alpha \cdot d\bar z^\beta,$$

where this time the summation is carried only from 1 to k. In particular, one can carry out all of the above calculations on the submanifold and obtain the following remarkable fact, often incorrectly referred to as Wirtinger's Theorem.

2.2.2 THEOREM. *If X is a Hermitian manifold with Hermitian metric form ω and $\iota : Y \hookrightarrow X$ is a k-dimensional submanifold then the associated Hermitian volume form of Y is*

$$dV_{g|_Y} = \frac{1}{k!}\iota^*\omega_g^k = \frac{1}{k!}\omega_{g|_Y}^k.$$

The original theorem of Wirtinger is the following.

2.2.3 THEOREM (Wirtinger). *If W is a $2k$-dimensional real submanifold of a Hermitian manifold (X, ω) then*

$$\mathit{Area}(W) \ge \int_W \frac{\omega^k}{k!}.$$

Moreover, if the area of W is finite then equality holds if and only if W is a complex manifold.

Theorem 2.2.3 is often important in the study of minimal surfaces. Since we will not need it, the proof of Theorem 2.2.3 is omitted.

EXERCISES

2.2.1. Let X be a compact complex manifold and let g be a metric whose metric form ω_g is a closed 2-form. Show that there are no non-zero $\bar\partial$-exact $(p, p-1)$-forms on X.

Lecture 3

Connections and Curvature

3.1 Connections

In this section we define and explore the notion of connection for a vector bundle on a manifold. The underlying field can be $\mathbb{R}$ or $\mathbb{C}$; when we use the term *linear*, we assume scalars lie in this field.

3.1.1 Basic definition

3.1.1 DEFINITION. *Let M be a manifold and let $E \to M$ be a complex vector bundle. A connection for $E \to M$ is a linear map*

$$D : \Gamma(M, E) \to \Gamma(M, T_M^* \otimes E),$$

satisfying the Leibniz rule

$$D(fs) = df \otimes s + fDs.$$

The Leibniz rule implies the following proposition.

3.1.2 PROPOSITION. *If D_1 and D_2 are two connections for a vector bundle $E \to M$ then*

$$D_1 - D_2 \in \Gamma(M, \mathscr{C}^\infty(T_M^* \otimes \mathrm{End}(E))).$$

Proposition 3.1.2 says that the space of connections on a vector bundle $E \to M$ is an affine space modeled on the vector space $\Gamma(M, \mathscr{C}^\infty(T_M^* \otimes \mathrm{End}(E)))$.

REMARK. There are several notions of connection, that have increasing generality. The type of connection we are considering here is often called an *affine connection*, presumably in reference to the affine linear structures of the fibers of a vector bundle. ◇

Definition 3.1.1 implies that a connection is a local operator, and may be restricted to small subsets. If one restricts to a sufficiently small subset $U \subset M$ then the vector bundle $E|_U \to U$ is isomorphic to the trivial bundle, i.e., it admits a frame.

With the choice of a frame $e_1, ..., e_r$ for $E|_U$ one has the trivial connection d defined by

$$d(s^i e_i) = ds^i \otimes e_i.$$

Note that this connection depends on the trivialization.

By Proposition 3.1.2 any other connection D for $E|_U$ is obtained from the trivial connection by adding to the latter a section A of $T_M^* \otimes \mathrm{Hom}(E|_U) \to U$:

$$D|_U = d + A.$$

The section is called the *connection form*. Since the trivial connection depends on the frame, so does the connection form A.

In terms of the frame $e_1, ..., e_r$ for $E|_U \to U$, one can write

$$Ae_i = A_i^j \otimes e_j,$$

and then by the Leibniz rule

$$D(s^i e_i) = ds^i \otimes e_i + s^i A_i^j \otimes e_j.$$

The matrix of 1-forms (A_i^j) is called the *connection matrix*.

Exercise 3.1.2 asks the reader to compute the transformation formula satisfied by the connection matrix under change of frame.

3.1.2 Induced connections

A connection for a vector bundle $E \to M$ induces connections on all vector bundles obtained from E via multilinear operations. Similarly, a collection of vector bundles with connections induces natural connections on various products of these vector bundles.

Dual connection

Given a connection D for a vector bundle $E \to M$, one defines the connection $D^\vee$ for the dual vector bundle $E^* \to M$ as follows. Given local sections s for E and α for E^*, one has a pairing $\langle s, \alpha \rangle$, which is a function on M. We require the dual connection $D^\vee$ to satisfy

$$d\langle s, \alpha \rangle = \langle Ds, \alpha \rangle + \langle s, D^\vee \alpha \rangle. \tag{3.1}$$

If we fix a frame $e_1, ..., e_r$ for E and denote by $\alpha^1, ..., \alpha^r$ its dual frame then the connection forms $A(D)$ and $A(D^\vee)$ satisfy

$$0 = d\delta_i^{\ j} = d\langle e_i, \alpha^j \rangle = \langle A(D)_i^{\ k} e_k, \alpha^j \rangle + \langle e_i, A(D^\vee)_\ell^j \alpha^\ell \rangle = A(D)_i^{\ j} + A(D^\vee)_i^{\ j}.$$

Thus the dual connection $D^\vee$ is completely determined by (3.1).

Product connections

Let $E_1 \to M$ and $E_2 \to M$ be two vector bundles equipped with connections D_1 and D_2 respectively, one would like to have a natural definition of connection for various products of V_1 and V_2. Given any product operation $\times$ (e.g. tensor, symmetric, or wedge product) one defines the product connection D for $E_1 \times E_2 \to M$ by the formula

$$D\left(s_1 \times s_2\right) = \left(D_1 s_1\right) \times s_2 + s_1 \times \left(D_2 s_2\right).$$

By induction, one can pass to any finite product of vector bundles.

3.1.3 EXAMPLE (Induced connections for determinant bundles). Let $E \to M$ be a vector bundle of rank r and let a connection D_E for E be given. Consider the complex line bundle

$$\det E \to M$$

whose transition functions are just the determinants of the corresponding transition functions for E. (We have met the determinant construction in the definition of the canonical bundle.) Fix a frame $e_1, ..., e_r$ for E. Then $e_1 \wedge ... \wedge e_r$ is a frame for $\det E$. Then

$$e_1 \wedge ... \wedge D_E e_j \wedge ... \wedge e_r = e_1 \wedge ... \wedge A(D_E)^k_j e_k \wedge ... \wedge e_r = A(D_E)^k_j \delta_{jk} e_1 \wedge ... \wedge e_j \wedge ... \wedge e_r,$$

and thus

$$D_{\det E}(e_1 \wedge ... \wedge e_r) = \left(\sum_j A(D_E)^j_j \right) e_1 \wedge ... \wedge e_r,$$

i.e., the connection matrix for $D_{\det E}$ is the trace of the connection matrix for D_E. ◇

3.1.3 Connections with additional symmetry

Metric Compatibility

3.1.4 DEFINITION. *Let $E \to M$ be endowed with a metric g. We say that a connection D for E is compatible with g if*

$$d(g(s,t)) = g(Ds,t) + g(s,Dt)$$

for all local sections s, t of E. ◇

If we view the metric g as a section of $E^* \otimes \overline{E^*} \to M$ then for any (not necessarily g-compatible) connection D for E one has

$$d(g(s,t)) = g(Ds,t) + g(s,Dt) + Dg(s,t)$$

(where, of course, the last term involves the induced connection). Thus the condition of metric compatibility may be written $Dg = 0$.

In general a given metric has many compatible connections. If D_1 and D_2 are two connections for E that are compatible with a metric g then their difference $\Theta := D_1 - D_2 \in \Gamma(M, T^*_M \otimes \mathrm{End}(E))$ satisfies

$$g(\Theta s, t) + g(s, \Theta t) = 0,$$

i.e., Θ is anti-symmetric (or anti-Hermitian if g is a Hermitian metric) with respect to g.

3.1.5 REMARK. If X is a complex manifold then the splitting $T_X^* \otimes_{\mathbb{R}} \mathbb{C} = T_X^{*1,0} \oplus T_X^{*0,1}$ induces the decomposition $\Theta = \Theta^{1,0} + \Theta^{0,1} \in (\mathrm{End}(E) \otimes T_X^{1,0}) \oplus (\mathrm{End}(E) \otimes T_X^{0,1})$, and the condition $\Theta^\dagger = -\Theta$ means that

$$(\Theta^{1,0})^\dagger = -\Theta^{0,1} \quad \text{and} \quad (\Theta^{0,1})^\dagger = -\Theta^{1,0}.$$

In particular, if $\Theta \in \Gamma(X, T_X^* \otimes \mathrm{End}(E))$ is of type $(1,0)$ then Θ must vanish identically. ◇

Symmetric connections

Given any smooth manifold M, we have the splitting

$$T_M^* \otimes T_M^* = \mathrm{Sym}^2(T_M^*) \oplus \Lambda^2(T_M^*),$$

Therefore every connection $D : \Gamma(M, \mathscr{C}^\infty(T_M^*)) \to \Gamma(M, \mathscr{C}^\infty(T_M^* \otimes T_M^*))$ for the cotangent bundle splits as

$$D = D^S + D^\Lambda.$$

On any manifold there is a natural operator sending 1-forms to 2-forms and satisfying the Leibniz rule with respect to the wedge-product, namely the exterior derivative. We can therefore make the following definition.

3.1.6 DEFINITION. *A connection D for T_M^* is said to be* symmetric *if $D^\Lambda = d$.* ◇

If D is a connection for T_M^* and we write

$$D(dx^i) = C^i_{jk} dx^k \otimes dx^j$$

in some local coordinate system then

$$\begin{aligned}
\Lambda^2\left(D(f_i dx^i)\right) &= \Lambda^2\left(\left(\frac{\partial f_j}{\partial x^k} + C^i_{jk}\right) dx^k \otimes dx^j\right) \\
&= \frac{1}{2}\left(\frac{\partial f_j}{\partial x^k} - \frac{\partial f_k}{\partial x^j}\right) dx^k \otimes dx^j + f_i \frac{(C^i_{jk} - C^i_{kj})}{2} dx^k \otimes dx^j \\
&= d(f_i dx^i) + f_i C^i_{jk} dx^k \wedge dx^j.
\end{aligned}$$

It follows that D is symmetric if and only if its connection matrix satisfies

$$C^i_{jk} = C^i_{kj}.$$

It is often useful to formulate the notion of symmetric connection for the tangent bundle. Since the tangent and cotangent bundle are dual, we shall call a connection for T_M *symmetric* if it is the dual of a symmetric connection for T_M^*. If ∇ is a connection for T_M dual to a given connection D for T_M^* then

$$\Gamma^i_{jk} = \left\langle \nabla_{\frac{\partial}{\partial x^k}}\left(\tfrac{\partial}{\partial x^j}\right), dx^i \right\rangle = -\left\langle \tfrac{\partial}{\partial x^j}, D_{\frac{\partial}{\partial x^k}}(dx^i) \right\rangle = -C^i_{jk},$$

and thus we see that ∇ is symmetric if and only if its connection matrix Γ^i_{jk} is symmetric, i.e.,

$$\Gamma^i_{jk} = \Gamma^i_{kj}.$$

Finally observe that if $\xi = \xi^i \frac{\partial}{\partial x^i}$ and $\eta = \eta^i \frac{\partial}{\partial x^i}$ are local vector fields, then

$$\nabla_\xi \eta - \nabla_\eta \xi = \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} + \xi^i \eta^j (\Gamma^k_{ij} - \Gamma^k_{ji}) \frac{\partial}{\partial x^k},$$

and thus ∇ is symmetric if and only if $\nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta]$.

Levi-Civita connections

On a Riemannian manifold there is exactly one symmetric, metric-compatible connection.

3.1.7 THEOREM (Levi-Civita). *On a Riemannian manifold (M, γ) there is a unique symmetric connection ∇ compatible with γ. In terms of the dual metric g for T^*_M, there exists a unique connection D such that*

$$d(g(s, s')) = g(Ds, s') + g(s, Ds') \quad \textit{and} \quad D^\Lambda = d.$$

Moreover ∇ and D are dual connections.

Proof of Levi-Civita's Theorem. Fix local coordinates $(x^1, ..., x^m)$ on a neighborhood U in M. Then $dx^1, ..., dx^m$ is a frame for T^*_M on the coordinate neighborhood U. Then with $\alpha = \alpha_i dx^i$, $g^{ij} = g(dx^i, dx^j)$ and $\omega^i{}_j = \omega^i{}_{jk} dx^k$, Levi-Civita's Theorem states that there is a unique solution $\omega^i{}_{jk}$ to the system of equations

$$g^{ia}\omega^j{}_{ak} + g^{jb}\omega^i{}_{bk} = \partial_k g^{ij} \quad \text{and} \quad \omega^i{}_{jk} = \omega^i{}_{kj}.$$

The first set of equations expresses metric compatibility, and the second, symmetry.

For notational ease, define

$$\omega^{ijk} = g^{ja} g^{kb} \omega^i{}_{ab} \quad \text{and} \quad \partial^k := g^{k\ell} \partial_\ell,$$

so that the equations to be solved are

$$\omega^{ijk} + \omega^{jik} = \partial^k g^{ij} \quad \text{and} \quad \omega^{ijk} = \omega^{ikj}. \tag{3.2}$$

To solve (3.2), observe that the right hand side of the first equation is symmetric in ij, so it makes sense to write $\omega^{ijk} = S^{ijk} + A^{ijk}$, where

$$S^{ijk} = \frac{1}{2}(\omega^{ijk} + \omega^{jik}) \quad \text{and} \quad A^{ijk} = \frac{1}{2}(\omega^{ijk} - \omega^{jik}).$$

(For fixed k S^{ijk} and A^{ijk} are the symmetric and antisymmetric parts of ω^{ijk}.) Thus

$$\omega^{ijk} = \frac{1}{2}\partial^k g^{ij} + A^{ijk},$$

and the second equation in (3.2) reads as

$$A^{ijk} - A^{ikj} = \frac{1}{2}(\partial^j g^{ik} - \partial^k g^{ij}). \tag{3.3}$$

In view of the symmetry of g^{ji} and the antisymmetry of A^{ijk} in ij, (3.3) yields

$$-A^{ijk} + A^{kji} = A^{jik} - A^{jki} = \frac{1}{2}(\partial^i g^{jk} - \partial^k g^{ij}).$$

Subtracting the latter from (3.3) and using the antisymmetry of A^{ijk} yields

$$A^{ijk} + (A^{ijk} + A^{kij} + A^{jki}) = \frac{1}{2}(\partial^j g^{ik} - \partial^i g^{jk}).$$

But

$$\begin{aligned} A^{ijk} + A^{jki} + A^{kij} &= \omega^{ijk} + \omega^{jki} + \omega^{kij} - \frac{1}{2}(\partial^k g^{ij} + \partial^i g^{jk} + \partial^j g^{ki}) \\ &= \omega^{ikj} - \frac{1}{2}\partial^j g^{ik} + \omega^{jik} - \frac{1}{2}\partial^k g^{ji} + \omega^{kji} - \frac{1}{2}\partial^i g^{kj} \\ &= A^{ikj} + A^{jik} + A^{kji} = -(A^{ijk} + A^{jki} + A^{kij}). \end{aligned}$$

Thus $A^{ijk} + A^{jki} + A^{kij} = 0$ and therefore

$$\omega^{ijk} := \tfrac{1}{2}\left(\partial^k g^{ij} + \partial^j g^{ik} - \partial^i g^{jk}\right)$$

is the (obviously unique) solution of (3.2). Lowering the indices, we have

$$\omega^i{}_{jk} = \frac{1}{2}\left(g_{jm}\partial_k g^{mi} + g_{km}\partial_j g^{mi} - g_{rj}g_{sk}g^{im}\partial_m g^{rs}\right).$$

Using the relation

$$0 = \partial_m(g_{rj}g^{rs}) = g^{rs}\partial_m g_{rj} + g_{rj}\partial_m g^{rs} \tag{3.4}$$

yields $g_{rj}g_{sk}g^{im}\partial_m g^{rs} = -g_{sk}g^{rs}\partial_m g_{rj} = -\partial_m g_{kj} = -\partial_m g_{jk}$, and therefore

$$\omega^i{}_{jk} = \frac{1}{2}\left(g_{jm}\partial_k g^{mi} + g_{km}\partial_j g^{mi} + g^{im}\partial_m g_{jk}\right). \tag{3.5}$$

To complete the proof, we must show that the connection $\nabla : \Gamma(M, T_M) \to \Gamma(M, T_M^* \otimes T_M)$ dual to the connection D is symmetric and compatible with the dual metric γ. This last step is left to the exercises (Exercise 3.1.3). $\square$

3.1.8 REMARK. The connection $D : \Gamma(M, T_M^*) \to \Gamma(M, T_M^* \otimes T_M^*)$ has associated to it the dual connection $\nabla : \mathscr{C}_M^\infty(T_M) \to \mathscr{C}_M^\infty(T_M \otimes T_M^*)$ via the relation

$$\left\langle \nabla\partial_j, dx^i \right\rangle + \left\langle \partial_j, Ddx^i \right\rangle = 0.$$

Contracting both sides with ∂_k and setting

$$\Gamma^i_{jk} := \left\langle \nabla_{\partial_k}\partial_j, dx^i \right\rangle,$$

we find that

$$\Gamma^i_{jk} = -\omega^i_{jk}.$$

Using the relation (3.4) and the symmetry of the metric, (3.5) yields

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left(\partial_k g_{mj} + \partial_j g_{mk} - \partial_m g_{jk}\right)$$

for the Christoffel symbols of ∇. $\diamond$

Complex connection

On a complex manifold X we have a splitting

$$T^*_X \otimes \mathbb{C} = T^{*1,0}_X \oplus T^{*0,1}_X.$$

It follows that, for a complex vector bundle $E \to X$, a connection D splits as

$$D = D^{1,0} + D^{0,1}. \tag{3.6}$$

If the vector bundle $E \to X$ is furthermore holomorphic then there is a canonical choice for the component $D^{0,1} : \Gamma(X, E) \to \Gamma(X, \Lambda^{0,1}_X \otimes E)$, namely the operator $\bar{\partial}$, introduced in Paragraph 1.4.3 (in particular, in Proposition/Definition 1.4.6).

3.1.9 DEFINITION. *A connection D for a holomorphic vector bundle $V \to X$ is said to be complex if $D^{0,1} = \bar{\partial}$ in terms of the splitting* (3.6).

Chern connection

The basic result about connections for holomorphic Hermitian vector bundles is the following analogue of Levi-Civita's theorem.

3.1.10 THEOREM. *On a holomorphic Hermitian vector bundle there exists a unique complex connection compatible with the Hermitian metric.*

Proof. We begin with uniqueness. Since the difference of two connections is a 0-th order differential operator, i.e., a matrix multiplier, if $D^{0,1} = \bar{\partial}$ then this multiplier is a matrix of $(1,0)$-forms. It follows from Remark 3.1.5 that a metric compatible complex connection is unique if it exists.

To prove existence, define $D^{1,0}$ by

$$h(D^{1,0}s, t) = \partial h(s,t) - h(s, \bar{\partial} t)$$

for all local sections s, t. Then

$$h(\bar{\partial} s, t) = \overline{h(t, \bar{\partial} s)} = \overline{\partial h(t,s) - h(D^{1,0}t, s)} = \bar{\partial} h(s,t) - h(s, D^{1,0}t),$$

and therefore the connection $D = D^{1,0} + \bar{\partial}$, which is clearly complex, satisfies

$$dh(s,t) = h(Ds,t) + h(s,Dt),$$

i.e., D is h-compatible. □

3.1.11 DEFINITION. *The unique metric-compatible complex connection for a Hermitian holomorphic vector bundle (E,h) is called the* Chern connection *for* (E,h).

3.1.4 The Kähler condition

The definition of Kähler metric

The tangent bundle T_X of a complex manifold X with Hermitian Riemannian metric g carries two natural connections. One connection is the Chern connection for the holomorphic Hermitian vector bundle (T_X, g), and the other connection is the Levi-Civita connection for (T_X, g). (Here g is viewed both as a Hermitian metric on $T_X^{1,0}$ and as a Riemannian metric on T_X, via the correspondence $s^{1,0} : T_X \to T_X^{1,0}$ discussed in Subsection 2.) In general, these two connections are different.

3.1.12 DEFINITION. *A metric g for which the Chern and Levi-Civita connections agree is said to be Kähler. A complex manifold admitting a Kähler metric is called a Kähler manifold.*

There are several equivalent ways to define Kähler metrics. Here we shall discuss only the two we will need.

Symplectic formulation

3.1.13 PROPOSITION. *A metric g with metric form ω is Kähler if and only if the $(1,1)$-form ω associated to g is closed.*

Proof. The Chern connection D for the metric g satisfies (c.f. Exercise 3.1.4)

$$D_{\frac{\partial}{\partial z^j}}(\tfrac{\partial}{\partial z^i}) =: \Gamma^k_{ij}\tfrac{\partial}{\partial z^k} = g^{k\bar{\ell}}\tfrac{\partial}{\partial z^j} g_{i\bar{\ell}}\tfrac{\partial}{\partial z^k}.$$

Since both the Levi-Civita and Chern connections are compatible with g, the Chern connection agrees with the Levi-Civita connection if and only if the Chern connection corresponds, under the isomorphism induced by $s^{1,0} : T_X \to T_X^{1,0}$, to a symmetric connection, i.e.,

$$D_{s^{1,0}w}(s^{1,0}v) = D_{s^{1,0}v}(s^{1,0}w) \tag{3.7}$$

whenever $v, w \in \left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}\right\}$. But since

$$s^{1,0}\tfrac{\partial}{\partial x^i} = \tfrac{\partial}{\partial z^i} \quad \text{and} \quad s^{1,0}\tfrac{\partial}{\partial y^i} = \sqrt{-1}\tfrac{\partial}{\partial z^i},$$

the relation (3.7) holds if and only if $\Gamma^k_{ij} = \Gamma^k_{ji}$. Therefore g is Kähler if and only if

$$\frac{\partial g_{i\bar{k}}}{\partial z^j} = \frac{\partial g_{j\bar{k}}}{\partial z^i}.$$

But the latter is equivalent to $\partial\omega = 0$, which in turn is equivalent to $d\omega = 0$ (since ω is real). To be more explicit, since $\omega = g_{i\bar{k}}\sqrt{-1}dz^i \wedge d\bar{z}^k$,

$$\begin{aligned} d\omega &= \sqrt{-1}\frac{\partial g_{i\bar{k}}}{\partial z^j} dz^j \wedge dz^i \wedge d\bar{z}^k + \sqrt{-1}\frac{\partial g_{i\bar{k}}}{\partial \bar{z}^\ell} d\bar{z}^\ell \wedge dz^i \wedge d\bar{z}^k \\ &= \sqrt{-1}\frac{\partial g_{i\bar{k}}}{\partial z^j} dz^j \wedge dz^i \wedge d\bar{z}^k + \overline{\sqrt{-1}\frac{\partial g_{k\bar{i}}}{\partial z^\ell} dz^\ell \wedge dz^k \wedge d\bar{z}^i} \\ &= \sqrt{-1}\sum_{i<j}\left(\frac{\partial g_{i\bar{k}}}{\partial z^j} - \frac{\partial g_{j\bar{k}}}{\partial z^i}\right) dz^j \wedge dz^i \wedge d\bar{z}^k \\ &\qquad + \overline{\sqrt{-1}\sum_{i<j}\left(\frac{\partial g_{i\bar{k}}}{\partial z^j} - \frac{\partial g_{j\bar{k}}}{\partial z^i}\right) dz^j \wedge dz^i \wedge d\bar{z}^k}. \end{aligned}$$

It follows that g is Kähler if and only if $d\omega = 0$, as desired. □

Locally Euclidean formulation

A very useful formulation of the Kähler condition is contained in the following theorem.

3.1.14 THEOREM. *The metric g is Kähler if and only if every point of X lies in a coordinate chart with coordinates z so that*

$$g = \sum_\alpha dz^\alpha \cdot\ d\bar{z}^\alpha + O(|z|^2).$$

The coordinates referred to in the theorem are called *Kähler coordinates*.

Proof. Observe that if two $(1,1)$-forms ω_1 and ω_2, defined in a neighborhood of 0 in $\mathbb{C}^n$, have the same Taylor expansion up to second order, then one has

$$(d\omega_1)_0 = (d\omega_2)_0.$$

Therefore if g is locally Euclidean to second order then $d\omega_g = 0$, and thus g is Kähler by Proposition 3.1.13.

Conversely, suppose ω is the $(1,1)$-form associated to a Kähler metric g, and let z be local coordinates such that

$$g_{i\bar{j}}(0) = \delta_{i\bar{j}} \tag{3.8}$$

(It is easy to find such, so-called *normal coordinates*.) Then the Taylor expansion of ω with respect to z is

$$\omega = \sqrt{-1}\left(\delta_{i\bar{j}} + a_{i\bar{j}k}z^k + a_{i\bar{j}\bar{\ell}}\bar{z}^\ell + O(|z|^2)\right) dz^i \wedge d\bar{z}^j.$$

The two properties of the Taylor coefficients $a_{i\bar{j}k}, a_{i\bar{j}\bar{k}}$ are

(a) $g_{i\bar{j}} = \overline{g_{j\bar{i}}} \Rightarrow a_{i\bar{j}\bar{k}} = \overline{a_{j\bar{i}k}}$,

(b) $d\omega = 0 \Rightarrow a_{i\bar{j}k} = a_{k\bar{j}i}$.

Since the result we seek is regarding a second order Taylor expansion, it suffices to find a quadratic biholomorphic local coordinate transformation. That is to say, we seek coordinates w defined by

$$z^k = w^k + \frac{1}{2} b^k_{\ell m} w^\ell w^m$$

(which do not modify condition (3.8)) such that

$$\omega = \frac{\sqrt{-1}}{2} \left(\delta_{i\bar{j}} + O(|w|^2) \right) dw^i \wedge d\bar{w}^j.$$

Since $w^\ell w^m = w^m w^\ell$, we may assume that $b^i_{jk} = b^i_{kj}$. Then

$$dz^k = dw^k + b^k_{\ell m} w^\ell dw^m,$$

and we have

$$\begin{aligned} -\sqrt{-1}\omega &= \delta_{i\bar{j}} \left(dw^i + b^i_{\ell m} w^\ell dw^m \right) \wedge \left(d\bar{w}^j + \overline{b^j_{rs}} \bar{w}^r d\bar{w}^s \right) \\ &\quad + \left(a_{i\bar{j}k} w^k + a_{i\bar{j}\bar{k}} \bar{w}^k \right) dw^i \wedge d\bar{w}^j + O(|w|^2) \\ &= \left(\delta_{i\bar{j}} + (a_{i\bar{j}k} + \delta_{\ell\bar{j}} b^\ell_{ki}) w^k + (a_{i\bar{j}\bar{k}} + \overline{\delta_{m\bar{i}} b^m_{kj}}) \bar{w}^k + O(|w|^2) \right) dw^i \wedge d\bar{w}^j. \end{aligned}$$

Thus, if we set $b^j_{ki} = -\delta^{j\bar{m}} a_{i\bar{m}k}$ (which is symmetric in ki by (a)) then

$$\overline{\delta_{mi} b^m_{kj}} = -\overline{\delta_{mi} \delta^{m\bar{n}} a_{j\bar{n}k}} = -\delta^{n\bar{i}} \overline{a_{j\bar{n}k}} = -a_{i\bar{j}\bar{k}}.$$

Thus

$$\omega = \sqrt{-1} \delta_{i\bar{j}} dw^i \wedge d\bar{w}^j + O(|w|^2),$$

so g is Euclidean to second order. □

3.1.5 Induced connection on twisted forms

Symmetric connections and exterior derivatives

Consider a differential 1-form $\alpha = \alpha_i dx^i$ on a manifold M. For a connection ∇ for $T^*_M \to M$,

$$\nabla\alpha = \frac{\partial \alpha_i}{\partial x^j} dx^j \otimes dx^i + \alpha_k \theta^k_{ij} dx^j \otimes dx^i,$$

where θ is the connection matrix of ∇. As we discussed earlier,

$$\Lambda^2(\nabla\alpha) = \frac{\partial \alpha_i}{\partial x^j} dx^j \wedge dx^i + \alpha_k \theta^k_{ij} dx^j \wedge dx^i. \tag{3.9}$$

We then introduced the notion of symmetric connection; by definition, a connection ∇ is symmetric if and only if

$$d\alpha = \Lambda^2(\nabla\alpha)$$

for any 1-form α. Of course, the formula (3.9) shows that ∇ is symmetric if and only if $\theta^k_{ij} = \theta^k_{ji}$.

Now suppose β is a differential r-form, i.e., a section of the product bundle $\Lambda^r(T^*_M)$. For a connection ∇ for $T^*_M \to M$, the product connection ∇_r acts on $\beta = \beta_{i_1\cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}$ (with our convention that β_I is skew-symmetric in I) by

$$\nabla_r\beta = \left(\frac{\partial \beta_I}{\partial x^{i_o}} + \sum_{j=1}^{r} \beta_{i_1\cdots(\ell)_j\cdots i_r}\theta^\ell_{i_o i_j} \right) dx^{i_o} \otimes dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

where the notation $(\ell)_j$ means that ℓ replaces i_j. If we take the $(r+1)^{\text{th}}$ skew-symmetric part of $\nabla_r\beta$ (thought of as an $(r+1)$-tensor) we obtain

$$\Lambda^{r+1}(\nabla_r\beta) = \left(\frac{\partial \beta_I}{\partial x^{i_o}} + \sum_{j=1}^{r} \beta_{i_1\cdots(\ell)_j\cdots i_r} \frac{(\theta^\ell_{i_o i_j} - \theta^\ell_{i_j i_o})}{2} \right) dx^{i_o} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r};$$

a calculation that uses the skew-symmetry of the β_I. (Note that

$$\Lambda^{r+1}(\nabla_r\beta) = \left(\frac{\partial \beta_I}{\partial x^{i_o}} + \sum_{j=1}^{r} \beta_{i_1\cdots(\ell)_j\cdots i_r}\theta^\ell_{i_o i_j} \right) dx^{i_o} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

but that the latter formula does not adhere to our convention of skew symmetry of the coefficients of a differential form.) Thus we see that a connection ∇ is symmetric if and only if

$$d = \Lambda^{r+1} \circ \nabla_r$$

holds for any integer r with $1 \le r \le n$.

Twisted exterior derivative

Let $V \to M$ be a vector bundle with connection D. We can define a twisted version of the exterior derivative for sections of

$$\Gamma(M, \mathscr{C}^\infty(T^*_M \otimes V))$$

or V*-valued* 1*-forms*. This twisted exterior derivative should produce a V-valued 2-form.

As in the previous paragraph, we fix a connection ∇ for T^*_M. For a V-valued 1-form α we compute that

$$(\nabla \otimes D)\alpha = \left(\frac{\partial \alpha^\nu_i}{\partial x^j} + \alpha^\mu_i \omega^\nu_{\mu j} + \alpha^\nu_k \theta^k_{ij} \right) dx^j \otimes dx^i \otimes \mathrm{e}_\nu$$

and

$$\Lambda^2((\nabla \otimes D)\alpha) = \left(\frac{\partial \alpha^\nu_i}{\partial x^j} + \alpha^\mu_i \omega^\nu_{\mu j} + \alpha^\nu_k \theta^k_{ij} \right) dx^j \wedge dx^i \otimes \mathrm{e}_\nu,$$

where ω and θ are the Christoffel symbols for D and ∇ respectively. Again if the connection ∇ for T^*_M is symmetric, then the anti-symmetric part

$$\Lambda^2((\nabla \otimes D)\alpha) = \left(\frac{\partial \alpha_i^\nu}{\partial x^j} + \alpha_i^\mu \omega_{\mu j}^\nu\right) dx^j \wedge dx^i \otimes \mathrm{e}_\nu$$

is independent of the connection ∇. Similarly, if β is a V-valued r-form then

$$\Lambda^{r+1}(\nabla_r \otimes D)\alpha)$$

is a V-valued $r+1$-form, which is again independent of ∇ as soon as ∇ is symmetric.

3.1.15 DEFINITION. *Let $V \to M$ be a vector bundle with connection D and let ∇ be a symmetric connection for T^*_M. The operator $D_1 : \Gamma(M, \mathscr{C}^\infty(T^*_M \otimes V)) \to \Gamma(M, \mathscr{C}^\infty(\Lambda^2(T^*_M) \otimes V))$ defined by*

$$D_1\alpha := \Lambda^2((\nabla \otimes D)\alpha)$$

(which is independent of ∇) is called the twisted exterior derivative associated to D. More generally, let ∇_r denote the induced product connection for $\Lambda^r(T^*_M) \to M$. The operator

$$D_r := \Lambda^{r+1} \circ (\nabla_r \otimes D) : \Gamma(M, \mathscr{C}^\infty(\Lambda^r(T^*_M) \otimes V)) \to \Gamma(M, \mathscr{C}^\infty(\Lambda^{r+1}(T^*_M) \otimes V))$$

is called the twisted r^{th} exterior derivative (for V-valued r-forms) associated to D. $\diamond$

If $e_1, ..., e_r$ is a frame for V and $x^1, ..., x^m$ is a local coordinate system on M, then for a section $\sigma \in \Gamma(M, V \otimes \Lambda^r(T^*_M))$ given locally by $\sigma = \sigma_I^\mu dx^I \otimes e_\mu$, one has (with $D = D_r$)

$$\begin{aligned} D\sigma &= \frac{\partial(\sigma_I^\mu)}{\partial x^j} dx^j \wedge dx^I \otimes e_\mu + \omega_\nu^\mu \wedge \sigma_I^\nu dx^I \otimes e_\mu \\ &= \frac{\partial(\sigma_I^\mu)}{\partial x^j} dx^j \wedge dx^I \otimes e_\mu + (-1)^r \sigma_I^\nu dx^I \wedge \omega_\nu^\mu \otimes e_\mu. \end{aligned}$$

Informally, we write

$$D\sigma = d\sigma + (-1)^r \sigma \wedge \omega.$$

EXERCISES

3.1.1. Prove proposition 3.1.2.

3.1.2. If two frames $e_1, ..., e_r$ and $\tilde{e}_1, ..., \tilde{e}_r$ determined two connection forms A and $\tilde{A}$ respectively, and if $G : U \to GL(r, \mathbb{C})$ is the matrix of functions relating the two frames by $Ge_i = \tilde{e}_i$, show that $\tilde{A} = (dG)G^{-1} + GAG^{-1}$.

3.1.3. Show that if D is a symmetric connection for $T^*_M \to M$ compatible with a metric g then the dual connection ∇ is symmetric and compatible with the metric γ for T_M dual to the metric g.

3.1.4. Let (E, h) be a Hermitian vector bundle. For a frame $e_1, ..., e_r$, write $s = s^\alpha e_\alpha$, $t = t^\beta e_\beta$ and $h_{\alpha\bar{\beta}} := h(e_\alpha, \bar{e}_\beta)$. Show that the Chern connection is given by the formula

$$Ds = \left(\partial s^\alpha + s^\gamma h^{\alpha\bar{\delta}} \partial h_{\gamma\bar{\delta}} + \bar{\partial} s^\alpha\right) e_\alpha.$$

3.1.5. Show that a Hermitian metric on a Riemann surface is automatically Kähler.

3.2 Curvature

3.2.1 Definition

3.2.1 DEFINITION. *Let $E \to M$ be a vector bundle with connection D and, in terms of some frame, connection matrix A. The curvatures of (E, D) are the operators*

$$\Theta_k := D_{k+1} D_k : \Gamma(M, \mathscr{C}^\infty(\Lambda^k(T^*_M) \otimes E)) \to \Gamma(M, \mathscr{C}^\infty(\Lambda^{k+2}(T^*_M) \otimes E)),$$

where D_j is the twisted exterior derivative associated to the connection D as in Definition 3.1.15.

Observe that if s is an E-valued k-form and f is a function then

$$\Theta_k(fs) = D(fDs + df \wedge s) = fD \circ Ds + df \wedge Ds - df \wedge Ds = f\Theta_k s,$$

so that $\Theta_k s$ is indeed an E-valued $(k+2)$-form. But even a little more is true.

3.2.2 PROPOSITION. *There exists an* $\mathrm{End}(E)$*-valued* 2*-form* $\Omega(D)$ *such that*

$$\Theta_k s = s \wedge \Omega(D)$$

for any $k = 0, 1, ...$ and any E-valued k-form s.

Proof. We work in a local trivialization, where we denote by A the connection matrix. We calculate that

$$\begin{aligned} \Theta_k s = D_{k+1} D_k s &= D_{k+1}(ds + (-1)^k s \wedge A) \\ &= d(ds + (-1)^k s \wedge A) + (-1)^{k+1}(ds + (-1)^k s \wedge A) \wedge A \\ &= (-1)^k(ds \wedge A + (-1)^k s \wedge dA) + (-1)^{k+1} ds \wedge A - (-1)^k s \wedge A \wedge A \\ &= s \wedge (dA - A \wedge A). \end{aligned}$$

Thus the k-independent local endomorphism

$$s \mapsto s \wedge (dA - A \wedge A)$$

agrees with $D \circ D$. Since $D \circ D$ is globally defined, the proposition is proved. □

3.2.2 Curvature of the Chern connection

Fix a holomorphic Hermitian vector bundle $(E, h) \to X$.

First observe that since $D = D^{1,0} + \bar\partial$ and $\bar\partial^2 = 0$,

$$D_1 \circ D = D_1^{1,0} \circ D^{1,0} + D_1^{1,0} \circ \bar\partial + \bar\partial_1 \circ D^{1,0}.$$

Thus far we have used the fact that the Chern connection is complex, but not that it is metric compatible. Metric compatibility reads as

$$\partial h(s,t) = h(D^{1,0}s, t) + h(s, \bar\partial t) \quad \text{and} \quad \bar\partial h(s,t) = h(\bar\partial s, t) + h(s, D^{1,0} t).$$

(Note that the first of these equations is consequence of the second via complex conjugation followed by interchange of the roles of s and t.) Since $\partial^2 = 0$,

$$0 = h(D_1^{1,0} \circ D^{1,0}s, t) + h(D^{1,0}s, \bar{\partial}t) - h(D^{1,0}s, \bar{\partial}t) + h(s, \bar{\partial}_1\bar{\partial}t) = h(D_1^{1,0} \circ D^{1,0}s, t),$$

and thus $D_1^{1,0} \circ D^{1,0} = 0$. Therefore

$$\Theta = D_1^{1,0} \circ \bar{\partial} + \bar{\partial}_1 \circ D^{1,0}.$$

In particular, the curvature maps sections to twisted $(1,1)$-forms, and therefore the curvature form of the Chern connection is a $(1,1)$-form.

3.2.3 PROPOSITION. *The curvature of the Chern connection of* $(E, h) \to X$ *is given by the formula*

$$\Omega_\beta^\alpha = \bar{\partial}(h^{\alpha\bar{\mu}}\partial h_{\beta\bar{\mu}}).$$

The proof is left as an exercise.

3.2.3 Curvature of a line bundle

Let $L \to M$ be a complex line bundle. If D is any connection for $L \to M$ then its curvature is a section of $\mathrm{End}(E) \otimes \Lambda_M^2 \to M$. Since the line bundle $\mathrm{End}(E) \to X$ is canonically trivial (see Exercise 3.2.2), the curvature of a line bundle is a well-defined 2-form on M.

In terms of any local connection form $A(D)$, the curvature of D is

$$d(A(D)).$$

Indeed, if the fibers are 1-dimensional then $A(D) \wedge A(D) = 0$. In particular, the form yields, via the isomorphism between $\mathrm{End}(L)$ and the trivial bundle, a globally-defined 2-form on M. Since locally this 2-form is $d(A(D))$, the curvature form is a closed form, but in general this form is not exact.

3.2.4 REMARK. The de Rham cohomology class

$$[\Theta(D)] \in H_{dR}^2(M, \mathbb{C})$$

of the curvature of any connection for L does not depend on the connection. This cohomology class, called the *Chern class* of the line bundle $L \to M$, is denoted $c_1(L)$. ◇

Suppose now that X is a complex manifold and the line bundle $L \to X$ is holomorphic. Let $\mathfrak{h}$ be a Hermitian metric for $L \to X$. If ξ is a holomorphic frame for L over an open set $U \subset X$ then one can define the function

$$\varphi^\xi := -\log \mathfrak{h}(\xi, \xi).$$

The curvature of the Chern connection of $\mathfrak{h}$ is

$$\Theta(\mathfrak{h}) = (\partial\bar{\partial} \log \varphi^{(\xi)}) \otimes \xi \otimes \xi^*,$$

where ξ^* is the frame for L^* dual to ξ. Since $\xi \otimes \xi^*$ is nowhere-zero, one can ignore it and define the curvature of the holomorphic line bundle to be $\partial\bar{\partial} \log \varphi^{(\xi)}$.

The reader should check that the left hand side of the latter equality is independent of the choice of holomorphic frame. Consequently some authors (probably only the present author[1]) uses the following global notation for Hermitian metrics of holomorphic line bundles.

NOTATION. In the sequel, a metric for a holomorphic line bundle will often be denoted $e^{-\varphi}$, and its curvature will be denoted $\partial\bar{\partial}\varphi$.

3.2.4 Curvature of determinant bundles

3.2.5 PROPOSITION. *Let $(E, D_E) \to M$ be a vector bundle of rank r with connection, and let $(\det E, D_{\det E}) \to M$ be its determinant line bundle. Then*

$$\Omega(D_{\det E}) = \mathrm{trace}(\Omega(D_E)).$$

The proof is left as an exercise.

Holomorphic hermitian vector bundles

Let $E \to X$ be a holomorphic vector bundle of rank r with Hermitian metric h. We have already observed that the (unique) Chern connection for the metric $\det h$ for the line bundle $\det E \to X$ has connection matrix

$$A = \frac{1}{\det h}\partial(\det h).$$

It follows that the curvature matrix of $\det h$ is

$$\Omega = dA - A \wedge A = \bar{\partial}(\partial \log \det h) = \partial\bar{\partial}(-\log \det h).$$

With calculations similar to those we used in the study of the connection, one can easily see that

$$\Omega = \text{trace}\bar{\partial}(\partial h h^{-1}),$$

a fact we already know from Proposition 3.2.5

The canonical bundle

Recall that the *canonical bundle* K_X of a complex manifold X is the line bundle $\det T^{*1,0}_X$. The local sections of K_X are $(n, 0)$-forms, where $n = \dim_{\mathbb{C}} X$.

If g is a Hermitian Riemannian metric on X, theorem 3.2.5 tells us that the curvature of the Chern connection for $(K_X, \det(g))$ is just the trace of the curvature of $(T^{*1,0}_X, g)$.

[1]In fact, many authors will write $e^{-\varphi}$ for a Hermitian metric of a holomorphic line bundle, but they mean that this is the local form of the metric; I like to use this notation in a global way, but my convention has its pitfalls as well.

For a general Hermitian metric g the latter Chern curvature has nothing to do with the curvature of the Levi-Civita connection for g. However if the metric g is Kähler, the curvature of the Chern connection for $(K_X, \det(g))$ is the negative of the so-called *Ricci curvature* of g:

$$\text{Ricci}(g) = -\text{trace}(\Omega(g)). \tag{3.10}$$

In the next paragraph, we will remind the reader of the definition, from Riemannian Geometry, of the Ricci curvature of a Riemannian metric. We will then show that (3.10) holds when the metric in question is Kähler.

3.2.5 Symmetry of Kähler curvature

Since the connection matrix of the Levi-Civita connection is symmetric, one can expect some symmetry in the curvature of this connection. If the metric is Kähler, the curvature of the Levi-Civita connection has even more symmetry. In this section we write down some of the symmetries of the Kähler curvature.

Curvature of the Levi-Civita connection

Let (M, g) be a Riemannian manifold, and let ∇ be its Levi-Civita connection. As we have seen, ∇ is uniquely determined by the two conditions

$$d(g(\xi,\eta))(\zeta) = g(\nabla_\zeta \xi, \eta) + g(\xi, \nabla_\zeta \eta) \quad \text{and} \quad \nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta]$$

For all vector fields ξ, η and ζ on M. It is customary to denote by R the curvature of the Levi-Civita connection. That is to say,

$$R(\xi,\eta)\zeta = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]}\zeta.$$

The curvature tensor is locally determined by

$$R_{ijk\ell} := g(R(\tfrac{\partial}{\partial x^k}, \tfrac{\partial}{\partial x^\ell})\tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^i}).$$

It is immediately clear that for any connection (Levi-Civita or not),

$$R_{ijk\ell} = -R_{ij\ell k}. \tag{3.11}$$

Since $d^2 = 0$, the curvature $\Theta(D)$ of any metric-compatible connection D must satisfy

$$g(\Theta(D)\xi, \eta) + g(\xi, \Theta(D)\eta) = d^2(g(\xi,\eta) = 0,$$

and the latter relation is equivalent to the identities

$$R_{ijk\ell} = -R_{jik\ell}. \tag{3.12}$$

The symmetry of the Levi-Civita connection implies that

$$\begin{aligned} R(\tfrac{\partial}{\partial x^k}, \tfrac{\partial}{\partial x^\ell})\tfrac{\partial}{\partial x^j} &= \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^\ell}} \tfrac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^\ell}} \nabla_{\frac{\partial}{\partial x^k}} \tfrac{\partial}{\partial x^j} \\ &= \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} \tfrac{\partial}{\partial x^\ell} - \nabla_{\frac{\partial}{\partial x^\ell}} \nabla_{\frac{\partial}{\partial x^j}} \tfrac{\partial}{\partial x^k} \\ &= R(\tfrac{\partial}{\partial x^k}, \tfrac{\partial}{\partial x^j})\tfrac{\partial}{\partial x^\ell} - R(\tfrac{\partial}{\partial x^\ell}, \tfrac{\partial}{\partial x^j})\tfrac{\partial}{\partial x^k} + \nabla_{\frac{\partial}{\partial x^j}} (\nabla_{\frac{\partial}{\partial x^\ell}} \tfrac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^k}} \tfrac{\partial}{\partial x^\ell}) \\ &= -R(\tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^k})\tfrac{\partial}{\partial x^\ell} - R(\tfrac{\partial}{\partial x^\ell}, \tfrac{\partial}{\partial x^j})\tfrac{\partial}{\partial x^k}. \end{aligned}$$

Thus we obtain the *First Bianchi Identity*

$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0. \tag{3.13}$$

As a consequence, we have

$$\begin{aligned} &\tfrac{1}{2}(R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk}) = 0, \;\; \tfrac{1}{2}(R_{jk\ell i} + R_{jik\ell} + R_{j\ell ik}) = 0, \\ &\tfrac{1}{2}(R_{k\ell ij} + R_{kj\ell i} + R_{kij\ell}) = 0, \;\; \tfrac{1}{2}(R_{\ell ijk} + R_{\ell kij} + R_{\ell jki}) = 0. \end{aligned}$$

and adding these four equations and using (3.11) and (3.12) yields

$$R_{ik\ell j} = R_{\ell jik}. \tag{3.14}$$

Curvature of the Kähler connection

We let Greek letters run through $\{1, ..., n\}$ and latin letters through $\{1, ..., n, \bar{1}, ..., \bar{n}\}$, and we use the notation

$$\partial_\alpha = \frac{\partial}{\partial z^\alpha} \quad \text{and} \quad \partial_{\bar{\alpha}} = \frac{\partial}{\partial \bar{z}^\alpha}, \quad 1 \le \alpha \le n. \tag{3.15}$$

Thus

$$\partial_i = \frac{\partial}{\partial z^i} \quad \text{and} \quad \partial_{\bar{i}} = \frac{\partial}{\partial \bar{z}^i} \qquad \text{for } i = 1, ..., n,$$

and

$$\partial_i = \frac{\partial}{\partial \bar{z}^i} \quad \text{and} \quad \partial_{\bar{i}} = \frac{\partial}{\partial z^i} \qquad \text{for } i = \bar{1}, ..., \bar{n}.$$

We denote by Γ^i_{jk} the Christoffel symbols of ∇, which are defined by the relation

$$\nabla_{\partial j} \partial_k = \Gamma^j_{ik} \partial_j.$$

Now let ∇ be the Kähler connection on a Kähler manifold (X, g). In Section 3.1.3 (and specifically in Remark 3.1.8) we derived the formula for the Christoffel symbols with respect to the real coordinates, and so those Christoffel are different from the symbols we have defined in the present paragraph. Nevertheless, as already mentioned, since

$$s^{1,0} \tfrac{\partial}{\partial x^i} = -\sqrt{-1} s^{1,0} \tfrac{\partial}{\partial y^i} = \tfrac{\partial}{\partial z^i},$$

one does retain the symmetries of the Levi-Civita connection when working on $T_X^{1,0}$ instead of T_X.

Since ∇ is the Chern connection of the Hermitian metric, its connection matrix is a matrix of $(1,0)$-forms, and must therefore satisfy

$$\Gamma^{\alpha}_{i\bar{\gamma}} = \Gamma^{\bar{\alpha}}_{i\gamma} = 0. \tag{3.16}$$

Additionally, appropriate use of complex conjugation yields

$$\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \overline{\Gamma^{k}_{ij}} \tag{3.17}$$

Let us introduce the following notation. Let R_i^j be the curvature tensor with respect to the local frame $\partial_1, ..., \partial_n, \partial_{\bar{1}}, ...\partial_{\bar{n}}$ for $T_X \otimes \mathbb{C}$ and write

$$R_{ij} = R_i^k g_{kj}.$$

The matrix entries R_{ij} are $(1,1)$-forms, and so we write

$$R_{ij} = R_{ijk\bar{\ell}} dz^k \wedge d\bar{z}^{\ell}.$$

(Note: this convention means $dz^{\bar{1}} = d\bar{z}^1$, etc.)

Warning: This notation differs from the notation of the previous paragraph, because we are using the frame (3.15), and not the real frame $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^m}$ used there. (Nevertheless, the first Bianchi Identity holds in this section as well, since it is defined by the same sorts of relations.)

Because the curvature is a $(1,1)$-form with skew-Hermitian symmetry, we have

$$R_{ij\alpha\beta} = R_{ij\bar{\alpha}\bar{\beta}} = 0 \quad \text{and} \quad R_{ij\bar{\alpha}\beta} = -R_{ij\beta\bar{\alpha}} = R_{ij\bar{\beta}\alpha}. \tag{3.18}$$

Now, since the Kähler connection is Chern, we have $(\nabla^{1,0})^2 = 0$ and $\bar{\partial}^2 = 0$, which reads as

$$R_{\alpha\beta ij} = R_{\bar{\alpha}\bar{\beta}ij} = 0. \tag{3.19}$$

Since the Kähler connection is also Levi-Civita, our work from the previous paragraph shows that

$$R_{ij\ell k} = R_{jik\ell} = -R_{k\ell ij}. \tag{3.20}$$

In particular, we have

$$R_{\alpha\bar{\beta}i\bar{j}} = R_{i\bar{j}\alpha\bar{\beta}} = R_{i\bar{\beta}\alpha\bar{j}}, \tag{3.21}$$

where the last equality is achieved as follows:

$$R_{i\bar{j}\alpha\bar{\beta}} = -R_{i\bar{\beta}\bar{j}\alpha} - R_{i\alpha\bar{\beta}\bar{j}} = -R_{i\bar{\beta}\bar{j}\alpha} = R_{i\bar{\beta}\alpha\bar{j}}.$$

Ricci curvature of the Kähler connection

Finally, we come to the promised discussion of Ricci curvature. Recall that for a Riemannian metric the Ricci curvature is the trace

$$\mathrm{Ricci}(g)_{ij} = g^{k\ell} R_{ik\ell j}.$$

As mentioned at the end of the previous paragraph, there is some symmetry to the Ricci curvature of a Kähler metric. Indeed, the Ricci curvature tensor satisfies

$$\mathrm{Ricci}(g)_{\alpha\beta} = \mathrm{Ricci}(g)_{\bar\alpha\bar\beta} = 0 \quad \text{and} \quad \mathrm{Ricci}(g)_{\alpha\bar\beta} = \overline{\mathrm{Ricci}(g)_{\beta\bar\alpha}}. \tag{3.22}$$

In fact, we claimed in the last paragraph that one has the formula

$$\mathrm{Ricci}(g)_{\alpha\bar\beta} = -\partial_\alpha \partial_{\bar\beta} \log\det\left(g_{\mu\bar\nu}\right), \tag{3.23}$$

where g is the Kähler metric in question. To see this formula, we use the above symmetries as follows.

$$-\mathrm{Ricci}(g)_{\alpha\bar\beta} := -g^{d\bar c} R_{\alpha\bar c d\bar\beta} \overset{(3.21)}{=} -g^{d\bar c} R_{\alpha\bar\beta d\bar c} \overset{(3.19)}{=} g^{d\bar c} R_{\alpha\bar\beta\bar c d} \overset{(*)}{=} g^{\delta\bar\gamma} R_{\alpha\bar\beta\bar\gamma\delta} = \partial_{z^\alpha}\partial_{\bar z^\beta} \log\det\left(g\right),$$

where $(*)$ holds because the metric is Hermitian, and so only has $(1,1)$-parts, and the last equality holds because, by Proposition 3.2.5,

$$\partial\bar\partial \log\det g = -\mathrm{trace}\ \bar\partial(\partial g g^{-1}).$$

3.2.6 Positivity

It is important to know when the curvature of the Chern connection for a Hermitian metric on a holomorphic vector bundle is 'positive'. Because the curvature $\Theta(\mathfrak{h})$ of the Chern connection of a metric $\mathfrak{h}$ for a holomorphic $E \to X$ is a $(1,1)$-form with values in $\mathrm{Hom}(E,E) \to X$, there are many ways to measure its positivity. The strongest notion of positivity is called *Nakano-positivity*, the weakest notion is called *Griffiths positivity*, and these two notions flank an intermediate sequence of conditions discovered by Demailly. We shall describe all of these in the present paragraph.

Quadratic form on $E \otimes T^{1,0}_X$

Using the metric $\mathfrak{h}$, one defines Hermitian forms $\{\ ,\ \}_{\mathfrak{h},\Theta(\mathfrak{h})}$ on the fibers of $E \otimes T^{1,0}_X$ by letting

$$\{v \otimes \xi, w \otimes \eta\}_{\mathfrak{h},\Theta(\mathfrak{h})} := \mathfrak{h}(\Theta(\mathfrak{h})_{\xi,\bar\eta} v, w) \tag{3.24}$$

for indecomposable tensors on a given fiber $E_x \otimes T^{1,0}_{X,x}$ and extending bilinearly to the entire fiber.

Definitions of positivity

3.2.6 DEFINITION. *Let $E \to X$ be a holomorphic vector bundle with smooth Hermitian metric $\mathfrak{h}$, and fix a smooth Hermitian metric g on X.*

(i) We say that $\mathfrak{h}$ has positive curvature in the sense of Griffiths at a point $x \in X$ if there exists $c > 0$ such that

$$\{v \otimes \xi, v \otimes \xi\}_{\mathfrak{h},\Theta(\mathfrak{h})} \geq c\mathfrak{h}(v,v)g(\xi,\xi)$$

for all $v \otimes \xi \in E_x \otimes T^{1,0}_{X,x}$.

(ii) We say that $\mathfrak{h}$ has positive curvature in the sense of Nakano at a point $x \in X$ if there exists $c > 0$ such that

$$\left\{\sum_{j=1}^{n} v_j \otimes \xi_j, \sum_{k=1}^{n} v_k \otimes \xi_k\right\}_{\mathfrak{h},\Theta(\mathfrak{h})} \geq c\sum_{j=1}^{n} \mathfrak{h}(v_j,v_j)g(\xi_j,\xi_j)$$

for all $v_1 \otimes \xi_1, ..., v_n \otimes \xi_n \in E_x \otimes T^{1,0}_{X,x}$, where $n = \min(\dim_{\mathbb{C}} Y, \mathrm{Rank} E)$.

(iii) Let m be an integer between 1 and $\min(\dim_{\mathbb{C}} Y, \mathrm{Rank} E)$. We say that $\mathfrak{h}$ has m-positive curvature in the sense of Demailly at a point $x \in Y$ if there exists $c > 0$ such that

$$\left\{\sum_{j=1}^{m} v_j \otimes \xi_j, \sum_{k=1}^{m} v_k \otimes \xi_k\right\}_{\mathfrak{h},\Theta(\mathfrak{h})} \geq c\sum_{j=1}^{m} \mathfrak{h}(v_j,v_j)g(\xi_j,\xi_j)$$

for all $v_1 \otimes \xi_1, ..., v_m \otimes \xi_m \in E_x \otimes T^{1,0}_{Y,x}$. ◇

One defines non-negative curvature by taking $c = 0$, and to define negative and non-positive curvature one simply changes the sign of c and reverses the inequalities.

If V and M are two vector spaces then the elements of $E \otimes M$ are linear maps from M^* to V. Any such linear map has a rank, which is equal to the dimension of its image. This rank is by definition the rank of a tensor $T \in V \otimes M$.

Thus a holomorphic vector bundle $E \to X$ with Hermitian metric $\mathfrak{h}$ is m-positive at $x \in X$ if for any Hermitian metric g on X there exists a positive constant c such that

$$\{T,T\}_{\mathfrak{h},\Theta(\mathfrak{h})} \geq c|T|^2_{g,\mathfrak{h}}$$

for every $T \in E_x \otimes T^{1,0}_{X,x}$ whose rank is at most m.

Duality

The Hermitian holomorphic vector bundle $(E, \mathfrak{h})$ is Griffiths-positive if and only if its dual $(E^*, \mathfrak{h}^*)$ is Griffiths-negative, but that this statement ceases to be true for the stronger notions of positivity. More precisely, the Riesz Representation Theorem provides a $\mathbb{C}$-conjugate linear isometry $R : E^* \to E$ defined by

$$\langle \alpha, v\rangle = \mathfrak{h}(v, R\alpha).$$

Since

$$\left\langle \nabla^{1,0}\alpha, v\right\rangle + \left\langle \alpha, \nabla^{1,0}v\right\rangle = \partial \left\langle \alpha, v\right\rangle = \partial\mathfrak{h}(v, R\alpha) = \mathfrak{h}(\nabla^{1,0}v, R\alpha) + \mathfrak{h}(v, \bar\partial R\alpha)$$
$$= \left\langle \alpha, \nabla^{1,0}v\right\rangle + \left\langle R^{-1}\bar\partial R\alpha, v\right\rangle,$$

one sees that

$$\nabla^{1,0}_{\xi}\alpha = R^{-1}\bar\partial_{\bar\xi}R\alpha, \quad \text{and similarly,} \quad \bar\partial_{\bar\xi}\alpha = R^{-1}\nabla^{1,0}_{\xi}R\alpha.$$

It follows that $\Theta(\mathfrak{h}^*) = R^{-1}\Theta(\mathfrak{h})R$, and therefore that

$$\Theta(\mathfrak{h}^*)_{\xi_1,\bar\xi_2} = -R^{-1}\Theta(\mathfrak{h})_{\xi_2,\bar\xi_1}R.$$

If we now choose $v_1, ..., v_m \in E_x$ and $\xi_1, ..., \xi_m \in T^{1,0}_{Y,x}$ then we find

$$\left\{\sum_{j=1}^m (Rv_j)\otimes\xi_j, \sum_{k=1}^m (Rv_k)\otimes\xi_k\right\}_{\mathfrak{h}^*,\Theta(\mathfrak{h}^*)}$$
$$= \sum_{j,k=1}^m \mathfrak{h}^*(\sqrt{-1}\Theta(\mathfrak{h}^*)_{\xi_j,\bar\xi_k}R^{-1}v_j, R^{-1}v_k) = -\sum_{j,k=1}^m \mathfrak{h}^*(R^{-1}\sqrt{-1}\Theta(\mathfrak{h})_{\xi_k,\bar\xi_j}v_j, R^{-1}v_k)$$
$$= -\sum_{j,k=1}^m h(\sqrt{-1}\Theta(\mathfrak{h})_{\xi_k,\bar\xi_j}v_j, v_k) = -\sum_{j,k=1}^m \left\{v_j\otimes\xi_k, v_k\otimes\xi_j\right\}_{\mathfrak{h},\Theta(\mathfrak{h})}.$$

And unless the rank of $v_1\otimes\xi_1 + \cdots + v_m\otimes\xi_m$ is at most 1, the last quadratic form is not the form measuring negativity of the curvature.

Positivity of line bundles

When the rank of $E \to X$ is 1, i.e., when $E \to X$ is a line bundle, all of the notions of positivity coincide. Indeed, when the fiber of E is 1-dimensional, any map $E_x \otimes T^{1,0}_{X,x} \to E_x \otimes T^{1,0}_{X,x}$ has rank at most one 1.

In the rank one case one therefore drops all adjectives and speaks of positivity of the curvature. In this case the curvature of a Hermitian metric $e^{-\varphi}$ then is

$$\partial\bar\partial\varphi = \frac{\partial^2\varphi}{\partial z^i\partial\bar z^j}dz^i\wedge d\bar z^j,$$

and hence the curvature of $e^{-\varphi}$ is (semi-)positive if and only if the Hermitian matrix

$$\left(\frac{\partial^2\varphi}{\partial z^i\partial\bar z^j}\right)_{i,j=1}^{\dim_{\mathbb{C}}(X)}$$

is positive (semi-)definite.

Criterion for Griffiths positivity

In Chapter 11 the following proposition will be very useful.

3.2.7 PROPOSITION. *The metric h for $V \to X$ is non-positive in the sense of Griffiths if and only if for any holomorphic section s of $V \to X$ the function*

$$\log h(s,s)$$

is a plurisubharmonic function on X.

Proof. We calculate that

$$\partial\bar{\partial}\log h(s,s) = -\frac{h(\Omega(h)s,s)}{h(s,s)} + \frac{h(s,s)h(\nabla^{1,0}s,\nabla^{1,0}s) - h(\nabla^{1,0}s,s)\wedge h(s,\nabla^{1,0}s)}{h(s,s)^2}. \tag{3.25}$$

The second term on the right hand side of (3.25) is non-negative because of the Cauchy-Schwarz Inequality, so we see that if h is nonpositive in the sense of Griffiths then the right hand side of (3.25) is non-negative, i.e., $\log h(s,s)$ is plurisubharmonic.

To see the converse, it is clearly enough to work locally, i.e., to assume the vector bundle V is trivial (but with non-trivial metric). Under the condition of triviality, given any vector $v \in V_x$ there exists a holomorphic section s_v of $V \to X$ such that $s_v(x) = v$ and $\nabla^1 s_v(x) = 0$. Plugging the section s_v into (3.25) yields

$$(\partial\bar{\partial}\log h(s_v,s_v))(x) = -\frac{h(\Omega(h)(x)v,v)}{h(v,v)},$$

which shows that if $\partial\bar{\partial}\log h(s_v,s_v)$ is plurisubharmonic then h is non-positive in the sense of Griffiths. □

EXERCISES

3.2.1. Prove Proposition 3.2.3.

3.2.2. Show that if $L \to M$ is a complex line bundle then the line bundle $\mathrm{End}(L) \to M$ is trivial.

3.2.3. Show that if $L \to X$ is a holomorphic line bundle and $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two Hermitian metrics for L then the curvature forms

$$\Theta(\mathfrak{h}_1) \quad \text{and} \quad \Theta(\mathfrak{h}_2)$$

of their Chern connections are cohomologous.

3.2.4. Prove Proposition 3.2.5.

3.2.5. Show that the Hermitian quadratic form (3.24) is well-defined.

3.2.6. Confirm the computation (3.25), and show that for any vector $v \in V_x$ there exists a holomorphic section s_v of $V \to X$ near x such that $s_v(x) = v$ and $\nabla^1 s_v(x) = 0$.

3.2.7. Let X be a complex manifold of dimension n and let $E \to X$ be a holomorphic vector bundle of rank r with smooth Hermitian metric h. Set

$$k := \min(r, n).$$

For a local coordinate z, write

$$\Upsilon^{i\bar{j}} := (-1)^{n^2/2} dz^1 \wedge ... \wedge dz^{i-1} \wedge dz^{i+1} \wedge ... \wedge dz^n \wedge d\bar{z}^1 \wedge ... \wedge d\bar{z}^{j-1} \wedge d\bar{z}^{j+1} \wedge ... \wedge d\bar{z}^n.$$

(a) Prove the following test for Nakano negitivity: the Chern connection for (E, h) has Nakano-negative curvature at a point $p \in X$ if and only if for every local coordinate system z at p and every k-tuple of holomorphic sections $(f_1, ..., f_k)$ the (n, n)-form

$$\bar{\partial}\partial \left(\sum_{i,j=1}^{k} h(f_i, f_j) \Upsilon^{i\bar{j}} \right)$$

is a negative multiple of Lebesgue measure $dV(z)$ near p.

(b) Formulate a test for Nakano positivity.

Part II

L^2 Estimates for $\bar{\partial}$

Lecture 4

$\mathbf{L^2}$ Estimates for $\bar{\partial}$ in complex dimension 1

In this section we prove Hörmander's Theorem for complex manifolds of dimension 1. We shall give two proofs. The first is longer, and will be given only for domains in the complex plane. However, the weighted estimates obtained will hold for very general weights. The second proof is rather short, and applies to general Riemann surfaces, but the estimates established hold only for smooth weights.

4.1 Domains in $\mathbb{C}$

For the rest of the section, fix an open connected set Ω in $\mathbb{C}$ and a smooth function $\varphi \in \mathscr{C}^\infty(\Omega)$. Classically the function φ is called the *weight function*, or simply the *weight*.

Hilbert spaces and operators

For the moment, we add hypotheses to Ω and φ: we assume that $\Omega \subset\subset \mathbb{C}$, that $\partial\Omega$ is smooth and that $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$. With these hypotheses in place, let us proceed to define some Hilbert spaces.

Space of functions For smooth functions $f, g : \Omega \to \mathbb{C}$ we define the inner product

$$(f, g)_o := \int_\Omega f\bar{g}e^{-\varphi}dA,$$

and we let $\mathscr{H}^o_\varphi$ denote the Hilbert space closure of $\mathscr{C}^\infty(\overline{\Omega})$ with respect to the norm induced by this inner product. (Note that $\mathscr{C}^\infty(\overline{\Omega})$ is contained in $\mathscr{H}^o_\varphi$ because Ω is bounded and $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$.)

Space of $(0,1)$-forms For $(0,1)$-forms $\alpha = fd\bar{z}$ and $\beta = gd\bar{z}$ we define the inner product

$$(\alpha, \beta)_1 := \frac{1}{2\sqrt{-1}} \int_\Omega \alpha \wedge \bar{\beta}e^{-\varphi} = (f, g)_o,$$

and we let $\mathscr{H}^1_\varphi$ denote the Hilbert space closure of $\mathscr{C}^\infty(\overline{\Omega})d\bar{z}$ with respect to the norm induced by this inner product.

The $\bar\partial$ operator

Next we turn to the definition of the Hilbert space $\bar\partial$ operator. The $\bar\partial$ operator as we know it up to now sends a smooth function f to the $(0,1)$-form $\bar\partial f = \frac{\partial f}{\partial \bar z} d\bar z$. We extend $\bar\partial$ to a densely defined operator (denoted with the same symbol)

$$\bar\partial : \mathscr{H}^o_\varphi \to \mathscr{H}^1_\varphi$$

defined as follows: the domain of $\bar\partial$ consists of all $f \in \mathscr{H}^o_\varphi$ such that the $(0,1)$-current $\bar\partial f$ is represented by integration against (an automatically unique) $\alpha \in \mathscr{H}^1_\varphi$, i.e., there exists $\alpha \in \mathscr{H}^1_\varphi$ such that

$$\frac{1}{2\sqrt{-1}} \int_\Omega \alpha \wedge \psi dz = - \int_\Omega f \frac{\partial \psi}{\partial z} dA \quad \text{for all } \psi \in \mathscr{C}^\infty_o(\Omega), \tag{4.1}$$

and then we define $\bar\partial f := \alpha$ for $f \in \text{Domain}(\bar\partial)$.

4.1.1 PROPOSITION. *The densely defined operator $\bar\partial : \mathscr{H}^o_\varphi \to \mathscr{H}^1_\varphi$ is closed.*

Proof. The operator $\bar\partial$ is closed, i.e., $\text{Graph}(\bar\partial)$ is closed, if and only if for any $f_j \in \text{Domain}(\bar\partial)$ such that $f_j \to f$ in $\mathscr{H}^o_\varphi$ and $\bar\partial f_j \to \alpha$ in $\mathscr{H}^1_\varphi$, $\alpha = \bar\partial f$ in the sense of currents, i.e., (4.1) holds. Let us fix such a sequence f_j. Then for all $g \in \mathscr{C}^\infty_o(\Omega)$

$$\begin{aligned}(\alpha, g d\bar z)_1 &= \frac{1}{2\sqrt{-1}} \int_\Omega \alpha \wedge \overline{g d\bar z} e^{-\varphi} = \lim \frac{1}{2\sqrt{-1}} \int_\Omega \bar\partial f_j \wedge \overline{g d\bar z} e^{-\varphi} \\ &= \lim \int_\Omega f_j \overline{\left(-e^{\varphi} \frac{\partial}{\partial z}(e^{-\varphi} g)\right)} e^{-\varphi} dA = \int_\Omega f \overline{\left(-e^{\varphi} \frac{\partial}{\partial z}(e^{-\varphi} g)\right)} e^{-\varphi} dA.\end{aligned}$$

But this equation means that (4.1) holds for $\psi = e^{-\varphi} g$. Since $g \mapsto e^{-\varphi} g$ is an isomorphism of $\mathscr{C}^\infty_o(\Omega)$, we are done. □

The Hilbert space and formal adjoints of $\bar\partial$

For reasons that will become clear soon, one wants to define the Hilbert space adjoint $\bar\partial^*$ of the densely defined operator $\bar\partial$. The domain of $\bar\partial^*$ is

$$\text{Domain}(\bar\partial^*) := \{\alpha \in \mathscr{H}^1_\varphi \, ; \, \exists C_\alpha > 0 \text{ such that } |(\alpha, \bar\partial g)_1| \le C_\alpha ||g||_o \text{ for all } g \in \text{Domain}(\bar\partial)\},$$

and $\bar\partial^* \alpha$ is the unique element of $\mathscr{H}^o_\varphi$ corresponding to the linear functional

$$\ell_\alpha : \mathscr{C}^\infty_o(\Omega) \ni g \mapsto (\bar\partial g, \alpha) \in \mathbb{C}$$

under the Riesz Representation Theorem. In particular,

$$(\bar\partial^* \alpha, g)_o = (\alpha, \bar\partial g)_1.$$

It is a general fact that the Hilbert space adjoint of a densely defined (resp. closed) operator is closed (resp. densely defined), and that the double adjoint of a closed densely defined operator is the operator itself. These elementary facts make it possible to formulate an adjoint (weak) version of the $\bar{\partial}$ equation.

To establish an estimate we will need later on, it is useful to have a formula for $\bar{\partial}^*$ on a dense subspace of $\mathscr{H}^1_\varphi$. With such a formula as our goal, let $\alpha = f d\bar{z}$ be a smooth $(0,1)$-form on $\overline{\Omega}$ and let $g \in \mathscr{C}^\infty(\overline{\Omega})$. Recalling our temporary assumption that Ω is bounded, $\partial\Omega$ is smooth and $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$, we compute that

$$\begin{aligned}
(\alpha, \bar{\partial} g)_1 &= \frac{1}{2\sqrt{-1}} \int_\Omega \alpha \wedge \overline{\bar{\partial} g} e^{-\varphi} \\
&= \frac{1}{2\sqrt{-1}} \int_\Omega \alpha \wedge d\bar{g} e^{-\varphi} \\
&= \frac{1}{2\sqrt{-1}} \int_\Omega \bar{g} d(e^{-\varphi}\alpha) - \frac{1}{2\sqrt{-1}} \int_\Omega d(e^{-\varphi}\alpha\bar{g}) \\
&= \frac{1}{2\sqrt{-1}} \int_\Omega \bar{g} \frac{\partial}{\partial z}\left(f e^{-\varphi}\right) dz \wedge d\bar{z} - \frac{1}{2\sqrt{-1}} \int_{\partial\Omega} f\bar{g} e^{-\varphi} d\bar{z} \\
&= \left(-e^{\varphi} \tfrac{\partial}{\partial z}(e^{-\varphi} f), g\right)_o + \frac{\sqrt{-1}}{2} \int_{\partial\Omega} \bar{g} e^{-\varphi} \alpha.
\end{aligned}$$

4.1.2 DEFINITION. *The operator* $\vartheta : \mathscr{C}^\infty(\Omega) d\bar{z} \to \mathscr{C}^\infty(\Omega)$ *defined by*

$$\vartheta(f d\bar{z}) := -e^{\varphi} \tfrac{\partial}{\partial z}(e^{-\varphi} f) = -\frac{\partial f}{\partial z} + \frac{\partial \varphi}{\partial z} f$$

is called the formal adjoint *of* $\bar{\partial}$.

4.1.3 REMARK. As one can see from the calculation preceding Definition 4.1.2, the formal adjoint of $\bar{\partial}$ can equally be defined as the operator $\vartheta : \mathscr{C}^\infty(\Omega) d\bar{z} \to \mathscr{C}^\infty(\Omega)$ that satisfies

$$(\vartheta\alpha, \psi)_o = (\alpha, \bar{\partial}\psi)_1$$

for all $\psi \in \mathscr{C}^\infty_o(\Omega)$. $\diamond$

With the formal adjoint in hand, we have the following proposition.

4.1.4 PROPOSITION. *Let* $\Omega \subset \mathbb{C}$ *be a bounded domain with smooth boundary and let* $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$. *Then a form* $\alpha \in \mathscr{C}^\infty(\overline{\Omega}) d\bar{z}$ *lies in* $\mathrm{Domain}(\bar{\partial}^*)$ *if and only if* $\alpha|_{\partial\Omega} \equiv 0$, *and in this case*

$$\bar{\partial}^* \alpha = \vartheta\alpha.$$

Proof. Let $\alpha \in \mathscr{C}^\infty(\overline{\Omega}) d\bar{z}$. For $g \in \mathscr{C}^\infty(\overline{\Omega})$ we have calculated that

$$(\alpha, \bar{\partial} g)_1 = (\vartheta\alpha, g)_o + \frac{\sqrt{-1}}{2} \int_{\partial\Omega} \bar{g} e^{-\varphi} \alpha. \tag{4.2}$$

Hence if $\alpha|_{\partial\Omega} \equiv 0$ then

$$|(\alpha, \bar{\partial} g)_1| = |(\vartheta\alpha, g)_o| \le ||\vartheta\alpha||_o ||g||_o,$$

which shows that $\alpha \in \mathrm{Domain}(\bar{\partial}^*)$, and the formula $(\alpha, \bar{\partial} g)_1 = (\vartheta\alpha, g)_o$ implies that $\bar{\partial}^*\alpha = \vartheta\alpha$.

Conversely, suppose $\alpha \in \mathrm{Domain}(\bar{\partial}^*)$. Fix functions $\chi_j \in \mathscr{C}^\infty_o(\Omega)$ such that $0 \le \chi_j \le 1$ and for each compact set $K \subset \Omega$ there exists $J \in \mathbb{N}$ such that if $j \ge J$ then $\chi_j|_K \equiv 1$. Then by the definition of $\mathrm{Domain}(\bar{\partial}^*)$

$$|(\alpha, \bar{\partial}(\chi_j g))_1| \le C_\alpha ||\chi_j g||_o$$

for some constant C_α that is independent of g or j. Therefore by Dominated Convergence

$$(\alpha, \bar{\partial} g)_1 = \lim_j (\alpha, \bar{\partial}(\chi_j g))_1 = \lim_j (\vartheta\alpha, \chi_j g)_o = (\vartheta\alpha, g)$$

for all $g \in \mathscr{H}^o_\varphi$. In particular, if $g \in \mathscr{C}^\infty(\overline{\Omega})$ then by (4.2)

$$\int_{\partial\Omega} \bar{g} e^{\varphi} \alpha = 0,$$

so $\alpha|_{\partial\Omega} \equiv 0$. $\square$

The formal identity and the basic estimate

We continue to assume that Ω is connected, bounded, and smoothly bounded, and that $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$. Let $\alpha = f d\bar{z}$ be a smooth $(0,1)$-form on Ω. We compute that

$$\begin{aligned}
\bar{\partial}\vartheta\alpha &= \bar{\partial}\left(-\frac{\partial f}{\partial z} + \frac{\partial \varphi}{\partial z} f\right) \\
&= \left(-\frac{\partial^2 f}{\partial z \partial \bar{z}} + \frac{\partial \varphi}{\partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} f\right) d\bar{z} \\
&= -e^{\varphi}\frac{\partial}{\partial z}\left(e^{-\varphi}\frac{\partial f}{\partial \bar{z}}\right) d\bar{z} + \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \alpha
\end{aligned}$$

It is tempting to write the term $-e^{\varphi}\frac{\partial}{\partial z}\left(e^{-\varphi}\frac{\partial f}{\partial \bar{z}}\right)$ as $\vartheta\bar{\partial}\alpha$, but since α is a $(0,1)$-form $\bar{\partial}\alpha = 0$. Therefore we need a better interpretation for this term if we are to extend these ideas to higher dimensions. The insight that seems most natural is to view α not as a differential form, but rather as a section of a complex vector bundle.

Recall that the $\bar{\partial}$-operator, when acting for sections, is well-defined only on a holomorphic vector bundle. If we think of the form $\alpha = f d\bar{z}$ not as a form, but rather as a section of the (trivial, so in this case, holomorphic) vector bundle $T^{*0,1}_\Omega$, then we can act with $\bar{\partial}$ on this section, and produce the section of $(T^{*0,1}_\Omega)^{\otimes 2}$ given by

$$\bar{\nabla}\alpha = \frac{\partial f}{\partial \bar{z}} d\bar{z}^{\otimes 2}.$$

(Later we will see that there is a way to define $\bar{\nabla}$ even when $T^{*0,1}_\Omega$ is not trivial, as long as one has a Hermitian Riemannian metric on Ω. The present case corresponds to the Euclidean metric on our domain $\Omega \subset \mathbb{C}$.)

Let us define an inner product on smooth sections of $(T^{*0,1}_{\Omega})^{\otimes 2}$ by

$$(gd\bar{z}^{\otimes 2}, hd\bar{z}^{\otimes 2})_2 := \int_{\Omega} f\bar{g}e^{-\varphi}dA.$$

With this inner product in hand, one can compute (Exercise 4.2.2) that the formal adjoint of the operator $\bar{\nabla} : \Gamma(\Omega, T^{*0,1}_{\Omega}) \to \Gamma(\Omega, (T^{*0,1}_{\Omega})^{\otimes 2})$ is given by the formula

$$\bar{\nabla}^*(gd\bar{z}^{\otimes 2}) = -e^{\varphi}\frac{\partial}{\partial z}(e^{-\varphi}g)d\bar{z}. \tag{4.3}$$

Thus one obtains the following theorem.

4.1.5 THEOREM. *Suppose* Ω *is a bounded, smoothly bounded domain and* $\varphi \in \mathscr{C}^{\infty}(\overline{\Omega})$. *Then*

$$\bar{\partial}\vartheta\alpha = \bar{\nabla}^*\bar{\nabla}\alpha + \frac{\partial^2\varphi}{\partial z\partial\bar{z}}\alpha \tag{4.4}$$

for any smooth $(0,1)$*-form* α *on* Ω.

Now let α be a smooth form on $\overline{\Omega}$ lying in the domain of $\bar{\partial}^*$. Then α vanishes on $\partial\Omega$, so

$$(\bar{\partial}\vartheta\alpha, \alpha)_1 = (\bar{\partial}\bar{\partial}^*\alpha, \alpha)_1 = ||\bar{\partial}^*\alpha||^2_o \quad \text{and} \quad (\bar{\nabla}^*\bar{\nabla}\alpha, \alpha)_1 = ||\bar{\nabla}\alpha||^2_2.$$

Theorem 4.1.5 therefore implies the following key result.

4.1.6 THEOREM (Bochner-Kodaira Identity). *Suppose* Ω *is a bounded, smoothly bounded domain and* $\varphi \in \mathscr{C}^{\infty}(\overline{\Omega})$. *Then for any smooth* $(0,1)$*-form* α *in the domain of* $\bar{\partial}^*$ *one has the identity*

$$||\bar{\partial}^*\alpha||^2_o = ||\bar{\nabla}\alpha||^2_2 + \frac{1}{2\sqrt{-1}}\int_{\Omega} \frac{\partial^2\varphi}{\partial z\partial\bar{z}}\alpha \wedge \bar{\alpha}e^{-\varphi}. \tag{4.5}$$

Density of smooth forms

The domain of $\bar{\partial}^*$ is a dense subspace of $\mathscr{H}^1_{\varphi}$, but we can endow it with another norm, namely

$$|||\alpha|||^2 := ||\alpha||^2_1 + ||\bar{\partial}^*\alpha||^2_o.$$

Let us denote by $\mathscr{F}$ the inner product space with norm $|||\cdot|||$ whose underlying vector space is $\text{Domain}(\bar{\partial}^*)$.

4.1.7 REMARK. Since $\bar{\partial}^*$ is also a closed operator, $\mathscr{F}$ is in fact a Hilbert space. We will not use the completeness of $|||\cdot|||$ in this paragraph. ◇

We have the following theorem.

4.1.8 THEOREM. *The smooth forms in* $\text{Domain}(\bar{\partial}^*)$ *are dense in* $\mathscr{F}$.

We begin with some important preliminaries. Fix a form $\alpha \in \mathscr{F}$. First, we extend α by 0 to $\mathbb{C} - \Omega$. This extended form, which we denote by $\tilde{\alpha}$, can be seen as a form in $L^2_{\ell oc}(\mathbb{C})$, and we want to say that, in some sense, $\tilde{\alpha}$ is still in the domain of $\bar{\partial}^*$. The trouble is that $\bar{\partial}^*$ itself, the Hilbert space adjoint of $\bar{\partial}$, is intimately tied to the domain Ω. However, the formal adjoint is given by some formula, namely $\bar{\partial}^* = L$, where

$$L(f d\bar{z}) = -\frac{\partial f}{\partial z} + \frac{\partial \varphi}{\partial z} f.$$

Since $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$, we can assume that $\varphi \in \mathscr{C}^\infty_o(\mathbb{C})$ and define the operator L as a densely defined operator on $L^2(\mathbb{C}, e^{-\varphi} dA)$. We can now make a more reasonable statement about $\tilde{\alpha}$.

4.1.9 PROPOSITION. *The form α is in the domain of $\bar{\partial}^*$ if and only if $L\tilde{\alpha} \in \mathscr{H}^o_\varphi$ in the sense of distributions, and in that case*

$$L\tilde{\alpha} = \bar{\partial}^* \alpha.$$

Proof. For a $(0,1)$-form α in $\mathscr{H}^1_\varphi$ we have the identity

$$(\alpha, \bar{\partial} h)_1 := \frac{1}{2\sqrt{-1}} \int_\Omega \alpha \wedge \overline{\bar{\partial} h} e^{-\varphi} = \frac{1}{2\sqrt{-1}} \int_{\mathbb{C}} \tilde{\alpha} \wedge \overline{\bar{\partial} h} e^{-\varphi} = (L\tilde{\alpha})(h)$$

where the last equation on the right hand side is the meaning of $L\tilde{\alpha}$ in the sense of distributions. Thus if $L\tilde{\alpha} \in \mathscr{H}^o_\varphi$ then $\alpha \in \mathrm{Domain}(\bar{\partial}^*)$, and then $\bar{\partial}^* \alpha = L\tilde{\alpha}$.

Conversely, suppose $\alpha \in \mathrm{Domain}(\bar{\partial}^*)$. Then $\bar{\partial}^* \alpha$ is well-defined in $\mathscr{H}^o_\varphi$ and

$$(L\tilde{\alpha})(h) = (\alpha, \bar{\partial} h)_1 = (\bar{\partial}^* \alpha, h)_o,$$

which means that $L\tilde{\alpha} = \bar{\partial}^* \alpha$ in the sense of distributions. □

Our next goal is to localize the problem using partitions of unity. Let us choose a collection of open disks $\{U_j\}$ that cover Ω, and another collection of open disks $\{V_k\}$ that cover $\partial\Omega$, and are sufficiently small that ∂V_k and $\partial\Omega$ intersect transversely. Because $\partial\Omega$ is compact, we can choose a finite subset $V_1, ..., V_{N_1}$ that cover $\partial\Omega$. Then $K := \Omega - \bigcup_{j=1}^{N_1} V_j$ is closed and bounded, so compact, and thus we can choose $U_1, ..., U_{N_o}$ that cover K. All together, $\{U_j, V_k\}$ cover $\overline{\Omega}$. We denote this cover by $\{W_i\}$ when we don't want to distinguish between the U_j and the V_k. Let us choose a partition of unity $\{\chi_\nu\}$ subordinate to this cover, i.e., for each index i there exists ν such that $\mathrm{Support}(\chi_\nu) \subset\subset W_i$, and $\sum_\nu \chi_\nu \equiv 1$ on the union of the W_i.

We can now write

$$\alpha = \sum_\nu \chi_\nu \alpha,$$

and each $\chi_\nu \alpha$ is compactly supported in some W_i. We need the following lemma.

4.1.10 LEMMA. *If $\alpha \in \mathrm{Domain}(\bar{\partial}^*)$ and $\chi \in \mathscr{C}^\infty(\overline{\Omega})$ then $\chi\alpha \in \mathrm{Domain}(\bar{\partial}^*)$.*

Proof. For any $g \in \text{Domain}(\bar\partial)$ we have

$$(\chi\alpha, \bar\partial g)_1 = (\alpha, \bar\partial(\chi g))_1 + \frac{1}{2\sqrt{-1}} \int_\Omega \bar g \frac{\partial \chi}{\partial z} f e^{-\varphi} dA,$$

where $\alpha = f d\bar z$. Thus since $\alpha \in \text{Domain}(\bar\partial^*)$

$$|(\chi\alpha, \bar\partial g)_1| \le C_\alpha ||\chi g||_o + ||\tfrac{\partial \chi}{\partial z} f||_o ||g||_o \le \left(C_\alpha \sup_{\overline{\Omega}} |\chi| + ||\alpha||_1 \sup_{\overline{\Omega}} \left| \frac{\partial \chi}{\partial z} \right| \right) ||g||_o,$$

which is what we wanted to show. □

Thus we are reduced to studying a $(0,1)$-form α in the domain of $\bar\partial^*$ and whose support is either a compact subset of U_j, or is the intersection of Ω and a compact subset of V_k.

In the first, interior case, we can use the usual mollifier method: let $\psi \in \mathscr{C}^\infty_o(\mathbb{D})$ and set $\psi_\delta(z) = \delta^{-2}\psi(z/\delta)$. Then for $\delta > 0$ sufficiently small $\alpha^\delta := \alpha * \psi_\delta$ is smooth with compact support in U_j and therefore lies in the domain of $\bar\partial^*$. Moreover, α^δ and $\bar\partial^*\alpha^\delta$ can be made arbitrarily close to α and $\bar\partial^*\alpha$ in $\mathscr{H}^1_\varphi$ and $\mathscr{H}^o_\varphi$ respectively by choosing $\delta > 0$ sufficiently small.

In the second, boundary case, we need to be a little more careful. Let K_k be the intersection of a cone in $\mathbb{C}$ with a small disk centered at the origin. If the cone angle and the radius of the disk are sufficiently small, and if the cone's axis is at an appropriate angle (for example, approximately parallel to the normal direction of some boundary point of Ω in V_k), then for each point $z \in \partial\Omega \cap V_k$ the truncated cone $z + K_k$ lies in the complement of Ω, while the truncated cone $z - K_k$ lies entirely in Ω, except for its vertex, which is of course on the boundary. (We must make sure that the disk V_k is sufficiently small, which we may assume is the case, without loss of generality.) Choose a function $\psi^{(k+)} \in \mathscr{C}^\infty_o(K_k)$ such that $\int_{\mathbb{C}} \psi^{(k+)} dA = 1$, and let $\psi^{(k-)}(z) = \psi^{(k+)}(-z)$.

Now suppose that $\alpha \in \text{Domain}(\bar\partial^*)$ is compactly supported in V_k. Let $\alpha^\delta := \alpha * \psi^{(k-)}_\delta$. Writing $\alpha = f d\bar z$, we have

$$\alpha^\delta(z) = d\bar z \cdot \int_{\mathbb{C}} f(z - \delta\zeta)\psi^{(k-)}(\zeta) dA(\zeta).$$

In particular, if $z \in \partial\Omega$ and $\zeta \in \text{Support}(\psi^{(k-)})$ then $-\zeta \in K_k$ and thus $z - \delta\zeta \notin \Omega$, which means that $f(z - \delta\zeta) = 0$. In other words, α^δ vanishes on $\partial\Omega$, and is therefore in the domain of $\bar\partial^*$.

By standard real analysis $\alpha^\delta \to \mathbf{1}_\Omega \alpha =: \tilde\alpha$ in $L^2(\mathbb{C})$, and $\bar\partial^*\alpha^\delta \to L\tilde\alpha$ in the sense of distributions. Thus by Proposition 4.1.9 $\bar\partial^*\alpha^\delta \to \bar\partial^*\alpha$ in $\mathscr{H}^o_\varphi$.

End of the proof of Theorem 4.1.8. Having written $\alpha = \sum_\nu \chi_\nu \alpha$, we can find smooth forms $(\chi_\nu\alpha)^\delta$ as just outlined, with each $(\chi_\nu\alpha)^\delta$ vanishing on $\partial\Omega$ and therefore lying in the domain of $\bar\partial^*$, and moreover such that

$$(\chi_\nu\alpha)^\delta \to \chi_\nu\alpha \quad \text{and} \quad \bar\partial^*(\chi_\nu\alpha)^\delta \to \bar\partial^*(\chi_\nu\alpha)$$

in $\mathscr{H}^1_\varphi$ and $\mathscr{H}^o_\varphi$ respectively. The proof is complete. □

4.1.11 COROLLARY. *The identity* (4.5) *holds for all* α *in the domain of* $\bar\partial^*$.

Hörmander's Theorem on domains in $\mathbb{C}$

We are now ready to state and prove Hörmander's Theorem.

4.1.12 THEOREM (Hörmander's Theorem for domains in $\mathbb{C}$). *Let $\Omega \subset \mathbb{C}$ be a domain and let $\varphi \in L^1_{\ell oc}(\Omega)$ be a function such that $\varphi(z) - c|z|^2$ is subharmonic in Ω for some $c > 0$. Then for any measurable function f on Ω such that*

$$\int_\Omega |f|^2 e^{-\varphi} dA < +\infty$$

there exists a locally integrable function u such that

$$\frac{\partial u}{\partial \bar{z}} = f$$

in the sense of currents, and

$$\int_\Omega |u|^2 e^{-\varphi} dA \le \frac{1}{c} \int_\Omega |f|^2 e^{-\varphi} dA.$$

Proof. Choose domains Ω_j, $j = 1, 2, ...$, with smooth boundary, such that

$$\Omega_j \subset\subset \Omega_{j+1} \quad \text{and} \quad \bigcup_j \Omega_j = \Omega.$$

Let us fix j for now. At the end of the proof we will let $j \to \infty$.

Define the function $\varphi_\delta := \varphi * (\delta^{-2}\psi(\cdot/\delta))$, where ψ is a radial function on $\mathbb{C}$ with compact support and total integral 1, and $\delta > 0$ is less than the Euclidean distance from Ω_j to $\mathbb{C} - \Omega$. Then φ_δ is smooth, subharmonic, and decreasing to φ, and we have

$$\frac{\partial^2}{\partial z \partial \bar{z}}(\varphi_\delta(z) - c|z|^2) = \int_{\mathbb{C}} \left(\varphi(\zeta) \frac{\partial^2 \psi}{\partial \zeta \partial \bar{\zeta}} \left(\frac{z-\zeta}{\delta} \right) - c\psi \left(\frac{z-\zeta}{\delta} \right) \right) \frac{dA(\zeta)}{\delta^2} \ge 0.$$

On the domain Ω_j we have the Hilbert spaces

$$\mathscr{H}^o_j := \mathscr{H}^o_{\varphi_\delta} \quad \text{and} \quad \mathscr{H}^1_j := \mathscr{H}^1_{\varphi_\delta},$$

as well as the closed, densely defined operators $\bar{\partial} : \mathscr{H}^o_j \to \mathscr{H}^1_j$ and the Hilbert space adjoint $\bar{\partial}^* : \mathscr{H}^1_j \to \mathscr{H}^o_j$. Then Corollary 4.1.11 applies, and thus from (4.5) we obtain

$$||\bar{\partial}^* \beta||^2_o \ge c||\beta||^2_1$$

for all $(0,1)$-forms β in the domain of $\bar{\partial}^*$. Let

$$\alpha := f d\bar{z}.$$

Then for all β in the domain of $\bar{\partial}^*$ we have

$$|(\alpha, \beta)_1|^2 \le ||\alpha||^2_1 ||\beta||^2_1 \le \frac{||\alpha||^2_1}{c} ||\bar{\partial}^* \beta||^2_o. \tag{4.6}$$

We claim that the estimate (4.6) provides a solution of the $\bar{\partial}$ equation with estimates. To see this claim let $\lambda : \mathrm{Image}(\bar{\partial}^*) \to \mathbb{C}$ be the linear functional defined by

$$\lambda(\bar{\partial}^* \beta) := (\beta, \alpha)_1.$$

The estimate (4.6) means that λ is continuous on $\mathrm{Image}(\bar{\partial}^*)$ (and hence on its closure). Setting λ equal to 0 on the orthogonal complement $\mathscr{H}_j^o \ominus \overline{\mathrm{Image}(\bar{\partial}^*)}$ makes λ a continuous linear functional on $\mathscr{H}_j^o$ with operator norm $||\lambda||_*^2 \le c^{-1}||\alpha||_1^2$. By the Riesz Representation Theorem there exists $u_{\delta,j} \in \mathscr{H}_j^o$ such that $\lambda(h) = (h, u_{\delta,j})$ and $||u_{\delta,j}|| = ||\lambda||_*$. Applying the first identity to $h = \bar{\partial}^* \beta$ shows that

$$\bar{\partial} u_{\delta,j} = \alpha, \qquad \text{i.e.,} \quad \frac{\partial u_{\delta,j}}{\partial \bar{z}} = f,$$

and the equality $||u_{\delta,j}|| = ||\lambda||_*$, together with the monotonicity of φ_δ with respect to δ, yields

$$\int_{\Omega_j} |u_{\delta,j}|^2 e^{-\varphi_\delta} dA \le \frac{1}{c} \int_{\Omega_j} |f|^2 e^{-\varphi_\delta} dA \le \frac{1}{c} \int_{\Omega_j} |f|^2 e^{-\varphi} dA \le \frac{1}{c} \int_{\Omega} |f|^2 e^{-\varphi} dA.$$

In fact, since φ_δ is decreasing in δ, for all $\delta_o > 0$ sufficiently small and all $\delta \le \delta_o$ we have

$$\int_{\Omega_j} |u_{\delta,j}|^2 e^{-\varphi_{\delta_o}} dA \le \int_{\Omega_j} |u_{\delta,j}|^2 e^{-\varphi_\delta} dA \le \frac{1}{c} \int_{\Omega} |f|^2 e^{-\varphi} dA.$$

Thus all the $\{u_{\delta,j}\}$ lie in a ball of radius $c^{-1/2}||f||$ in $\mathscr{H}^o_{\varphi_{\delta_o}}$. By Alaoglu's Theorem there is a sequence $\delta_k \to 0$ such that $u_{\delta_k,j}$ converges in $\mathscr{H}^o_{\varphi_{\delta_o}}$ and

$$\int_{\Omega_j} |u_{\delta_k,j}|^2 e^{-\varphi_{\delta_o}} dA \le \frac{1}{c} \int_{\Omega} |f|^2 e^{-\varphi} dA.$$

On the other hand, for all $k \ge \ell$

$$\int_{\Omega_j} |u_{\delta_k,j}|^2 e^{-\varphi_{\delta_\ell}} dA \le \int_{\Omega_j} |u_{\delta_k,j}|^2 e^{-\varphi_{\delta_k}} dA \le \frac{1}{c} \int_{\Omega} |f|^2 e^{-\varphi} dA,$$

so we can again extract a convergent subsequence in $\mathscr{H}^o_{\varphi_{\delta_\ell}}$. Taking the diagonal subsequence, we find that it converges in $\mathscr{H}^o_{\varphi_{\delta_\ell}}$ for all ℓ. Thus we have a limit u_j such that

$$\int_{\Omega_j} |u_j|^2 e^{-\varphi_{\delta_k}} dA \le \frac{1}{c} \int_{\Omega} |f|^2 e^{-\varphi} dA.$$

By the Monotone Convergence Theorem

$$\int_{\Omega_j} |u_j|^2 e^{-\varphi} dA \le \frac{1}{c} \int_{\Omega} |f|^2 e^{-\varphi} dA.$$

Note that since u_j is a limit of functions $u_{k,j}$ satisfying $\bar{\partial} u_j = \alpha$ in the weak sense, i.e.,

$$(u_{k,j}, \bar{\partial}^* \varphi)_o = (\alpha, \varphi)_1$$

for any smooth $(0,1)$-form α with compact support in Ω_j, we can take limits to see that

$$\bar\partial u_j = \alpha$$

in the weak sense.

Let us now extend the functions u_j from Ω_j to Ω by setting them equal to 0 outside Ω_j. Then we have the estimates

$$\int_\Omega |u_j|^2 e^{-\varphi} dA \le \frac{1}{c}\int_\Omega |f|^2 e^{-\varphi} dA.$$

We can now apply the same technique of Alaoglu's Theorem to the index j, and extract a subsequence converging to a function $u \in \mathscr{H}^o_\varphi$ on Ω such that

$$\int_\Omega |u|^2 e^{-\varphi} dA \le \frac{1}{c}\int_\Omega |f|^2 e^{-\varphi} dA.$$

Now, for each smooth $(0,1)$-form ψ with compact support in Ω_{j_o} and any $j \ge j_o$ we have

$$(u_j, \bar\partial^* \psi)_o = (\alpha, \psi)_1.$$

Thus we can let $j \to \infty$ and obtain the equation

$$\frac{\partial u}{\partial \bar z} = f.$$

The proof of Hörmander's Theorem is complete. □

4.2 L^2 estimates for $\bar\partial$ on Riemann surfaces

There is another proof of Theorem 4.1.12 that, though rather similar at in the most crucial aspects, is rather more streamlined in its technicalities. In this paragraph we present this proof, as well as a slight generalization going beyond domains in $\mathbb{C}$ to the setting of Riemann surfaces.

This generalization forces us to introduce the geometric setting of Hörmander's Theorem. Indeed, compact Riemann surfaces have no nonconstant subharmonic functions, but all Riemann surfaces have holomorphic line bundles with metrics of positive curvature. Thus we formulate and prove Hörmander's Theorem for Hilbert spaces of sections of holomorphic line bundles, rather than functions.

4.2.1 THEOREM (Hörmander's Theorem for Riemann surfaces). *Let X be a Riemann surface with Hermitian metric g whose metric form is ω. Let $L \to X$ be a holomorphic line bundle with smooth metric $e^{-\varphi}$ such that*

$$\sqrt{-1}(\partial\bar\partial\varphi + \mathrm{R}(g)) \ge c\omega,$$

where $R(g)$ is the curvature of the Levi-Civita connection for g. Then for any L-valued measurable $(0,1)$-form α on X such that

$$\frac{1}{2\sqrt{-1}}\int_X \alpha \wedge \bar\alpha e^{-\varphi} < +\infty$$

there exists a measurable section u of $L \to X$ such that $\bar\partial u = \alpha$ in the sense of currents, and

$$\int_X |u|^2 e^{-\varphi}\omega \le \frac{1}{c}\frac{1}{2\sqrt{-1}}\int_X \alpha \wedge \bar\alpha e^{-\varphi}. \tag{4.7}$$

Proof. We use the square norms appearing on the left and right hand sides of (4.7) respectively to define Hilbert spaces $\mathscr{H}^o$ and $\mathscr{H}^1$ for sections of $L \to X$ and L-valued $(0,1)$-forms on X respectively.

Let β be a smooth L-valued $(0,1)$-form on X and let h be a smooth section of $L \to X$ with compact support. We define the operator $\vartheta : \Gamma_o(X, T^{*0,1}_X) \to \mathscr{C}^\infty_o(X)$, by

$$(\vartheta\beta, h)_o = (\beta, \bar\partial h)_1,$$

where $\Gamma_o(X, T^{*0,1}_X)$ is the vector space of compactly supported smooth $(0,1)$-forms on X. Then

$$\begin{aligned}(\beta, \bar\partial h)_1 &= \frac{1}{2\sqrt{-1}}\int_X \beta \wedge \overline{\bar\partial h} e^{-\varphi}\\ &= \frac{\sqrt{-1}}{2}\int_X d(e^{-\varphi}\beta\bar h) - \frac{1}{2\sqrt{-1}}\int_X \partial(e^{-\varphi}\beta)\bar h = \frac{1}{2\sqrt{-1}}\int_X \left(-e^{\varphi}\partial(e^{-\varphi}\beta)\right)\bar h e^{-\varphi}.\end{aligned}$$

Writing $\beta = f d\bar z$ and $\omega = e^{-\psi}\frac{\sqrt{-1}}{2}dz \wedge d\bar z$ in some local coordinate z on which the line bundle $L \to X$ is trivial, we have

$$\frac{1}{2\sqrt{-1}}e^{\varphi}\partial(e^{-\varphi}\beta) = \frac{1}{2\sqrt{-1}}e^{\varphi}\frac{\partial}{\partial z}(e^{-\varphi}f)dz \wedge d\bar z = -e^{\varphi+\psi}\frac{\partial}{\partial z}(e^{-\varphi}f)\omega.$$

Thus

$$\vartheta\beta = -e^{\varphi+\psi}\frac{\partial}{\partial z}(e^{-\varphi}f) = e^{\psi}\left(-\frac{\partial f}{\partial z} + \frac{\partial\varphi}{\partial z}f\right).$$

The reader can check that the right hand side is independent of the coordinate system chosen.

4.2.2 REMARK. The aforementioned check can be done by seeing how the right hand side transforms with a change of coordinates, but there is a geometric way to see it as well. Given the section β of $L \otimes T^{*0,1}_X \to X$, the metric $e^{-\varphi}$, which identifies the fibers of L with the conjugates of their duals, allows us to define a section $e^{-\varphi}\beta$ of the line bundle $\overline{L}^* \otimes T^{*0,1}_X \to X$. Since the latter line bundle is anti-holomorphic, it has a well-defined ∂ operator, so we have a section $\partial(e^{-\varphi}\beta)$ of $\overline{L}^* \otimes T^{*1,0}_X \otimes T^{*0,1}_X \to X$. But the line bundle $T^{*1,0}_X \otimes T^{*0,1}_X \to X$ has a nowhere zero section, namely the metric g, so it is trivial. We thus obtain a section $e^{\psi}\partial(e^{-\varphi}\beta)$ of $\overline{L}^* \to X$. Finally, using the metric $e^{-\varphi}$ again, we have a section $e^{\varphi+\psi}\partial(e^{-\varphi}\beta)$ of $L \to X$, and this section is denoted $\vartheta\beta$. $\diamond$

In the same vein as the Remark 4.2.2, given our section β of $T^{*0,1}_X \otimes L \to X$, there exists an L-valued $(1,0)$-vector field V_β (meaning V_β is a section of $T^{1,0}_X \otimes L \to X$) such that

$$\beta = V_\beta \lrcorner \omega,$$

where, for a vector ξ and a form α, $\xi \lrcorner \alpha$ denotes the contraction of α with ξ. In local coordinates, if $\beta = f d\bar{z}$ for a local section f of L then

$$V_\beta = f e^{\psi} \frac{\partial}{\partial z}.$$

Since V_β is a section of a holomorphic line bundle, it makes sense to compute $\bar{\partial} V_\beta$ as a section of $T^{1,0}_X \otimes T^{*0,1}_X \otimes L$. We think of $\bar{\partial} V_\beta$ as a $(1,0)$-vector field with values in $T^{*0,1}_X \otimes L$, and as such there is a section of $(T^{*0,1}_X)^{\otimes 2} \otimes L$, which we denote $\bar{\nabla}\beta$ and which we think of as a $(0,1)$-form with values in $T^{*0,1}_X \otimes L$, such that

$$\bar{\partial} V_\beta \lrcorner \omega = \bar{\nabla}\beta.$$

In local coordinates, we have

$$\bar{\nabla}(f d\bar{z}) = e^{-\psi} \frac{\partial}{\partial \bar{z}} \left(e^{\psi} f\right) d\bar{z}^{\otimes 2} = \left(\frac{\partial f}{\partial \bar{z}} + \frac{\partial \psi}{\partial \bar{z}} f\right) d\bar{z}^{\otimes 2}.$$

We can define an inner product structure on the space of compactly supported smooth sections of $L \otimes (T^{*0,1}_X)^{\otimes 2} \to X$ defined as follows. If $\xi_i = h_i d\bar{z}^{\otimes 2}$ then

$$(\xi_1, \xi_2)_2 := \int_X e^{\psi} h_1 \bar{h}_2 e^{-\varphi} = \int_X \langle \xi_1, \xi_2 \rangle_g e^{-\varphi} \omega,$$

where $\langle \cdot, \cdot \rangle_g$ is the metric for $(T^{*0,1}_X)^{\otimes 2} \to X$ induced by the metric g for $T^{1,0} \to X$. (Locally, if $\xi_i = h_i d\bar{z}^{\otimes 2}$ then $\langle \xi_1, \xi_2 \rangle_g = h_1 \bar{h}_2 e^{2\psi}$.) We can then calculate the adjoint operator $\bar{\nabla}^*$ defined by

$$(\bar{\nabla}^* \xi, \beta)_1 = (\xi, \bar{\nabla}\beta)_2.$$

The reader can check that

$$\bar{\nabla}^*(f d\bar{z}^{\otimes 2}) = -e^{\psi+\varphi} \frac{\partial}{\partial z} (e^{-\varphi} f) d\bar{z}.$$

Again the reader can check directly that the right hand side is an L-valued $(0,1)$-form, but we encourage a derivation along the lines of Remark 4.2.2.

We compute that

$$\begin{aligned}
\bar{\partial}\vartheta\beta &= \frac{\partial}{\partial \bar{z}} \left(e^{\psi} \left(-\frac{\partial f}{\partial z} + \frac{\partial \varphi}{\partial z} f \right) \right) d\bar{z} \\
&= e^{\psi} \left(-\frac{\partial^2 f}{\partial z \partial \bar{z}} - \frac{\partial f}{\partial z} \frac{\partial \psi}{\partial \bar{z}} + \frac{\partial \varphi}{\partial z} \frac{\partial f}{\partial \bar{z}} + \left(\frac{\partial \varphi}{\partial z} \frac{\partial \psi}{\partial \bar{z}} + \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \right) f \right) d\bar{z},
\end{aligned}$$

while

$$\begin{aligned}
\bar{\nabla}^* \bar{\nabla}\beta &= -e^{\psi+\varphi} \frac{\partial}{\partial z} \left(e^{-\varphi} \left(\frac{\partial f}{\partial \bar{z}} + \frac{\partial \psi}{\partial \bar{z}} f \right) \right) d\bar{z} \\
&= e^{\psi} \left(-\frac{\partial^2 f}{\partial z \partial \bar{z}} - \frac{\partial \psi}{\partial \bar{z}} \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \frac{\partial \varphi}{\partial z} + \left(\frac{\partial \varphi}{\partial z} \frac{\partial \psi}{\partial \bar{z}} - \frac{\partial^2 \psi}{\partial z \partial \bar{z}} \right) f \right) d\bar{z}.
\end{aligned}$$

It follows that

$$\bar{\partial}\vartheta\beta - \bar{\nabla}^*\bar{\nabla}\beta = e^{\psi}\left(\frac{\partial^2\varphi}{\partial z\partial\bar{z}} + \frac{\partial^2\psi}{\partial z\partial\bar{z}}\right)\beta,$$

and thus

$$||\vartheta\beta||_o^2 = ||\bar{\nabla}\beta||_2^2 + \int_X \left\langle \Theta_g(\partial\bar{\partial}\varphi + \mathrm{R}(g))\beta, \beta\right\rangle_g e^{-\varphi}\omega.$$

The hypotheses of Theorem 4.2.1 therefore imply that

$$||\bar{\partial}^*\beta||_o^2 \geq c||\beta||_1^2$$

for all $\beta \in \Gamma_o(X, L \otimes T_X^{0,1})$.

Now let α be the L-valued $(0,1)$-form in the hypotheses of Theorem 4.2.1. Define the linear functional $\lambda_\alpha : \bar{\partial}^*(\Gamma_o(X, L \otimes T_X^{0,1})) \to \mathbb{C}$ by

$$\lambda_\alpha(\bar{\partial}^*\beta) := (\alpha, \beta)_1.$$

Then

$$||\lambda_\alpha||^2 := \sup_{||\bar{\partial}^*\beta||_o = 1} |(\alpha, \beta)_1|^2 \leq \frac{1}{c}||\alpha||_1^2 < +\infty,$$

so λ_α is bounded on the (non-closed) subspace $\bar{\partial}^*(\Gamma_o(X, L \otimes T_X^{0,1}))$ of $\mathscr{H}^o$. By the Hahn-Banach Theorem (or simply by extending by 0 in the orthogonal complement) we can assume that the linear functional $\lambda_\alpha \in \mathscr{H}^{o*}$ is bounded with norm at most $c^{-1/2}||\alpha||$. By the Riesz Representation Theorem there is a section $u \in \mathscr{H}^o$ such that $\lambda_\alpha(v) = (u, v)_o$ and

$$\int_X |u|^2 e^{-\varphi}\omega \leq \frac{1}{c}\frac{1}{2\sqrt{-1}}\int_X \alpha \wedge \bar{\alpha}e^{-\varphi}.$$

Restricting to $\bar{\partial}^*(\Gamma_o(X, L \otimes T_X^{0,1}))$ shows that

$$(u, \bar{\partial}^*\beta)_o = (\alpha, \beta)_1,$$

i.e., that $\bar{\partial}u = \alpha$ in $\mathscr{H}^1$. The proof of Theorem 4.2.1 is complete. □

4.2.1 Regularity of solutions

In Theorems 4.1.12 and 4.2.1 the following question is left unanswered. Suppose that α is a smooth form on Ω and that u is the solution of the equation $\bar{\partial}u = \alpha$. Is u smooth? The answer to this question is *yes*. Indeed, any two solutions of the $\bar{\partial}$ equation differ by a holomorphic function, and so either all of the solutions are smooth or none are smooth. But for domains in $\mathbb{C}$ there is a standard solution, using Green's Theorem, to the inhomogeneous Cauchy-Riemann equations. This solution is given by an integral formula, and one can read off it that the solution is smooth when the data is smooth. Hence the positive answer. These ideas are more fully developed in Exercise 4.2.4.

EXERCISES

4.2.1. Prove that the Hilbert space adjoint $\bar{\partial}^*$ of the $\bar{\partial}$ operator, defined on page 57, is closed and densely defined, and that $(\bar{\partial}^*)^* = \bar{\partial}$ as densely defined operators.

4.2.2. Establish Formula (4.3) for the formal adjoint of the operator $\bar{\nabla}$.

4.2.3. Let φ be a strictly plurisubharmonic function (i.e., such that $\Delta\varphi \geq c > 0$ in the sense of currents) on a smoothly bounded domain $\Omega \subset\subset \mathbb{C}$. Show that if α is a smooth $(0,1)$-form on Ω that vanishes on $\partial\Omega$ and satisfies $\bar{\partial}^*\alpha = 0$ then $\alpha = 0$. Show that there is a smooth $(0,1)$-form $\alpha \not\equiv 0$ on Ω such that $\bar{\partial}^*\alpha \equiv 0$.

4.2.4. Let $\Omega \subset\subset \mathbb{C}$ be a domain with smooth boundary.

(a) Use Green's Theorem to show that for any $z \in \Omega$ and any function $f \in \mathscr{C}^\infty(\overline{\Omega})$ one has the formula

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Omega} \frac{f(\zeta)d\zeta}{z-\zeta} - \frac{1}{\pi} \int_\Omega \frac{\partial f}{\partial\bar{\zeta}} \frac{\partial A(\zeta)}{z-\zeta}.$$

(b) Show that if $g \in \mathscr{C}^k_o(\Omega)$ then the function

$$u(z) := -\frac{1}{\pi} \int_\Omega \frac{g(\zeta)\partial A(\zeta)}{z-\zeta}$$

satisfies the equation $\frac{\partial u}{\partial\bar{z}} = g$ in Ω.

(c) Let X be a Riemann surface and let α be a smooth $(0,1)$-form on X. Show that any function u such that $\bar{\partial}u = \alpha$ is smooth.

Lecture 5

The Bochner-Kodaira Identity

We fix a Kähler manifold (X, g) and a holomorphic vector bundle $E \to X$ with Hermitian metric $\mathfrak{h}$. We let ω denote the metric form of g. We denote by Θ the curvature of the Chern connection associated to $(E, \mathfrak{h})$ and by R the curvature operator of the Kähler connection associated to (X, g). These curvature operators induce operators on E-valued (p, q)-forms, whose definitions will be apparent from the derivation. We shall denote the induced operators by the same letters.

5.1 The Hilbert spaces

Because of Remark 1.4.7 we shall work only with E-valued (n, q)-forms.

Let φ be an (n, q)-form with values in E. If we choose local coordinates z and a local frame $e_1, ..., e_r$ for E, then we may write

$$\varphi = \varphi^{\alpha}_{\bar{J}} dz^1 \wedge ... \wedge dz^n \wedge d\bar{z}^J \otimes e_{\alpha}.$$

Let

$$g_{i\bar{j}} := g(\partial_i, \partial_{\bar{j}}) \quad \text{and} \quad \mathfrak{h}_{\alpha\bar{\beta}} = \mathfrak{h}(e_{\alpha}, e_{\beta}).$$

As usual, $g^{i\bar{j}}$ and $\mathfrak{h}^{\alpha\bar{\beta}}$ denote the inverse matrices; they are the matrices of the metrics for the dual bundles, with respect to the dual frame.

The metrics g and $\mathfrak{h}$ induce a metric on $E \otimes \Lambda^{n,q}_X$. In terms of the above metrics the new metric is given locally by

$$\langle \varphi, \psi \rangle := \frac{1}{q!} \varphi^{\alpha}_{\bar{J}} \overline{\psi^{\beta}_{\bar{L}}} \mathfrak{h}_{\alpha\bar{\beta}} g^{L\bar{J}} \det(g^{i\bar{j}}),$$

where $g^{L\bar{J}} = g^{\ell_1\bar{j}_1} \cdots g^{\ell_q\bar{j}_q}$. It is easy to check that the right hand side is independent of all the frames that were chosen. When we want to indicate the dependence on the metrics g and h, we shall write $\langle \varphi, \psi \rangle_{g,\mathfrak{h}}$.

5.1.1 REMARK. It is sometimes convenient to employ the notation $\psi^{\bar{I}}_{\bar{\alpha}} := \psi^{\beta}_{\bar{L}} h_{\beta\bar{\alpha}} g^{J\bar{L}}$, with respect to which one has $\langle \varphi, \psi \rangle = \frac{1}{q!} \varphi^{\alpha}_{\bar{J}} \overline{\psi^{J}_{\bar{\alpha}}}$. $\diamond$

5.1.2 DEFINITION. *Let $(E, \mathfrak{h}) \to X$ be a Hermitian complex vector bundle and let dV be a volume element, i.e., a smooth (n,n)-form with no zeros, on a Hermitian manifold (X,g).*

(i) For two compactly supported E-valued (p,q)-forms φ and ψ, we define the inner product

$$(\varphi, \psi) = \int_X \langle \varphi, \psi \rangle \, dV_g.$$

With this inner product, the space of compactly supported smooth E-valued (p,q)-forms is an inner product space.

(i) We let $L^2_{p,q}(g, \mathfrak{h})$ denote the Hilbert space completion of the inner product space of smooth compactly supported E-valued (p,q)-forms, with the inner product $(\cdot, \cdot)$.

5.1.3 REMARK. Note that $dV_g = \det(g_{i\bar{j}})dV(z)$, and therefore the (n,n)-form

$$\{\varphi, \psi\} := \frac{1}{q!} \varphi^{\alpha}_{\bar{J}} \overline{\psi^{\beta}_{\bar{L}}} \mathfrak{h}_{\alpha\bar{\beta}} g^{L\bar{J}} \tag{5.1}$$

In the case $q = 0$ the (n,n)-form $\{\varphi, \psi\}$ depends only on the metric $\mathfrak{h}$, and not on the metric g.

5.2 Two $\bar{\partial}$-operators and their formal adjoints

A smooth E-valued (n,q)-form can be viewed in two ways. In the first, usual way, such a form is a (n,q)-form with values in a holomorphic vector bundle. From the second perspective, a (n,q)-form is a section of the vector bundle $\Lambda^{n,q}_X \otimes E \to X$; a complex vector bundle that is not holomorphic. In this paragraph we will define natural $\bar{\partial}$ operators for these two types of sections.

Let us begin with the first perspective. We fix a basis $\{e_\alpha\}$ of local holomorphic sections of $E \to X$. To indicate the type of the forms we are using, it is helpful to include subscripts for our multi-indices. Let us denote by ∇ the Chern connection associated to E-valued (p,q)-forms on our Kähler manifold (X,g).

5.2.1 PROPOSITION. *Let $\varphi = \varphi^{\alpha}_{\bar{J}_q} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^{J_q} \otimes e_\alpha$ be an E-valued (n,q)-form. Then*

$$\bar{\partial}\varphi = (-1)^n \sum_{k=0}^{q} (-1)^k (\nabla_{\bar{j}_k} \varphi)^{\alpha}_{\bar{j}_0 \ldots \hat{\bar{j}}_k \ldots \bar{j}_q} dz^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^J \otimes e_\alpha.$$

Proof. By definition of $\bar{\partial}$ we have

$$\bar{\partial}\varphi = (\bar{\partial}\varphi^{\alpha}_{\bar{J}_q}) \wedge dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^{J_q} \otimes e_\alpha.$$

Then

$$\begin{aligned}
\bar{\partial}\varphi &= (-1)^n \partial_{\bar{j}} \varphi^{\alpha}_{\bar{j}_0 \ldots \bar{j}_q} \wedge dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^j \wedge d\bar{z}^{J_q} \otimes e_\alpha \\
&= (-1)^n \sum_{k=0}^{q} (-1)^k \partial_{\bar{j}_k} \varphi^{\alpha}_{\bar{j}_0 \ldots \hat{\bar{j}}_k \ldots \bar{j}_q} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^{j_0} \wedge \ldots \wedge d\bar{z}^{j_q} \\
&= (-1)^n \sum_{k=0}^{q} (-1)^k (\nabla_{\bar{j}_k} \varphi)^{\alpha}_{\bar{j}_0 \ldots \hat{\bar{j}}_k \ldots \bar{j}_q} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^{j_0} \wedge \ldots \wedge d\bar{z}^{j_q},
\end{aligned}$$

where the last equality holds since $\nabla^{0,1} = \bar{\partial}$ for the Chern connection. $\square$

We denote by $\mathfrak{d}$ the formal adjoint of $\bar{\partial}$. Thus $\mathfrak{d}$ is defined by the relation

$$(\mathfrak{d}\varphi, \psi) = (\varphi, \bar{\partial}\psi)$$

for all smooth twisted forms φ, and all smooth twisted forms ψ with compact support. Note that $\mathfrak{d}$ sends smooth E-valued (n, q)-forms to smooth E-valued $(n, q-1)$-forms.

5.2.2 PROPOSITION. *Let $\varphi = \varphi^{\alpha}_{\bar{J}_q} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^{J_q} \otimes e_\alpha$ be an E-valued (n, q)-form. Then*

$$\mathfrak{d}\varphi = (-1)^{n+1} g^{i\bar{j}} (\nabla_i \varphi)^{\alpha}_{\bar{j}\bar{j}_1 \ldots \bar{j}_{q-1}} dz^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_{q-1}} \otimes e_\alpha.$$

Proof. One has

$$\begin{aligned}
&(\mathfrak{d}\varphi, \psi) = (\varphi, \bar{\partial}\psi) \\
&= \frac{1}{n!q!} \int_X \varphi^{\alpha}_{\bar{j}_1 \ldots \bar{j}_q} (-1)^n \sum_{k=1}^{q} (-1)^{k+1} \overline{(\nabla_{\overline{j'_k}} \psi)^{\beta}_{\overline{j'_1} \ldots \widehat{\overline{j'_k}} \ldots \overline{j'_q}}} g^{J'_q \overline{J_q}} h_{\alpha\bar{\beta}} \\
&= \frac{1}{n!q!} \int_X (-1)^{n+1} \sum_{k=1}^{q} (-1)^{k+1} g^{j'_k \bar{j}_k} \left(\nabla_{j'_k} \varphi\right)^{\alpha}_{\bar{j}_1 \ldots \bar{j}_q} \overline{\psi^{\beta}_{\overline{j'_1} \ldots \widehat{\overline{j'_k}} \ldots \overline{j'_q}}} g^{j'_1 \bar{j}_1} \ldots \widehat{g^{j'_k \bar{j}_k}} \ldots g^{j'_q \bar{j}_q} h_{\alpha\bar{\beta}} \\
&= \frac{1}{n!(q-1)!} \int_X \left((-1)^{n+1} g^{i\bar{j}} \left(\nabla_i \varphi\right)^{\alpha}_{\bar{j}\bar{j}_1 \ldots \bar{j}_{q-1}} \right) \overline{\psi^{\beta}_{\overline{j'_1} \ldots \overline{j'_{q-1}}}} g^{J'_{q-1} \overline{J_{q-1}}} h_{\alpha\bar{\beta}},
\end{aligned}$$

Where the second-to-last inequality follows from the metric compatibility of the connection. $\square$

Now let us turn to the second perspective of a V-valued (n, q)-form seen as a section of the complex vector bundle $K_X \otimes \bigwedge^q T^{*0,1}_X \otimes E \to X$. As we have pointed out, the latter vector bundle is not holomorphic. However, because X is equipped with a metric, there is a natural way to map sections of $K_X \otimes \bigwedge^q T^{*0,1}_X \otimes E \to X$ to sections of the holomorphic vector bundle $K_X \otimes \bigwedge^q T^{1,0}_X \otimes V \to X$. In terms of local data, if we write $\varphi = \varphi^{\alpha}_{\bar{J}} dz^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^J \otimes e_\alpha$ then

$$\mathfrak{F}\varphi := g^{I\bar{J}} \varphi^{\alpha}_{\bar{J}} dz^1 \wedge \ldots \wedge dz^n \otimes \frac{\partial}{\partial z^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial z^{i_q}} \otimes e_\alpha =: g^{I\bar{J}} \varphi^{\alpha}_{\bar{J}} dz^1 \wedge \ldots \wedge dz^n \otimes \frac{\partial}{\partial z^I} \otimes e_\alpha$$

is a section of $K_X \otimes \bigwedge^q T^{1,0}_X \otimes E \to X$, and clearly the map $\varphi \mapsto \mathfrak{F}\varphi$ is a 1-1 correspondence that depends only on the pointwise values of φ. The map $\mathfrak{F}$ extends naturally to a 1-1 correspondence $\mathfrak{F}_k$ of $K_X \otimes \bigwedge^q T^{1,0}_X \otimes E$-valued $(0, k)$-forms, i.e., sections of $\bigwedge^k (T^{*0,1}_X) \otimes K_X \otimes \bigwedge^q T^{1,0}_X \otimes E \to X$, by acting trivially on the last factor, i.e., for $J = (j_1, \ldots, j_k)$ set

$$\mathfrak{F}_k \left(d\bar{z}^J \otimes dz^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^I \otimes e_\alpha \right) = d\bar{z}^J \otimes \mathfrak{F} \left(dz^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^I \otimes e_\alpha \right).$$

The vector bundle $K_X \otimes \bigwedge^q T^{1,0}_X \otimes E \to X$, being a holomorphic vector bundle, is equipped with $\bar{\partial}$, and thus we have a well-defined $K_X \otimes \bigwedge^q T^{1,0}_X \otimes E$-valued $(0, 1)$-form $\bar{\partial}\mathfrak{F}\varphi$.

5.2.3 DEFINITION. *The operator* $\bar\nabla : \Gamma(X, (\Lambda_X^{n,q}) \otimes E) \to \Gamma(X, T_X^{*0,1} \otimes (\Lambda_X^{n,q}) \otimes E)$ *is defined by*

$$\bar\nabla\varphi := \mathfrak{F}_1^{-1}\bar\partial\mathfrak{F}\varphi.$$

In terms of local data

$$\bar\nabla\varphi := g_{I\bar L}\frac{\partial}{\partial\bar z^k}\left(g^{I\bar J}\varphi^\alpha_{\bar J}\right)d\bar z^k\otimes dz^1\wedge\ldots\wedge dz^n\wedge d\bar z^L\otimes e^\alpha = (\nabla_{\bar k}\varphi)^\alpha_{\bar I}d\bar z^k\otimes dz^1\wedge\ldots\wedge dz^n\wedge d\bar z^I\otimes e_\alpha,$$

where ∇ denotes the Kähler-Chern connection for E-valued tensors.

We can compute the formal adjoint $\bar\nabla^*$ of $\bar\nabla$ with respect to the inner products induced on sections of $(\Lambda^{n,q}T_X^*) \otimes E$ and $T_X^{*0,1} \otimes (\Lambda^{n,q}T_X^*) \otimes E)$ by the metrics $\mathfrak{h}$ and g. For smooth sections φ and ψ of $K_X \otimes (\Lambda^q T_X^{*0,1}) \otimes T_X^{*0,1} \otimes V$ and $K_X \otimes \Lambda^q T_X^{*0,1} \otimes V$ respectively, with ψ compactly supported, and we write

$$\varphi = \varphi^\alpha_{\bar J,\bar j}d\bar z^j \otimes dz^1 \wedge \ldots \wedge dz^n \wedge d\bar z^J \otimes e_\alpha$$

then

$$(\varphi, \bar\nabla\psi) = \int_X g^{k\bar j}\varphi^\alpha_{\bar J,\bar j}\overline{(\nabla_{\bar k}\psi)^{\bar J}_\alpha}dV(z) = \int_X \left(-g^{k\bar j}(\nabla_k\varphi)^\alpha_{\bar J,\bar j}\right)\overline{\psi^{\bar J}_\alpha}dV(z),$$

where $dV(z) = (\frac{\sqrt{-1}}{2})^n dz^1 \wedge d\bar z^1 \wedge \ldots \wedge dz^n \wedge d\bar z^n$ and we have used the fact that the Chern connection is compatible with the metric. Thus

$$(\bar\nabla^*\varphi)^\alpha_{\bar J} = -g^{k\bar j}(\nabla_k\varphi)^\alpha_{\bar J,\bar j}$$

When viewing φ as a section rather than a form, one sets $\bar\nabla^*\varphi = 0$.

We can therefore associated to $\bar\nabla$ its Laplace-Beltrami operator $\bar\nabla^*\bar\nabla$ on sections of the vector bundle $(\Lambda_X^{n,q}) \otimes E \to X$. Our computations show that

$$\bar\nabla^*\bar\nabla = -g^{k\bar j}\nabla_k\nabla_{\bar j}. \tag{5.2}$$

5.3 The formal identity

The following identity is crucial in establishing L^2 estimates for solutions of the $\bar\partial$ equation.

5.3.1 THEOREM (The formal Bochner-Kodaira Identity). *Let* (X, g) *be a Kähler manifold and let* $E \to X$ *be a holomorphic vector bundle with Hermitian metric* $\mathfrak{h}$. *Denote by* $\Box = \bar\partial\mathfrak{d} + \mathfrak{d}\bar\partial$ *the Laplace-Beltrami operator associated to* $\bar\partial$. *Then one has the formal identity*

$$\Box = \bar\nabla^*\bar\nabla + \Theta_g(\mathfrak{h}), \tag{5.3}$$

where $\Theta_g(\mathfrak{h})$ *is the operator on* E*-valued* (n, q)*-forms induced by the curvature of the metric* $\mathfrak{h}$.

Proof. Using Propositions 5.2.1 and 5.2.2, we calculate that

$$
\begin{aligned}
(\mathfrak{d}\bar{\partial}\varphi)^{\alpha}_{\bar{j}_1\cdots\bar{j}_q} &= (-1)^{n+1} g^{i\bar{j}} \nabla_i \left((\bar{\partial}\varphi)^{\alpha}_{\bar{j}\bar{J}} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^J \otimes e_\alpha \right) \\
&= (-1)^{n+1} g^{i\bar{j}} \nabla_i \Big((-1)^n (\nabla_{\bar{j}}\varphi)^{\alpha}_{\bar{J}} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^J \otimes e_\alpha \\
&\qquad + (-1)^n \sum_{k=1}^{q} (-1)^k (\nabla_{\bar{j}_k}\varphi)^{\alpha}_{\bar{j}\bar{j}_1\cdots\widehat{\bar{j}_k}\cdots\bar{j}_q} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^J \otimes e_\alpha \Big) \\
&= \left((-g^{i\bar{j}} \nabla_i \nabla_{\bar{j}} \varphi)^{\alpha}_{\bar{j}_1\cdots\bar{j}_q} + \sum_{k=1}^{q} (g^{i\bar{j}} \nabla_i \nabla_{\bar{j}_k} \varphi)^{\alpha}_{\bar{j}_1\cdots(\bar{j})_k\cdots\bar{j}_q} \right) dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^J \otimes e_\alpha,
\end{aligned}
$$

and

$$
\begin{aligned}
(\bar{\partial}\mathfrak{d}\varphi)^{\alpha}_{\bar{j}_1\cdots\bar{j}_q} &= (-1)^n \sum_{k=1}^{q} (-1)^{k+1} (\nabla_{\bar{j}_k}(\mathfrak{d}\varphi))^{\alpha}_{\bar{j}_1\cdots\widehat{\bar{j}_k}\cdots\bar{j}_q} \\
&= (-1)^n \sum_{k=1}^{q} (-1)^{k+1} (\nabla_{\bar{j}_k}((-1)^{n+1} g^{i\bar{j}} \nabla_i \varphi))^{\alpha}_{\bar{j}\bar{j}_1\cdots\widehat{\bar{j}_k}\cdots\bar{j}_q} \\
&= -\sum_{k=1}^{q} (\nabla_{\bar{j}_k}(g^{i\bar{j}} \nabla_i \varphi^\alpha))_{\bar{j}_1\cdots(\bar{j})_k\cdots\bar{j}_q} \\
&= -\sum_{k=1}^{q} g^{i\bar{j}} (\nabla_{\bar{j}_k} \nabla_i \varphi)^{\alpha}_{\bar{j}_1\cdots(\bar{j})_k\cdots\bar{j}_q},
\end{aligned}
$$

Thus

$$
(\Box\varphi)^{\alpha}_{\bar{J}_q} = -g^{i\bar{j}} (\nabla_i \nabla_{\bar{j}} \varphi)^{\alpha}_{\bar{J}_q} + \sum_{k=1}^{q} g^{i\bar{\ell}} ([\nabla_i, \nabla_{\bar{j}_k}]\varphi)^{\alpha}_{\bar{j}_1\cdots(\bar{\ell})_k\cdots\bar{j}_q}.
$$

Finally, since $[\nabla_i, \nabla_{\bar{j}}] = \Theta(\mathfrak{h})_{i\bar{j}}$ is the curvature,

$$
\sum_{k=1}^{q} g^{i\bar{\ell}} \left[\nabla_i, \nabla_{\bar{j}_k}\right] \varphi^{\alpha}_{\bar{j}_1\cdots(\bar{\ell})_k\cdots\bar{j}_q} = \sum_{k=1}^{q} g^{i\bar{\ell}} \Theta(\mathfrak{h})^{\alpha}_{\beta i\bar{j}_k} \varphi^{\beta}_{\bar{j}_1\cdots(\bar{\ell})_k\cdots\bar{j}_q}.
$$

The right hand side of the last equation is by definition the operator $\Theta_g(\mathfrak{h})$ induced in E-valued (n,q)-forms by the curvature of $\mathfrak{h}$:

$$
\Theta_g(\mathfrak{h})\varphi := \sum_{k=1}^{q} g^{i\bar{\ell}} \Theta(\mathfrak{h})^{\alpha}_{\beta i\bar{j}_k} \varphi^{\beta}_{\bar{j}_1\cdots(\bar{\ell})_k\cdots\bar{j}_q} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \otimes e_\alpha. \tag{5.4}
$$

The proof is complete. □

Exercises

5.3.1. Let X be a compact complex manifold and let $L \to X$ be a holomorphic line bundle with a smooth Hermitian metric h whose Chern connection has positive curvature. Show that if a smooth, L-valued $(n, 1)$-form α satisfies $\Box\alpha = 0$ then $\alpha = 0$.

Lecture 6

L^2 Estimates on Compact Kähler manifolds

In this section we present our first, and simplest, result on L^2 estimates for solutions of $\bar{\partial}$. Before beginning, we should say a word on the meaning of the solution of $\bar{\partial}$ in spaces of not necessarily smooth sections, such as $L^2_{p,q}(g, \mathfrak{h})$. In fact, the $\bar{\partial}$ operator can act on currents, and in particular on any locally integrable sections. In the space the distribution equation $\bar{\partial}u = \varphi$, $\varphi \in L^2_{p,q}(g, \mathfrak{h})$, formulates as follows. A section $u \in L^2_{p,q-1}(g, \mathfrak{h})$ is a *weak solution* of the equation $\bar{\partial}u = \varphi$ if

$$(u, \mathfrak{d}\psi) = (\varphi, \psi)$$

for all smooth, compactly supported $\psi \in L^2_{p,q}(g, \mathfrak{h})$. In coming paragraphs we shall go further, discussing the structure of $\bar{\partial}$ as a closed, densely defined operator on $L^2_{p,q}(g, \mathfrak{h})$.

6.1 Hörmander's Theorem on compact Kähler manifolds

6.1.1 THEOREM. *Let (X, g) be a compact Kähler manifold and let $F \to X$ be a holomorphic vector bundle with Hermitian metric h. Fix p, q with $0 \le p \le n$ and $1 \le q \le n$. Assume that the curvature of the Chern connection of the metric vector $g^{(p)} \otimes \det(g) \otimes h$ for the vector bundle $\bigwedge^p T^{1,0}_X \otimes K^*_X \otimes F$ satisfies*

$$\Theta_g(g^{(p)} \otimes \det g \otimes h) \ge c \cdot \mathrm{Id} \tag{6.1}$$

for some positive constant c. Then for every F-valued $\bar{\partial}$-closed (p, q)-form φ satisfying

$$\int_X |\varphi|^2_{g,h} dV_g < +\infty$$

there exists a F-valued $(p, q-1)$-form u satisfying

$$\bar{\partial}u = \varphi \quad \textit{and} \quad \int_X |\varphi|^2_{g,h} dV_g \le \frac{1}{c} \int_X |\varphi|^2_{g,h} dV_g.$$

6.1.2 REMARK. Note that the condition (6.1) holds if the Hermitian metric $g^{(p)} \otimes \det g \otimes h$ has Nakano-positive curvature. However, as we shall see, if $q > 1$ then (6.1) can hold under weaker assumptions. $\diamond$

Proof of Theorem 6.1.1. Let $E := \bigwedge^p T^{1,0}_X \otimes K^*_X \otimes F$. Then an F-valued (p,q)-form identifies with an E-valued (n,q)-form. Moreover, the metric $\mathfrak{h} := g^{(p)} \otimes \det g \otimes h$ identifies the corresponding L^2 structures, and the curvature assumption is equivalent to the existence of a positive constant c such that

$$\Theta(\mathfrak{h}) \geq c \cdot \mathrm{Id}.$$

Let φ be an E-valued (n,q)-form with finite L^2 norm. For any smooth, E-valued (n,q)-form ψ we have the estimate

$$||\psi||^2 \leq \frac{1}{c}||\bar\partial\psi||^2 + ||\bar\partial^*\psi||^2. \tag{6.2}$$

Now consider the bilinear form $(\cdot,\cdot)_{\mathscr{H}}$ defined on smooth sections of $\Lambda^{n,q}_X \otimes E \to X$ by

$$(\psi_1,\psi_2)_{\mathscr{H}} := (\bar\partial\psi_1, \bar\partial\psi_2) + (\bar\partial^*\psi_1, \bar\partial^*\psi_2).$$

The inequality (6.2) implies that $||\cdot||_{\mathscr{H}}$ is a norm. Define $\mathscr{H}$ to be the Hilbert space completion of $\Gamma(X, \Lambda^{p,q}T^*_X \otimes E)$ with respect to the norm $||\cdot||_{\mathscr{H}}$.

Let $\lambda_\varphi : L^2_{p,q}(g,h) \to \mathbb{C}$ be the linear functional defined by

$$\lambda_\varphi(\psi) := (\psi, \varphi).$$

By Cauchy-Schwarz

$$|\lambda_\varphi(\psi)|^2 \leq \frac{||\varphi||^2}{c}||\psi||^2_{\mathscr{H}},$$

and thus $\lambda_\varphi \in \mathscr{H}^*$. By the Riesz Representation Theorem there exists $v \in \mathscr{H}$ such that

$$||v||^2_{\mathscr{H}} \leq \frac{||\varphi||^2}{c} \quad \text{and} \quad (v,\psi)_{\mathscr{H}} = \lambda_\varphi(\psi), \quad \psi \in \mathscr{H}.$$

The latter means that

$$(\bar\partial v, \bar\partial\psi) + (\bar\partial^* v, \bar\partial^*\psi) = (\varphi, \psi), \quad \psi \in \mathscr{H}.$$

In particular, the latter holds for smooth ψ.

Now, since $\bar\partial\varphi = 0$ we have

$$0 = (\varphi, \bar\partial^*\psi) = (\bar\partial v, \bar\partial\bar\partial^*\psi) + (\bar\partial^* v, \bar\partial^*\bar\partial^*\psi) = (\bar\partial v, \bar\partial\bar\partial^*\psi),$$

which implies that $\bar\partial\bar\partial^*\bar\partial v = 0$ in the sense of currents. Therefore $||\bar\partial^*\bar\partial v||^2 = (\bar\partial\bar\partial^*\bar\partial v, \bar\partial v) = 0$, and thus $(\bar\partial v, \bar\partial\psi) = 0$. Hence

$$(\bar\partial^* v, \bar\partial^*\psi) = (\varphi, \psi),$$

which means that $u := \bar\partial^* v$ is a solution of $\bar\partial u = \varphi$. Moreover

$$||u||^2 = ||\bar\partial^* v||^2 = ||v||^2_{\mathscr{H}} \leq \frac{||\varphi||^2}{c}.$$

The proof is therefore complete. □

6.2 A few useful applications

Regularity and cohomology vanishing

Theorem 6.1.1 provides L^2 solutions to the $\bar\partial$ equation with L^2 data, but leaves open an important question: If the data φ is a smooth E-valued (p,q)-form, is the solution provided by Theorem 6.1.1 also smooth.

The answer to this question is affirmative. However, except in the case $q = 1$, the reason is a bit subtle. For general q, the solution we provided is of the form $u = \mathfrak{d}v$, and any section of this form is orthogonal to the $\bar\partial$-closed twisted forms. Since any two solutions differ by a $\bar\partial$-closed twisted form, our solution is the one of minimal norm. Consequently v is a (weak) solution of the equation $\Box v = \varphi$. The operator $\Box$ is an elliptic system, and general machinery, which we have not discussed, implies the desired smoothness. However, when $q > 1$ there are *always* other solutions of the equation $\bar\partial u = \varphi$ that are not smooth.

The case $q = 1$ is slightly different. In this case, *all* solutions are either smooth or not. Indeed, any two solutions differ by a holomorphic section. So to show regularity, one has only to show that there exists *some* smooth solution. The smoothness of the solution is a local problem: using a partition of unity, one can write $\varphi = \sum \chi_j\varphi_j$, and if we can find smooth solutions with compact support u_j of the equations $\bar\partial u_j = \varphi_j$ then the section $u = \sum u_j$ provides the needed smooth solution. To then find the smooth solution of the localized equation, one can work in $\mathbb{C}^n$. In this case there are a number of available techniques, and at least one of them, which is based on Green's Theorem, is rather elementary; it is often taught in first courses in complex analysis. The approach is to produce an integral formula for the solution, and the smoothness of the solution can be read off from the formula. Therefore, for $q = 1$, any section u of $F \to X$ that satisfies $\bar\partial u = \varphi$ is smooth when this is the case for φ.

As a corollary of this discussion, one obtains the following important proposition.

6.2.1 PROPOSITION. *Under the hypotheses of Theorem 6.1.1, the Dolbeault cohomology groups*

$$H^{p,q}_{\bar\partial}(X,F) := \frac{\mathrm{Kernel}\left(\bar\partial : \Gamma(X,\Lambda^{p,q}_X \otimes F) \to \Gamma(X,\Lambda^{p,q+1}_X \otimes F)\right)}{\bar\partial\left(\Gamma(X,\Lambda^{p,q-1}_X \otimes F)\right)}$$

are trivial, i.e., $H^{p,q}_{\bar\partial}(X,F) = \{0\}$.

Kodaira Vanishing

Let $L \to X$ be a holomorphic line bundle on a complex manifold X of complex dimension n, and let $e^{-\varphi}$ be a smooth Hermitian metric for L. The positivity of the curvature $\partial\bar\partial\varphi$ of $e^{-\varphi}$ means precisely that the $(1,1)$-form

$$\omega := \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi$$

is a Kähler form. (Thus, *a posteriori*, every complex manifold admitting a line bundle with a metric of positive curvature is necessarily Kähler.) We therefore can, and do, use the Kähler form ω to define our Kähler metric g on X.

Suppose β is a smooth, L-valued (n,q)-form. In terms of a local frame ξ for L, if we write $\beta = \beta_{\bar{J}} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^J$ then we compute that

$$\begin{aligned}
&\Theta_g(e^{-\varphi})\beta \\
&= 2\pi \sum_{k=1}^{q} g^{i\bar{\ell}} g_{i\bar{j}_k} \beta_{\bar{j}_1 \ldots (\bar{\ell})_k \ldots \bar{j}_q} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \\
&= 2\pi q \beta
\end{aligned}$$

By Proposition 6.2.1 one has the following theorem.

6.2.2 THEOREM (Kodaira Vanishing Theorem). *Let X be a compact Kähler manifold and let $L \to X$ be a holomorphic line bundle admitting a smooth metric of positive curvature. Then*

$$H^{n,q}_{\bar{\partial}}(X, L) = \{0\}.$$

Kodaira-Nakano Vanishing

Another direct corollary of Proposition 6.2.1 is the following generalization of Kodaira Vanishing. The proof is left to the reader.

6.2.3 THEOREM (Kodaira-Nakano Vanishing Theorem). *Let X be a compact Kähler manifold and let $E \to X$ be a holomorphic vector bundle admitting a smooth metric whose curvature is positive in the sense of Nakano. Then*

$$H^{n,q}_{\bar{\partial}}(X, E) = \{0\}.$$

Vanishing theorems for $q > 1$

Let (X, ω) be a Kähler manifold. Suppose $L \to X$ is a line bundle with smooth Hermitian metric $e^{-\varphi}$ whose curvature $\theta = \sqrt{-1}\partial\bar{\partial}\varphi$ is not necessarily positive. Let

$$\lambda_1(p) \le \lambda_2(p) \le \ldots \le \lambda_n(p)$$

denote the eigenvalues of θ_p relative to ω_p, thought of as Hermitian forms on $T^{1,0}_{X,p}$. (These eigenvalues depend on the point p, but nature of this dependence will not be important to us.) Fix a correspoinding ω-orthonormal basis $\xi_1(p), \ldots, \xi_n(p)$ of eigenvectors for $T^{1,0}_{X,p}$, and let $\alpha^i \in T^{1,0*}_{X,p}$ be defined by

$$\alpha^i(v) = \omega(\xi_i, v), \quad v \in T^{1,0}_{X,p}.$$

Given an L-valued (n,q)-form β, we can write $\beta = \beta_{\bar{J}} \alpha^1 \wedge \ldots \wedge \alpha^n \wedge \bar{\alpha}^J \otimes e$. We compute that

$$\begin{aligned}
\Theta_g(e^{-\varphi})\beta &= \sum_{k=1}^{q} g^{i\bar{\ell}} \theta_{i\bar{j}_k} \beta_{\bar{j}_1 \ldots (\bar{\ell})_k \ldots \bar{j}_q} \alpha^1 \wedge \ldots \wedge \alpha^n \wedge \bar{\alpha}^J \otimes e \\
&= \sum_{k=1}^{q} \delta^{i\bar{\ell}} \lambda_i \delta_{i\bar{j}_k} \beta_{\bar{j}_1 \ldots (\bar{\ell})_k \ldots \bar{j}_q} \alpha^1 \wedge \ldots \wedge \alpha^n \wedge \bar{\alpha}^J \otimes e \\
&= (\lambda_{j_1} + \ldots + \lambda_{j_g}) \beta_{\bar{J}} \alpha^1 \wedge \ldots \wedge \alpha^n \wedge \bar{\alpha}^J \otimes e.
\end{aligned}$$

Since $\omega = \sqrt{-1}\sum_i \alpha^i \wedge \bar{\alpha}^i$ and $\theta = \sqrt{-1}\sum_i \lambda_i \alpha^i \wedge \bar{\alpha}^i$, we have

$$\theta \wedge \omega^{q-1} = (\sqrt{-1})^q (q-1)! \sum_{|J|=q} (\lambda_{j_1} + \cdots + \lambda_{j_q}) \alpha^{j_1} \wedge \bar{\alpha}^{j_1} \wedge \cdots \wedge \alpha^{j_q} \wedge \bar{\alpha}^{j_q}$$

and

$$\omega^q = (\sqrt{-1})^q q! \sum_{|J|=q} \alpha^{j_1} \wedge \bar{\alpha}^{j_1} \wedge \cdots \wedge \alpha^{j_q} \wedge \bar{\alpha}^{j_q}.$$

By Theorem 6.1.1 we have the following theorem.

6.2.4 THEOREM. *Let (X, ω) be a compact Kähler manifold and let $L \to X$ be a holomorphic line bundle admitting a smooth metric with curvature θ. Fix $q \in \{1, ..., n\}$ and suppose*

$$\theta \wedge \omega^{q-1} \geq c\omega^q$$

for some positive constant c. Then

$$H^{n,q}_{\bar{\partial}}(X, L) = \{0\}.$$

There are analogous vanishing theorems for vector bundles, but they are a little more elaborate, involving the notion of m-positivity discussed in Chapter 1. We shall not formulate them here.

Lecture 7

L^2 Estimates on Complete Kähler manifolds

7.1 Extending the $\bar{\partial}$ operator to Hilbert spaces

7.1.1 Closed, densely defined operators

Let H_1 and H_2 be Hilbert spaces. We are interested in linear operators from subspaces of H_1 into H_2. Two important things to keep in mind about linear operators are

- each operator $T : H_1 \to H_2$ comes with its own domain of definition $\mathrm{Domain}(T)$, and
- $T : \mathrm{Domain}(T) \to H_2$ need not be continuous (with respect to the relative topology of $\mathrm{Domain}(T) \subset H_1$).

It may sometimes be possible to extend an operator to a larger domain. If this is so, and S is such an extension, then the Graph

$$\text{Graph}(T) := \{(x, Tx)\ ;\ x \in \text{Domain}(T)\} \subset H_1 \times H_2$$

of T is a subspace of the graph Graph(S) of S.

7.1.1 DEFINITION. *An operator $T : H_1 \to H_2$ is said to be*

1. *densely defined if* $\mathrm{Domain}(T)$ *is a dense subspace of* H_1*, and*
2. *closed if* $\mathrm{Graph}(T)$ *is a closed subspace of* $H_1 \times H_2$*.*

7.1.2 REMARK. If an operator is bounded on a Hilbert space (or more generally on a Banach space), then the Closed Graph Theorem tells us that it is closed. $\diamond$

As the reader will recall, a Hilbert space is isomorphic to its dual. The explicit theorem yielding the isomorphism is a rather elementary version of the Riesz Representation Theorem. The result states that the map sending $v \in H$ to the bounded linear functional $\lambda_v \in H^*$ define by

$$\lambda_v w := \langle w, v \rangle\,,$$

is a conjugate linear isomorphism that is also an isometry:

$$||\lambda_v||_{H^*} = ||v||.$$

The next objective on our agenda is to define the adjoint of a linear operator $T : H_1 \to H_2$ as an operator $T^* : H_2 \to H_1$. To define the adjoint operator, we begin by defining its domain. We let Domain(T^*) consist of all $\eta \in H_2$ such that

$$\mathscr{L}_\eta : x \mapsto \langle Tx, \eta \rangle_2$$

is a continuous linear functional on Domain(T). By the continuity of $\mathscr{L}_\eta$, one can extend $\mathscr{L}_\eta$ to the closure of Domain(T). We can also extend $\mathscr{L}_\eta$ to the orthogonal complement of Domain(T) by setting

$$\mathscr{L}_\eta(y) = 0 \quad \text{for all } y \in \text{Domain}(T)^\perp.$$

By the Riesz representation theorem $\mathscr{L}_\eta$ is represented by a unique element $T^*\eta$ of H_1 satisfying

$$\langle Tx, \eta \rangle_2 = \langle x, T^*\eta \rangle_1 , \quad x \in \text{Domain}(T).$$

The linearity of $\eta \mapsto T^*\eta$ is trivial to prove.

7.1.3 REMARK. Extending $\mathscr{L}_\eta$ by zero to Domain$(T)^\perp$ might initially seem rather arbitrary. However, if λ is any extension of $\mathscr{L}_\eta$ and λ_o denotes the extension by zero to Domain$(T)^\perp$ then $\lambda - \lambda_o$ vanishes on Domain(T). Thus, writing $u = v + w \in \text{Domain}(T) \oplus \text{Domain}(T)^\perp$,

$$||\lambda||^2 = \sup_{|v|^2=|w|^2=1} |\lambda(v+w)|^2 = \sup_{|v|^2=|w|^2=1} |\lambda_o(v) + (\lambda - \lambda_o)(w)|^2 = ||\lambda_o||^2 + ||\lambda - \lambda_o||^2.$$

We see that λ_o is the extension of minimal norm, which does seem rather more natural. ◇

7.1.4 PROPOSITION. *If $T : H_1 \to H_2$ is densely defined (resp. closed) then $T^* : H_2 \to H_1$ is closed (resp. densely defined).*

Proof. We denote by $\pi_i : H_1 \times H_2 \to H_i$ the projection to the i^{th} factor, $i = 1, 2$, and by $\langle\ ,\ \rangle_{1,2} = \pi_1^* \langle\ ,\ \rangle_1 + \pi_2^* \langle\ ,\ \rangle_2$ the inner product on $H_1 \times H_2$. Let $F : H_1 \times H_2 \to H_1 \times H_2$ be given by

$$F(\xi, \eta) = (-\xi, \eta).$$

Observe that $(\xi, \eta) \perp F(\text{Graph}(T))$ if and only if

$$\langle x, \xi \rangle_1 = \langle Tx, \eta \rangle_2$$

for all $x \in$ Domain(T). Note that while the latter means that $\eta \in$ Domain(T^*), we cannot conclude that $\xi = T^*\eta$ unless we also know that $\xi \perp$ Domain$(T)^\perp$.

Now suppose Domain(T) is dense. Then Domain$(T)^\perp = \{0\}$, so $(\xi, \eta) \perp F(\text{Graph}(T))$ if and only if $\eta \in$ Domain(T^*) and $T^*\eta = \xi$. In other words,

$$F(\text{Graph}(T))^\perp = \text{Graph}(T^*),$$

and thus T^* is closed.

Next suppose T is closed. Let $\eta \perp$ Domain(T^*). Then the vector $(0, \eta)$ lies in $\text{Graph}(T^*)^\perp = \overline{F(\text{Graph}(T))} = F(\overline{\text{Graph}(T)}) = F(\text{Graph}(T))$. Thus $\eta = T0 = 0$, and from Lemma 7.1.5 below and the Riesz Representation Theorem we conclude that T^* is densely defined. □

7.1.5 LEMMA. *Let A be a subspace of a topological vector space H. Then $\overline{A} = H$ if and only if every continuous linear functional on H that vanishes on A is zero.*

Proof. If $\overline{A} = H$ then clearly every continuous linear functional on H that vanishes on A is zero. Conversely, let $v \in H - \overline{A}$. The linear functional on the closed subspace $\mathbb{C}v + \overline{A}$ of H defined by $tv + a \mapsto t$ is continuous and vanishes on A. By the Hahn-Banach Theorem, this linear functional extends to a continuous linear functional $\ell \in H'$. Evidently $\ell|_A \equiv 0$ and $\ell \not\equiv 0$. □

7.1.6 PROPOSITION. *If T is a closed, densely defined operator, then $T^{**} = T$.*

Proof. We know that T^* is also closed and densely defined, and we have

$$\langle x, T^*\eta \rangle = \langle Tx, \eta \rangle$$

whenever $x \in \mathrm{Domain}(T)$ and $\eta \in \mathrm{Domain}(T^*)$. From the boundedness of $\eta \mapsto \langle Tx, \eta \rangle$ we therefore conclude that $\mathrm{Domain}(T) \subset \mathrm{Domain}(T^{**})$.

Let $\theta \in \mathrm{Domain}(T^{**})$. Since T is densely defined there exist $\mathrm{Domain}(T) \ni x_j \to \theta$. Then for all $\eta \in \mathrm{Domain}(T^*)$ we have

$$\langle T^{**}\theta, \eta \rangle = \langle \theta, T^*\eta \rangle = \lim \langle x_j, T^*\eta \rangle = \lim \langle Tx_j, \eta \rangle .$$

Thus $Tx_j \to T^{**}\theta$ in H_2, and therefore $\mathrm{Graph}(T) \ni (x_j, Tx_j) \to (\theta, T^{**}\theta)$ in $H_1 \times H_2$. Since T is closed, $(\theta, T^{**}\theta) \in \mathrm{Graph}(T)$, and thus $\theta \in \mathrm{Domain}(T)$ and $T^{**}\theta = T\theta$. □

7.1.2 The maximal extension of $\bar\partial$

Let (X, g) be a Hermitian manifold and let $E \to X$ be a holomorphic vector bundle with smooth Hermitian metric h. By our definition of $L^2_{p,q}(g, h)$ the subspace $\Gamma_o(X, \Lambda^{p,q}_X \otimes E)$ of $L^2_{p,q}(g, h)$ consisting of smooth compactly supported E-valued (p, q)-forms is dense.

We have the operator $\bar\partial : \Gamma_o(X, \Lambda^{p,q}T^*_X \otimes E) \to \Gamma_o(X, \Lambda^{p,q+1}T^*_X \otimes E)$ defined on smooth sections, and we extend $\bar\partial$ to a densely-defined operator $L^2_{p,q}(g, h) \to L^2_{p,q+1}(g, h)$ as follows.

Since every element of $L^2_{p,q}(g, h)$ is trivially locally integrable, we can extend $\bar\partial$ as on operator on all of $L^2_{p,q}(g, h)$ in the sense of currents: for $\varphi \in L^2_{p,q}(g, h)$,

$$\bar\partial\varphi(\psi) := (\varphi, \mathfrak{d}\psi), \quad \psi \in \Gamma_o(X, \Lambda^{p,q+1}T^*_X \otimes E).$$

We can construct a densely defined extension of $\bar\partial$ if we take the domain of the extension of $\bar\partial$ to be any subspace H of $L^2_{p,q}(g, h)$ containing $\Gamma_o(X, \Lambda^{p,q}T^*_X \otimes E)$ such that for each $\varphi \in H$ the E-valued $(p, q+1)$-current $\bar\partial\varphi$ is represented by integration against some $F_\varphi \in L^2_{p,q+1}(g, h)$:

$$\bar\partial\varphi(\psi) = (F_\varphi, \psi) \quad \text{for all } \psi \in \Gamma_o(X, \Lambda^{p,q+1}_X \otimes E).$$

(We will shorten the terminology and simply say that $\bar\partial\varphi \in L^2_{p,q+1}(g, h)$ in the sense of currents, or simply that $\bar\partial\varphi \in L^2_{p,q+1}(g, h)$.) Each such subspace H yields a different densely defined operator.

In this chapter, we will take the so-called *maximal extension* $T_{p,q}$ of $\bar\partial$ defined by the domain

$$\mathrm{Domain}(T_{p,q}) := \{\varphi \in L^2_{p,q}\ ;\ \bar\partial\varphi \in L^2_{p,q+1}(g, h)\}.$$

7.1.7 PROPOSITION. *The operator $T = T_{p,q}$ is closed.*

Proof. Let $\{\varphi_j\} \subset \mathrm{Domain}(T)$ be a sequence that converges to some $\varphi \in \mathrm{Domain}(T)$ such that $T\varphi_j \to \Phi$ in $L^2_{p,q+1}(g,h)$. Then for all $\eta \in \Gamma_o(X, \Lambda^{p,q+1}_X \otimes E)$ we have

$$(T\varphi - \Phi, \eta) = \lim(T\varphi - T\varphi_j, \eta) = \lim(\varphi - \varphi_j, \bar{\partial}^*\eta) = 0.$$

Thus $T\varphi = \Phi$. □

7.1.3 The Hilbert space adjoint $T^*_{p,q}$

The maximal extension $T_{p,q}$ of $\bar{\partial}$ has a closed, densely defined Hilbert space adjoint $T^*_{p,q}$ whose domain is

$$\mathrm{Domain}(T^*_{p,q}) = \{\alpha \in L^2_{p,q+1}(g,h)\ ;\ |(\alpha, T_{p,q}\psi)| \le C_\alpha ||\psi|| \text{ for all } \psi \in \mathrm{Domain}(T_{p,q})\}.$$

Below we will need the following simple proposition.

7.1.8 PROPOSITION. *If $\chi \in \mathscr{C}^\infty_o(X)$ and $\varphi \in \mathrm{Domain}(T^*_{p,q}) \cap \mathrm{Domain}(T_{p,q+1})$ then*

$$\chi\varphi \in \mathrm{Domain}(T^*_{p,q}) \cap \mathrm{Domain}(T_{p,q+1}).$$

Moreover,

$$T_{p,q+1}(\chi\varphi) = \chi T_{p,q+1}\varphi + \bar{\partial}\chi \wedge \varphi \quad \textit{and} \quad T^*_{p,q}(\chi\varphi) = \chi T^*\varphi - \mathrm{grad}^{0,1}\bar{\chi} \lrcorner \varphi,$$

where, for a function f, $\mathrm{grad}^{0,1} f$ is the $(0,1)$-vector field defined by

$$g(\xi, \mathrm{grad}^{0,1}\bar{f}) = \partial f(\xi), \qquad \xi \in T^{1,0}_X.$$

Proof. We compute that, as currents,

$$\bar{\partial}(\chi\varphi) = \bar{\partial}\chi \wedge \varphi + \chi T_{p,q+1}\varphi,$$

so clearly $\chi\varphi \in \mathrm{Domain}(T_{p,q+1})$ and the formula for $T_{p,q+1}$ holds. Next, if $\psi \in \mathrm{Domain}(T_{p,q})$ then the calculation just completed shows that $\chi\psi \in \mathrm{Domain}(T_{p,q})$. Thus

$$\begin{aligned}
|(\chi\varphi, T_{p,q}\psi)| &= |(\varphi, T_{p,q}(\bar{\chi}\psi) - \bar{\partial}\bar{\chi} \wedge \psi)| \\
&\le C_\varphi ||\bar{\chi}\psi|| + ||\varphi|| \cdot ||\bar{\partial}\bar{\chi} \wedge \psi|| \\
&\le (C_\varphi \sup|\chi| + ||\varphi|| \sup_X |\partial\chi|)||\psi||,
\end{aligned}$$

which shows that $\chi\varphi \in \mathrm{Domain}(T^*_{p,q})$ and

$$(\chi\varphi, T_{p,q}\psi) = (\varphi, T_{p,q}(\bar{\chi}\psi)) - (\varphi, \bar{\partial}\bar{\chi} \wedge \psi) = (\chi T^*_{p,q}\varphi, \psi) - (\mathrm{grad}^{0,1}\bar{\chi} \lrcorner \varphi, \psi),$$

and the formula for $T^*_{p,q}$ follows. □

7.2 Complete Kähler manifolds

7.2.1 Complete Riemannian manifolds and exhaustion functions

Recall that in Riemannian geometry one has the notion of a complete (connected) Riemannian manifold (M, g): The Riemannian metric g induces a distance function, with the distance $\delta_g(x, y)$ between two points $x, y \in M$ being the infimum of the lengths of any two paths connecting those two points. (In general, a curve realizing this infimum, which is called the minimizing geodesic, need not exist.) The underlying manifold with this distance function is a metric space, and we say that a Riemannian manifold is complete if this induced metric space is complete.

The celebrated Hopf-Rinow Theorem says that the completeness property of the Riemannian manifold is equivalent to the condition that for each $x_o \in M$ the function

$$\psi_o : M \ni x \mapsto \delta_g(x, x_o) \in [0, \infty)$$

is proper. In general the function ψ_o is not smooth, but it is Lipschitz with constant 1, and thus it is almost everywhere differentiable. Moreover, $|d\psi_o|_g \leq 1$. We can therefore smooth ψ_o to a function ψ satisfying

$$|d\psi|_g \leq 2 \quad \text{and} \quad |\psi(x) - \psi_o(x)| \leq 1.$$

The function $\psi : M \to [-1, \infty)$ is then also proper.

It is not hard to show, on the other hand, that the Riemannian manifold (M, g) is complete if there exists a smooth proper function $\psi : M \to [A, \infty)$ on M such that $|d\psi|_g$ is uniformly bounded.

7.2.1 REMARK. Note that the constant A plays no role in the completeness of (M, g), but we must have it if we insist on using the word 'proper'. In complex analysis, it is customary to avoid this trivial issue by introducing the notion of an exhaustion function. $\diamond$

We summarize the discussion in the following proposition.

7.2.2 PROPOSITION. *A Riemannian manifold (M, g) is complete if and only if there exists an exhaustion function $\psi \in \mathscr{C}^\infty(M)$ such that $|d\psi|_g \leq 1$.*

7.2.2 Approximation of twisted (p, q)-forms on complete Kähler manifolds

For the rest of the chapter, we employ the following convention: we fix p and q, and let

$$T := T_{p,q-1} \quad \text{and} \quad S := T_{p,q}.$$

Since the operator of primary interest is T, while the operator S comes in to take care of the compatibility condition for the data, it is useful to distinguish these two operators more clearly, as this notation does.

On a complete Kähler manifold forms in $\text{Domain}(S) \cap \text{Domain}(T^*)$ can be approximated by smooth forms, with respect to the norm

$$\varphi \mapsto ||\varphi|| + ||T^*\varphi|| + ||S\varphi||. \tag{7.1}$$

7.2.3 THEOREM. *Let (X,ω) be a complete Kähler manifold and let $E \to X$ be a holomorphic Hermitian vector bundle. Then for any E-valued (p,q)-form $\varphi \in \mathrm{Domain}(T^*) \cap \mathrm{Domain}(S)$ there exist smooth, compactly supported E-valued (p,q)-forms $\{\varphi_k\}$ such that*

$$\lim_{k\to\infty} ||\varphi - \varphi_k|| + ||T^*\varphi - \bar\partial^*\varphi_k|| + ||S\varphi - \bar\partial\varphi_k|| = 0.$$

Proof. Fix an exhaustion function $\psi \in \mathscr{C}^\infty(X)$ such that $|d\psi|_\omega \le 1$ and a function $\chi \in \mathscr{C}^\infty(\mathbb{R})$ such that $\chi(r) = 1$ for $r \le 0$ and $\chi(r) = 0$ for $r \ge 1$. Let $f_k(x) := \chi(\psi(x) - k + 1)$. Then $f_k \in \mathscr{C}^\infty_o(X)$, and in fact f_k is supported in the compact set $X_k := \{x \in X\ ;\ \psi(x) \le k\}$. For any $\varphi \in \mathrm{Domain}(T^*) \cap \mathrm{Domain}(S)$ Proposition 7.1.8 implies that the compactly supported form

$$\Phi_k := f_k\varphi$$

lies in $\mathrm{Domain}(S) \cap \mathrm{Domain}(T^*)$ and satisfies

$$S\Phi_k = f_k S\varphi + \bar\partial f_k \wedge \varphi \quad \text{and} \quad T^*\Phi_k = f_k T^*\varphi + (\mathrm{grad}^{0,1} f_k) \lrcorner \varphi.$$

We estimate that

$$||S\Phi_k - S\varphi|| \le ||(1-f_k)S\varphi|| + (\sup_X |df_k|)||\mathbf{1}_{X-X_k}\varphi|| \le C(||\mathbf{1}_{X-X_k}\varphi|| + ||\mathbf{1}_{X-X_k}S\varphi||),$$

and the right hand side converges to 0 as $k \to \infty$ because φ and $S\varphi$ are in L^2. Since

$$||\mathrm{grad}^{0,1} f_k|| = ||\partial f_k|| = \sqrt{2}||df_k||,$$

a similar calculation shows that $||T^*\Phi_k - T^*\varphi|| \to 0$ as $k \to \infty$. Therefore

$$||\varphi - \Phi_k|| + ||T^*\varphi - T^*\Phi_k|| + ||S\varphi - S\Phi_k|| \to 0 \quad \text{as } k \to \infty.$$

Using a partition of unity $\{\chi_i\}$, we can write $\Phi_k = \Phi_k^1 + \cdots + \Phi_k^N$ with each $\Phi_k^i := \chi_i f_k \varphi$ supported in some coordinate chart U^i, and, again by Proposition 7.1.8, lying in the domain of T^*. Using mollifiers in the coordinate chart U_i we can approximate Φ_k^i by a compactly supported smooth form φ_k^i that converges to Φ_k^i in the Sobolev 1-norm. Thus, with $\varphi_k := \varphi_k^1 + \cdots + \varphi_k^N$,

$$||\Phi_k - \varphi_k|| + ||T^*\Phi_k - \bar\partial^*\varphi_k|| + ||S\Phi_k - \bar\partial\varphi_k|| \lesssim \sum_{i=1}^N ||\Phi_k^i - \varphi_k|| + ||d(\Phi_k^i - \varphi_k)|| \to 0$$

as $k \to \infty$. Thus by the triangle inequality $||\varphi - \varphi_k|| + ||T^*\varphi - \bar\partial^*\varphi_k|| + ||S\varphi - \bar\partial\varphi_k|| \to 0$ as $k \to \infty$, and the proof is complete. □

7.2.4 REMARK. The norm (7.1) is often called the *graph norm*, since it is the restriction of the norm of $L^2_{p,q} \oplus L^2_{p,q-1} \oplus L^2_{p,q+1}$ to the graph of the operator $T^* \oplus S$. ⋄

7.2.3 The basic estimate for complete Kähler metrics

We can now prove the following version of the Basic Estimate.

7.2.5 THEOREM (The Basic Estimate: Complete Kähler metric case). *Let (X, g) be a complete Kähler manifold and let $E \to X$ be a holomorphic vector bundle with Hermitian metric h. Then for all E-valued (p, q)-forms $\varphi \in$ Domain$(T^*) \cap$ Domain(S) such that $\Theta_g(g^{(p)} \otimes \det g \otimes h)\varphi \in L^2_{p,q}(g, h)$ one has the estimate*

$$||T^*\varphi||^2 + ||S\varphi||^2 \geq (\Theta_g(g^{(p)} \otimes \det g \otimes h)\varphi, \varphi).$$

Proof. By Theorem 7.2.3 the smooth compactly supported E-valued forms are dense, and thus the result follows from integration by parts applied to the formal Bochner-Kodaira Identity (Theorem 5.3.1) and the obvious inequality $||\bar{\nabla}\varphi||^2 \geq 0$. □

7.3 Hörmander's Theorem

7.3.1 Hörmander's Theorem for Complete Kähler metrics

We shall now use Theorem 7.2.5 to establish the following result.

7.3.1 THEOREM (Hörmander's Theorem; complete metric case). *Let (X, g) be a complete Kähler manifold and let $V \to X$ be a holomorphic vector bundle with Hermitian metric h. Assume that*

$$\Theta_g(g^{(p)} \otimes \det g \otimes h) \geq c\mathrm{Id}_{\Lambda^{p,q}_X \otimes V}$$

for some $c > 0$. Then for each V-valued (p, q)-form φ such that

$$\bar{\partial}\varphi = 0 \quad \textit{and} \quad \int_X |\varphi|^2_{h,g}\omega^n < +\infty$$

there exists a V-valued $(p, q-1)$-form u such that

$$\bar{\partial}u = \varphi \quad \textit{and} \int_X |u|^2_{h,g}\omega^n \leq \frac{1}{c}\int_X |\varphi|^2_{h,g}\omega^n.$$

Proof. Since $\mathrm{Kernel}(S)$ is a closed subspace of $L^2_{p,q}(h, g)$ we can replace the latter by the former. Restricting the Basic Estimate to $\mathrm{Kernel}(S)$ then gives us that for all $\psi \in$ Domain$(T^*)\cap$Kernel(S),

$$|(\varphi, \psi)|^2 \leq ||\varphi||^2||\psi||^2 \leq \frac{||\varphi||^2}{c}||T^*\varphi||^2, \tag{7.2}$$

where the last inequality follows from Theorem 7.2.5. The Functional Analysis Lemma **??** with $H_1 = L^2_{p,q}(h, g)$ and $H_2 = \mathrm{Kernel}(S)$ then yields a (p, q)-form u such that $Tu = \varphi$ and $||u|| \leq c^{-1}||\varphi||$, as desired. □

7.3.2 Comparison of L^2 norms from comparison of metrics

Let X be a complex manifold and $V \to X$ a holomorphic vector bundle with Hermitian metric h. Fix Hermitian metrics g and γ with metric forms ω and θ respectively, and assume $\gamma \geq g$.

Choose an ω-orthonormal basis of $(1,0)$-forms $\alpha^1, ..., \alpha^n$ such that

$$\theta = \lambda_1 \sqrt{-1}\alpha^1 \wedge \bar{\alpha}^1 + \cdots + \lambda_n \sqrt{-1}\alpha^n \wedge \bar{\alpha}^n.$$

Then $\lambda_i \geq 1$ for $1 \leq i \leq n$.

Given a V-valued (p,q)-form η, one can locally write

$$\eta = \eta_{I\bar{J}} \alpha^I \wedge \bar{\alpha}^J$$

with $\eta_{I\bar{J}}$ local sections of V. With the notation $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_p}$ and $\lambda_{\bar{J}} = \lambda_{j_1} \cdots \lambda_{j_q}$, the norms of η with respect to the metrics (h,g) and (h,γ) are respectively

$$|\eta|^2_{h,g} = \frac{1}{p!q!} \sum_{|I|=p,|J|=q} |\eta_{I\bar{J}}|^2_h \quad \text{and} \quad |\eta|^2_{h,\gamma} = \frac{1}{p!q!} \sum_{|I|=p,|J|=q} \frac{|\eta_{I\bar{J}}|^2_h}{\lambda_I \lambda_{\bar{J}}}.$$

In particular,

$$|\eta|^2_{h,\gamma} \leq |\eta|^2_{h,g}.$$

On the other hand, the volume forms for the two metrics are

$$dV_g = (\sqrt{-1})^n \alpha^1 \wedge \bar{\alpha}^1 \wedge \cdots \alpha^n \wedge \bar{\alpha}^n \quad \text{and} \quad dV_\gamma = \lambda_1 \cdots \lambda_n (\sqrt{-1})^n \alpha^1 \wedge \bar{\alpha}^1 \wedge \cdots \alpha^n \wedge \bar{\alpha}^n,$$

so $dV_g \leq dV_\gamma$, and hence it is in general hard to compare the L^2-norms of (p,q)-forms with respect to these two metrics.

However, there is one exceptional but important case: the case $p = n$. In this case

$$\begin{aligned} |\eta|^2_{h,\gamma} dV_\gamma &= \frac{1}{n!q!} \sum_{|I|=n,|J|=q} \frac{|\eta_{I\bar{J}}|^2_h}{\lambda_{\bar{J}}} (\sqrt{-1})^n \alpha^1 \wedge \bar{\alpha}^1 \wedge \cdots \alpha^n \wedge \bar{\alpha}^n \\ &\leq \frac{1}{n!q!} \sum_{|I|=n,|J|=q} |\eta_{I\bar{J}}|^2_h (\sqrt{-1})^n \alpha^1 \wedge \bar{\alpha}^1 \wedge \cdots \alpha^n \wedge \bar{\alpha}^n \\ &= |\eta|^2_{h,g} dV_g. \end{aligned}$$

Consequently one has the comparison of L^2-norms: if η is a V-valued (n,q)-form and $\gamma \geq g$ then

$$||\eta||^2_{h,\gamma} \leq ||\eta||^2_{h,g}.$$

This monotonicity of norms is very useful if one wants to generalize Hörmander's Theorem 7.3.1 to the setting in which the manifold X is complete Kähler but the metric g is not necessarily complete. Indeed, of g is a Hermitian (resp. Kähler) metric and g_* is a complete Hermitian (resp. Kähler) metric then for every $\varepsilon > 0$ the metric $g_\varepsilon := g + \varepsilon g_*$ is a complete Hermitian (resp. Kähler) metric that dominates g.

There is also of course the problem that the monotonicity of L^2-norms only works when $p = n$, but this matter is dealt with by another clever trick.

7.3.3 Hörmander's Theorem for complete Kähler manifolds

We can now remove the hypothesis of completeness for the metric g in Theorem 7.3.1.

7.3.2 THEOREM (Hörmander's Theorem). *Let X be a complete Kähler manifold with metric g that is not necessarily complete, and let $E \to X$ be a holomorphic vector bundle with Hermitian metric h. Assume that*

$$\Theta_g(g^{(p)} \otimes \det g \otimes h) \ge c\mathrm{Id}_{\Lambda^{p,q}_X \otimes E}$$

for some $c > 0$. Then for each E-valued (p,q)-form φ such that

$$\int_X |\varphi|^2_{h,g}\omega^n < +\infty \quad \textit{and} \quad \bar\partial\varphi = 0$$

in the sense of distributions, there exists a E-valued $(p,q-1)$-form u such that

$$\bar\partial u = \varphi \quad \textit{and} \int_X |u|^2_{h,g}\omega^n \le \frac{1}{c}\int_X |\varphi|^2_{h,g}\omega^n.$$

Proof. Fix a complete Kähler metric g_* and qrite $g_\varepsilon := g + \varepsilon g_*$. If $\varepsilon > 0$ is sufficiently small then

$$\Theta_{g_\varepsilon}(g^{(p)} \otimes \det g \otimes h) \ge c_\varepsilon \mathrm{Id}_{\Lambda^{p,q}_X \otimes E}$$

with $c_\varepsilon > 0$ and $\lim_{\varepsilon\to 0} c_\varepsilon = c$.

In order to exploit the monotonicity of L^2-norms with respect to the Kähler metrics, one uses the following trick to identify E-valued (p,q)-forms with $\tilde E$-valued (n,q)-forms for some holomorphic vector bundle $\tilde E$. We have used this trick at the bundle level, but not at the metric level.

Let $\tilde E := \Lambda^{p,0}T^*_X \otimes K^*_X \otimes E$ and set $\tilde h = g^{(p)} \otimes \det g \otimes h$. Then $\tilde h$ is a metric for $\tilde E$ satisfying

$$\Theta_{g_\varepsilon}(\tilde h) \ge c_\varepsilon,$$

and moreover for any E-valued (p,q)-form, a.k.a. $\tilde E$-valued $(n,\tilde q)$-form, η

$$|\eta|^2_{h,g} = |\eta|^2_{\tilde h,g}.$$

We emphasize that the metric $\tilde h$ involves the metric g and not the metric g_ε, and so in this sense $\tilde h$ is '*ε-static*'.

Now let $\varphi \in L^2_{p,q}(h,g) = L^2_{n,q}(\tilde h, g)$. Then by monotonicity

$$\int_X |\varphi|^2_{\tilde h,g_\varepsilon} dV_{g_\varepsilon} \le \int_X |\varphi|^2_{\tilde h,g} dV_g = \int_X |\varphi|^2_{h,g} dV_g < +\infty,$$

so by Theorem 7.3.1 there exists $u_\varepsilon \in L^2_{n,q}(\tilde h, g_\varepsilon)$ such that

$$\bar\partial u_\varepsilon = \varphi \quad \text{and} \quad \int_X |u_\varepsilon|^2_{\tilde h,g_\varepsilon} dV_{g_\varepsilon} \le \frac{1}{c_\varepsilon}\int_X |\varphi|^2_{\tilde h,g_\varepsilon} dV_{g_\varepsilon} \le \frac{1}{c_\varepsilon}\int_X |\varphi|^2_{\tilde h,g} dV_g.$$

Now, for $0 < \varepsilon < \varepsilon_o$, $g_\varepsilon \le g_{\varepsilon_o}$, and hence by monotonicity again

$$\int_X |u_\varepsilon|^2_{\tilde{h},g_{\varepsilon_o}} dV_{g_{\varepsilon_o}} \le \int_X |u_\varepsilon|^2_{\tilde{h},g_\varepsilon} dV_{g_\varepsilon} \le \frac{1}{c_\varepsilon}\int_X |\varphi|^2_{h,g} dV_g.$$

Thus since $c_\varepsilon \to c$, $\{u_\varepsilon\}$ lies in a fixed ball inside $L^2_{n,q}(\tilde{h}, g_{\varepsilon_o})$. By Alaoglu's Theorem we can choose a subsequence $u_{o,j_o} := u_{\varepsilon_{j_o}}$ that converges in $L^2_{n,q}(\tilde{h}, g_{\varepsilon_o})$. Choosing $\varepsilon_1 < \varepsilon_o$, we have a convergent subsequence

$$\{u_{1,j_1}\ ;\ j_1 = 1, 2, ...\} \subset \{u_{o,j_o}\ ;\ j_o = 1, 2, ...\} \cap L^2_{n,q}(\tilde{h}, g_{\varepsilon_1}).$$

Continuing inductively, we choose $\varepsilon_{k+1} < \varepsilon_k$ and a convergent subsequence

$$\{u_{k+1,j_{k+1}}\ ;\ j_{k+1} = 1, 2, ...\} \subset \{u_{k,j_k}\ ;\ j_k = 1, 2, ...\} \cap L^2_{n,q}(\tilde{h}, g_{\varepsilon_{k+1}}).$$

Assuming further that $\varepsilon_k \to 0$, the sequence $\{u_j := u_{j,1}\}$ converges to some

$$u \in \bigcap_{k\ge o} L^2_{n,q}(\tilde{h}, g_{\varepsilon_k}).$$

Since $|u|^2_{\tilde{h},g_{\varepsilon_k}} dV_{g_{\varepsilon_k}}$ is an increasing sequence (in k), by the Monotone Convergence Theorem

$$\int_X |u|^2_{h,g} dV_g = \int_X |u|^2_{\tilde{h},g} dV_g = \lim_{k\to\infty}\int_X |u|^2_{\tilde{h},g_{\varepsilon_k}} dV_{g_{\varepsilon_k}} \le \frac{1}{c}\int_X |\varphi|^2_{\tilde{h},g} dV_g = \frac{1}{c}\int_X |\varphi|^2_{h,g} dV_g.$$

Finally,

$$(u, \bar\partial^*\psi) = \lim(u_j, \bar\partial^*\psi) = \lim(\varphi, \psi) = (\varphi, \psi)$$

for any compactly supported ψ. Thus $\bar\partial u = \varphi$, and the proof is complete. □

7.3.4 Weakly pseudoconvex Kähler manifolds

An important class of complete Kähler manifolds in complex analysis is the class of weakly pseudoconvex manifolds.

7.3.3 DEFINITION. *A complex manifold X is said to be weakly pseudoconvex if X has a smooth plurisubharmonic exhaustion function.*

7.3.4 THEOREM. *If X is a weakly pseudoconvex Hermitian (resp. Kähler) manifold then X has a complete Hermitian (resp. Kähler) metric.*

Proof. Fix a Hermitian form ω and a smooth plurisubharmonic exhaustion function ψ. Then

$$\tilde\omega = \omega + \sqrt{-1}\partial\bar\partial e^{2\psi}$$

is Hermitian, and it is Kähler if ω is Kähler. Moreover

$$\tilde\omega = \omega + 4e^{2\psi}\sqrt{-1}\partial\psi \wedge \bar\partial\psi + 2e^{2\psi}\sqrt{-1}\partial\bar\partial\psi \ge 4\sqrt{-1}\partial(e^\psi) \wedge \bar\partial(e^\psi).$$

Note that for any real-valued function ρ, $d\rho = \partial\rho + \bar{\partial}\rho = \partial\rho + \overline{\partial\rho}$, so that

$$|d\rho|^2_{\tilde{\omega}} = g^*_{\tilde{\omega}}(\partial\rho + \overline{\partial\rho}, \partial\rho + \overline{\partial\rho}) = g^*_{\tilde{\omega}}(\partial\rho, \partial\rho) + g^*_{\tilde{\omega}}(\overline{\partial\rho}, \overline{\partial\rho}) = 2g^*_{\tilde{\omega}}(\partial\rho, \partial\rho) = 2|\partial\rho|^2_{\tilde{\omega}}.$$

Thus

$$|d(e^\psi)|^2_{\tilde{\omega}} = 2|\partial(e^\psi)|^2_{\tilde{\omega}} \le \tfrac{1}{2}.$$

Since the exponential function is increasing and proper (as a function from $\mathbb{R}$ to $(0, \infty)$), e^ψ is also a smooth exhaustion function. By Proposition 7.2.2, $(X, \tilde{\omega})$ is complete. □

7.3.5 REMARK. In the latter proof, if we can replace the exponential function by $\chi \circ \psi$ where $\chi : [0, \infty) \to [0, \infty)$ is any non-constant convex increasing function. (Observe that a bounded increasing convex function on $[0, \infty)$ is necessarily constant, and thus in fact $\chi \circ \psi$ is also a plurisubharmonic exhaustion.) We then compute that

$$\tilde{\omega} = \omega + \sqrt{-1}(\chi' \circ \psi)\partial\bar{\partial}\psi + \chi''(\psi)\sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \ge \chi''(\psi)\sqrt{-1}\partial\psi \wedge \bar{\partial}\psi = \sqrt{-1}\partial(h \circ \psi) \wedge \bar{\partial}(h \circ \psi),$$

where h is a real-valued function satisfying $(h')^2 = \chi''$, i.e.,

$$h(x) = C + \int_o^x \sqrt{\chi''(t)}dt.$$

Note that as χ is convex, h is necessarily increasing, so in order for $h \circ \psi$ to be an exhaustion, it is necessary and sufficient that h is an exhaustion of $\mathbb{R}$, i.e., $\lim_{x \to +\infty} h(x) = +\infty$. ◇

Thus Hörmander's Theorem holds on weakly pseudoconvex manifolds.

7.3.5 Skoda's estimate

Theorem 7.3.2 has a generalization, with almost the same proof, that can be useful in certain applications. The result is still about solving the $\bar{\partial}$-equation with L^2 estimates, but the estimates are for data that that lies in L^2 with respect to different norms. The generalization was first observed by Skoda, and then generalized by Demailly, though it is often still referred to as Hörmander's Theorem in the literature.

Before stating this version of Hörmander's Theorem, let us introduce some notation. Given a positive definite Hermitian $(1, 1)$-form Υ, we can define a pointwise norm on E-valued (p, q)-forms θ by

$$|\theta|^2_{\Upsilon,h,g} := \theta^\alpha_{I,\bar{J}}\overline{\theta^\beta_{K,\bar{L}}}h_{\alpha\bar{\beta}}g^{I\bar{K}}\Upsilon^{\ell_1\bar{j}_1}\cdots\Upsilon^{\ell_q\bar{j}_q},$$

where locally, $\theta = \theta^\alpha_{I,\bar{J}}e_\alpha \otimes dz^I \wedge d\bar{z}^J$, $\Upsilon = \Upsilon_{i\bar{j}}\sqrt{-1}dz^i \wedge d\bar{z}^j$, and $\Upsilon^{k\bar{\ell}}$ is the inverse of the matrix $(\Upsilon_{i\bar{j}})$. In other words, this norm is much like the norm using only h and g, except that in the $(0, 1)$-directions we replace the metric g by the Hermitian form Υ.

With these observations in hand, we can now state Skoda's estimate.

7.3.6 THEOREM (Demailly-Hörmander-SkodaTheorem). *Let (X, g) be Kähler manifold and let $E \to X$ be a holomorphic vector bundle with Hermitian metric h. Assume that X admits a complete Kähler metric (which need not be g). Suppose that*

$$\left\langle \Theta_g(g^{(p)} \otimes \det g \otimes h)\xi, \xi \right\rangle_{g,h} \geq \langle \Upsilon_g \xi, \xi \rangle_{g,h}, \qquad \xi \in \Lambda^{p,q}_X \otimes E,$$

where

$$\Upsilon_g(\xi^\alpha_{I\bar{J}} dz^I \wedge d\bar{z}^J \otimes e_\alpha) := \sum_{k=1}^{q} g^{i\bar{\ell}} \Upsilon_{i\bar{j}_k} \xi^\alpha_{I\bar{j}_1 \cdots (\bar{\ell})_k \cdots \bar{j}_q}.$$

Then for each E-valued (p, q)-form φ such that

$$\bar{\partial}\varphi = 0 \quad \textit{and} \quad \int_X |\varphi|^2_{\Upsilon,g,h} dV_g < +\infty$$

there exists an E-valued $(p, q-1)$-form u such that

$$\bar{\partial} u = \varphi \quad \textit{and} \int_X |u|^2_{g,h} dV_g \leq \int_X |\varphi|^2_{\Upsilon,g,h} dV_g.$$

Sketch of proof. The proof of Hörmander's Theorem goes through in exactly the same way if one proves the pointwise estimate

$$|\langle \varphi, \psi \rangle|^2 \leq |\varphi|^2_{\Upsilon,g,h} \langle \Upsilon_g \psi, \psi \rangle_{g,h}.$$

The fact that such an inequality holds for a Hermitian form is a simple exercise in linear algebra: it holds for diagonal operators by using the rescaled Cauchy-Schwarz Inequality

$$|a \cdot \bar{b}| \leq |(\lambda_1^{-1} a^1, ..., \lambda_N^{-1} a^N)|^2 \cdot |(\lambda_1 b^1, ..., \lambda_N b^N)|^2,$$

and the general case is proved by diagonalizing Υ.

While the argument used when passing from complete metrics to non-complete metrics requires some checking, it is not difficult to see after a careful look, as the reader can confirm. □

7.4 Singular Hermitian metrics: an overview

In the statement of Hörmander's Theorem for domains in $\mathbb{C}$ (Theorem 4.1.12) we assumed only that the weight function φ is strictly subharmonic. Thinking of $e^{-\varphi}$ as a metric for the trivial line bundle on $\mathbb{C}$, one sees that Theorem 4.1.12 does not follow from Theorem 7.3.2 because the latter theorem assumes the metric h for $E \to X$ is smooth.

For a general vector bundle there is as yet no accepted notion of a singular Hermitian metric such that Theorem 7.3.2 holds; this question lies in an active area of research. The exceptional situation is when the rank of E is 1, i.e., when $E \to X$ is a line bundle. The following notion was introduced by Demailly.

7.4.1 DEFINITION. *Let $L \to X$ be a holomorphic line bundle. A singular Hermitian metric for L is a measurable section $h \in \Gamma(X, L^* \otimes \overline{L^*})$ such that for any local holomorphic frame ξ of L over $U \subset X$*

1. (Hermitian) $h(\xi, \xi) : U \to [0, \infty]$, *and*
2. (Singular) $\varphi^\xi := -\log h(\xi, \xi) \in L^1_{\ell oc}(U)$.

The name *singular Hermitian metric* is slightly misleading; it should probably have been called a *possibly singular Hermitian metric*, since smooth metrics satisfy the criteria of Definition 7.4.1.

The main point of the definition is that one can define the $(1,1)$-current $\partial\bar{\partial}\varphi^\xi$ by

$$\partial\bar{\partial}\varphi^\xi(\eta) := \int_U \varphi^\xi \bar{\partial}\partial\eta \quad \text{for all smooth compactly supported } (n-1, n-1)\text{-forms } \eta \text{ on } U$$

In the smooth case this $(1,1)$-current is represented by integration against the smooth form $\partial\bar{\partial}\varphi^\xi$, and the latter is independent of the choice of frame ξ. This form is of course the curvature of the Chern connection for L, h. One can therefore make the following definition.

7.4.2 DEFINITION. *The current*

$$\Theta(h) := \partial\bar{\partial}\varphi^\xi$$

is called the curvature current of the singular Hermitian metric h.

To have singular metrics with positive[1] curvature current simply means that the local functions φ^ξ are plurisubharmonic. Strict positivity means these functions are strictly plurisubharmonic. These notions, which are local, also do not depend on the choice of local frame ξ for L.

Quite often, however, positivity is not quite the correct hypothesis. For this purpose one wants to have a class of singular Hermitian metrics whose curvature currents do not become 'too negative'.

7.4.3 DEFINITION. *Let X be a Kähler manifold with Kähler form ω. A function $\psi : X \to [-\infty, \infty)$ is said to be ω-plurisubharmonic if*

$$\sqrt{-1}\partial\bar{\partial}\psi + \omega$$

is a positive $(1,1)$-current. More generally, the function ψ is said to be quasi-plurisubharmonic if for any $x \in X$ there exist an open set U containing X, a smooth function $\psi_{U,\mathrm{sm}}$ and a plurisubharmonic function $\psi_{U,\mathrm{psh}}$ such that

$$\psi|_U = \psi_{U,\mathrm{psh}} + \psi_{U,\mathrm{sm}}.$$

[1] In the context of currents the word 'positive' often means 'non-negative', especially when the authors are French. The nomenclature comes from measure theory, in which a positive measure can still be supported on a proper subset.

In many situations, Hörmander's Theorem 7.3.2 (again, with the rank of E equal to 1) holds with singular Hermitian metrics in place of smooth ones. The idea of the proof is to regularize the metrics with the right monotonicity: one seeks a sequence of smooth metrics h_j that increases pointwise to the singular metric h, such that the curvature of each of the metrics in the approximating sequence carries enough positivity to apply Hörmander's Theorem. With such a sequence of metrics in hand, one can deduce the singular metric version of Hörmander's Theorem from Theorem 7.3.2 for the smooth metrics h_j and standard limit and weak-$*$ compactness theorems from real analysis. This is precisely what was done in the last part of the proof of Theorem 4.1.12. (A key point here is that the bounds in Hörmander's Theorem do not depend on the manifold X, the vector bundle E, and not even on the metrics weights themselves, but only on the positivity conditions that these metrics satisfy.)

As for finding appropriate monotonic smooth approximations to a singular Hermitian metric, there are many interesting cases in which this can be done.

- The classical example occurs when X is a bounded domain in $\mathbb{C}^n$ and $L \to X$ is the trivial line bundle. In this case we can write our metric h as $e^{-\varphi}$ where φ is a globally defined function. If φ is quasi-plurisubharmonic and $\Omega_{\varepsilon_o} := \{z \in \Omega\ ;\ B_{\varepsilon_o}(z) \subset \Omega\}$ is the set of points of Ω of distance at least ε_o from $\partial\Omega$ then there exists $C > 0$ such that $\varphi + C|z|^2$ is plurisubharmonic on Ω_{ε_o}. One can then define, for $\varepsilon \in (0, \varepsilon_o)$, the function $\varphi_\varepsilon : \Omega_{\varepsilon_o} \to \mathbb{R}$ by
$$\varphi_\varepsilon(z) := -C|z|^2 + \frac{1}{\varepsilon^{2n}} \int_{\mathbb{C}^n} (\varphi(\zeta) + C|\zeta|^2))\chi(\varepsilon^{-1}(z-\zeta))dV(\zeta),$$
where χ is a $U(n)$-invariant function in $\mathbb{C}^n$ satisfying $\int_{\mathbb{C}^n} \chi dV = 1$. The function φ_ε decreases monotonically to φ as $\varepsilon \searrow 0$, and for each $\varepsilon > 0$ the function $z \mapsto \varphi_\varepsilon(z) + C|z|^2$ is plurisubharmonic.

- One can also carry out this sort of regularization on a Stein manifold, though in that case the argument is more subtle.

- There are some manifolds, such as projective manifolds, that carry an analytic hypersurface whose complement is a Stein manifold. The typical example is that of projective manifolds. In such manifolds one works on the complement of this analytic hypersurface. Often the L^2 estimates will allow for extension of holomorphic sections across the analytic hypersurface. Indeed, the L^2 norm dominates the pointwise norm, so a uniform L^2 bound implies locally uniform bounds, and one can apply Riemann's Theorem on removable singularities.

- For a general complete Kähler manifold, even a weakly pseudoconvex one, this sort of regularization is not necessarily possible. It is possible to regularize metrics by giving up some positivity– this is a well-known and beautiful technique due to Demailly– but the loss of positivity is often a problem since, unlike Stein or Projective manifolds, one does not have an ambient positive line bundle to add in the extra positivity.

  However, if on a complete Kähler manifold a given singular metric h has a monotonic approximation by smooth metrics without loss of positivity then for this metric h one can establish Hörmander's Theorem.

EXERCISES

Part III

Twisted Methods

Lecture 8

Donnelly-Fefferman-Ohsawa Technique

In this section we shall obtain an improvement of Hörmander's Theorem on complex manifolds that are equipped with certain special geometric features.

8.1 The twisted basic estimate

In Section 5 we established the Bochner-Kodaira Identity (5.3.1): If X is a Kähler manifold with Kähler metric g and $E \to X$ is a holomorphic vector bundle with smooth Hermitian metric $\mathfrak{h}$ then for any smooth E-valued (n,q)-form α one has the formal identity

$$\Box\alpha = \bar{\nabla}^*\bar{\nabla}\alpha + \Theta_g(\mathfrak{h})\alpha.$$

If α has compact support then by pairing the last identity with α and integrating by parts one has

$$\int_X |\bar{\partial}^*\alpha|^2 + \int_X |\bar{\partial}\alpha|^2 = \int_X |\bar{\nabla}\alpha|^2 + \int_X \langle\Theta_g(\mathfrak{h})\alpha, \alpha\rangle\,,$$

form which the so-called *basic estimate*

$$\int_X |\bar{\partial}^*\alpha|^2 + \int_X |\bar{\partial}\alpha|^2 \geq \int_X \langle\Theta_g(\mathfrak{h})\alpha, \alpha\rangle \tag{8.1}$$

trivially follows. If the Kähler metric g is complete, as we shall assume from here on is the case, then (8.1) holds for all $\alpha \in \mathrm{Domain}(\bar{\partial}) \cap \mathrm{Domain}(\bar{\partial}^*)$.

There are several equivalent ways to approach the twisted basic estimate. We shall take the one developed by Siu, but reformulate it slightly. The first step is to consider a metric of the form

$$\mathfrak{h} = e^{-\eta}h$$

where h is a Hermitian metric for $E \to X$ and η is a smooth function. By Propositions 5.2.2 together with the definition of the Chern connection,

$$\bar{\partial}^*_{\mathfrak{h}}\alpha = \bar{\partial}^*_h\alpha + (\mathrm{grad}^{0,1}\eta)\lrcorner\alpha \quad \text{and} \quad \Theta(\mathfrak{h}) = \Theta(h) + \partial\bar{\partial}\eta,$$

where

$$\mathrm{grad}^{0,1}\eta := g^{i\bar{j}}\frac{\partial \eta}{\partial z^i}\frac{\partial}{\partial \bar{z}^j}$$

is the $(0,1)$-vector field obtained from $\partial\eta$ via the isomorphism $T^{*1,0}_X \cong T^{0,1}_X$ induced by the metric g, and $\lrcorner$ is contraction of forms by vectors. Then

$$|\bar{\partial}\alpha|^2_{g,\mathfrak{h}} = e^{-\eta}|\bar{\partial}\alpha|^2_{g,h} = |\sqrt{e^{-\eta}}\bar{\partial}\alpha|^2_{g,h},$$

$$\langle \Theta_g(\mathfrak{h})\alpha, \alpha\rangle_{g,\mathfrak{h}} = \left\langle e^{-\eta}\left(\Theta(h) + \partial\bar{\partial}\eta \otimes \mathrm{Id}_E\right)_g \alpha, \alpha\right\rangle_{g,h}$$

and

$$\begin{aligned}
|\bar{\partial}^*_{\mathfrak{h}}\alpha|^2_{g,\mathfrak{h}} &= e^{-\eta}\left|\bar{\partial}^*_h\alpha + (\mathrm{grad}^{0,1}\eta)\lrcorner\alpha\right|^2_{g,h} \\
&= e^{-\eta}\left(\left|\bar{\partial}^*_h\alpha\right|^2_{g,h} + \left|(\mathrm{grad}^{0,1}\eta)\lrcorner\alpha\right|^2_{g,h} + 2\mathrm{Re}\,\left\langle \bar{\partial}^*_h\alpha, (\mathrm{grad}^{0,1}\eta)\lrcorner\alpha\right\rangle_{g,h}\right) \\
&\le e^{-\eta}\left((1+a)\left|\bar{\partial}^*_h\alpha\right|^2_{g,h} + (1+a^{-1})\left|(\mathrm{grad}^{0,1}\eta)\lrcorner\alpha\right|^2_{g,h}\right) \\
&= |\sqrt{e^{-\eta}(1+a)}\bar{\partial}^*_h\alpha|^2_{g,h} + \left\langle \frac{1+a}{a}e^{-\eta}\left(\partial\eta\wedge\bar{\partial}\eta\otimes \mathrm{Id}_E\right)_g\alpha, \alpha\right\rangle_{g,h},
\end{aligned}$$

where a is any positive function. The inequality follows from Cauchy-Schwarz and the estimate $2\alpha\beta \le C^{-1}\alpha^2 + C\beta^2$. From the basic estimate (8.1) we therefore have the a priori estimate

$$\begin{aligned}
&\int_X \left|\sqrt{e^{-\eta}(1+a)}\bar{\partial}^*_h\alpha\right|^2_{g,h} + \int_X \left|\sqrt{e^{-\eta}}\bar{\partial}\alpha\right|^2_{g,h} \\
&\qquad \ge \int_X \left\langle e^{-\eta}\left(\Theta(h) + \left(\partial\bar{\partial}\eta - \tfrac{1+a}{a}\partial\eta\wedge\bar{\partial}\eta\right)\otimes \mathrm{Id}_E\right)_g\alpha, \alpha\right\rangle_{g,h}.
\end{aligned}$$

Observing that the domains of $\bar{\partial}$ and $\bar{\partial}^*_{\mathfrak{h}}$ are the same as the domains of the operators

$$S := \sqrt{e^{-\eta}}\bar{\partial} \quad \text{and} \quad T^* := \left(\bar{\partial}_h \circ \sqrt{e^{-\eta}(1+a)}\right)^*$$

respectively, one has the following theorem.

8.1.1 THEOREM (Twisted basic estimate). *Let X be a Kähler manifold with complete Kähler metric g and let $E \to X$ be a holomorphic vector bundle with Hermitian metric h. Then*

$$\begin{aligned}
&\int_X |T^*\alpha|^2_{g,h} + \int_X |S\alpha|^2_{g,h} \\
&\qquad \ge \int_X \left\langle e^{-\eta}\left(\Theta(h) + \left(\partial\bar{\partial}\eta - \tfrac{1+a}{a}\partial\eta\wedge\bar{\partial}\eta\right)\otimes \mathrm{Id}_E\right)_g\alpha, \alpha\right\rangle_{g,h}
\end{aligned}$$

for every form $\alpha \in \mathrm{Domain}((\bar{\partial}\circ\sqrt{\tau + A})^*_h) \cap \mathrm{Domain}(\sqrt{\tau}\circ\bar{\partial})$.

8.2 Donnelly-Fefferman-Ohsawa estimate for twisted $\bar{\partial}$

With Theorem 8.1.1 in hand, one can use the same method of proof of Skoda's estimate to establish the following theorem.

8.2.1 THEOREM (Donnelly-Fefferman-Ohsawa Estimate). *Let X be a complete Kähler manifold with not necessarily complete Kähler metric g, let $E \to X$ be a holomorphic vector bundle with Hermitian metric h and let $\eta : X \to \mathbb{R}$ and $a : X \to (0, \infty)$ be functions with η $\mathscr{C}^2$-smooth. Assume the operator*

$$\Psi := e^{-\eta} \left(\Theta(h) + \left(\partial\bar{\partial}\eta - \tfrac{1+a}{a}\partial\eta \wedge \bar{\partial}\eta\right) \otimes \mathrm{Id}_E\right)_g : \Lambda^{n,q}_X \otimes E \to \Lambda^{n,q}_X \otimes E$$

is invertible on each fiber. Then for each E-valued (n, q)-form φ satisfying

$$S\varphi = 0 \quad \text{and} \quad \int_X \left\langle \Psi^{-1}\varphi, \varphi \right\rangle_{g,h} < +\infty$$

there exists an E-valued $(n, q-1)$-form u such that

$$Tu = \varphi \quad \text{and} \quad \int_X |u|^2_{g,h} \leq \int_X \left\langle \Psi^{-1}\varphi, \varphi \right\rangle_{g,h}.$$

8.3 Ohsawa's $\bar{\partial}$ estimate

8.3.1 THEOREM (Ohsawa's $\bar{\partial}$ estimate). *Let X be a complete Kähler manifold with not necessarily complete Kähler metric g, let $E \to X$ be a holomorphic vector bundle with Hermitian metric $\mathfrak{h}$ and let $\eta : X \to \mathbb{R}$ be a $\mathscr{C}^2$-smooth function. Assume the operator*

$$\Phi_\delta := \left(\Theta(\mathfrak{h}) + \left(2\partial\bar{\partial}\eta - (1+\delta)\partial\eta \wedge \bar{\partial}\eta\right) \otimes \mathrm{Id}_E\right)_g : \Lambda^{n,q}_X \otimes E \to \Lambda^{n,q}_X \otimes E$$

is invertible for some $\delta > 0$. Then for each E-valued (n, q)-form φ satisfying

$$\bar{\partial}\varphi = 0 \quad \text{and} \quad \int_X \left\langle \Phi_\delta^{-1}\varphi, \varphi \right\rangle_{g,\mathfrak{h}} < +\infty$$

there exists an E-valued $(n, q-1)$-form U such that

$$\bar{\partial}U = \varphi \quad \text{and} \quad \int_X |U|^2_{g,\mathfrak{h}} \leq \frac{1+\delta}{\delta} \int_X \left\langle \Phi_\delta^{-1}\varphi, \varphi \right\rangle_{g,\mathfrak{h}}.$$

8.3.2 REMARK. In the typical application of Theorem 8.3.1 one takes η to be a strictly plurisubharmonic function such that

$$\partial\bar{\partial}\eta \geq (1+\delta)\partial\eta \wedge \bar{\partial}\eta, \tag{8.2}$$

and in this case the assumption on the metric $\mathfrak{h}$ is that the metric $\mathfrak{h}e^{-\eta}$ is Nakano positive. The latter condition holds as long as $\Theta(\mathfrak{h}) \geq -\partial\bar{\partial}\eta \otimes \mathrm{Id}_E$, and therefore the curvature hypotheses can be significantly weaker than those of Hörmander's Theorem. Of course, one obtains a non-trivial result only if there are non-constant functions η satisfying (8.2). We shall explore this condition a little further in the next paragraph. ◇

Proof of Theorem 8.3.1. We shall apply Theorem 8.2.1 with $h := e^{-\eta}\mathfrak{h}$ and $a = 1/\delta$. Then

$$\Theta(h) = \Theta(e^{-\eta}\mathfrak{h}) = \Theta(\mathfrak{h}) + \partial\bar{\partial}\eta$$

and therefore

$$\Psi := e^{-\eta}\left(\Theta(h) + \left(\partial\bar{\partial}\eta - \tfrac{1+a}{a}\partial\eta \wedge \bar{\partial}\eta\right) \otimes \mathrm{Id}_E\right)_g = e^{-\eta}\Phi_\delta$$

is invertible. By Theorem 8.2.1 there exists u such that $\bar{\partial}(\sqrt{(1+\delta^{-1})e^{-\eta}} \cdot u) = \varphi$ and

$$\int_X |u|^2_{g,h} \le \int_X \left\langle \Psi^{-1}\varphi, \varphi\right\rangle_{g,h}.$$

Setting $U := \sqrt{(1+\delta^{-1})e^{-\eta}} \cdot u$, we find that

$$\int_X |U|^2_{g,\mathfrak{h}} = \frac{1+\delta}{\delta}\int_X |u|^2_{g,h} \le \frac{1+\delta}{\delta}\int_X \left\langle \Psi^{-1}\varphi, \varphi\right\rangle_{g,h} = \frac{1+\delta}{\delta}\int_X \left\langle \Phi_\delta^{-1}\varphi, \varphi\right\rangle_{g,\mathfrak{h}},$$

and the proof is complete. □

8.4 Functions with self-bounded gradient

If one wants the hypotheses of Theorem 8.2.1 to yield a gain in positivity then it makes sense to ask for functions η and a such that

$$\partial\bar{\partial}\eta - \frac{a+1}{a}\partial\eta \wedge \bar{\partial}\eta > 0.$$

Of course, any function satisfying this positivity requirement is obviously plurisubharmonic, but the condition is even stronger. Indeed, since $\frac{a}{1+a} < 1$, one has the estimate

$$\partial\eta \wedge \bar{\partial}\eta < \partial\bar{\partial}\eta,$$

and hence the function $\xi := -e^{-\eta}$ satisfies

$$\partial\bar{\partial}\xi = \partial(e^{-\eta}\bar{\partial}\eta) = e^{-\eta}(\partial\bar{\partial}\eta - \partial\eta \wedge \bar{\partial}\eta) > 0,$$

i.e., ξ is a negative, strictly plurisubharmonic function.

8.4.1 DEFINITION (McNeal). *A function $f : X \to \mathbb{R}$ is said to have self-bounded gradient if*

$$\partial f \wedge \bar{\partial} f \le \partial\bar{\partial} f,$$

or equivalently, if $-e^{-f}$ is plurisubharmonic.

To complex analysts the existence of a negative plurisubharmonic is a familiar restriction that first appears in an important result regarding Green's functions: the latter exist on a domain $X \subset \mathbb{C}$ (or more generally, on a non-compact Riemann surface) if and only if the Riemann surface has a non-constant negative subharmonic function.

In higher dimensions functions with self-bounded gradient have been used by a number of authors to obtain a number of different types of estimates for $\bar{\partial}$.

8.4.2 EXAMPLE. If $X := \{z \in \mathbb{C}^n \; ; \; |z| < 1\}$ is the unit ball and

$$\eta := -\log(1 - |z|^2)$$

then

$$-e^{-\eta} = |z|^2 - 1$$

is strictly plurisubharmonic, and therefore η has self-bounded gradient.

Note that $\sqrt{-1}\partial\bar{\partial}\eta$ is, up to a constant, the Kähler form of the Poincaré metric, the Bergman metric, the Kobayashi metric and the Caratheodory metric. All of these metrics are so-called Einstein metrics, i.e., their Ricci curvature is a negative constant multiple of their Kähler form. ◇

8.4.3 EXAMPLE. Let X be a complex manifold and let $f : X \to \mathbb{C}^n$ be a holomorphic map such that $||f(x)|| < 1$ for all $x \in X$. Then the function

$$\eta := -\log(1 - ||f||^2)$$

has self-bounded gradient, since it is the pullback by f of the function $-\log(1 - |z|^2)$ of the previous example. ◇

8.4.4 EXAMPLE. If $X = \{\zeta \in \mathbb{C} \; ; \; 0 < |\zeta| < 1\}$ is the punctured unit disk and

$$\eta := \log(\log|\zeta|^{-2})$$

then

$$-e^{-\eta} = \frac{1}{\log|\zeta|^2}$$

satisfies

$$\partial\bar{\partial}(-e^{-\eta}) = -\partial\left(\frac{d\bar{\zeta}}{\bar{\zeta}\log|\zeta|^2}\right) = \frac{d\zeta \wedge d\bar{\zeta}}{|\zeta|^2(\log|\zeta|^2)^2},$$

and hence η has self-bounded gradient. Note again that

$$\sqrt{-1}\partial\bar{\partial}\eta = \sqrt{-1}\partial\left(\frac{d\bar{\zeta}}{-\bar{\zeta}\log|\zeta|^2}\right) = \frac{\sqrt{-1}d\zeta \wedge d\bar{\zeta}}{|\zeta|^2(\log|\zeta|^2)^2}$$

is the Kähler form of the unique (up to a constant factor) metric for X whose Gaussian curvature is a negative constant. ◇

8.4.5 EXAMPLE. Let X be a complex manifold and assume there exists a non-constant holomorphic function $T \in \mathcal{O}(X)$ such that $\sup_X |T| \le 1$. Let

$$Z := \{x \in X \; ; \; T(x) = 0\}$$

by the analytic hypersurface defined by T. Then the function

$$\eta := \log\log\frac{1}{|T|^2}$$

has self-bounded gradient on the manifold $Y := X - Z$. Indeed, $T : Y \to \{0 < |\zeta| < 1\}$ is a holomorphic map and the function η is just the pullback of the function $\zeta \mapsto \log\log\frac{1}{|\zeta|^2}$ of the previous example. ◇

8.4.6 EXAMPLE. A complex manifold is said to be *hyperconvex* if it has a negative plurisubharmonic exhaustion function. (Usually one asks for the exhaustion function to be continuous as well, but this is not a fixed convention.) For example, every strictly pseudoconvex domain has a negative, strictly plurisubharmonic exhaustion function.

Letting ψ be a negative plurisubharmonic function, which such domains automatically support, one sets

$$\eta := -\log(-\psi).$$

Clearly $-e^{-\eta} = \psi$ is plurisubharmonic, and therefore η has self-bounded gradient. $\diamond$

8.4.7 EXAMPLE. Let X be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$. One can define the Hilbert space

$$\mathscr{H}_X := \left\{ f \in \Gamma_{\mathcal{O}}(X, K_X) \; ; \; \int_X |f|^2 < +\infty \right\},$$

where $|f|^2 = \sqrt{-1}^{n^2/2} f \wedge \bar{f}$. For any orthonormal (Riesz) basis $f_1, f_2, ...$ of $\mathscr{H}_X$ one can define the so-called Bergman function

$$B_X(z) := \sum_{j=1}^{\infty} \frac{|f_j(z)|^2}{dV(z)},$$

where $dV(z)$ is Lebesgue measure. The Bergman function is independent of the choice of orthonormal basis for $\mathscr{H}_X$.

Suppose $\psi : \mathbb{C}^n \to \mathbb{R}$ is a smooth function such that

(i) $X = \psi^{-1}(-\infty, 0)$,

(ii) $|d\psi| \equiv 0$ on ∂X and

(iii) ψ is strictly plurisubharmonic on a neighborhood of the closure of X.

(A function satisfying (i) is called a *defining function* for X, and if (ii) also holds then it is called a *Levi defining function*.) A famous result of C. Fefferman [Fe-1974] states that the Bergman kernel has an asymptotic expression

$$B_X(z) = F(z)(-\psi(z))^{-(n+1)} + G(z)\log(-\psi(z)), \qquad z \in X,$$

where F and G are smooth functions that do not vanish in a neighborhood of the boundary of X.

In particular, the function

$$\eta(z) := \frac{1}{n+1} \log B_X(z) \tag{8.3}$$

is strictly plurisubharmonic. Moreover,

$$\eta(z) = -\log(-\psi(z)) + \log\left(F(z) + G(z)\psi(z)^{n+1}\log(-\psi(z))\right),$$

and therefore $\eta + \log(-\psi)$ is a smooth function in a neighborhood of ∂X. Since the expression for B_X is only relevant near ∂X, one can choose the functions F and G so that $\phi := \eta + \log(-\psi)$ is smooth on a neighborhood of the closure of X.

Now, the function $\eta_o := -\log(-\psi)$ satisfies

$$\partial\bar{\partial}\eta_o - \partial\eta_o \wedge \bar{\partial}\eta_o = \frac{\partial\bar{\partial}\psi}{-\psi} \quad \text{and} \quad \partial\eta_o \wedge \bar{\partial}\eta_o = \frac{\partial\psi \wedge \bar{\partial}\psi}{(-\psi)^2},$$

and therefore $\eta = \eta_o + \phi$ satisfies

$$\begin{aligned}
\partial\bar{\partial}\eta - \partial\eta \wedge \bar{\partial}\eta &= (-\psi)^{-1}\partial\bar{\partial}\psi - 2\mathrm{Re}\,\partial\eta_o \wedge \bar{\partial}\phi + \partial\bar{\partial}\phi - \partial\phi \wedge \bar{\partial}\phi \\
&\geq (-\psi)^{-1}\partial\bar{\partial}\psi - (-\psi)\partial\eta_o \wedge \bar{\partial}\eta_o - (-\psi)^{-1}\partial\phi \wedge \bar{\partial}\phi + \partial\bar{\partial}\phi - \partial\phi \wedge \bar{\partial}\phi \\
&= (-\psi)^{-1}(\partial\bar{\partial}\psi - \partial\psi \wedge \bar{\partial}\psi - \partial\phi \wedge \bar{\partial}\phi) + \partial\bar{\partial}\phi - \partial\phi \wedge \bar{\partial}\phi \\
&\geq \frac{-C}{(-\psi)}\partial\bar{\partial}|z|^2
\end{aligned}$$

for some sufficiently large constant $C > 0$.

Although the function (8.3) does not have self bounded gradient in general (though it can, sometimes), the estimate $\partial\bar{\partial}\eta - \partial\eta \wedge \bar{\partial}\eta \geq \frac{-C}{(-\psi)}\partial\bar{\partial}|z|^2$ can still be very useful in obtaining L^2 estimates for $\bar{\partial}$. ◇

EXERCISES

Lecture 9

Extension Theorems

The problem of interpolation of data is fundamental in many areas of mathematics and the hard sciences. In our setting, one might like to interpolate holomorphic functions, or perhaps sections of a holomorphic vector bundle, that are specified on a complex submanifold.

In this lecture we will discuss the interpolation problem from a complex submanifold (in fact, often a hypersurface) in a complex manifold. We will restrict our attention to the case of smooth hypersurfaces, though standard techniques can be used to extend all of our results so as to yield extension from singular hypersurfaces.

Additionally, since we will be very interested in using singular metrics, we will confine ourselves to the case of extension problems for holomorphic line bundles. When the metrics are smooth, many of the techniques can be extended to the higher rank case.

9.1 Extension without estimates

Let X be a complex manifold, let $S \subset X$ be a complex and let $L \to X$ be a holomorphic line bundle. Given a holomorphic section $f \in \Gamma_{\mathcal{O}}(S, L|_S)$, one wants to know if there is a holomorphic extension of f to X, i.e., if there is a holomorphic section $F \in \Gamma_{\mathcal{O}}(X, L)$ such that

$$F|_S = f.$$

In general, no such extension need exist.

9.1.1 EXAMPLE. Let $X = \mathbb{P}_1$, let $L \to X$ be the trivial bundle and let $S = \{[1,0],[0,1]\}$. Then the function $f : S \to \mathbb{C}$ defined by $f([1,0]) = 0$ and $f([0,1]) = 1$ is holomorphic on S (in fact, since S is discrete any function on S is holomorphic) but there is no holomorphic extension of f to X. Indeed, since X is compact, every holomorphic function on X is constant. ⋄

9.1.1 Strongly pseudoconvex manifolds

On the other hand, if S is a closed submanifold of $\mathbb{C}^n$ and $f \in \mathcal{O}(X)$ then there is *always* an extension of f to $\mathbb{C}^n$, i.e., a function $F \in \mathcal{O}(\mathbb{C}^n)$ such that $F|_S = f$.

More generally, a manifold X is said to be *strongly pseudoconvex* if there is a smooth, strictly plurisubharmonic exhaustion function, i.e., a function $\rho : X \to \mathbb{R}$ such that

$$\sqrt{-1}\partial\bar{\partial}\rho > 0 \quad \text{and} \quad \Omega_c := \{x \in X \ ; \ \rho(x) < c\} \subset\subset X$$

for all $c \in \mathbb{R}$. (The manifold $\mathbb{C}^n$ is strongly pseudoconvex, as one can see by taking $\rho(z) = |z|^2$.) Then one has the following theorem.

9.1.2 THEOREM. *Let X be a strongly pseudoconvex manifold and let $S \subset X$ be a closed submanifold. Then for all $f \in \mathcal{O}(S)$ there exists $F \in \mathcal{O}(X)$ such that $F|_S = f$.*

Proof. We fix once and for all a strictly plurisubharmonic exhaustion function ρ on X, and we let $\omega := \sqrt{-1}\partial\bar{\partial}\rho$ be our Kähler form for X.

Let k be the dimension of S. Choose a collection $\{U_j \ ; \ j = 1, 2, ...\}$ of open sets $U_j \subset\subset X$, and local coordinates $z_j = (x_j, y_j)$ such that

(i) each point of X is contained in finitely many U_j,

(ii) the map $(x_j, y_j) : U_j \to \mathbb{D}^n$ is a holomorphic diffeomorphism,

(iii) $S \cap U_j = \{y_j^1 = ... = y_j^{n-k} = 0\}$ and

(iv) $S \subset \bigcup_{j=1}^{\infty} U_j$.

(Thus $x_j = (x_j^1, ..., x_j^k)$ are local coordinates on $S \cap U_j$, and $n = \dim_{\mathbb{C}} X$.) Let $U_0 = X - S$.

Define functions $F_j \in \mathcal{O}(U_j)$, $j \in \mathbb{N}$, by

$$F_0(x) = 0, \quad F_j(x_j, y_j) := f(x_j), \ j = 1, 2, ... \quad \text{and} \quad g_{ij} := F_i - F_j \in \mathcal{O}(U_i \cap U_j).$$

Then $g_{ij}|_{S \cap U_i \cap U_j} \equiv 0$. We seek functions $g_i \in \mathcal{O}(U_i)$ such that

$$g_i|_{S \cap U_i} \equiv 0 \quad \text{and} \quad g_i - g_j = g_{ij} \text{ on } U_i \cap U_j. \tag{9.1}$$

If such g_i are found then the function $F \in \mathcal{O}(X)$ given by

$$F := F_i - g_i \text{ on } U_i$$

is well-defined, since on $U_i \cap U_j$ one has $F_i - g_i = g_{ij} + F_j - (g_{ij} + g_j) = F_j - g_j$. Moreover for $x \in S$, if $x \in U_j$ then $F(x) = F_j(x) - g_j(x) = f(x) - 0 = f(x)$, and hence we have our extension.

Let $\{\chi_j\}$ be a partition of unity subordinate to $\{U_j\}$. Define

$$\tilde{g}_i := \sum_m \chi_m g_{ik}.$$

Then $\tilde{g}_i|_{S\cap U_i} \equiv 0$ and

$$\tilde{g}_i - \tilde{g}_j = \sum_m \chi_k(g_{im} - g_{jm}) = \sum_m \chi_m g_{ij} = g_{ij} \in \mathcal{O}(U_i \cap U_j).$$

Of course, $\tilde{g}_i$ are not holomorphic, but since their differences are holomorphic, the $(0,1)$-form α defined by

$$\alpha := \bar{\partial}\tilde{g}_j \quad \text{on } U_j$$

is globally defined on X. We seek a solution $u \in \mathscr{C}^\infty(X)$ of the equation $\bar{\partial}u - \alpha$ such that $u|_S \equiv 0$. If we find such a function u then the functions $g_i := \tilde{g}_i - u$ satisfy (9.1) and the proof is finished.

To obtain our function u we shall use Hörmander's Theorem with singular metrics. The idea is to construct a weight function ψ such that $e^{-\psi}$ is not locally integrable at any point of S. If u has finite L^2 norm with respect to such a weight then u must vanish along S. Of course, the form α must also be L^2 with respect to the weight, so we cannot make this weight *too singular* along S; it has to be just right.

Next we claim that

$$\int_{U_j} \frac{|\alpha|^2_\omega}{|y_j|^{2(n-k)}} dV_\omega < +\infty.$$

Indeed, since for each m the function g_{jm} vanishes along $U_j \cap U_m$, there exist holomorphic functions $f_{mj,\ell}$ such that

$$g_{jm} = \sum_{\ell=1}^{n-k} f_{jm,\ell} y_j^\ell.$$

Thus

$$\tilde{g}_j = \sum_m \sum_{\ell=1}^{n-k} \chi_m f_{jm,\ell} y_j^\ell,$$

and

$$\bar{\partial}\alpha = \sum_m \sum_{\ell=1}^{n-k} (\bar{\partial}\chi_m) f_{jm,\ell} y_j^\ell.$$

Thus by Cauchy-Schwarz

$$\int_{U_j} \frac{|\alpha|^2_\omega}{|y_j|^{2(n-k)}} dV_\omega \le \sum_m \int_{U_j} \left(\sum_{\ell=1}^{n-k} |f_{jm,\ell}|^2\right) \frac{|\bar{\partial}\chi_m|^2_\omega}{|y_j|^{2(n-k-1)}} dV_\omega < +\infty, \tag{9.2}$$

where the finiteness follows because $|y|^{2-2d}$ is integrable in any relatively compact neighborhood of 0 in $\mathbb{C}^d$.

Now consider the weight

$$\psi := h \circ \rho + \sum_j \chi_j \log |y_j|^{2(n-k)},$$

where h is a smooth, convex, sufficiently rapidly increasing function. Then

$$\partial\bar{\partial}h\circ\rho = (h'\circ\rho)\partial\bar{\partial}\rho + (h''\circ\rho)\partial\rho\wedge\bar{\partial}\rho \geq (h'\circ\rho)\omega,$$

and

$$\begin{aligned}\partial\bar{\partial}\left(\sum_j \chi_j \log|y_j|^{2(n-k)}\right) &= \sum_j \chi_j \partial\bar{\partial}\log|y_j|^{2(n-k)} \\ &+ \sum_j \left((\partial\bar{\partial}\chi_j)\log|y_j|^{2(n-k)} + (n-k)|y_j|^{-2}\left(\partial\chi_j\wedge(y_j\cdot d\bar{y}_j) + (y_j\cdot dy_j)\wedge\bar{\partial}\chi_j\right)\right) \\ &\geq \sum_j \left((\partial\bar{\partial}\chi_j)\log|y_j|^{2(n-k)} + (n-k)|y_j|^{-2}\left(\partial\chi_j\wedge(y_j\cdot d\bar{y}_j) + (y_j\cdot dy_j)\wedge\bar{\partial}\chi_j\right)\right).\end{aligned}$$

We claim that the right hand side is locally bounded. To see this booundedness, fix $j \geq 1$, and denote by $k_1, ..., k_N$ the set of k such that $U_j \cap U_k \neq \emptyset$. Because the local coordinates y_j cut out S, there exist smooth functions $h^{\mu}_{j\ell,\nu}$ such that

$$y^{\nu}_{k_\ell} = \sum_{\mu=1}^{n-k} \exp(h^{\nu}_{j\ell,\mu}) y^{\mu}_j, \quad 1 \leq \nu \leq n-k.$$

Then on U_j we have

$$\begin{aligned}\sum_{\ell=1}^{N}(\partial\bar{\partial}\chi_{k_\ell})\log|y_{k_\ell}|^{2(n-k)} &= \sum_{\ell=1}^{N}(\partial\bar{\partial}\chi_{k_\ell})\log|y_j|^{2(n-k)} + \text{a smooth function} \\ &= \left(\partial\bar{\partial}\sum_{\ell=1}^{N}\chi_{k_\ell}\right)\log|y_j|^{2(n-k)} + \text{a smooth function} \\ &= \left(\partial\bar{\partial}(1)\right)\log|y_j|^{2(n-k)} + \text{a smooth function} \\ &= \text{a smooth function.}\end{aligned}$$

The proof of boundedness of the other terms is similar, and is left to the reader. Additionally, the above argument shows that

$$\psi = \log|y_j|^{2(n-k)} + \text{a smooth function}$$

in a neighborhood of $S \cap U_j$. Therefore

(a) $e^{-\psi}$ is not locally integrable near any point of S, and

(b) by (9.2)

$$\int_{U_j} |\alpha|^2_\omega e^{-\psi} dV_\omega < +\infty.$$

Now choose h increasing so rapidly– here it is crucial that ρ is an exhaustion– that

$$\sqrt{-1}\partial\bar{\partial}\psi + \mathrm{Ricci}(\omega) \geq \omega \quad \text{and} \quad \int_X |\alpha|^2_\omega e^{-\psi} dV_\omega < +\infty.$$

By Hörmander's Theorem there is a function $u \in \mathscr{C}^\infty(X)$ such that $\bar{\partial}U = \alpha$ and

$$\int_X |u|^2 e^{-\psi} dV_\omega < +\infty.$$

Because u is smooth and $e^{-\psi}$ is not locally integrable at any point of S, u must vanish identically along S. The proof is therefore complete. □

Note that if the function h in the proof of Theorem 9.1.2 is chosen correctly then the function f to be extended satisfies the estimate

$$\int_S |f|^2 e^{-\psi} dV_\omega < +\infty, \tag{9.3}$$

and therefore we have shown the following: for every $f \in \mathcal{O}(S)$ satisfying (9.3) there exists $F \in \mathcal{O}(X)$ such that

$$F|_S = f \quad \text{and} \quad \int_X |F|^2 e^{-\psi} dV_\omega < +\infty.$$

An application of the Closed Graph Theorem shows that in fact there exists a constant $C > 0$, independent of f, such that

$$\int_X |F|^2 e^{-\psi} dV_\omega \leq C \int_S |f|^2 e^{-\psi} dV_\omega.$$

However C *does* depend on S, X, ω and ψ, and hence this extension theorem is entirely too specialized.

As we will see in Section 10.2, it is extremely useful to have a result that relies much less on the data. In the next section we state and prove such a result.

9.1.2 Stein manifolds

Stein manifolds are fundamental objects in complex analysis and geometry. There are several equivalent definitions, which we shall now review.

9.1.3 DEFINITION. *A complex manifold X is said to be Stein if the following conditions hold.*

(HC) *The manifold X is holomorphically convex, i.e., for any closed discrete subset $S \subset X$ there exists a holomorphic function $f \in \mathcal{O}(X)$ such that*

$$\sup_S \lim |f| = +\infty.$$

(SP) *The algebra $\mathcal{O}(X)$ separates points, i.e., if $x, y \in X$ are distinct points then there exists $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$.*

(ST) *The algebra $\mathcal{O}(X)$ separates tangents, i.e., for any $x \in X$ and $\alpha \in T^{*1,0}_{X,x}$ there exists $f \in \mathcal{O}(X)$ such that $df(x) = \alpha$.*

Note that any closed submanifold of $\mathbb{C}^n$ has these three properties, and in fact one can realize each property with affine-linear functions. Thus closed submanifolds of $\mathbb{C}^n$ are Stein. In fact, a celebrated theorem of R. Narasimhan concludes the converse.

9.1.4 THEOREM (R. Narasimhan). *Every Stein manifold can be embedded as a closed submanifold of a finite-dimensional complex vector space.*

Narasimhan's theorem is rather deep and technical, and unfortunately lies outside the range of our goals. Perhaps more relevant to our presentation is the following theorem of Grauert.

9.1.5 THEOREM (Grauert). *A complex manifold is Stein if and only if it is strongly pseudoconvex.*

To prove that strongly pseudoconvex manifolds are Stein, one constructs holomorphic functions that realize the three properties of Stein manifolds. Properties (HC) and (PS) follow immediately from Theorem 9.1.2 applied to the 0-dimensional submanifolds $S = \{p_j\}$ and $S = \{x, y\} \subset X$ respectively. Property (TC) follows from a slight modification of the proof of Theorem 9.1.2 for the 0-dimensional manifold $S = \{x\} \subset X$; we leave it to the exercises.

The other direction of Grauert's Theorem is a consequence of the following stronger, often very useful result (though we will not make use of it in these notes).

9.1.6 THEOREM. *Let X be a Stein manifold, K a compact subset of X, and U a neighborhood of $\widehat{K}_{\mathcal{O}(X)}$. Then there is a strictly plurisubharmonic exhaustion $u \in \mathscr{C}^\infty(X)$ such that $u < 0$ on K but $u > 0$ on $X - U$. In particular, X is strongly pseudoconvex.*

Proof. Since X is holomorphically convex, we can find compact sets $K_1 = K, K_2, ...$, such that for all $i \geq 1$

$$K_i = \widehat{(K_i)}_{\mathcal{O}(X)}, \quad K_i \subset \text{interior}(K_{i+1}) \quad \text{and} \quad \bigcup_{i \geq 1} K_i = X.$$

Next, choose open sets U_i such that $K_i \subset U_i \subset K_{i+1}$, and in addition $U_1 \subset U$.

For each $i \geq 1$, choose functions $f_{ik} \in \mathcal{O}(X)$, $k = 1, ..., k_i$ such that

$$\sup_{K_i} |f_{ik}| < 1 \quad \text{and} \quad \max_k |f_{ik}(z)| > 1$$

for all $z \in K_{i+2} - U_i$. By taking large powers of the f_{ik} if necessary, we can assume that

$$\sup_{K_i} \sum_{k=1}^{k_i} |f_{ik}|^2 < 2^{-i} \quad \text{and} \quad \inf_{K_{i+2} - U_i} \sum_{k=1}^{k_i} |f_{ik}|^2 > i.$$

Moreover, by property (3) of Stein manifolds, we can also make sure that at each point of K_i n of the functions f_{ik}, $1 \leq k \leq k_i$, form a local coordinate system.

Now set

$$u := -1 + \sum_{i=1}^{\infty} \sum_{k=1}^{k_i} |f_{ik}|^2.$$

The sum converges uniformly on each K_i, but moreover u is smooth. (The simplest way to see the smoothness of u is to consider the function

$$F(z, \bar{\zeta}) := -1 + \sum_{i=1}^{\infty} \sum_{k=1}^{k_i} f_{ik}(z)\overline{f_{ik}(\zeta)}$$

on the complex manifold $X \times X^{\dagger}$, where $X^{\dagger}$ is the complex manifold with the complex conjugate structure of X. For the same reasons as above, F converges locally uniformly on $X \times X^{\dagger}$ and is therefore holomorphic. Restricting to the real submanifold $z = \zeta$ establishes the smoothness (and even real-analyticity) of u.) Next, $u > i - 1$ on $X - U_i$. Finally,

$$\sqrt{-1}\partial\bar{\partial}u(\xi, \xi) = \sum_{i=1}^{\infty} \sum_{k=1}^{k_i} |\partial f_{ik}(\xi)|^2.$$

Since at each point of X at least one of these terms is strictly positive for non-zero ξ, we see that u is strictly plurisubharmonic. The proof is complete. □

9.1.3 Compact complex manifolds with positively curved line bundles

In the previous section we studied complex manifolds that admit a strictly plurisubharmonic exhaustion function. Such manifolds are never compact, since on a compact complex manifold there are no non-constant plurisubharmonic functions. There are, however, such functions locally, and therefore it makes sense to ask if there is a holomorphic line bundle with a smooth Hermitian metric whose curvature is strictly positive. Such line bundles are often called *positive*.

As we will see in the next paragraph, not every compact complex manifold admits a positively curved Hermitian holomorphic line bundle. In the present paragraph we will prove that line bundles with smooth Hermitian metrics of sufficiently positive curvature have many holomorphic sections. The precise statement is as follows.

9.1.7 THEOREM. *Let X be a compact complex manifold and let $H \to X$ be a holomorphic line bundle with smooth Hermitian metric $e^{-\varphi}$ whose curvature $\partial\bar{\partial}\varphi$ is positive, i.e., such that $\sqrt{-1}\partial\bar{\partial}\varphi$ is a Kähler form. Then for every holomorphic line bundle $L \to X$ finite subset $\{x_1, ..., x_N\} \subset X$ and positive integers $d_1, ..., d_N$ there exists an integer*

$$m_o = m_o(X, H, L, x_1, ..., x_N, d_1, ..., d_N)$$

such that for any polynomials $p_1, ..., p_N \in \mathscr{O}(\mathbb{C}^n)$ of degrees $\deg(p_i) = d_i$, any local coordinates z_i near x_i, $1 \le 1 \le N$, and holomorphic frames ξ_i for H and η_i for L near x_i, $1 \le i \le N$, and any integer $m \ge m_o$ there exists a section $s \in \Gamma_{\mathscr{O}}(X, L^{\otimes m})$ such that

$$s(z_i) = \left(p_i(z_i) + O\left(|z_i|^{d_i+1}\right)\right) \xi_i^{\otimes m} \otimes \eta_i, \quad 1 \le i \le N.$$

In other words for a sufficiently positive line bundle one can specify for a holomorphic section the Taylor polynomials of any given degree at any finite set of points.

Proof. Choose positive numbers $\varepsilon_1, ..., \varepsilon_N$ such that the local coordinate patches $|z_i| < 3\varepsilon_i$ are pairwise disjoint, and fix a function $\chi \in \mathscr{C}^\infty_o([0,\infty))$ such that $\chi|_{[0,1]} \equiv 1$ and $\mathrm{Support}(\chi) \subset [0,2]$. Consider the smooth section $\tilde{s}$ of $H^{\otimes m} \otimes L \to X$ defined by

$$\tilde{s}(x) := \chi(|z_i|^2/\varepsilon_i^2)p(z_i)\xi_i^{\otimes m} \otimes \eta_i \text{ on } U_i := \{|z_i| < 2\varepsilon_i\}, \quad \text{i = 1,...,N},$$

and $\tilde{s} = 0$ away from the coordinate neighborhoods U_i. Note that $\tilde{s}$ is holomorphic on the open sets $V_i := \{|z_i| < \varepsilon_i\}$. and therefore the $H^{\otimes m} \otimes L$-valued $(0,1)$-form $\alpha := \bar{\partial}\tilde{s}$ is smooth and supported on the union $A_1 \cup ... \cup A_N$ of the annuli

$$A_i := \overline{U_i} - V_i, \quad 1 \le i \le N.$$

Fix the Kähler form $\omega := \sqrt{-1}\partial\bar{\partial}\varphi$ for X and any smooth metric $e^{-\psi}$ for $L \to X$, and define the function

$$\rho := \sum_{i=1}^{N} \chi(|z_i|/\varepsilon_i) \log |z_i|^{2(d_i+n)}.$$

Evidently ρ is smooth away from the points x_i and $\rho|_{U_i} = (d_i + n)\log|z_i|^2$ is plurisubharmonic. It follows that

$$\int_X |\alpha|^2_\omega e^{-(m\varphi+\psi+\rho)} dV_\omega < +\infty$$

and that there is a constant $C > 0$ such that

$$\sqrt{-1}\partial\bar{\partial}\rho \ge -C\omega.$$

By taking m_o so large that

$$m_o\sqrt{-1}\partial\bar{\partial}\varphi + \partial\bar{\partial}\psi + \partial\bar{\partial}\rho - \sqrt{-1}\partial\bar{\partial}\log\det\omega \ge \omega,$$

we see that for any $m \ge m_o$ the curvature of the metric metric $e^{-m\varphi+\psi+\rho}dV_\omega$ for the line bundle $H^{\otimes m} \otimes L \otimes K_X^*$ is more than ω. By Hörmander's Theorem there exists a section u of $H^{\otimes m} \otimes L$ such that

$$\bar{\partial}u = \alpha \quad \text{and} \quad \int_X |u|^2 e^{-(m\varphi+\psi+\rho)} dV_\omega \le \int_X |\alpha|^2_\omega e^{-(m\varphi+\psi+\rho)} dV_\omega.$$

Any section u satisfying $\bar{\partial}u = \alpha$ is necessarily smooth. Since $e^{-\rho} \sim |z_i|^{-2(d_i+n)}$ on U_i, it follows that u vanisheds to order $d_i + 1$ at x_i. Consequently the section

$$s := \tilde{s} - u$$

is holomorphic and has the desired properties. □

9.1.4 The Kodaira Embedding Theorem

In this paragraph we establish the following fundamental theorem of algebraic geometry.

9.1.8 THEOREM (Kodaira Embedding Theorem). *A compact complex manifold X is projective if and only if there is a positively curved holomorphic line bundle on X.*

Theorem 9.1.7 is the key ingredient in the proof of Theorem 9.1.8, as we shall soon see.

The easy direction: necessity of the existence of a positively curved line bundle

One direction of Theorem 9.1.8 is rather straight-forward. If a complex manifold M has a positively curved Hermitian holomorphic line bundle then any submanifold S of M also has such a line bundle, namely the restriction to S of the line bundle on M. Thus to prove the easy direction of Theorem 9.1.8 we need only show that $\mathbb{P}_n$ has a positively curved Hermitian holomorphic line bundle. We have already met the line bundle in question: this is the hyperplane line bundle $\mathbb{H} \to \mathbb{P}_n$ of Example 1.3.9, whose sections are in $1-1$ correspondence with linear functions on $\mathbb{C}^{n+1}$ (c.f. Exercise 1.3.5). Let us fix a basis $z^0, ..., z^n \in (\mathbb{C}^{n+1})^\vee$. Evidently at each point of $\mathbb{P}_n$ at least one of these sections does not vanish. Using this basis we define the so-called *Fubini-Study metric* $e^{-\varphi_{FS}}$ for $\mathbb{H} \to \mathbb{P}_n$ as follows. Let $\ell \in \mathbb{P}_n$ be a 1-dimensional subspace of $\mathbb{C}^{n+1}$. Any $\lambda \in \mathbb{H}_\ell$ is a linear functional on ℓ, and therefore we define

$$|\lambda|^2 e^{-\varphi_{FS}(\ell)} = \frac{|\lambda(v)|^2}{\sum_{j=1}^n |z^i(v)|^2} \quad \text{for any } v \in \ell - \{0\}.$$

Note that the right hand side is independent of the choice of v in $\ell - \{0\}$.

Let us compute the curvature of $e^{-\varphi_{FS}}$. Without loss of generality we may work in the open set U_o and with coordinates $\zeta = (\zeta^1, ..., \zeta^n)$, where $\zeta^i = z^i/z^0$. In this open set the section z^o vanishes nowhere, and therefore trivializes $\mathbb{H}$. Evidently

$$|z^0|^2 e^{-\varphi_{FS}} = \frac{1}{1+|\zeta|^2},$$

and therefore the curvature is

$$\sqrt{-1}\partial\bar{\partial}\log(1+|\zeta|^2),$$

which is a strictly positive $(1,1)$-form.

Maps into projective spaces

Before understanding the structure of all maps to projective spaces, it is helpful to construct some examples of such maps. The following set of examples, though rather abstract, turns out to be fundamental.

9.1.9 EXAMPLE. Let X be a complex manifold and let $H \to X$ be a holomorphic line bundle. Fix a finite-dimensional subspace $W \subset \Gamma_{\mathcal{O}}(X, H)$ of holomorphic sections of $H \to X$. For each $x \in X$ one can define

$$\phi_W(x) := \{s \in W \; ; \; s(x) = 0\} \subseteq W.$$

Of course, it can happen that every section of W vanishes at x, i.e., $\phi_{|W|}(x) = W$, but if for a given $x \in X$ one has a section $s \in W$ such that $s(x) \neq 0$ then $\phi_{|W|}(x)$ is a hyperplane in W.

9.1.10 DEFINITION. *The set*

$$\mathrm{Bs}(W) := \{x \in X \; ; \; s(x) = 0 \textit{ for all } s \in W\}$$

is called the base locus of W. *One says* W *is basepoint-free if* $\mathrm{Bs}(W) = \emptyset$.

By using the canonical identification of a hyperplane $H \subset W$ with the line in $W^\vee$ consisting of all linear functionals that vanish on H one has a map

$$\phi_W : X - \mathrm{Bs}(W) \to \mathbb{P}(W^\vee).$$

Thus we have a large collection of maps into projective space. In words, this map sends a point x to the line in $\mathbb{P}(W^\vee)$ determined by the kernel of linear function $\mathscr{E}_x^W : W \to H_x$ of evaluation of sections at the point x. ◇

Remarkably, every holomorphic map into a projective space is almost (but not quite) of the form described in Example 9.1.9. The precise result is as follows.

9.1.11 PROPOSITION. *Let* $F : X \to \mathbb{P}(V)$ *be a holomorphic map. Then there exist*

(a) a holomorphic line bundle $H_F \to X$,

(b) a finite-dimensional subspace $W_F \subset \Gamma_{\mathscr{O}}(X, H_F)$*, and*

(c) an injective linear map $P_F : W_F^\vee \to V$

such that

$$F = \wp_F \circ \phi_{W_F},$$

where $\wp_F : \mathbb{P}(W_F^\vee) \ni [w] \mapsto [P_F w] \in \mathbb{P}(V)$ *is the injective map induced by* P_F.

In other words, up to inclusion by projective subspaces, every holomorphic map to a projective space is of the form given in Example 9.1.9.

Proof of Proposition 9.1.11. Let $F : X \to \mathbb{P}(V)$ be given. The image $F(X)$ is contained in a minimal projective subspace $\mathbb{P}(V_o) \subset \mathbb{P}(V)$; this projective subspace corresponds to the smallest linear subspace $V_o \subset V$ all 1-dimensional subspaces $\ell \subset V$ such that $\ell \in F(X)$ (when ℓ is viewed as a point of $\mathbb{P}(V)$. Let us write

$$F_o : X \to \mathbb{P}(V_o)$$

for the map obtained after the range of F is changed from $\mathbb{P}(V)$ to $\mathbb{P}(V_o)$, i.e., $F = \iota_o \circ F_o$, where $\iota_o : \mathbb{P}(V_o) \hookrightarrow \mathbb{P}(V)$ is the inclusion.

The subspace V_o is the intersection of all the hyperplanes in V that are the zero loci of sections $s \in V^\vee = \Gamma_{\mathscr{O}}(\mathbb{P}(V), \mathbb{H})$ (c.f. Exercise 1.3.5) vanishing identically on $F(X)$. The set of such sections s is a subspace S_F of $V^\vee$, and the quotient space

$$V^\vee / S_F$$

is naturally isomorphic to the dual space $V_o^\vee = \Gamma_{\mathcal{O}}(\mathbb{P}(V_o), \mathbb{H})$. Moreover, every section $s_o \in V_o^\vee$ that vanishes on $F(X) \subset \mathbb{P}(V_o)$ vanishes identically. It follows that the map

$$F_o^* : \Gamma_{\mathcal{O}}(\mathbb{P}(V_o), \mathbb{H}) \to \Gamma_{\mathcal{O}}(X, F^*\mathbb{H})$$

is injective[1].

We let

$$L_F := F_o^*\mathbb{H}, \quad W_F := F_o^*(\Gamma_{\mathcal{O}}(\mathbb{P}(V_o), \mathbb{H})) \quad \text{and} \quad P_F := F_o^* \circ \pi_{S_F},$$

where $\pi_{S_F} : V^\vee \to V^\vee / S_F \cong V_o^\vee$ is the quotient map. With these objects, all of the claimed properties are established. □

9.1.12 REMARK. For the reader that prefers n-tuples to vectors and matrices to linear functionals, the map F has a useful description. Suppose $F(X)$ lies in a proper projective subspace $\mathbb{P}(V_o) \subset \mathbb{P}(V)$. Choosing a basis $z^0, ..., z^n$ for $V^\vee$ such that $z^0, ..., z^k$ is a basis for W, we get projective coordinates $[z^0, ..., z^n]$ for $\mathbb{P}(V)$ and $[z^0, ..., z^k]$ for $\mathbb{P}(W^\vee)$. In terms of these coordinates we may write F as

$$F = [F^0, ..., F^k, 0, ..., 0].$$

The map $G : \mathbb{P}(W^\vee) \to \mathbb{P}(V)$ is given by

$$G([z^0, ..., z^k]) = [z^0, ..., z^k, 0, ..., 0]$$

and the hyperplane in $W^\vee$ is corresponding to $\mathbb{C}F(x) \in V$ is, with respect to the basis $z^0, ..., z^n$, the set of all vectors $c = (c_o, ..., c_k) \in \mathbb{C}^{k+1} \cong W^\vee$ satisfying

$$c \cdot (F^0(x), ..., F^k(x)) = 0.$$

Note also that $F^i \in \Gamma_{\mathcal{O}}(X, F^*\mathbb{H})$. ◇

9.1.13 PROPOSITION. *Let $H \to X$ be a holomorphic line bundle and let $W \subset \Gamma_{\mathcal{O}}(X, H)$ be basepoint free.*

1. *The map $\phi_W : X \to \mathbb{P}(W^\vee)$ is injective if and only if for any pair of distinct points $x, y \in X$ there exists a section $s \in W$ such that $s(x) = 0$ and $s(y) \neq 0$.*

2. *The map $\phi_W : X \to \mathbb{P}(W^\vee)$ is an immersion if and only if for any $x \in X$, $\alpha \in T_X^{*1,0}$ and $v \in H_x$ there exists $s \in W$ such that $s(x) = 0$ and $ds(x) = \alpha \otimes v$.*

Proof. If $s(x) = 0 \neq s(y)$ then $\phi_W(x) \ni s \notin \phi_W(y)$, so ϕ_W separates x and y. Conversely if $\phi_W(x) = \phi_W(y)$ then every section that vanishes at y also vanishes at x. Thus 1 is proved. We leave 2 as an exercise. □

Proof of Theorem 9.1.8. Let $L \to X$ be a positively curved Hermitian holomorphic line bundle. By Proposition 9.1.13 it suffices to show that there exists a holomorphic line bundle $H \to X$ such that $W = \Gamma_{\mathcal{O}}(X, H)$ is basepoint-free, separates points (i.e. satisfies Item 1) and separates tangents (i.e. satisfies Item 2). But by Theorem 9.1.7 the line bundle $H = L^{\otimes m}$ has these properties as soon as m is large enough. □

[1]Recall that if $E \to Y$ is a holomorphic vector bundle and $G : X \to Y$ is a holomorphic map then the pullback $G^*E \to X$ of E by G is by definition the restriction of the Cartesian projection $X \times E \to X$ to the submanifold $G^*E := \{(x, v) \in X \times E\ ;\ v \in E_{G(x)}\}$ of $X \times E$.

9.2 L^2 Extension

Let us finally begin the journey to our universal extension theorem.

9.2.1 Adjunction

In order to prove our universal extension theorem, we need to have an appropriate normalization of the data. Part of this preparation involves the so-called *adjunction* construction, which we now explain.

Let X be a complex manifold and let $Z \subset X$ be smooth complex hypersurface, i.e., a complex submanifold of codimension 1. Then there exists an open cover $\{U_j\}$ of X, and holomorphic functions $T_j \in \mathcal{O}(U_j)$ such that

$$Z \cap U_j = \{T_j = 0\} \quad \text{and} \quad dT_j(x) \neq 0 \text{ for all } x \in U_j \cap Z.$$

Denote by $L_Z \to X$ the holomorphic line bundle associated to Z, i.e., the line bundle defined by the transition functions

$$g_{ij} = \frac{T_i}{T_j},$$

which have no zeros on $U_i \cap U_j$. Since $T_i = g_{ij} T_j$, the functions T_j define a section $T \in \Gamma_{\mathcal{O}}(X, L_Z)$, and evidently the zero locus of T is precisely Z, counting multiplicity. Note also that since

$$dT_i = g_{ij} dT_j + dg_{ij} T_j,$$

along Z the sections dT_j provide a well-defined holomorphic section dT of $\left(T^{*1,0}_X \otimes L_Z\right)|_Z$. Moreover, since the L_Z-valued holomorphic 1-forms dT_j annihilate the tangent spaces of Z, dT is a holomorphic section of the rank-1 subbundle

$$N^*_{X/Z} \otimes L_Z \to Z$$

of $\left(T^{*1,0}_X \otimes L_Z\right)|_Z$. The line bundle $N^*_{X/Z}$, whose fibers consist of $(1,0)$-forms that annihilate the $(1,0)$-tangent spaces of Z, is called the conormal bundle of Z in X.

Since Z is smooth the section dT is nowhere zero, and therefore the line bundle $N^*_{X/Z} \otimes L_Z$ is trivial. In particular, we see that for a smooth hypersurface Z the line bundle $L_Z \to X$ restricts to Z as the dual of the conormal bundle, i.e., the line bundle

$$N_{X/Z} := (H^*_{X/Z})^*.$$

This line bundle, which is naturally isomorphic to the quotient $(T^{1,0}_X|_Z)/T^{1,0}_Z$, is called the *normal bundle*. In other words we have proved the following proposition.

9.2.1 PROPOSITION (Adjunction Formula). *The holomorphic line bundle $L_Z \to X$ of a smooth hypersurface Z is an extension to X of the normal bundle of Z.*

By taking determinants, one can see that the Adjunction Formula is equivalent to the formula

$$K_Z = (K_X \otimes L_Z)|_Z,$$

and the identification sends an $(n-1,0)$-form f on Z to the L_Z-valued n form $f \wedge dT$, defined along Z.

9.2.2 Statement of the L^2 Extension Theorem

9.2.2 THEOREM. *Let X be a Stein manifold and let $Z \subset X$ be a smooth complex hypersurface. Assume there exists a section $T \in \Gamma_{\mathcal{O}}(X, L_Z)$ and a singular Hermitian metric $e^{-\lambda}$ for $L_Z \to X$ such that*

$$Z = \{T = 0\} \quad \textit{and} \quad \sup_X |T|^2 e^{-\lambda} \le 1. \tag{9.4}$$

Let g be a Kähler metric on X and let $L \to X$ be a holomorphic line bundle with singular Hermitian metric $e^{-\varphi}$, and assume there exists a constant $\delta \in (0,1]$ such that

$$\partial\bar{\partial}\varphi + \mathrm{Ricci}(g) \ge (1 + t\delta)\partial\bar{\partial}\lambda \quad \textit{for all } t \in [0,1]. \tag{9.5}$$

Then for each $f \in \Gamma_{\mathcal{O}}(Z, L)$ such that

$$\int_Z \frac{|f|^2 e^{-\varphi} dA_g}{|dT|_g^2 e^{-\lambda}} < +\infty$$

there exists $F \in \Gamma_{\mathcal{O}}(X, L)$ such that

$$F|_Z = f \quad \textit{and} \quad \int_X |F|^2 e^{-\varphi} dV_g \le \frac{24\pi}{\delta} \int_Z \frac{|f|^2 e^{-\varphi} dA_g}{|dT|_g^2 e^{-\lambda}},$$

where $dA_g = \frac{\omega^{n-1}}{(n-1)!}|_Z$ is the area form associated to the submanifold Z.

In fact, there are many other versions of the L^2 extension theorem, and all of them can be established by slight modifications of the proof of Theorem 9.2.2 that we shall give below. We give a brief discussion of some of these, and of the problems that arise in their consideration.

1. One version that can be established involves replacing the line bundle $L \to X$ with a vector bundle of higher rank. The trouble with such a version (and the reason we did not state it here) is that there is at present no manageable definition of a singular Hermitian metric for holomorphic vector bundles of higher rank. The difficulty is that there seems to be no such definition in which the curvature can be defined as a current; this problem makes it difficult to see how to handle condition (10.6) in the higher rank case.

2. In another version of Theorem 9.2.2 that is very useful, one relaxes the hypothesis that X is Stein. Here again the trouble is that on a general (even complete) Kähler manifold there is no way to approximate singular Hermitian metrics by smooth ones without losing too much positivity. There are, however, some Kähler manifolds where this approximation is in some sense possible.

   9.2.3 DEFINITION. *A complex manifold X is said to be* essentially Stein *if there is a (possibly singular) complex hypersurface V such that the manifold $X - V$ is Stein. A submanifold $Z \subset X$ is said to be an essentially Stein submanifold if the hypersurface V whose complement is Stein can be chosen so that $Z \not\subset V$.*

The most interesting examples of an essentially Stein manifold are projective manifolds, and so-called *projective families*, i.e., spaces $\mathscr{X}$ for which there are

(a) a proper holomorphic submersion $f : \mathscr{X} \to B$ onto some complex manifold B (i.e., $\mathscr{X} \to B$ is a holomorphic family), and

(b) a holomorphic line bundle $\mathscr{A} \to \mathscr{X}$ with a smooth Hermitian metric of positive curvature.

In particular, condition (a) implies that the fibers of f, all of which are complex submanifolds of $\mathscr{X}$, are pairwise diffeomorphic, and condition (b) implies, via the Kodaira Embedding Theorem, that each fiber is a projective manifold.

If one has L^2 estimates for sections of holomorphic line bundles (or for that matter, vector bundles) on a Stein complement of a hypersurface then those sections can be extended across the hypersurface. Indeed, a priori the sections could have only poles or removable singularities along the divisor, but the finiteness of the L^2 norm forbids poles.

3. Yet another version of Theorem 9.2.2 considers different L^2 norms for the sections of the line bundle $L \to Z$, or of the line bundle $L \to X$, or perhaps both. One version of such a result was worked out by Jeff McNeal and the author; the interested reader can go to [MV-2007] or [MV-2015] for more information.

4. One also wants to extend holomorphic sections from submanifolds of higher codimension. Extension theorems of this sort do exist, but they require specifying different sorts of data. Again, proofs are not so different from what is presented here, but the notion of positivity that is required in place of condition (10.6) becomes more complicated to state, and harder to understand geometrically. There are short cuts that can be taken which yield useful and interesting results– see, for example, [BL-2016]– but the most general case, which was first treated by Ohsawa in [O-2001], is still not in its final form, in the author's opinion.

9.2.3 Proof of Theorem 9.2.2

We fix once and for all the section $f \in \Gamma_{\mathscr{O}}(Z, L|_Z)$ to be extended, i.e., which satisfies

$$\int_Z \frac{|f|^2 e^{-\varphi} dA_g}{|dT|_g^2 e^{-\lambda}} < +\infty.$$

Since X is Stein, it is strongly pseudoconvex. Using the strictly plurisubharmonic exhaustion function, one finds domains

$$\Omega_1 \subset\subset \Omega_2 \subset\subset ... \subset\subset X \quad \text{such that } \bigcup_j \Omega_j = X \quad \text{and} \quad Z \pitchfork \partial\Omega_j \text{ for all } j.$$

We shall work on these domains first, and then then take limits of our results.

The domain Ω_j is trivially a strongly pseudoconvex manifold. By Exercise **??** there exists $F_o \in \Gamma_{\mathcal{O}}(X, L)$ such that

$$F_o|_Z = L.$$

Therefore

$$\int_{\Omega_j} |F_o|^2 e^{-\varphi} dV_g < +\infty,$$

but at present we have no estimate on this L^2 norm; in particular, one cannot let $j \to \infty$ and conclude, or even expect, that finiteness persists. After all, there are many extensions of f and most of them will not have finite L^2 norm.

We must therefore correct F_o somehow away from $Z \cap \Omega_j$. To do so, let $t \in (0,1)$ (we will eventually send t to 0) and let $\chi \in \mathscr{C}^\infty_o([0,\infty))$ be a positive function supported in $[0,1]$ and satisfying

$$0 \le \chi \le 1, \quad \chi \equiv 1 \text{on } [0,t] \quad \text{and} \quad |\chi'| \le 1 + ct,$$

where $c > 1$. (Eventually we will let $c \to 1$ and then $t \to 0$.) We define $\chi_\varepsilon := \chi(|T|^2 e^{-\lambda}/\varepsilon^2)$. Then for $\varepsilon > 0$ sufficiently small the section

$$\tilde{F}_\varepsilon := \chi_\varepsilon F_o$$

of $L \to \Omega_j$ is smooth, and holomorphic in a tubular neighborhood of $Z \cap \Omega_j$.

We wish to correct $\tilde{F}_\varepsilon$ to be a holomorphic extension of f on Ω_j that has better estimates than F_o. To find such a correction we will solve the equation $\bar{\partial} u = \alpha_\varepsilon$, where

$$\alpha_\varepsilon := \bar{\partial} \chi_\varepsilon F_o.$$

Note that α_ε is supported on the annular tube $\{\varepsilon^2 t \le |T|^2 e^{-\lambda} \le \varepsilon^2\}$.

To solve this equation we will use the twisted estimates of Donnelly-Fefferman-Ohsawa, i.e., Theorem 8.2.1, which we state again for the situation at hand (i.e., a vector bundle of rank 1 and a singular Hermitian metric), and for ease of reading.

9.2.4 THEOREM (Donnelly-Fefferman-Ohsawa Estimate; singular rank 1 case). *With the notation of Theorem 9.2.2, let Ω_j be as above. Let $e^{-\psi}$ be a singular Hermitian metric for $L|_{\Omega_j}$ and let $\eta : X \to \mathbb{R}$ and $a : X \to (0,\infty)$ be functions with η $\mathscr{C}^2$-smooth, such that*

$$\Psi := e^{-\eta}\left(\partial\bar{\partial}\psi + \text{Ricci}(g) + \partial\bar{\partial}\eta - \frac{1+a}{a}\partial\eta \wedge \bar{\partial}\eta\right) \ge \Theta \ge 0.$$

Then for each $L|_{\Omega_j}$-valued $(0,1)$-form φ such that

$$\bar{\partial}\varphi = 0 \quad \textit{and} \quad \int_{\Omega_j} |\varphi|^2_\Theta e^{-\psi} dV_g < +\infty$$

there exists a measurable section U of $L|_{\Omega_j}$ such that

$$\bar{\partial}\left(\sqrt{e^{-\eta}(1+a)}U\right) = \varphi \quad \textit{and} \quad \int_{\Omega_j} |U|^2 e^{-\psi} dV_g \le \int_{\Omega_j} |\varphi|^2_\Theta e^{-\psi} dV_g.$$

9.2.5 REMARK. As we have suggested earlier, Theorem 9.2.4 follows from Theorem 8.2.1 on Stein manifolds by approximation of singular metrics with smooth ones. In the present situation there is also the issue that the $(1,1)$-form Θ is not assumed to be strictly positive. The issue can be dealt with by perturbing the metric $e^{-\psi}$ to be strictly positively curved; since one is working on a domain $\Omega_j \subset\subset X$ one can add a small multiple of the plurisubharmonic exhaustion to ψ. $\diamond$

To make use of Theorem 9.2.4 we must choose a singular Hermitian metric $e^{-\psi}$, a smooth function η and a positive function a such that $\Psi \geq \Theta$ for some non-negative Θ for which the integral

$$\int_{\Omega_j} |\alpha_\varepsilon|^2_\Theta e^{-\psi} dV_g$$

is finite. Moreover, in order to make sure that the correction

$$F_\varepsilon := \chi_\varepsilon F_o - e^{-\eta}(1+a)U$$

remains an extension, the metric $e^{-\psi}$ must be singular along $Z \cap \Omega_j$. We therefore choose

$$\psi := \varphi + \log |T|^2 e^{-\lambda}.$$

The next task is to choose η and a. To simplify the estimates to come, it is convenient to introduce the auxiliary functions

$$v := \log |T|^2 e^{-\lambda} : X \to [-\infty, 0) \quad \text{and} \quad s := \gamma - \delta \log(e^v + \varepsilon^2) : X \to [1, \infty),$$

where δ is as in Theorem 9.2.2 and $\gamma := 1 + \delta \log(1+\varepsilon^2) \in (1, \infty)$.

With v and s in hand, we set

$$\eta = -\log(2 + \log(2e^{s-1} - 1)).$$

At first glance, the choice of η is bewildering. We do not have a great explanation for this choice, which comes out of necessity for obtaining good estimates later on. Perhaps the most important clue is that this function is very similar to the self-bounded gradient function of Example 8.4.4 on the punctured unit disk; the idea is that a tubular neighborhood of the hypersurface Z looks very much like the product of Z with the disk, and if we are creating a singularity near Z, a function of this form will give us some gain in positivity.

Finally the function a is chosen only to simplify the expression for Ψ so that it becomes manageable. For this reason it is convenient to define $H(x) := 2 + \log(2e^{x-1} - 1)$ (so that $\eta = -\log H(s)$). Then

$$\partial\bar{\partial}\eta - \frac{1+a}{a}\partial\eta \wedge \bar{\partial}\eta = \frac{H'(s)}{H(s)}(-\partial\bar{\partial}s) - \frac{1}{aH(s)}\left(H''(s)a + \frac{H'(s)^2}{H(s)}\right)\partial s \wedge \bar{\partial}s,$$

and so we choose

$$a = \frac{-(H'(s))^2}{H''(s)H(s)},$$

which yields

$$\partial\bar{\partial}\eta - \frac{1+a}{a}\partial\eta\wedge\bar{\partial}\eta = \frac{H'(s)}{H(s)}(-\partial\bar{\partial}s).$$

Note that for $x \geq 1$,

$$H(x) \geq 1, \quad H'(x) = \frac{2e^{x-1}}{2e^{x-1}-1} = 1 + \frac{1}{2e^{x-1}-1} \quad \text{and} \quad H''(x) = \frac{-2e^{x-1}}{(2e^{x-1}-1)^2},$$

so that a is a positive function– one of our requirements. Note also that $e^{-\eta}$ and a are smooth, a fact we shall use later.

We compute that

$$\begin{aligned}
-\partial\bar{\partial}s &= \delta\partial\bar{\partial}\log(e^v+\varepsilon^2) = \partial\left(\frac{\delta e^v\bar{\partial}v}{e^v+\varepsilon^2}\right)\\
&= \frac{\delta e^v}{(e^v+\varepsilon^2)}\partial\bar{\partial}v + \delta\frac{4\varepsilon^2\partial(e^{v/2})\wedge\bar{\partial}(e^{v/2})}{((e^{v/2})^2+\varepsilon^2)^2}\\
&= \delta\frac{e^v}{(e^v+\varepsilon^2)}\left(\partial\bar{\partial}\log|T|^2 - \partial\bar{\partial}\lambda\right) + \delta\frac{4\varepsilon^2\partial(e^{v/2})\wedge\bar{\partial}(e^{v/2})}{((e^{v/2})^2+\varepsilon^2)^2}\\
&= -\frac{\delta e^v}{(e^v+\varepsilon^2)}\partial\bar{\partial}\lambda + \delta\frac{4\varepsilon^2\partial(e^{v/2})\wedge\bar{\partial}(e^{v/2})}{((e^{v/2})^2+\varepsilon^2)^2}.
\end{aligned}$$

In the last equality we have used the fact that $\partial\bar{\partial}\log|T|^2$ is supported on Z, where $e^v = |T|^2e^{-\lambda}$ vanishes.

Now, the function

$$[1,\infty) \ni s \mapsto 2 + \log(2e^{s-1}-1) - \frac{2e^{s-1}}{2e^{s-1}-1} = 1 + \log(2e^{s-1}-1) - \frac{1}{2e^{s-1}-1}$$

has derivative $\frac{2e^{s-1}}{2e^{s-1}-1} + \frac{2e^{s-1}}{(2e^{s-1}-1)^2} = \frac{(2e^{s-1})^2}{(2e^{s-1}-1)^2} > 0$, and therefore it takes its minimum at $s = 1$, where it vanishes. Consequently

$$\frac{H'(s)}{H(s)}\cdot\frac{e^v}{e^v+\varepsilon^2} \leq \frac{\frac{2e^{s-1}}{2e^{s-1}-1}}{2+\log(2e^{s-1}-1)},$$

and we find that

$$\begin{aligned}
&\partial\bar{\partial}\psi + \mathrm{Ricci}(g) + \partial\bar{\partial}\eta - \frac{1+a}{a}\partial\eta\wedge\bar{\partial}\eta = \partial\bar{\partial}\psi + \mathrm{Ricci}(g) + \frac{H'(s)}{H(s)}(-\partial\bar{\partial}s)\\
&= \left(1 - \frac{H'(s)}{H(s)}\cdot\frac{e^v}{e^v+\varepsilon^2}\right)(\partial\bar{\partial}\psi + \mathrm{Ricci}(g)) + \frac{H'(s)}{H(s)}\cdot\frac{e^v}{e^v+\varepsilon^2}\left(\partial\bar{\partial}\psi + \mathrm{Ricci}(g) - \delta\partial\bar{\partial}\lambda\right)\\
&\qquad +\delta\frac{H'(s)}{H(s)}\frac{4\varepsilon^2\partial(e^{v/2})\wedge\bar{\partial}(e^{v/2})}{((e^{v/2})^2+\varepsilon^2)^2}\\
&\geq \delta\frac{H'(s)}{H(s)}\frac{4\varepsilon^2\partial(e^{v/2})\wedge\bar{\partial}(e^{v/2})}{((e^{v/2})^2+\varepsilon^2)^2},
\end{aligned}$$

where the last inequality follows from the definition of ψ and the curvature hypothesis (10.6) of Theorem 9.2.2. Since $H'(s) \geq 1$,

$$\Psi = e^{-\eta}\left(\partial\bar{\partial}\psi + \mathrm{Ricci}(g) + \partial\bar{\partial}\eta - \frac{1+a}{a}\partial\eta \wedge \bar{\partial}\eta\right) \geq \delta \frac{4\varepsilon^2 \partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{((e^{v/2})^2 + \varepsilon^2)^2}.$$

We take Θ to be the right-hand side of the inequality, and prepare to estimate the L^2 norm of α_ε. Now,

$$\alpha_\varepsilon = \bar{\partial}\chi(\varepsilon^{-2}e^v)F_o = \chi'(\varepsilon^{-2}e^v)F_o\varepsilon^{-2}\bar{\partial}e^v = \chi'(\varepsilon^{-2}e^v)F_o 2\varepsilon^{-2}e^{v/2}\bar{\partial}(e^{v/2}),$$

so

$$|\alpha_\varepsilon|^2_\Theta e^{-\psi} = \frac{1}{\delta\varepsilon^6}|\chi'(\varepsilon^{-2}e^v)|^2|F_o|^2 e^{-\psi}e^v(e^v+\varepsilon^2)^2 = \frac{1}{\delta\varepsilon^6}|\chi'(\varepsilon^{-2}e^v)|^2|F_o|^2 e^{-\varphi}(e^v+\varepsilon^2)^2.$$

Consequently

$$\begin{aligned}
\int_{\Omega_j} |\alpha_\varepsilon|^2_\Theta e^{-\psi} dV_g &= \frac{1}{\delta\varepsilon^6}\int_{\Omega_j} |\chi'(\varepsilon^{-2}e^v)|^2|F_o|^2 e^{-\varphi}(e^v+\varepsilon^2)^2 dV_g \\
&= \frac{1}{\delta\varepsilon^6}\int_{\{e^v<\varepsilon^2\}} |\chi'(\varepsilon^{-2}e^v)|^2|F_o|^2 e^{-\varphi}(e^v+\varepsilon^2)^2 dV_g \\
&\leq \frac{4(1+ct)^2}{\delta\varepsilon^2}\int_{\{e^v<\varepsilon^2\}} |F_o|^2 e^{-\varphi} dV_g.
\end{aligned}$$

In particular, note that

$$\limsup_{\varepsilon\to 0}\int_{\Omega_j} |\alpha_\varepsilon|^2_\Theta e^{-\psi} dV_g \leq \frac{4\pi}{\delta}(1+ct)^2 \int_Z \frac{|f|^2 e^{-\varphi} dA_g}{|dT|^2_g e^{-\lambda}}.$$

By Theorem 9.2.4 there exists a section U_ε of $L \to \Omega_j$ such that

$$\bar{\partial}(\sqrt{e^{-\eta}(1+a)}U_\varepsilon) = \alpha_\varepsilon \quad \text{and} \quad \int_{\Omega_j} |U_\varepsilon|^2 e^{-\psi} dV_g \leq (1+o(1))\frac{8\pi}{\delta}(1+ct)^2 \int_Z \frac{|f|^2 e^{-\varphi} dA_g}{|dT|^2 e^{-\lambda}}$$

as $\varepsilon \sim 0$.

Since α_ε is smooth, the function $\sqrt{e^{-\eta}(1+a)}U_\varepsilon$ is smooth, and therefore so is U_ε. Since $e^{-\psi} = e^{-\varphi}/(|T|^2 e^{-\lambda})$ is not locally integrable on Z, the section U_ε must vanish at all points of Z. Consequently the section

$$F_\varepsilon := \chi_\varepsilon F_o - \sqrt{e^{-\eta}(1+a)}U_\varepsilon$$

is an extension of f to Ω_j. Now,

$$\int_{\Omega_j} |F_\varepsilon|^2 e^{-\varphi} dV_g = (1+o(1))\int_{\Omega_j} e^{v-\eta}(1+a)|U_\varepsilon|^2 e^{-\psi} dV_g, \qquad \varepsilon \sim 0.$$

But since $s = \gamma - \delta \log(e^v + \varepsilon^2)$, $e^{-\eta} = H(s)$ and $a = \frac{-H'(s)^2}{H''(s)H(s)}$,

$$e^{v-\eta}(1+a) = e^{(\gamma-s)/\delta}\frac{H'(s)^2 - H''(s)H(s)}{-H''(s)} = e^{(\gamma-s)/\delta}\left(2e^{s-1} + 2 + \log(2e^{s-1}-1)\right).$$

A straight-forward calculus exercise shows that, for $s \geq 1$, the right hand side is bounded above by $6e^{\gamma-1} = 6(1+\varepsilon^2)^{-\delta}$.

We therefore have, for every $\varepsilon > 0$, a section $F_{\varepsilon,c,t} \in \Gamma_{\mathcal{O}}(\Omega_j, L)$ such that

$$F_{\varepsilon,c,t}|_{\Omega_j \cap Z} = f|_{\Omega_j \cap Z} \quad \text{and} \quad \int_{\Omega_j} |F_{\varepsilon,c,t}|^2 e^{-\varphi} dV_g \leq (1+o(1)) \frac{24\pi(1+ct)^2}{\delta} \int_Z \frac{|f|^2 e^{-\varphi} dA_g}{|dT|_g^2 e^{-\lambda}}.$$

We can now use Alaoglu's Theorem to take subsequential limits in ε, t and then j, and thereby obtain a limit $F \in \Gamma_{\mathcal{O}}(X, L)$ satisfying

$$F|_Z = f \quad \text{and} \quad \int_X |F|^2 e^{-\varphi} dV_g \leq \frac{24\pi}{\delta} \int_Z \frac{|f|^2 e^{-\varphi} dA_g}{|dT|_g^2 e^{-\lambda}}.$$

The proof of Theorem 9.2.2 is therefore complete. □

EXERCISES

Lecture 10

Some applications of the L^2 Extension Theorem

10.1 Beurling-Seip Theory of Interpolation

In the 1950's A. Beurling began to study interpolation and sampling problems on Hardy spaces; he did not publish his work until 1986; it appeared in his final published paper.

A few years after the appearance of Beurling's paper, K. Seip began to consider the analogous problem for Bergman spaces. Seip's motivation for consideration of the problem seems at least in part to have been linked to the following problem in mathematical solid state physics. Consider the so-called *Fock Space*

$$\mathscr{F}(\mathbb{C}) := \left\{ F \in \mathcal{O}(\mathbb{C}) \ ; \ \int_{\mathbb{C}} |F(z)|^2 e^{-|z|^2} dA(z) < +\infty \right\} = L^2(\mathbb{C}, e^{-|\cdot|^2} dA) \cap \mathcal{O}(\mathbb{C}).$$

To a lattice $\Lambda \subset \mathbb{C}$ (i.e., a free Abelian subgroup of maximal rank) one associates the *little Fock space*

$$\mathfrak{f}(\Lambda) := \left\{ f : \Lambda \to \mathbb{C} \ ; \ \sum_{\lambda \in \Lambda} |f(\lambda)|^2 e^{-|\lambda|^2} < +\infty \right\}.$$

The lattice Λ is said to be *interpolating* or *sampling* if the restriction map

$$\mathfrak{R}_\Lambda : \mathfrak{F} \ni F \mapsto F|_\Lambda \in \mathfrak{f}_\Lambda$$

is, respectively, surjective or injective. The problem was to find a lattice that is both interpolating and sampling, or to show there is no such lattice.

Mathematical physicists already knew that the quantity

$$D(\Lambda) := \limsup_{r \to \infty} \sup_{z \in \mathbb{C}} \frac{\# \Lambda \cap D_r(z)}{r^2},$$

called the *asymptotic density* of Λ, is ≤ 1 if Λ is interpolating, and ≥ 1 if Λ is sampling, and hence that if a lattice is both interpolating and sampling then it has asymptotic density 1.

10.1.1 Seip's Theorem

Seip showed that if Λ is an interpolating lattice then $D(\Lambda) < 1$ and that if Λ is a sampling lattice then $D(\Lambda) > 1$, thus supplying the negative solution to the aforementioned problem. In fact, Seip showed more. He considered locally finite subsets Γ (for which, unlike lattices, the numbers

$$D^+(\Gamma) := \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{\#(\Gamma \cap D(z,r))}{r^2} \quad \text{and} \quad D^-(\Gamma) := \liminf_{r\to\infty} \inf_{z\in\mathbb{C}} \frac{\#(\Gamma \cap D(z,r))}{r^2}$$

called the asymptotic upper and lower densities, are possibly distinct).

The notions of interpolation and sampling sets have to be modified slightly in order to get the proper result. In this setting a locally finite set Γ is interpolating if the restriction map, in addition to being surjective, is bounded. The set Γ is said to be sampling if the restriction map, in addition to being injective, is bounded with closed image.

, which can be provided with the same definition, in terms of the restriction map, of interpolation and sampling sets. and established, partly in joint work with Wallsten, the following theorem.

10.1.1 THEOREM. *A locally finite set Γ is*

1. *interpolating if and only if Γ is uniformly separated and $D^+(\Gamma) < 1$, and*
2. *sampling if and only if Γ is a finite union of uniformly separated sequences $\Gamma = \Gamma_1 \cup ... \cup \Gamma_N$ such that $D^-(\Gamma_1) > 1$.*

A locally finite set Γ is said to be uniformly separated if its separation radius

$$\rho(\Gamma) := \left\{ \frac{|\gamma - \mu|}{2} \; ; \; \gamma, \mu \in \Gamma, \; \gamma \neq \mu \right\}$$

is positive. The necessity of uniform separation is a relatively simple but instructive result which we shall explain momentarily.

10.1.2 A more general setting

Let W be a smooth hypersurface in $\mathbb{C}^n$. In interpolation theory one considers the Hilbert spaces

$$\mathscr{H}(\mathbb{C}^n, \varphi) := \left\{ f \in \mathcal{O}(\mathbb{C}^n) \; ; \; \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} \omega^n < +\infty \right\}$$

and

$$\mathfrak{H}(W, \varphi) := \left\{ f \in \mathcal{O}(W) \; ; \; \int_W |f|^2 e^{-\varphi} \omega^{n-1} < +\infty \right\},$$

where $\omega = \sqrt{-1}\partial\bar{\partial}|z|^2$ is the Kähler form of the Euclidean metric. One can then define the restriction operator

$$\mathfrak{R}_W : \mathscr{H}(\mathbb{C}^n, \varphi) \to \mathfrak{H}(W, \varphi)$$

as the operator sending a function F to its restriction to W.

10.1.2 DEFINITION. *The hypersurface W is then said to be* an interpolation set *if $\mathfrak{R}_W$ is bounded and surjective, and W is said to be* a sampling set *if $\mathfrak{R}_W$ is bounded, injective and has closed image.*

We will be primarily interested in the surjectivity of $\mathfrak{R}_W$; establishing such surjectivity is of course an L^2 extension result, and as such one might imagine a connection with Theorem 9.2.2. We emphasize, however, the fact that the L^2 norm defining the space $\mathfrak{H}(W, \varphi)$ is not the norm appearing in Theorem 9.2.2.

In what follows, we shall always assume that our weight φ is smooth and satisfies the bound

$$0 \le \sqrt{-1}\partial\bar{\partial}\varphi \le M\sqrt{-1}\partial\bar{\partial}|z|^2 \tag{10.1}$$

for some constant $M > 0$. The us of this curvature hypothesis, which is clearly satisfied by the Bargmann-Fock weights $\varphi(z) = c|z|^2$, $c > 0$, will become clear in due course.

The most obvious analogue of uniform separation is the notion of uniform flatness; it is defined as follows.

10.1.3 DEFINITION. *A smooth complex hypersurface $W \subset \mathbb{C}^n$ is said to be uniformly flat if there exists $\varepsilon > 0$ such that the set*

$$U_\varepsilon(W) := \{z \in \mathbb{C}^n \ ; \ \mathrm{dist}(z, W) < \varepsilon\}$$

is a tubular neighborhood of W, i.e., if $z_1, z_2 \in W$ and $v_i \in \mathbb{C}^n$ is perpendicular to T_{W,z_i}, $i = 1, 2$, and $z_1 + v_1 = z_2 + v_2$ then $\max(|v_1|, |v_2|) > \varepsilon$.

(Equivalently, any two Euclidean disks or radius $< \varepsilon$ and perpendicular to W at their centers do not intersect.)

10.1.4 DEFINITION. *Let $T \in \mathcal{O}(W)$ be a global holomorphic function whose zero locus is W, counting multiplicity*[1]*. For $r > 0$ define the function*

$$\lambda_r^T(z) := \frac{1}{\mathrm{Vol}(B_r(z))}\int_{B_r(z)} \log|T|^2 dV = \frac{1}{\mathrm{Vol}(B_r(0))}\mathbf{1}_{B_r(0)} * \log|T|^2.$$

(i) The function $S_r^W : W \to \mathbb{R}_+$ defined by

$$S_r^W(z) := |dT|^2_\omega e^{-\lambda_r^T}$$

is called the separation function of W.

(ii) The $(1,1)$-current

$$\Upsilon_r^W := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\lambda_r^T = \frac{1}{\mathrm{Vol}(B_r(0))}\mathbf{1}_{B_r(0)} * [W]$$

is called the total mass current *of W.*

[1]Recall that in general T is a section of the line bundle $L_W \to \mathbb{C}^n$ associated to W; here it is a function because every line bundle on $\mathbb{C}^n$ is trivial.

(Here $[W]$ is the current of integration over W, $*$ denotes convolution, and the second equality follows from the Poincaré-Lelong Formula.)

Note that

$$|T|^2 e^{-\lambda_r^T}, \quad |dT|^2_\omega e^{-\lambda_r^T}|_W \quad \text{and} \quad \partial\bar{\partial}\lambda_r^T$$

do not depend on the function T used to cut out W.

V. Pingali and the author have proved the following theorem.

10.1.5 THEOREM. *If W is uniformly flat then for each $r > 0$ there is a positive constant C_r such that*

$$S_r^W \ge C_r$$

on W.

In fact, if $n = 1$ the converse of Theorem 10.1.5 is also true. It is at present not known if the converse is true in general.

10.1.6 DEFINITION. *The asymptotic upper and lower densities of W with respect to the weight φ are the numbers*

$$D^+_\varphi(W) := \limsup_{r\to\infty} \sup_{z\in W} \left\{ \tfrac{1}{a} \; ; \; a > 0 \textit{ and } \sqrt{-1}\partial\bar{\partial}\varphi(v,v) \ge a\Upsilon_r^W(v,v) \textit{ for all } v \in \mathbb{C}^n \right\}$$

and

$$D^-_\varphi(W) := \liminf_{r\to\infty} \inf_{z\in W} \left\{ \frac{1}{a} \; ; \; a > 0 \textit{ and } \sqrt{-1}\partial\bar{\partial}\varphi(v,v) < a\Upsilon_r^W(v,v) \textit{ for some } v \in \mathbb{C}^n \right\}$$

In other words, $D^+_\varphi(W)$ is the supremum of all numbers $1/a$ such that $\varphi - a\lambda_r^T$ is plurisubharmonic for all $r >> 0$, and $D^-_\varphi(W)$ is the infimum of all numbers $1/a$ such that $\varphi - a\lambda_r^T$ is not plurisubharmonic for all $r >> 0$.

From Theorems 9.2.2 and 10.1.5 we immediately obtain the following theorem.

10.1.7 THEOREM. *Let $\varphi \in \mathbb{C}^n$ be a smooth function and let $W \subset \mathbb{C}^n$ be a smooth hypersurface. If $D^+_\varphi(W) < 1$ then for every $f \in \mathcal{O}(W)$ such that*

$$\int_W |f|^2 e^{-\varphi} \frac{dA_W}{S_r^W} < +\infty$$

there exists $F \in \mathscr{H}(\mathbb{C}^n, \varphi)$ such that $F|_W = f$. In particular, if W is also uniformly flat then W is an interpolation hypersurface.

In particular, we recover the positive direction of the interpolation part of Seip's Theorem. The positive direction of the sampling theorem is also true; it was proved by Ortega Cerdá, Schuster and the author [OSV-2006].

Remarkably, except in the 1-dimensional case, uniform flatness is not a necessary condition for interpolation. A number of examples have been produced by Pingali and the author [PV-2016, PV-2019].

There is very little done on interpolation theory in higher dimensions on manifolds other than $\mathbb{C}^n$. In dimension 1 there is a fair amount of work; we refer the reader to [V-2018, V-2016] for references and further reading.

In the next paragraph we discuss some of the ideas behind the necessity of uniform separation. This discussion is provided for the interested reader, and although very useful tools are presented, most of the material is slightly less naturally aligned with the general topics covered in this lecture series.

10.1.3 Weighted Bergman Inequalities

In preparation for a number of estimates we will need below, we now state and prove a very useful lemma on the existence of bounded solutions of $\sqrt{-1}\partial\bar{\partial}$ when the forcing term is bounded. More precisely, we have the following result, which was proved by Berndtsson and Ortega-Cerdà in the 1-dimensional case, and generalized by Lindholm to higher dimensions for the case where one has only $\mathscr{C}^0$-estimates. However, a modification of the proof of Berndtsson and Ortega-Cerdà easily gives the present form.

10.1.8 LEMMA. *There exists a constant $C > 0$ with the following property. Let ω be a $\mathscr{C}^2$-smooth, closed $(1,1)$-form on a neighborhood of the closed unit ball $\overline{B}$ such that*

$$-M\sqrt{-1}\partial\bar{\partial}|z|^2 \le \omega \le M\sqrt{-1}\partial\bar{\partial}|z|^2$$

for some positive constant M. Then there exist a function $\psi \in \mathscr{C}^2(B)$ such that

$$\sqrt{-1}\partial\bar{\partial}\psi = \omega \quad \textit{and} \quad \sup_B(|\psi| + |d\psi|) \le CM.$$

Proof. We can assume that in fact ω has compact support in $B(0,2)$ by multiplication with an appropriate cut-off function.

Let us first present the proof in the case $n = 1$. In this case, one simply takes

$$\psi(z) := \int_{B(0,2)} \log|z-\zeta|^2\omega(\zeta).$$

Note that $\omega = h\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}$ for some real-valued function h. A standard argument using integration-by-parts shows that

$$\frac{1}{\pi}\frac{\partial^2\psi}{\partial z\partial\bar{z}} = h.$$

The function ψ is clearly bounded by the constant

$$M \sup_{z\in B(0,1)} \int_{B(0,2)} \left|\log|\zeta - z|^2\right| dA$$

while the derivative is controlled by

$$M \sup_{z\in B(0,1)} \int_{B(0,2)} \frac{dA(\zeta)}{|z-\zeta|}.$$

Thus we have the stated result.

In higher dimensions, write $\omega = \omega_{i\bar{j}} \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^j$. Then as in the 1-dimensional case, the function

$$\psi(z) := \int_{B(0,2)} \omega_{1\bar{1}}(\zeta, z^2, ..., z^n) \log |z^1 - \zeta|^2 dA(\zeta)$$

then satisfies

$$\frac{1}{\pi} \frac{\partial^2 \psi}{\partial z^1 \partial \bar{z}^1} = \omega_{1\bar{1}}.$$

From the condition $d\omega = 0$, we see that, when i and j are both different from 1,

$$\begin{aligned}
\frac{1}{\pi} \frac{\partial^2 \psi}{\partial z^i \partial \bar{z}^j} &= \int_{B(0,2)} \frac{\partial^2 \omega_{1\bar{1}}}{\partial z^i \partial \bar{z}^j}(\zeta, z^2, ..., z^n) \log |z^1 - \zeta|^2 dA(\zeta) \\
&= \int_{B(0,2)} \frac{\partial^2 \omega_{i\bar{j}}}{\partial \zeta \partial \bar{\zeta}}(\zeta, z^2, ..., z^n) \log |z^1 - \zeta|^2 dA(\zeta) \\
&= \omega_{i\bar{j}}(z).
\end{aligned}$$

As before, ψ is bounded in $\mathscr{C}^1$-norm by CM. The proof is complete. □

One important application of Lemma 10.1.8 is the following result.

10.1.9 PROPOSITION. *Let $\omega = \sqrt{-1}\partial\bar{\partial}|z|^2$ and $\varphi \in PSH(2B)$ such that $\sqrt{-1}\partial\bar{\partial}\varphi \le M\omega$ for some positive constant M on $2B$, where B is the unit ball in $\mathbb{C}^n$. Then there is a constant $C > 0$ such that for all $f \in \mathcal{O}(2B)$ satisfying*

$$\int_B |f|^2 e^{-\varphi} \omega^n < +\infty,$$

$$|f(0)|^2 e^{-\varphi(0)} \le C \int_B |f|^2 e^{-\varphi} \omega^n \tag{10.2}$$

and

$$|d(|f|^{2r} e^{-r\varphi})|(0) \le C \left(\int_B |f|^2 e^{-\varphi} \omega^n \right)^r. \tag{10.3}$$

In particular, if $f \in \mathscr{H}(\mathbb{C}^n, \varphi)$ then weighted point evaluation is bounded with norm independent of the point. More generally, there exists $C > 0$ such that for all $x \in \mathbb{C}^n$,

$$(|f(x)|e^{-\varphi(x)} + \left|d(|f|^2 e^{-\varphi})(x)\right| \le C_M \left(\int_{\mathbb{C}^n} |f|^2 e^{-\varphi} \omega^n \right)^{1/2} < +\infty. \tag{10.4}$$

and thus

$$||f||_{\mathscr{C}^1(\mathbb{C}^n)} := \sup_{\mathbb{C}^n} (|f| + |df|) \le C_M \left(\int_{\mathbb{C}^n} |f|^2 e^{-\varphi} \omega^n \right)^{1/2} < +\infty.$$

Proof. The estimate (10.4) follows from the estimate (10.2) and (10.3) because the Euclidean form ω is translation-invariant. To prove (10.2), let ψ be a bounded plurisubharmonic function on B such that $\sqrt{-1}\partial\bar{\partial}\psi = \sqrt{-1}\partial\bar{\partial}\varphi$. Such a function exists by Lemma 10.1.8. Let $h = \psi - \varphi$. Then h is pluriharmonic, and therefore twice the real part of a holomorphic function H. Let $F = fe^{H-H(0)}$. We have

$$|f(0)|^2 e^{-\varphi(0)} = |F(0)|^2 e^{-\varphi(0)} \le \fint_B |F|^2 e^{-\varphi(0)} \omega^n = \fint |f|^2 e^{-\varphi} e^{\psi - \psi(0)} \omega^n \le C \int_B |f|^2 e^{-\varphi} \omega^n.$$

Next, observe that

$$\begin{aligned} d(|f|^{2r} e^{-r\varphi}) &= r\left(|f|^{2(r-1)}(\partial f \bar{f} + f\bar{\partial}\bar{f})e^{-r\varphi} - |f|^{2r} e^{-r\varphi} d\varphi\right) \\ &= r(|f|^2 e^{-\varphi})^{r-1}(\bar{f}(\partial f - f\partial\varphi) + f(\bar{\partial}\bar{f} - \bar{f}\bar{\partial}\varphi))e^{-\varphi}. \end{aligned}$$

We therefore have

$$|d(|f|^{2r} e^{-r\varphi})(0)| \lesssim ||f||^{2r-1} \left|\partial f(0) e^{-\varphi(0)} - f(0) e^{-\varphi(0)} \partial\varphi(0)\right|^2.$$

Now, $\varphi + h = \psi$ is bounded in $\mathscr{C}^1$-norm, and thus we have

$$\begin{aligned} \left|\partial f(0) e^{-\varphi(0)} - f(0) e^{-\varphi(0)} \partial\varphi(0)\right|^2 &\lesssim \left|\partial f(0) e^{-\varphi(0)} + f(0) e^{-\varphi(0)} \partial h(0)\right|^2 + |f(0)|^2 e^{-\varphi(0)} |\partial\psi(0)| \\ &= |d(fe^H)(0)|^2 e^{-\varphi(0)} + C||f||^2. \end{aligned}$$

By the Cauchy estimates and Lemma 10.1.8, we have

$$|d(fe^H)(0)|^2 e^{-\varphi(0)} \lesssim \int_B |fe^H|^2 e^{-\varphi(0)} \omega^n \lesssim \int_B |f|^2 e^{-\varphi} \omega^n.$$

Therefore (10.3) holds. The proof is complete. □

As a consequence of Proposition 10.1.9, we now prove that a locally finite subset $\Gamma \subset \mathbb{C}$ is

(i) a finite union of uniformly separated locally finite sets if and only if $\mathfrak{R}_\Gamma : \mathfrak{F}(\mathbb{C}) \to \mathfrak{f}_\Gamma$ is bounded, and

(ii) uniformly separated if $\mathfrak{R}_\Gamma : \mathfrak{F}(\mathbb{C}) \to \mathfrak{f}_\Gamma$ is surjective.

Proof of (i). Let $F \in \mathfrak{F}(\mathbb{C})$. If Γ is a finite union of uniformly separated locally finite sets then for any $r > 0$ there is an integer N such that no more than N of the disks $\{D_r(\gamma)\ ;\ \gamma \in \Gamma\}$ intersect. By Proposition 10.1.9

$$\sum_{\gamma \in \Gamma} |F(\gamma)|^2 e^{-|\gamma|^2} \le C \sum_{\gamma \in \Gamma} |F(z)|^2 e^{-|z|^2} dA(z) \le CN \int_{\mathbb{C}} |F(z)|^2 e^{-|z|^2} dA(z),$$

so $\mathfrak{R}_\Gamma$ is bounded.

Conversely, suppose $\mathfrak{R}_\Gamma$ is bounded. Let $z \in \mathbb{C}$. Choose a function $F \in \mathfrak{F}(\mathbb{C})$ such that $|F(z)|^2 e^{-|z|^2} = 1$. Then there exists a constant μ, independent of z, such that

$$\int_{\mathbb{C}} |F(\zeta)|^2 e^{-|\zeta|^2} dA(\zeta) \le \mu.$$

We shall prove this fact below (Lemma 10.1.10). Let $r \in (0,1)$; shortly we will further restrict r. For every $\gamma \in \Gamma \cap D_r(z)$ one has

$$\begin{aligned}
\left| |F(\gamma)|^2 e^{-|\gamma|^2} - 1 \right| &= \left| |F(\gamma)|^2 e^{-|\gamma|^2} - |F(z)|^2 e^{-|z|^2} \right| \\
&= \int_0^1 \frac{d}{dt} |F(tz + (1-t)\gamma)|^2 e^{-|tz+(1-t)\gamma|^2} dt \\
&\le r \sup_{D_r(z)} d(|F|^2 e^{-|\cdot|^2}) \\
&\le r \sup_{D_1(z)} d(|F|^2 e^{-|\cdot|^2}) \le rC\mu,
\end{aligned}$$

where C is independent of r, γ or z. The last inequality follows from (10.3) of Proposition 10.1.9. Now choose $r < \frac{1}{2C\mu}$. Then

$$|F(\gamma)|^2 e^{-|\gamma|^2} \ge \frac{1}{2}$$

for all $\gamma \in D_r(z)$, and so

$$\#(\Gamma \cap D_r(z)) \le 2 \sum_{\gamma \in \Gamma \cap D_r(z)} |F(\gamma)|^2 e^{-|\gamma|^2} \le 2 \sum_{\gamma \in \Gamma} |F(\gamma)|^2 e^{-|\gamma|^2} \le 2\mu ||\mathfrak{R}_\Gamma||^2$$

Thus the number of points of $\Gamma \cap D_r(z)$ is finite, and bounded above by a number that does not depend on z. Thus (i) is proved. □

Proof of (ii). Suppose $\mathfrak{R}_\Gamma$ is surjective. Let $E_\Gamma : \mathfrak{f}(\Gamma) \to \mathfrak{F}(\mathbb{C})$ be the operator that assigns to $g \in \mathfrak{f}(\Gamma)$ the extension whose norm is minimal among all $\mathfrak{R}_\Gamma^{-1}\{g\}$. We claim that E_Γ is bounded. By the Closed Graph Theorem it suffices to show that the graph of E_Γ is closed. To that end, let $g_j \to g$ in $\mathfrak{f}(\Gamma)$ and let $F = \lim E_\Gamma(g_j)$ in $\mathfrak{F}(\mathbb{C})$. The sub-mean value property shows that convergence in L^2 implies locally uniform convergence, and consequently F is an extension of g. We wish to show that F is the extension of g having minimal norm. The latter condition holds if and only if F is orthogonal to all functions in $\mathfrak{F}(\mathbb{C})$ that vanish along Γ. Let G be such a function. Then

$$\int_{\mathbb{C}} F\bar{G} e^{-|\cdot|^2} dA = \lim \int_{\mathbb{C}} E_\Gamma(g_j) \bar{G} e^{-|\cdot|^2} dA = \lim 0 = 0,$$

and thus $F = E_\Gamma(g)$. Therefore E_Γ is bounded.

Fix a point $\gamma_o \in \Gamma$ and consider the function $f : \Gamma \to \mathbb{C}$ defined by

$$f(\gamma_o) = e^{|\gamma_o|^2/2} \quad \text{and} \quad f(\mu) = 0 \text{ for all } \mu \in \Gamma - \{\gamma_o\}.$$

Then

$$\sum_{\gamma\in\Gamma} |f(\gamma)|^2 e^{-|\gamma|^2} = 1.$$

Let $F := E_\Gamma(f)$. Then $F|_\Gamma = f$ and $||F|| \le ||E_\Gamma||$ is independent of γ_o.

Now let $\gamma \in \Gamma - \{\gamma_o\}$. Then

$$\begin{aligned}\frac{1}{|\gamma-\gamma_o|} &= \left|\frac{|F(\gamma)|^2 e^{-|\gamma|^2} - |F(\gamma_o)|^2 e^{-|\gamma_o|^2}}{|\gamma-\gamma_o|}\right| \\ &= \frac{1}{|\gamma-\gamma_o|}\left|\int_0^1 \frac{d}{dt}|F(t\gamma_o+(1-t)\gamma)|^2 e^{-|t\gamma_o+(1-t)\gamma|^2}dt\right| \\ &\le \sup_{\mathbb{C}} d(|F|^2 e^{-|\cdot|^2}) \le C||F||^2 \le C||E_\Gamma||^2.\end{aligned}$$

The proof is complete. □

10.1.10 LEMMA. *Let $\psi \in \mathscr{C}^2(\mathbb{C})$ be a smooth weight function such that $\partial\bar\partial\psi \ge 4c\partial\bar\partial|\cdot|^2$ for some $c > 0$. Then there exists a constant μ with the following property: for every $z \in \mathbb{C}$ there exists $F \in \mathcal{O}(\mathbb{C})$ such that*

$$|F(z)|^2 e^{-\psi(z)} = 1 \quad \textit{and} \quad \int_{\mathbb{C}} |F|^2 e^{-\psi} dA \le \mu.$$

Proof. Let $T(\zeta) := \sqrt{c}(\zeta - z)$ and let $\lambda(\zeta) = \log(1 + c|\zeta - z|^2)$. Then

$$\{T = 0\} = \{z\}, \quad |T|^2 e^{-\lambda} \le 1 \quad \text{and} \quad \partial\bar\partial\lambda(\zeta) = \frac{c}{(1+c|\zeta-z|^2)^2}\partial\bar\partial|\zeta|^2.$$

Then

$$\partial\bar\partial\varphi(\zeta) - (1+\tfrac{1}{2})\partial\bar\partial\lambda(\zeta) \ge c\partial\bar\partial|\zeta|^2,$$

so by Theorem 9.2.2 there exists $F \in \mathcal{O}(\mathbb{C})$ such that

$$F(z) = e^{\frac{1}{2}\psi(z)} \quad \text{and} \quad \int_{\mathbb{C}} |F|^2 e^{-\psi} dA \le 48\pi,$$

and the proof is complete. □

10.1.11 REMARK. Theorem 9.2.2 was used in the proof of Lemma 10.1.10 primarily for the sake of convenience. One can also use Hörmander's Theorem, though doing so requires some ingenuity. But the point is that the twisted methods need not be used here. ◇

10.2 Deformation Invariance of Plurigenera

Plurigenera, the invariance problem and Siu's extension theorem

10.2.1 DEFINITION. *Let Y be a complex manifold. For each $m \in \mathbb{N}_{\ge 1}$ the number*

$$P_m(Y) := \mathit{dim}_{\mathbb{C}}\Gamma_{\mathcal{O}}(Y, K_Y^{\otimes m})$$

is called the mth plurigenus of Y.

For a holomorphic family $\pi : X \to \mathbb{D}$ with fibers $X_t := \pi^{-1}(t)$, we define the function

$$\mu_m : \mathbb{D} \ni t \mapsto P_m(X_t) \in \mathbb{N}.$$

One says that the m-genera are invariant for this family if the function μ_m is constant.

Our objective in the present section is to prove that, for a projective family, all the m-genera are invariant, i.e., the function μ_m is constant, or equivalently, continuous. We shall prove the continuity of μ_m by showing that (i) μ_m is upper semi-continuous and (ii) μ_m is lower semi-continuous. These properties are local, so it's enough to show them at $0 \in \mathbb{D}$.

The upper semi-continuity holds more generally for any holomorphic family. Let us, then, fix a holomorphic family $\pi : \mathscr{X} \to \mathbb{D}$, and some Hermitian metric ω on $\mathscr{X}$. To prove that μ_m is upper semi-continuous at 0, which means that

$$\limsup_{t \to 0} \mu_m(t) \le \mu_m(0).$$

We begin with the following lemma.

10.2.2 LEMMA. *Let $\pi : \mathscr{X} \to \mathbb{D}$ be a holomorphic family and let $L \to \mathscr{X}$ be a holomorphic line bundle with smooth Hermitian metric $e^{-\varphi}$. Let $t_j \in \mathbb{D}(0, \varepsilon) - \{0\}$ be a sequence of complex numbers converging to 0 in the unit disk. Let $s_j \in \Gamma_{\mathscr{O}}(X_{t_j}, K_{X_{t_j}} \otimes L)$ be a sequence of pluricanonical sections satisfying*

$$\int_{X_{t_j}} |s_{t_j}|^2 e^{-\varphi} = 1.$$

Then there is a section $s_o \in \Gamma_{\mathscr{O}}(X_o, K_{X_o} \otimes L)$ such that

$$\int_{X_o} |s_o|^2 e^{-\varphi} = 1,$$

and a subsequence $\{t_{j_k}\}$, such that for any coordinate chart $U \subset \mathscr{X}$ on which ξ is a frame for $L|_U$, satisfying

$$\pi(U) = \mathbb{D}(0, \varepsilon) \quad \textit{and} \quad (z, \pi) : U \xrightarrow{\cong_{\mathscr{O}}} B \times \mathbb{D}(0, \varepsilon)$$

for some coordinate unit ball $B \subset X_o$ and holomorphic map $z : U \to B$, if we write

$$s_{t_j}(x) = f_j(z) dz^1 \wedge ... \wedge dz^n \otimes \xi \quad \textit{and} \quad s_o(x) = f_o(z) dz^1 \wedge ... \wedge dz^n \otimes \xi$$

then

$$\lim_{k \to \infty} f_{j_k}(z) = f_o(z)$$

uniformly on the set $|z| \le 1/2$.

10.2.3 REMARK. Note that by the implicit function theorem for holomorphic submersions, each point of X_o is contained inside the sort of product coordinate chart used in the statement of the lemma. ◇

Proof of Lemma 10.2.2. In the coordinate chart, let us write $|\xi|^2 e^{-\varphi} = e^{-\varphi(z,t)}$. Then

$$|s_{t_j}|^2 e^{-\varphi} = |f_j(z)|^2 e^{-\varphi(z,t_j)} dV(z).$$

Note that since the metric $e^{-\varphi}$ is globally defined and smooth, the function $e^{-\varphi(z,t)}$ is uniformly bounded above and below by positive constants on $|z| \le \frac{3}{4}$, and also uniformly in $|t| \le \varepsilon$. Therefore we have

$$|f_j(z)|^2 \le C_\delta \int_{B_\delta(z)} |f_j|^2 e^{\varphi(z,t_j)} dV(z) \le C_\delta \int_{X_{t_j}} |s_{t_j}|^2 e^{-\varphi} = C_\delta.$$

for some delta sufficently small, and all $|z| \le \frac{2}{3}$. By Montel's Theorem, the $\{f_j\}$ contain a subsequence that converges uniformly on $|z| \le 1/2$. By a diagonal argument we obtain a subsequence $\{s_{j_k}\}$ such that the restriction to each of these charts charts converges locally uniformly. The resulting sections automatically glue together to produce a section of $K_{X_o} \otimes L$, and the proof is finished. □

10.2.4 REMARK. The above proof also works when the fibers of $\mathscr{X} \to \pi$ are non-compact, i.e., when the map π is a submersion that might not be proper. (In the absence of properness, we don't know that the fibers are diffeomorphic.) ⋄

10.2.5 LEMMA. *Let $t_j \to 0$ be a sequence of complex numbers in the unit disk, and suppose* $\dim_{\mathbb{C}}(\Gamma_{\mathcal{O}}(X_{t_j}, K_{X_{t_j}} \otimes L) \ge N$ *for some* $N \in \mathbb{N}$. *Then* $\dim_{\mathbb{C}}(\Gamma_{\mathcal{O}}(X_o, K_{X_o} \otimes L) \ge N$.

Proof. The hypothesis assumes that for each t_j there is a subspace of $\Gamma_{\mathcal{O}}(X_{t_j}, K_{X_{t_j}} \otimes L)$ of dimension at least N. For each j, we fix such a subspace $V_j^N \subset \Gamma_{\mathcal{O}}(X_{t_j}, K_{X_{t_j}} \otimes L)$, and an orthonormal basis$\{s_1^{(j)}, ..., s_N^{(j)}\} \subset V_j^N$. After passing to a subsequence of the t_j if needed, we find by Lemma 10.2.2 sections $\{s_1, ..., s_N\} \subset \Gamma_{\mathcal{O}}(X_o, K_{X_{t_j}} \otimes L)$, each of unit norm, such that

$$\lim \int_{X_{t_j}} |s_i^{(j)}|^2 e^{-\varphi} = \lim \int_{X_o} |s_i|^2 e^{-\varphi}.$$

From the identity

$$\langle a, b \rangle = ||a+b||^2 - ||a-b||^2 + \sqrt{-1}(||a - \sqrt{-1}b||^2 - ||a + \sqrt{-1}b||^2),$$

which holds in any Hilbert space, we see (from the proof, rather than the statement of Lemma 10.2.2) that the limit sections $\{s_1, ..., s_N\}$ are also orthonormal. Thus there are at least N independent sections in $\Gamma_{\mathcal{O}}(X_o, K_{X_o} \otimes L)$. This completes the proof. □

As an immediate corollary, we obtain the following proposition.

10.2.6 PROPOSITION. *Let $\pi : \mathscr{X} \to \mathbb{D}$ be a holomorphic family of n-dimensional complex manifolds. Then for any $m \in \mathbb{N}$ the function μ_m is upper semi-continuous.*

Thus to prove the constancy of μ_m, it suffices to prove that every pluricanonical section on the central fiber X_o of a holomorphic family extends to a pluricanonical section on the family X.

10.2.7 THEOREM. *Let $\pi : \mathscr{X} \to \mathbb{D}$ be a projective family of fiber dimension n, let $L \to \mathscr{X}$ be a holomorphic with singular Hermitian metric $e^{-\kappa}$, and let $s \in \Gamma_{\mathcal{O}}(X_o, K_{X_o}^{\otimes m} \otimes L)$ satisfy*

$$\int_{X_o} |s|^2 \omega^{-n(m-1)} e^{-\kappa} < +\infty.$$

Then there is a section $S \in \Gamma_{\mathcal{O}}(X, K_X^{\otimes m} \otimes L)$ such that

$$S|_{X_o} = s \wedge (d\pi)^{\otimes m} \quad \textit{and} \quad \int_X |S|^2 \omega^{-(n+1)(m-1)} e^{-\kappa} < +\infty.$$

10.2.8 REMARK. The constancy of μ_m follows by taking $L = 0$. One can also define twisted plurigenera, which evidently are also invariant in families. ⋄

10.2.9 REMARK. It is known that, for general holomorphic families of compact complex manifolds, plurigenera are not invariant in families. At the time of writing of these notes, the case of the deformation invariance of plurigenera in Kähler families was still open. ⋄

We shall first prove the result when $\pi : \mathscr{X} \to \mathbb{D}$ is uniformly projective. At the end of the proof we will see that the extension has uniform estimates, and we will be able to establish the non-uniform case.

10.2.1 $\mathbf{L^2}$ extension of twisted canonical forms

Theorem 9.2.2 is the key tool in the proof of Theorem 10.2.7. We state a version of Theorem 9.2.2 for line bundle-valued holomorphic forms of top degree, or, *twisted canonical forms*.

10.2.10 THEOREM. *Let X be a Stein manifold and let $Z \subset X$ be a smooth complex hypersurface. Assume there exists a section $T \in \Gamma_{\mathcal{O}}(X, L_Z)$ and a singular Hermitian metric $e^{-\lambda}$ for $L_Z \to X$ such that*

$$Z = \{T = 0\} \quad \textit{and} \quad \sup_X |T|^2 e^{-\lambda} \le 1. \tag{10.5}$$

Let $L \to X$ be a holomorphic line bundle with singular Hermitian metric $e^{-\varphi}$, and assume there exists a constant $\delta \in (0, 1]$ such that

$$\partial\bar{\partial}\varphi \ge t\delta\partial\bar{\partial}\lambda \quad \textit{for all } t \in [0, 1]. \tag{10.6}$$

Then for each $f \in \Gamma_{\mathcal{O}}(Z, K_Z \otimes L)$ such that

$$\int_Z |f|^2 e^{-\varphi} < +\infty$$

there exists $F \in \Gamma_{\mathcal{O}}(X, K_X \otimes L_Z \otimes L)$ such that

$$F|_Z = f \wedge dT \quad \textit{and} \quad \int_X |F|^2 e^{-\varphi-\lambda} \le \frac{24\pi}{\delta} \int_Z |f|^2 e^{-\varphi}.$$

In particular, we shall make use of the following special case, in which Z is cut out by a bounded holomorphic function.

10.2.11 THEOREM. *Let X be a Stein manifold and $T : X \to \mathbb{D}$ a holomorphic function such that*

$$dT(x) \neq 0 \text{ for all } x \in Z := \{T = 0\}.$$

Let $L \to X$ be a holomorphic line bundle with singular Hermitian metric $e^{-\varphi}$ such that $\partial\bar{\partial}\varphi \geq 0$. Then for each $f \in \Gamma_{\mathcal{O}}(Z, K_Z \otimes L)$ such that

$$\int_Z |f|^2 e^{-\varphi} < +\infty$$

there exists $F \in \Gamma_{\mathcal{O}}(X, K_X \otimes L_Z \otimes L)$ such that

$$F|_Z = f \wedge dT \quad \text{and} \quad \int_X |F|^2 e^{-\varphi-\lambda} \leq 24\pi \int_Z |f|^2 e^{-\varphi}.$$

10.2.2 Positively twisted canonical sections

We fix a smooth metric $e^{-\gamma}$ for L and a holomorphic line bundle $A \to \mathscr{X}$ (which we may assume admits a smooth Hermitian metric with positive curvature) with the following property:

(GG) For each $0 \leq p \leq m-1$ there are sections $\{\tilde{\sigma}^{(p)}_j \;\; 1 \leq j \leq N_p\} \subset \Gamma_{\mathcal{O}}(\mathscr{X}, K^{\otimes p}_{\mathscr{X}} \otimes A)$ that generate the sheaf of germs of holomorphic sections of $K^{\otimes p}_{\mathscr{X}} \otimes A \to \mathscr{X}$.

Property (GG) may be assumed because $\mathscr{X} \to \mathbb{D}$ is uniformly projective. Let us also fix a smooth metric $e^{-\varphi}$ for $A \to \mathscr{X}$ such that $\omega := \sqrt{-1}\partial\bar{\partial}\varphi$ is a Kähler metric. Again by uniform projectivity, we may assume that $e^{-\varphi}$ extends to a neighborhood of $\mathscr{X}$ in some larger projective family, and thus

$$\int_{\mathscr{X}} \omega^{n+1} < +\infty.$$

Since $\kappa - \gamma$ is locally the sum of a smooth function and a plurisubharmonic function, the submean value property tells us that $e^{\kappa-\gamma}$ is locally bounded above. Thus by uniform projectivity we may assume that

$$\sup_{\mathscr{X}} e^{\kappa-\gamma} < +\infty.$$

We let $\sigma^{(p)}_j \in \Gamma_{\mathcal{O}}(X_o, K^{\otimes p}_{X_o} \otimes A|_{X_o})$ be defined by

$$\tilde{\sigma}^{(p)}_j|X_o = \sigma^{(p)}_j \wedge (d\pi)^{\otimes p}.$$

Then we have the following key proposition.

10.2.12 PROPOSITION. *There exist a constant $C > 0$ and sections*

$$\{\tilde{\sigma}^{(km+p)}_j \in \Gamma_{\mathcal{O}}(\mathscr{X}, K^{\otimes km+p}_{\mathscr{X}} \otimes L^{\otimes k} \otimes A)\;;\; 1 \leq j \leq N_p\}_{0 \leq p \leq m-1, k=0,1,2,\dots}$$

with the following properties.

(a) $\tilde{\sigma}_j^{(mk+p)}|_{X_o} = s^{\otimes k} \otimes \sigma_j^{(p)} \wedge (d\pi)^{\otimes(km+p)}$

(b) If $k \geq 1$,

$$\int_{\mathscr{X}} \frac{\sum_{j=1}^{N_o} |\tilde{\sigma}_j^{(mk)}|^2 e^{-\gamma}}{\sum_{j=1}^{N_{m-1}} |\tilde{\sigma}_j^{(mk-1)}|^2} \leq C.$$

(c) For $1 \leq p \leq m-1$,

$$\int_{\mathscr{X}} \frac{\sum_{j=1}^{N_p} |\tilde{\sigma}_j^{(mk+p)}|^2}{\sum_{j=1}^{N_{p-1}} |\tilde{\sigma}_j^{(mk+p-1)}|^2} \leq C.$$

Proof. To simplify the notation a little, let us denote by $\tilde{\sigma}^{(mk+p)}$ the N_p-tuples

$$\tilde{\sigma}^{(mk+p)} = (\tilde{\sigma}_1^{(mk+p)}, ..., \tilde{\sigma}_{N_p}^{(mk+p)}) \quad \text{and} \quad \sigma^{(mk+p)} = (\sigma_1^{(mk+p)}, ..., \sigma_{N_p}^{(mk+p)}),$$

and write

$$||\tilde{\sigma}^{(mk+p)}||^2 := \sum_{j=1}^{N_p} |\tilde{\sigma}^{(mk+p)}|^2 \quad \text{and} \quad ||\sigma^{(mk+p)}||^2 := \sum_{j=1}^{N_p} |\sigma^{(mk+p)}|^2.$$

Let

$$\widehat{C} := \sup_{\mathscr{X}} \left\{ \frac{||\tilde{\sigma}^{(0)}||^2 \omega^{(n+1)(m-1)}}{||\tilde{\sigma}^{(m-1)}||^2} \, , \, \frac{||\tilde{\sigma}^{(p+1)}||^2}{||\tilde{\sigma}^{(p)}||^2 \omega^{n+1}} \, ; \, 0 \leq p \leq m-2 \right\}.$$

We proceed using double induction on k and p.
($k = 0$) As far as extension there is nothing to prove; the sections $\sigma_j^{(p)}$ have extensions by assumption. Note that

$$\int_{\mathscr{X}} \frac{||\tilde{\sigma}^{(p+1)}||^2}{||\tilde{\sigma}^{(p)}||^2} \leq \widehat{C} \int_{\mathscr{X}} \omega^{n+1}.$$

($k \geq 1$) Assume the result has been proved for $k-1$.
(($p = 0$)): Consider the sections $s^{\otimes k} \otimes \sigma_j^{(0)}$ of $K_{X_o}^{\otimes mk} \otimes (L^{\otimes k} \otimes A)|_{X_o}$, and define the metric

$$\psi_{k,0} := \log ||\tilde{\sigma}^{(km-1)}||^2$$

for $K_{\mathscr{X}}^{\otimes mk-1} \otimes L^{\otimes k} \otimes A$. Observe that $\sqrt{-1}\partial\bar{\partial}\psi_{k,0} \geq 0$ and

$$\int_{X_o} |s^k \otimes \sigma_j^{(0)}|^2 e^{-(\psi_{k,0}+\kappa)} = \int_{X_o} \frac{|\sigma_j^{(0)}|^2}{||\sigma^{(m-1)}||^2} |s|^2 e^{-\kappa} < +\infty.$$

By the L^2 Extension Theorem 10.2.11 there exist sections

$$\tilde{\sigma}_j^{(km)} \in \Gamma_{\mathcal{O}}(\mathscr{X}, K_{\mathscr{X}}^{\otimes mk} \otimes L^{\otimes k} \otimes A), \quad 1 \leq j \leq N_o$$

such that

$$\tilde{\sigma}_j^{(km)}|X_o = s^{\otimes k} \otimes \sigma_j^{(0)} \wedge (d\pi)^{\otimes km}, \quad 1 \leq j \leq N_o,$$

and

$$\int_{\mathscr{X}} |\tilde{\sigma}_j^{(km)}|^2 e^{-(\psi_{k,0}+\kappa)} \le 48\pi \int_{X_o} |s|^2 \frac{|\sigma_j^{(0)}|^2}{||\sigma^{(m-1)}||^2} e^{-\kappa}.$$

Summing, we obtain

$$\begin{aligned}\int_{\mathscr{X}} \frac{||\tilde{\sigma}^{(km)}||^2 e^{-\gamma}}{||\tilde{\sigma}^{(km-1)}||^2} &\le \sup_{\mathscr{X}} e^{\kappa-\gamma} \int_{\mathscr{X}} \frac{||\tilde{\sigma}^{(km)}||^2 e^{-\kappa}}{||\tilde{\sigma}^{(km-1)}||^2} \\ &\le 48\pi \sup_{\mathscr{X}} e^{\kappa-\gamma} \int_{X_o} |s|^2 \frac{||\sigma^{(0)}||^2}{||\sigma^{(m-1)}||^2} e^{-\kappa} \\ &\le 48\pi \widehat{C} \sup_{\mathscr{X}} e^{\kappa-\gamma} \int_{X_o} |s|^2 \omega^{-n(m-1)} e^{-\kappa}.\end{aligned}$$

<u>$((1 \le p \le m-1))$</u>: Assume that we have obtained the sections $\tilde{\sigma}_j^{(km+p-1)},\ 1 \le j \le N_{p-1}$. Consider the non-negatively curved singular metric

$$\psi_{k,p-1} := \log ||\tilde{\sigma}^{(mk+p-1)}||^2$$

for $K_{\mathscr{X}}^{\otimes km+p-1} \otimes L^{\otimes k} \otimes A$. We have

$$\int_{X_o} |s^k \otimes \sigma_j^{(p)}|^2 e^{-\psi_{k,p-1}} \le \int_{X_o} \frac{|\sigma_j^{(p)}|^2}{\sum_{j=1}^{N_{p-1}} |\sigma_j^{(p-1)}|^2} < +\infty.$$

By the L^2 Extension Theorem 10.2.11 there exist sections

$$\tilde{\sigma}_j^{(km+p)} \in \Gamma_{\mathcal{O}}(\mathscr{X}, K_{\mathscr{X}}^{\otimes mk+p} \otimes L^{\otimes k} \otimes A), \quad 1 \le j \le N_p$$

such that

$$\tilde{\sigma}_j^{(km+p)} | X_o = s^{\otimes k} \otimes \sigma_j^{(p)} \wedge (d\pi)^{\otimes km+p}, \quad 1 \le j \le N_p,$$

and

$$\int_{\mathscr{X}} |\tilde{\sigma}_j^{(km+p)}|^2 e^{-\psi_{k,p-1}} \le 48\pi \int_{X_o} \frac{|\sigma_j^{(p)}|^2}{||\sigma^{(p-1)}||^2}.$$

Summing, we obtain

$$\int_{\mathscr{X}} \frac{||\tilde{\sigma}^{(km+p)}||^2}{||\tilde{\sigma}^{(km+p-1)}||^2} \le 48\pi \widehat{C} \int_{X_o} \omega^n.$$

Let

$$C = \widehat{C} \times \max\left\{ \int_{\mathscr{X}} \omega^{n+1}, 48\pi \sup_{\mathscr{X}} e^{\kappa-\gamma} \int_{X_o} |s|^2 \omega^{-n(m-1)} e^{-\kappa}, 48\pi \int_{X_o} \omega^n \right\}.$$

The proof is finished. □

10.2.3 Construction of the metric

Fix a smooth metric $e^{-\psi}$ for $A \to \mathscr{X}$. Consider the functions

$$\lambda_N := \log \left(||\tilde{\sigma}^{(km+p)}||^2 \omega^{-(n+1)(mk+p)} e^{-(k\gamma+\psi)} \right),$$

where $N - p = km$.

Observe that by Proposition 10.2.12 and the concavity of the logarithm, we have the bound

$$\frac{1}{\int_{\mathscr{X}} \omega^{n+1}} \int_{\mathscr{X}} (\lambda_N - \lambda_{N-1}) \omega^{n+1} \leq \log C.$$

(This inequality if the reverse of Jensen's inequality; it can be obtained by considering the convex function $-\log$.) It follows that the function

$$\Lambda_k = \frac{1}{k} \lambda_{mk}$$

satisfies the integral bound

$$\int_{\mathscr{X}} \Lambda_k \omega^{n+1} \leq mC'$$

for some uniform constant C'.

Now, locally, the functions λ_{km+p} are a sum of a smooth function and a subharmonic function. By applying the sub-mean value property for plurisubharmonic functions and making use of the uniform projectivity of the family, we find that

$$\Lambda_k(x) \leq C \int_{\mathscr{X}} \Lambda_k \leq mCC', \quad x \in \mathscr{X}.$$

It follows that the function

$$\Lambda(x) := \limsup_{y \to x} \limsup_{k \to \infty} \Lambda_k(y)$$

exists, and is locally the sum of a plurisubharmonic function and a continuous function on $\mathscr{X}$.

Consider the singular Hermitian metric $e^{-\mu}$ for $K_{\mathscr{X}}^{\otimes m} \otimes L \to \mathscr{X}$ defined by

$$e^{-\mu} = e^{-\Lambda} \omega^{-(n+1)m} e^{-\gamma}.$$

This singular metric is given by the formula

$$e^{-\mu(x)} = \liminf_{y \to x} \liminf_{k \to \infty} e^{-\mu_k},$$

where

$$e^{-\mu_k} = e^{-\Lambda_k} \omega^{-(n+1)m} e^{-\gamma}.$$

The curvature of $e^{-\mu_k}$ is thus

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\mu_k &= \sqrt{-1}\partial\bar{\partial} \log \sum_{j=1}^{N_o} |\tilde{\sigma}_j^{(mk)}|^2 - \frac{1}{k}\sqrt{-1}\partial\bar{\partial}\psi \\ &\geq -\frac{1}{k}\sqrt{-1}\partial\bar{\partial}\psi \end{aligned}$$

10.2.13 PROPOSITION. *The curvature of $e^{-\mu}$ is non-negative in the sense of currents.*

Proof. It suffices to work locally. Then we have that the function

$$\mu_k + \frac{1}{k}\psi$$

is plurisubharmonic. But

$$\limsup_{y \to x} \limsup_{k \to \infty} \mu_k + \frac{1}{k}\psi = \limsup_{y \to x} \limsup_{k \to \infty} \mu_k = \mu.$$

Thus μ is plurisubharmonic, as desired. □

10.2.4 Examination of $e^{-\mu}$ on the central fiber: conclusion of the proof

Since

$$\Lambda_k|_{X_o} = \log(|s|^2 \omega^{-(nm)} e^{-\gamma}) + \frac{1}{k}\left(\log \sum_{j=1}^{N_o} |\sigma_j^{(0)}|^2 e^{-\psi} + \log \frac{\sqrt{-1} d\pi \wedge d\bar{\pi}}{\omega}\right),$$

we obtain the inequality

$$e^{-\mu}|_{X_o} \le \frac{1}{|s|^2}.$$

Therefore

$$\int_{X_o} |s|^2 e^{-\frac{(m-1)\mu+\kappa}{m}} \le \int_{X_o} |s|^{2/m} e^{-\frac{\kappa}{m}} \le \left(\int_{X_o} \omega^n\right)^{\frac{m-1}{m}} \left(\int_{X_o} |s|^2 \omega^{-n(m-1)} e^{-\kappa}\right)^{\frac{1}{m}} < +\infty,$$

where the second inequality is a consequence of Hölder's Inequality.

An application of the L^2 extension theorem concludes the proof of Theorem 10.2.7 in the uniform case.

To pass to the general case, note that the L^2 Extension Theorem 10.2.11 yields a holomorphic section $S \in \Gamma_{\mathcal{O}}(\mathscr{X}, K_{\mathscr{X}}^{\otimes m} \otimes L)$ satisfying $S|_{X_o} = s \wedge d\pi^{\otimes m}$ and

$$(10.7) \qquad \int_{\mathscr{X}} |S|^2 e^{-\frac{(m-1)\mu+\kappa}{m}} \le 48\pi \left(\int_{X_o} \omega^n\right)^{\frac{m-1}{m}} \left(\int_{X_o} |s|^2 \omega^{-n(m-1)} e^{-\kappa}\right)^{\frac{1}{m}}.$$

We can therefore use Montel's Theorem to pass from the uniformly projective case to the general case of projective families. This completes the proof of Theorem 10.2.7. □

10.2.5 An $L^{2/m}$-estimate for m-canonical sections

Let us call a holomorphic family $\pi : X \to \mathbb{D}$ *uniformly projective* if there exists a projective family $\tilde{\pi} : \tilde{X} \to \mathbb{D}_{1+r}(0)$ such that $X \subset\subset \tilde{X}$ and $\tilde{\pi}|_X = \pi$. Then in fact we have actually proved the following theorem.

10.2.14 THEOREM. *Let $\pi : \mathscr{X} \to \mathbb{D}$ be a uniformly projective family, and let $L \to \mathscr{X}$ be a holomorphic line bundle with singular Hermitian metric $e^{-\kappa}$ whose curvature is a positive $(1,1)$-current. Then for any holomorphic section $s \in \Gamma_{\mathcal{O}}(X_o, K_{X_o}^{\otimes m} \otimes L)$ satisfying*

$$\int_{X_o} |s|^{2/m} e^{-\frac{1}{m}\kappa} < +\infty$$

there exists a holomorphic section $S \in \Gamma_{\mathcal{O}}(\mathscr{X}, K_{\mathscr{X}}^{\otimes m} \otimes L)$ such that

$$S|_{X_o} = s \otimes d\pi^{\otimes m} \quad \textit{and} \quad \int_{\mathscr{X}} |S|^{2/m} e^{-\frac{1}{m}\kappa} \le 48\pi \int_{X_o} |s|^{2/m} e^{-\frac{1}{m}\kappa}.$$

Proof. First suppose that Ω is a pseudoconvex domain in a Stein manifold $\tilde{X}$, that $L \to X$ is a holomorphic line bundle with smooth Hermitian metric $e^{-\kappa}$, and that $\pi : \tilde{X} \to \mathbb{D}$ is a holomorphic submersion such that π restricts to the closure of Ω as a proper map. Assume that we have a section $s \in \Gamma_{\mathcal{O}}(\tilde{X}_o, K_{\tilde{X}_o}^{\otimes m} \otimes L)$ that we wish to extend to Ω with $L^{2/m}$-estimates, and which we can normalize to satisfy

$$\int_{\Omega_o} |s|^{2/m} e^{-\kappa/m} = 1.$$

(Of course, $\Omega_o := \pi^{-1}(0) \cap \Omega$.)

A look at the proof of Theorem 10.2.7 shows that if there is a metric $e^{-\mu}$ for $K_{\Omega}^{\otimes m} \otimes L \to \Omega$ satisfying

$$e^{-\mu}|_{\Omega_o} \le |s|^{-2} \quad \text{and} \quad \sqrt{-1}\partial\bar{\partial}\mu \ge 0$$

there there is a holomorphic section $S_1 \in \Gamma_{\mathcal{O}}(\Omega, K_{\tilde{X}}^{\otimes m} \otimes L)$ satisfying

$$S_1|_{\Omega_o} = s \wedge d\pi^{\otimes m} \quad \text{and} \quad C_1 := \int_{\Omega} |S_1|^2 e^{-\frac{(m-1)\mu+\kappa}{m}} < +\infty.$$

(Conversely, if such a section S_1 exists, one can take $e^{-\mu} = (|S|^2 e^{-\kappa})^{-1}$, so extension is equivalent to the existence of such a metric.) Moreover, in the proof of Theorem 10.2.7 we constructed one such metric $e^{-\mu}$, which in the present setting can be assumed to satisfy

$$\int_{\Omega} e^{(\mu-\kappa)/m} < +\infty.$$

Now, by Hölder's Inequality,

$$\int_{\Omega} |S_1|^{2/m} e^{-\frac{\kappa}{m}} = \int_{\Omega} \left(|S_1|^2 e^{-\frac{(m-1)\mu+\kappa}{m}} \right)^{1/m} e^{\frac{(m-1)(\mu-\kappa)}{m^2}} \le C_1^{1/m} \left(\int_{\Omega} e^{(\mu-\kappa)/m} \right)^{\frac{m-1}{m}} =: A_1.$$

Without loss of generality we may assume $A_1 > 48\pi$. Then we have shown that given any metric $e^{-\mu}$ as above, there exists a section S_1 such that

$$S_1|_{\Omega_o} = s \wedge d\pi^{\otimes m} \quad \text{and} \int_\Omega |S_1|^{2/m} e^{-\frac{\kappa}{m}} \le A_1$$

for some constant $A_1 > 48\pi$.

Let us now define the metric

$$\mu_1 := \frac{(m-1)\log|S_1|^2 + \kappa}{m}.$$

Then

$$\int_{\Omega_o} |s|^2 e^{-\mu_1} = \int_{\Omega_o} |s|^{2/m} e^{-\kappa/m} = 1,$$

and thus by the L^2 Extension Theorem 10.2.11 there exists a section $S_2 \in \Gamma_{\mathcal{O}}(\Omega, K_\Omega^{\otimes m} \otimes L)$ such that

$$S_2|_{\Omega_o} = s \otimes d\pi^{\otimes m} \quad \text{and} \quad \int_\Omega |S_2|^2 e^{-\mu_1} \le 48\pi.$$

Thus by Hölder's Inequality

$$\int_\Omega |S_2|^{2/m} e^{-\frac{\kappa}{m}} \le \left(\int_\Omega |S_2|^2 e^{-\mu_1}\right)^{1/m} \left(\int_\Omega e^{\frac{\mu_1}{m-1}}\right)^{(m-1)/m} \le (48\pi)^{1/m} A_1^{(m-1)/m} =: A_2.$$

Continuing by induction, we obtain sections $S_j \in \Gamma_{\mathcal{O}}(\Omega, K_\Omega^{\otimes m} \otimes L)$ such that

$$S_j|_{\Omega_o} = s \otimes d\pi^{\otimes m} \quad \text{and} \quad \int_\Omega |S_j|^{2/m} e^{-\frac{\kappa}{m}} \le A_j,$$

where $A_j = (48\pi)^{1/m} A_{j-1}^{(m-1)/m} < A_{j-1}$. Since A_j is decreasing and larger than 48π, A_j converges to some A, and taking limits of the inductive relation yields

$$A = (48\pi)^{1/m} A^{(m-1)/m}.$$

Thus $A = 48\pi$. We have thus constructed, for every $\varepsilon > 0$, a sequence of sections S_j, $j \ge j_o$ such that

$$\int_\Omega |S_j|^{2/m} e^{-\kappa/m} \le A + \varepsilon.$$

Using arguments that are by now familiar, we can apply Montel's Theorem to let $\varepsilon \to 0$, pass to a singular metric $e^{-\kappa}$, and let $\Omega \to \tilde{X}$. Next, we can take $\tilde{X}$ to be the Stein manifold obtained from a uniform projective family $\mathscr{X} \to \mathbb{D}$ by removing a hyperplane section, and use the L^2 estimates to extend S across the hyperplane section. Finally, we can again use Montel to pass to the case of any projective family. The proof is therefore complete. □

EXERCISES

Lecture 11

Berndtsson's Theorem on Plurisubharmonic Variation

11.1 Berndtsson's Theorem

11.1.1 Variations of Hilbert spaces

Let (X, ω) be a Kähler manifold and let $L \to X$ be a holomorphic line bundle. We equip L with a smooth Hermitian metric $e^{-\varphi}$, and thus we can define the Hilbert space

$$\mathscr{H}(\varphi) := \left\{ f \in \Gamma_{\mathcal{O}}(X, L)\ ;\ \int_X |f|^2 e^{-\varphi} dV_\omega < +\infty \right\},$$

where dV_ω is the volume form on X induced by the metric ω. Our goal is to study what happens to the Hilbert space $\mathscr{H}(\varphi)$ as we vary the metric $e^{-\varphi}$ in the sense of the next definition.

11.1.1 DEFINITION. *Let $\Omega \subset \mathbb{C}^n$ be an open connected set. A family of metrics for $L \to X$ parameterized by Ω is a metric $e^{-\varphi}$ for the line bundle $\mathbf{p}^* L \to X \times \Omega$, where $\mathbf{p} : X \times \Omega \to X$ is the projection to the first factor.*

For each $t \in \Omega$ there is a natural isomorphism of line bundles

$$\iota_t : L \to \mathbf{p}^* L|_{X \times \{t\}},$$

and we write

$$e^{-\varphi_t} := \iota_t^* e^{-\varphi}$$

for the metric induced on L by this isomorphism. We can then define the Hilbert spaces

$$\mathscr{H}(\varphi_t) = \left\{ f \in \Gamma_{\mathcal{O}}(X, L)\ ;\ ||f||_t^2 := \int_X |f|^2 e^{-\varphi_t} dV_\omega < +\infty \right\}.$$

We therefore obtain a fibration $\mathscr{H}(\varphi) \to \Omega$ whose fiber $\mathscr{H}(\varphi)_t$ over t is the Hilbert space $\mathscr{H}(\varphi_t)$. We want to know when this fibration is (locally) trivial.

11.1.1. Let $X = \mathbb{C}^n$ with its Euclidean metric $\sqrt{-1}\partial\bar{\partial}|z|^2$ and let $\varphi(z,t) = |t|^2|z|^2$. Show that for $t \neq 0$ the Hilbert spaces $\mathscr{H}(\varphi)_t$ are all equal, as subsets of $\mathcal{O}(\mathbb{C}^n)$. What is $\mathscr{H}(\varphi)_o$?

To avoid the situation that occurs in Exercise 11.1.1, namely that the Hilbert spaces change as vector spaces when we change the base parameter t, we assume from now on that X is a bounded pseudoconvex domain in some larger Stein manifold $\tilde{X}$, that L is the restriction to X of a holomorphic line bundle on $\tilde{X}$, and that the family of smooth metrics $e^{-\varphi}$ also extends to $\tilde{X}$.

Under these assumptions, the vector spaces $\mathscr{H}(\varphi)_t$, seen as subsets of $\Gamma_{\mathcal{O}}(X, L)$, are independent of t. Of course, their norms vary. In fact, for each $t, s \in \Omega$ the function $e^{-\varphi_s}/e^{-\varphi_t}$ is bounded above by a positive constant $C_{s,t}$ (and therefore below by $1/C_{t,s}$), and thus

$$\frac{1}{C_{t,s}} \int_X |f|^2 e^{-\varphi_t} dV_\omega \leq \int_X |f|^2 e^{-\varphi_s} dV_\omega \leq C_{s,t} \int_X |f|^2 e^{-\varphi_t} dV_\omega,$$

so $\mathscr{H}(\varphi)_t$ and $\mathscr{H}(\varphi)_s$ are quasi-isometric as Hilbert spaces (i.e. they have equivalent norms). It follows that their dual vector spaces are also quasi-isometric. (Note, also, that under these hypotheses these Hilbert spaces are *always* infinite-dimensional.) But more important for us is the fact that in this case, the subspace

$$\mathscr{H}(\varphi)_t \subset \Gamma_{\mathcal{O}}(X, L)$$

is independent of t.

We can now define the trivial bundle

$$\mathscr{H}(\varphi) := \mathscr{H}(\varphi)_o \times \Omega \to \Omega$$

and define a Hilbert metric on $\mathscr{H}(\varphi)$ by endowing the fiber $\mathscr{H}(\varphi)_o \times \{t\}$ with the norm $||\cdot||_t$; thus the fiber is $\mathscr{H}(\varphi)_t$ as a Hilbert space.

11.1.2 DEFINITION. *Let $L \to X$ be a holomorphic line bundle and let $e^{-\varphi}$ be a family of Hermitian metrics for L.*

(S) A section of $\mathscr{H}(\varphi) \to \Omega$ is a section $\mathfrak{f} \in \Gamma(X \times \Omega, \mathbf{p}^*L)$ such that $\iota_t^*\mathfrak{f} \in \mathscr{H}(\varphi)_t$ for each $t \in \Omega$. The section $\mathfrak{f}$ is said to be holomorphic if $\mathfrak{f} \in \Gamma_{\mathcal{O}}(X \times \Omega, \mathbf{p}^*L)$. In this case, we write

$$\mathfrak{f} \in \Gamma_{\mathcal{O}}(\Omega, \mathscr{H}(\varphi)).$$

(Thus all sections are holomorphic on the fibers, and a holomorphic section means it is holomorphic in the base variable as well.)

(B) Let $\mathscr{H}(\varphi)^* \to \Omega$ denote the dual bundle, i.e., the trivial bundle $\mathscr{H}(\varphi)_o^* \times \Omega \to \Omega$ with the Hilbert norms*

$$||\xi||_{t*} := \sup_{f \in \mathscr{H}(\varphi)_t - \{0\}} \frac{|\langle \xi, f \rangle|}{||f||_t}$$

on the fibers $\mathscr{H}(\varphi)_t^*$.

(S) A section of $\mathscr{H}(\varphi)^* \to \Omega$ is a map $\xi : \mathscr{H}(\varphi) \to \mathbb{C}$ such that*

$$\xi_t := \xi|_{\mathscr{H}(\varphi)_t} \in \mathscr{H}(\varphi)^*_t.$$

The section ξ of $\mathscr{H}(\varphi)^* \to \Omega$ is said to be holomorphic if for each $f \in \mathscr{H}(\varphi)_o$ the function

$$\Omega \ni t \mapsto \langle \xi_t, f \rangle \in \mathbb{C}$$

is holomorphic. The set of holomorphic sections is denoted $\Gamma_{\mathcal{O}}(\Omega, \mathscr{H}(\varphi))$.

11.1.3 REMARK. Note that since $\mathscr{H}(\varphi)_t = \mathscr{H}(\varphi)_o$ as subspaces of $\Gamma_{\mathcal{O}}(X, L)$, every $f \in \mathscr{H}(\varphi)_o$ induces a constant (and thus, holomorphic) section $\mathfrak{f}_f$ of $\mathscr{H}(\varphi) \to \Omega$ defined by

$$\iota_t^* \mathfrak{f}_f = f, \quad t \in \Omega.$$

We shall abusively denote this section by f, rather than $\mathfrak{f}_f$. ◇

11.1.2. Show that $\xi \in \Gamma_{\mathcal{O}}(\Omega, \mathscr{H}(\varphi)^*)$ if and only if the function

$$\Omega \ni t \mapsto \langle \xi_t, \iota_t^* \mathfrak{f} \rangle \in \mathbb{C}$$

is holomorphic for each holomorphic section $\mathfrak{f} \in \Gamma_{\mathcal{O}}(\Omega, \mathscr{H}(\varphi))$.

11.1.4 REMARK. Fix a Hilbert basis $\{f_1, f_2, ...\}$ for $\mathscr{H}(\varphi)_o$. Then for any section $\mathfrak{g}$ of $\mathscr{H}(\varphi) \to \Omega$ one has the *Fourier series*

$$\iota_t^* \mathfrak{g} = \sum_i c_i(t) f_i$$

Note that

$$\frac{\partial}{\partial \bar{t}} \iota_t^* \mathfrak{g} = \sum_i \frac{\partial c_i(t)}{\partial \bar{t}} f_i = \iota_t^* \left(\tfrac{\partial}{\partial \bar{t}} \lrcorner \bar{\partial} \mathfrak{g} \right).$$

The last equality holds because $\mathfrak{f}$ is holomorphic on the fibers of $\mathbf{p}$. Thus

$$\iota_t^* \bar{\partial} \mathfrak{g} = \left(\sum_i \frac{\partial c_i(t)}{\partial \bar{t}} f_i \right) \otimes d\bar{t},$$

where the $\bar{\partial}$ operator on the left is the one on $\mathbf{p}^* L \to X \times \Omega$. This fact parses well with the definition of the $\bar{\partial}$-operator for a holomorphic vector bundle. ◇

11.1.5 EXAMPLE. Let $\mathfrak{g} \in \Gamma_{\mathcal{O}}(\Omega, \mathscr{H}(\varphi))$. Then, with

$$\xi_t^{\mathfrak{g}}(\mathfrak{f}) := (\iota_t^* \mathfrak{f}, \iota_t^* \mathfrak{g})_t,$$

$\xi^{\mathfrak{g}}$ is a section of $\mathscr{H}(\varphi)^* \to \Omega$, but this section is almost never holomorphic.

11.1.6 EXAMPLE. Let $x \in X$ and let $\tilde{\xi}^x$ be defined by

$$\tilde{\xi}_t(\mathfrak{f}) := \iota_t^* \mathfrak{f}(x).$$

Then $\tilde{\xi}^x : \mathscr{H}(\varphi)_t \to L_x$ is a linear map, and it is bounded provided we equip the complex line L_x with the metric $e^{-\varphi_t(x)}$. Indeed, the boundedness is the inequality

$$|f(x)|^2 e^{-\varphi_t(x)} \le C_x ||f||_t^2,$$

which is often called *Bergman's Inequality*, and is proved as follows: first, when X is a domain in $\mathbb{C}^n$ and the weight φ is 0, then the inequality is standard (and could be proved, for instance, from Cauchy's Theorem). Then the inequality holds locally for any smooth metric $e^{-\varphi}$ by the obvious estimate $C^{-1} dV_o \le e^{-\varphi} dV_\omega \le C dV_o$ combined with the unweighted Bergman inequality, where dV_o is the Euclidean volume form in the local coordinate. Finally, the L^2 norm on a small ball is obviously controlled by the L^2 norm over all of X.

11.1.3. Fill in the details of the proof of Bergman's Inequality.

To obtain a linear functional from $\tilde{\xi}^x$ we need a non-zero vector for L_x^*, and we choose a vector $\mathbf{e}$ whose norm with respect to the dual metric e^{φ_t} is 1: $|\mathbf{e}|^2 e^{\varphi_t(x)} = 1$. (This vector is of course unique up to a unimodular factor.) We then set

$$\xi^x := \mathbf{e} \otimes \tilde{\xi}^x.$$

The section ξ^x is now a bounded linear functional on each fiber $\mathscr{H}(\varphi)_t$, and the holomorphic dependence on t is clear because for each $f \in \mathscr{H}(\varphi)_o$ $\xi^x f = f(x)$ is independent of t.

By the Riesz Representation Theorem, for each $x \in X$ there is a section $K_t^x \in \mathscr{H}(\varphi)_t$ such that

$$(f, K_t^x)_t = f(x), \quad f \in \mathscr{H}(\varphi)_t.$$

Writing

$$K_t(x, z) := \overline{K_t^x(z)},$$

we see that $K_t(x, \cdot) \in \Gamma\left(X^\dagger, L^\dagger\right)$ for each $x \in X$. (Here, for a complex manifold Y the notation $Y^\dagger$ means the manifold with the complex conjugate structure.) Moreover, since

$$\left(f, \overline{K_t(x, \cdot)}\right)_t = \int_X f(z) K_t(x, z) e^{-\varphi_t(z)} dV_\omega(z) = f(x),$$

we see that the section $x \mapsto K_t(x, z)$ of $L \to X$ is holomorphic for each $z \in X$. It follows that

$$K_t \in \Gamma_{\mathcal{O}}\left(X \times X^\dagger, L \boxtimes L^\dagger\right),$$

where for (holomorphic) line bundles $\pi_1 : L_1 \to X_1$ and $\pi_2 : L_2 \to X_2$,

$$L_1 \boxtimes L_2 := \pi_1^* L_1 \otimes \pi_2^* L_2.$$

The section K_t is called the *Bergman kernel* of $\mathscr{H}(\varphi)_t$, and we shall meet it again soon.

11.1.4. Show that if $\{g_j\}$ is any orthonormal basis for $\mathscr{H}(\varphi)_t$ then

$$K_t(x,y) = \sum_j g_j(x) \otimes \overline{g_j(y)}.$$

Show also that

$$K_t(x,x)e^{-\varphi_t(x)} = \sup\left\{|f(x)|^2 e^{-\varphi_t(x)}\ ;\ f \in \mathscr{H}(\varphi)_t \text{ and } ||f||_t = 1\right\}.$$

(Hint: Given $x \in X$, if $S_x := \{f \in \mathscr{H}(\varphi)_t\ ;\ f(x) = 0\}$, show that $S_x^\perp$ has dimension 0 or 1.)

11.1.2 Statement of Berndtsson's Theorem

We are now ready to state Berndtsson's Theorem.

11.1.7 THEOREM (Berndtsson's Theorem on Plurisubharmonic Variation). *Let X be a relatively compact, pseudoconvex domain in a Stein Kähler manifold$(\tilde{X}, \omega)$ and let $L \to \tilde{X}$ be a holomorphic line bundle. Let $\Omega \subset\subset \mathbb{C}^n$ be an open connected set. Let $e^{-\varphi}$ be a family of Hermitian metrics for $L \to \tilde{X}$ (in the sense of Definition 11.1.1). If the Hermitian $(1,1)$-form*

$$\partial\bar{\partial}\varphi + \mathbf{p}^*\omega$$

is non-negative (resp. positive) then the curvature of Chern connection for the holomorphic vector bundle $\mathscr{H}(\varphi) \to \Omega$, with its L^2 metric $||\cdot||_t^2$, is non-negative (resp. positive) in the sense of Nakano.

Note that the result is local with respect to the base Ω, so that from now on we can (and will) assume that Ω is the unit ball.

We shall not prove Theorem 11.1.7 in its full generality, but rather assume that the base domain Ω is 1-dimensional, i.e., the unit disk. In this situation Griffiths and Nakano positivity are the same. In view of Proposition 3.2.7, the case of Theorem 11.1.7 in which $\Omega \subset \mathbb{C}$ is equivalent to the following theorem.

11.1.8 THEOREM. *Assume that for the metric $e^{-\varphi}$ for $\mathbf{p}^*L \to X \times \mathbb{D}$ we have*

$$\partial\bar{\partial}\varphi + \mathbf{p}^*\mathrm{Ricci}(\omega) \geq 0 \textit{ (resp. } > 0),$$

where $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$. Let ξ be a holomorphic section of $\mathscr{H}(\varphi) \to \mathbb{D}$ that is not identically zero. Then the function

$$\mathbb{D} \ni t \mapsto \log ||\xi_t||_{t*}^2$$

is subharmonic (resp. strictly subharmonic).

Though we will not explain precisely why, the proof of Theorem 11.1.7 is not more complicated than that of Theorem 11.1.8; it just requires a little more notation and discussion. We have chosen to skip the more general proof because we will only look at applications of Theorem 11.1.8.

11.1.3 The Bergman projection

As we will see very soon, the proof of Berndtsson's Theorem requires a more systematic understanding of solutions of the $\bar{\partial}$-equation with minimal norm.

Fixing our X as in the previous paragraph, consider the larger Hilbert space $L^2(e^{-\varphi_t})$ defined to be the Hilbert space closure of the vector space of all the smooth sections of $L \to X$ with compact support, the closure being taken with respect to the norm

$$||f||_t := \left(\int_X |f|^2 e^{-\varphi_t} dV_\omega \right)^{1/2}.$$

Using the sub-mean value property and a Montel type argument, one shows that $\mathscr{H}(\varphi)_t$ is a closed subspace of $L^2(e^{-\varphi_t})$. Consequently there is an orthogonal projection

$$P_t : L^2(e^{-\varphi_t}) \to \mathscr{H}(\varphi)_t,$$

called the Bergman projection.

There is an important link between the Bergman projection and the minimal solution of the $\bar{\partial}$-equation. Indeed, suppose u_1, u_2 are solutions of the equation $\bar{\partial} u = \alpha$. Then $u_1 - u_2$ is holomorphic. It follows that the solution of minimal norm is orthogonal to the holomorphic subspace. Consequently, given solution u of $\bar{\partial} u = \alpha$, the solution

$$u_{\min} := u - P_t(u) \in L^2(e^{-\varphi_t})$$

is the solution of minimal norm.

In particular, the norm of this solution is no larger than the norm of the solution provided by Skoda's version of Hörmander's Theorem when the latter applies. Consequently we have the following result.

11.1.9 PROPOSITION. *Fix $t \in \mathbb{D}$. If the curvature of the metric $e^{-\varphi_t}$ for $L \to X$ satisfies*

$$\partial\bar{\partial}\varphi_t + \mathrm{Ricci}(\omega) \geq \Theta$$

for some non-negative $(1,1)$-form Θ on X then

$$\int_X |u - P_t(u)|^2 e^{-\varphi_t} dV_\omega \leq \int_X |\bar{\partial} u|^2_\Theta e^{-\varphi_t} dV_\omega$$

for all u such that the right hand side is finite.

11.1.10 REMARK. Note that the orthogonal projection operator $P_t : L^2(e^{-\varphi_t}) \to \mathscr{H}(\varphi)_t$ is an integral operator. Indeed, if $f \in \mathscr{H}(\varphi)_t$ then $P_t f = f$. Thus for each $x \in X$ and $f \in \mathscr{H}(\varphi)_t$, $P_t f(x) = \langle \xi^x, f \rangle$, where ξ^x was defined in Example 11.1.6. It follows that

$$P_t f(x) = \int_X f(z) K_t(x,z) e^{-\varphi_t(z)} dV_\omega(z), \quad f \in L^2(e^{-\varphi_t}),\ x \in X,$$

where $K_t \in \Gamma_{\mathcal{O}}\left(X \times X^\dagger, L \boxtimes L^\dagger\right)$ is the Bergman kernel. Indeed, if $f = P_t f + f_t^\perp$ is the orthogonal decomposition of f in $\mathscr{H}(\varphi)_t \oplus \mathscr{H}(\varphi)_t^\perp$ then the holomorphicity of the Bergman kernel K_t implies that $(f_t^\perp, K_t^x)_t = 0$. $\diamond$

11.1.4 Proof of Theorem 11.1.8

An estimate for the Laplacian of $-\log ||\mathfrak{f}||_t^2$

For simplicity, let us use the notation

$$\mathfrak{f}_t := \iota_t^* \mathfrak{f}$$

for a section $\mathfrak{f}$ of $\mathscr{H}(\varphi) \to \mathbb{D}$.

11.1.11 LEMMA. *If* $\mathfrak{f} \in \Gamma_{\mathcal{O}}(\mathbb{D}, \mathscr{H}(\varphi))$ *is holomorphic up to the boundary then*

$$\frac{\partial}{\partial t} ||\mathfrak{f}||_t^2 = \left(\frac{\partial \mathfrak{f}_t}{\partial t} - P_t \left(\frac{\partial \varphi_t}{\partial t} \mathfrak{f}_t \right), \mathfrak{f}_t \right)_t .$$

Proof. We calculate that

$$\frac{\partial}{\partial t} \int_X |\mathfrak{f}_t|^2 e^{-\varphi_t} dV_\omega = \int_X \left\langle \frac{\partial \mathfrak{f}_t}{\partial t} - \frac{\partial \varphi_t}{\partial t} \mathfrak{f}_t, \mathfrak{f}_t \right\rangle e^{-\varphi_t} dV_\omega = \int_X \left\langle \frac{\partial \mathfrak{f}_t}{\partial t} - P_t \left(\frac{\partial \varphi_t}{\partial t} \mathfrak{f}_t \right), \mathfrak{f}_t \right\rangle e^{-\varphi_t} dV_\omega,$$

where the last equality holds because for any smooth section u of $L \to X$ the section $u - P_t u$ is orthogonal to $\mathscr{H}(\varphi)_t$. □

11.1.12 DEFINITION. *We define the operator* $\nabla^{1,0}_{\frac{\partial}{\partial t}}$ *on sections of* $\mathscr{H}(\varphi) \to \mathbb{D}$ *by*

$$\iota_t^* (\nabla^{1,0}_{\frac{\partial}{\partial t}} \mathfrak{f}) = (\nabla^{1,0}_{\frac{\partial}{\partial t}} \mathfrak{f})_t := \frac{\partial \mathfrak{f}_t}{\partial t} - P_t \left(\frac{\partial \varphi_t}{\partial t} \mathfrak{f}_t \right).$$

11.1.13 LEMMA. *Let* $\mathfrak{f} \in \Gamma_{\mathcal{O}}(\mathbb{D}, \mathscr{H}(\varphi))$ *be holomorphic up to the boundary. Then*

$$\begin{aligned} &\frac{\partial^2}{\partial t \partial \bar{t}} ||\mathfrak{f}_t||_t^2 \\ &= - \int_X \frac{\partial^2 \varphi_t}{\partial t \partial \bar{t}} |\mathfrak{f}_t|^2 e^{-\varphi_t} dV_\omega + \left|\left| (\nabla^{1,0}_{\frac{\partial}{\partial t}} \mathfrak{f})_t \right|\right|_t^2 + \left|\left| \left(\frac{\partial \mathfrak{f}_t}{\partial t} - \frac{\partial \varphi_t}{\partial t} \mathfrak{f}_t \right) - P_t \left(\frac{\partial \mathfrak{f}_t}{\partial t} - \frac{\partial \varphi_t}{\partial t} \mathfrak{f}_t \right) \right|\right|_t^2 \\ &= - \int_X \frac{\partial^2 \varphi_t}{\partial t \partial \bar{t}} |\mathfrak{f}_t|^2 e^{-\varphi_t} dV_\omega + \left|\left| (\nabla^{1,0}_{\frac{\partial}{\partial t}} \mathfrak{f})_t \right|\right|_t^2 + \left|\left| P_t \left(\frac{\partial \varphi_t}{\partial t} \mathfrak{f}_t \right) - \frac{\partial \varphi_t}{\partial t} \mathfrak{f}_t \right|\right|_t^2 . \end{aligned}$$

Proof. In the proof of Lemma 11.1.11 we established the equality

$$\frac{\partial}{\partial t} \int_X |\mathfrak{f}_t|^2 e^{-\varphi_t} dV_\omega = \int_X \left\langle \frac{\partial \mathfrak{f}_t}{\partial t} - \frac{\partial \varphi_t}{\partial t} \mathfrak{f}_t, \mathfrak{f}_t \right\rangle e^{-\varphi_t} dV_\omega.$$

Starting from here, we compute that

$$\frac{\partial^2}{\partial t \partial \bar{t}} ||\mathfrak{f}_t||_t^2 = \int_X \left\langle - \frac{\partial^2 \varphi_t}{\partial t \partial \bar{t}} \mathfrak{f}_t, \mathfrak{f}_t \right\rangle e^{-\varphi_t} dV_\omega + \int_X \left| \frac{\partial \mathfrak{f}_t}{\partial t} - \frac{\partial \varphi_t}{\partial t} f \right|^2 e^{-\varphi_t} dV_\omega,$$

Now, by Pythagoras' Theorem, for any smooth section u one has

$$\int_X |u|^2 e^{-\varphi_t} dV_\omega = \int_X |P_t u|^2 dV_\omega + \int_X |u - P_t u|^2 dV_\omega,$$

so with $u = \frac{\partial \mathfrak{f}_t}{\partial t} - \frac{\partial \varphi_t}{\partial t} f$ the result follows. □

11.1.14 LEMMA. *Let $f \in \mathscr{H}(\varphi)_o$ and assume, moreover, that*

$$\sqrt{-1}\partial\bar{\partial}\varphi_t + \mathrm{Ricci}(\omega) \geq \Theta$$

for some positive Hermitian form ω on X. Then

$$\left|\left|\frac{\partial\varphi_t}{\partial t}f - P_t\left(\frac{\partial\varphi_t}{\partial t}f\right)\right|\right|^2_t \leq \int_X \left|\bar{\partial}_X\left(\frac{\partial\varphi_t}{\partial t}\right)\right|^2_\Theta |f|^2 e^{-\varphi_t}dV_\omega.$$

Proof. Since $\bar{\partial}_X\left(\frac{\partial\varphi_t}{\partial t}f\right) = -\left(\bar{\partial}_X\frac{\partial\varphi_t}{\partial t}\right)f$, the result follows immediately from Proposition 11.1.9. □

Next we calculate that

$$-||\mathfrak{f}_t||^2_t\frac{\partial^2}{\partial t\partial\bar{t}}\log||\mathfrak{f}_t||^2_t = -\frac{\partial^2||\mathfrak{f}_t||^2_t}{\partial t\partial\bar{t}} + \frac{1}{||\mathfrak{f}_t||^2_t}\left|\frac{\partial||\mathfrak{f}_t||^2_t}{\partial t}\right|^2.$$

By Lemma 11.1.11

$$\frac{1}{||\mathfrak{f}_t||^2_t}\left|\frac{\partial||\mathfrak{f}_t||^2_t}{\partial t}\right|^2 = \frac{|((\nabla^{1,0}_{\frac{\partial}{\partial t}}\mathfrak{f})_t,\mathfrak{f}_t)_t|^2}{||\mathfrak{f}_t||^2_t}$$

and if we also assume that $\sqrt{-1}\partial\bar{\partial}\varphi_t \geq \omega$, then by Lemmas 11.1.13 and 11.1.14

$$\begin{aligned}-\frac{\partial^2||\mathfrak{f}_t||^2_t}{\partial t\partial\bar{t}} &= \int_X \frac{\partial^2\varphi_t}{\partial t\partial\bar{t}}|\mathfrak{f}_t|^2e^{-\varphi_t}dV_\omega - \left|\left|(\nabla^{1,0}_{\frac{\partial}{\partial t}}\mathfrak{f})_t\right|\right|^2_t - \left|\left|\frac{\partial\varphi_t}{\partial t}\mathfrak{f}_t - P_t\left(\frac{\partial\varphi_t}{\partial t}\mathfrak{f}_t\right)\right|\right|^2_t \\ &\geq \int_X\left(\frac{\partial^2\varphi_t}{\partial t\partial\bar{t}} - \left|\bar{\partial}_X\left(\frac{\partial\varphi_t}{\partial t}\right)\right|^2_\Theta\right)|\mathfrak{f}_t|^2e^{-\varphi_t}dV_\omega - \left|\left|\left(\nabla^{1,0}_{\frac{\partial}{\partial t}}\mathfrak{f}_t\right)\right|\right|^2_t.\end{aligned}$$

We have therefore proved the following theorem.

11.1.15 THEOREM. *Let $\mathfrak{f} \in \Gamma_{\mathcal{O}}(\mathbb{D},\mathscr{H}(\varphi))$ be holomorphic up to the boundary and assume, moreover, that $\sqrt{-1}\partial\bar{\partial}\varphi_t + \mathrm{Ricci}(\omega) \geq \Theta$ for some positive Hermitian form ω on X. Then*

$$\begin{aligned}\frac{\partial^2}{\partial t\partial\bar{t}}\log\frac{1}{||\mathfrak{f}_t||^2_t} &\geq \frac{1}{||\mathfrak{f}_t||^2_t}\int_X\left(\frac{\partial^2\varphi_t}{\partial t\partial\bar{t}} - \left|\bar{\partial}_X\left(\frac{\partial\varphi_t}{\partial t}\right)\right|^2_\Theta\right)|\mathfrak{f}_t|^2e^{-\varphi_t}dV_\omega \\ &\quad + \frac{1}{||\mathfrak{f}_t||^2_t}\left(\frac{|((\nabla^{1,0}_{\frac{\partial}{\partial t}}\mathfrak{f})_t,\mathfrak{f}_t)_t|^2}{||\mathfrak{f}_t||^2_t} - \left|\left|(\nabla^{1,0}_{\frac{\partial}{\partial t}}\mathfrak{f})_t\right|\right|^2_t\right) \qquad (11.1)\end{aligned}$$

End of the proof of Theorem 11.1.8

Let us assume first that $\Theta := \sqrt{-1}(\partial\bar{\partial}\varphi_t + \mathrm{Ricci}(\omega))$ is a Kähler form for each $t \in \mathbb{D}$. We claim that if $\partial\bar{\partial}\varphi + \mathbf{p}^*\mathrm{Ricci}(\omega)$ is non-negative $X \times \mathbb{D}$ then

$$\frac{\partial^2\varphi_t}{\partial t\partial\bar{t}} - \left|\bar{\partial}_X\left(\frac{\partial\varphi_t}{\partial t}\right)\right|^2_\Theta \geq 0. \qquad (11.2)$$

Indeed, with respect to the product structure of $X \times \mathbb{D}$ we compute that

$$\partial\bar{\partial}\varphi = \frac{\partial^2 \varphi_t}{\partial t \partial \bar{t}} dt \wedge d\bar{t} + \partial_X \frac{\partial \varphi_t}{\partial \bar{t}} \wedge dt - \bar{\partial}_X \frac{\partial \varphi_t}{\partial t} \wedge dt + \partial\bar{\partial}\varphi_t,$$

and therefore

$$\begin{aligned} &\frac{(\sqrt{-1}(\partial\bar{\partial}\varphi + \mathbf{p}^*\mathrm{Ricci}(\omega)))^{n+1}}{(n+1)!} \\ &= \left(\frac{\partial^2 \varphi_t}{\partial t \partial \bar{t}}\right) \frac{\Theta^n}{n!} \wedge dt \wedge d\bar{t} - \left(\partial_X \frac{\partial \varphi_t}{\partial \bar{t}}\right) \wedge \left(\bar{\partial}_X \frac{\partial \varphi_t}{\partial t}\right) \wedge \frac{\Theta^{n-1}}{(n-1)!} \\ &= \left(\frac{\partial^2 \varphi_t}{\partial t \partial \bar{t}} - \left|\bar{\partial}_X \left(\frac{\partial \varphi_t}{\partial t}\right)\right|^2_{\Theta}\right) \frac{\Theta^n}{n!} \wedge dt \wedge d\bar{t}, \end{aligned}$$

which establishes (11.2) in this case. In the general case, since X is a bounded pseudoconvex domain in a Stein manifold, X has a negative strictly plurisubharmonic function $u \in \mathscr{C}^\infty(\overline{X})$. Replacing $e^{-\varphi_t}$ by $e^{-(\varphi_t + \varepsilon u)}$ (which does not change the underlying vector space of the Hilbert space), we have

$$\frac{\partial^2 \varphi_t}{\partial t \partial \bar{t}} - \left|\bar{\partial}_X \left(\frac{\partial \varphi_t}{\partial t}\right)\right|^2_{\Theta + \varepsilon\sqrt{-1}\partial\bar{\partial}u} \geq 0,$$

and letting $\varepsilon \to 0$ yields the desired positivity.

Let us fix a holomorphic section ξ of $\mathscr{H}(\varphi)^* \to \mathbb{D}$ and assume that

$$\sqrt{-1}(\partial\bar{\partial}\varphi_t + \mathrm{Ricci}(\omega)) =: \Theta \geq 0.$$

First, we note that by Theorem 11.1.15, if $\mathfrak{f} \in \Gamma_{\mathcal{O}}(\mathbb{D}, \mathscr{H}(\varphi))$ satisfies

$$(\nabla^{1,0}_{\frac{\partial}{\partial t}} \mathfrak{f})_t = 0$$

for some $t \in \mathbb{D}$, then for that same t we have

$$\left(\frac{\partial^2}{\partial \tau \partial \bar{\tau}} \log \frac{|\langle \xi_\tau, \mathfrak{f}_\tau \rangle|^2}{||\mathfrak{f}_\tau||^2}\right)_{\tau = t} \geq \frac{1}{||\mathfrak{f}_t||_t^2} \int_X \left(\frac{\partial^2 \varphi_t}{\partial t \partial \bar{t}} - \left|\bar{\partial}_X \left(\frac{\partial \varphi_t}{\partial t}\right)\right|^2_{\Theta}\right) |\mathfrak{f}_t|^2 e^{-\varphi_t} dV_\omega.$$

Next, observe that by the Riesz representation Theorem, for each fixed $t \in \mathbb{D}$ there exists a section $f \in \mathscr{H}(\varphi)_t$ such that

$$||f||_t = ||\xi_t||_{t*} \quad \text{and} \quad \langle \xi_t, g \rangle = (f, g)_t \text{ for all } g \in \mathscr{H}(\varphi)_t.$$

In particular,

$$||\xi_t||_{t*}^2 = ||f||_t^2 = \langle \xi_t, f \rangle = |\langle \xi_t, f \rangle| = \frac{|\langle \xi_t, f \rangle|^2}{||f||_t^2},$$

and moreover

$$||\xi_t||_{t*}^2 \geq \frac{|\langle \xi_t, g \rangle|^2}{||g||_t^2} \quad \text{for all } g \in \mathscr{H}(\varphi)_t.$$

We emphasize that t is fixed, and we assume from here on that for this particular t, the linear functional ξ_t is not zero.

Now define the section $\mathfrak{f} \in \Gamma_{\mathscr{O}}(\mathbb{D}, \mathscr{H}(\varphi))$ by

$$\mathfrak{f}_\tau := f - (\tau - t)(\nabla^{1,0}_{\frac{\partial}{\partial \tau}} f)_t.$$

Then $\mathfrak{f}_t = f$ and

$$(\nabla^{1,0}_{\frac{\partial}{\partial t}} \mathfrak{f})_t = \lim_{\tau \to t} \left((\nabla^{1,0}_{\frac{\partial}{\partial t}} f)_\tau - (\nabla^{1,0}_{\frac{\partial}{\partial t}} f)_t - (\tau - t)(\nabla^{1,0}_{\frac{\partial}{\partial t}} (\nabla^{1,0}_{\frac{\partial}{\partial t}} f)_t)_\tau \right) = 0.$$

Therefore

$$\frac{1}{||\mathfrak{f}_t||^2_t} \left(\frac{|((\nabla^{1,0}_{\frac{\partial}{\partial t}} \mathfrak{f})_t, \mathfrak{f}_t)_t|^2}{||\mathfrak{f}_t||^2_t} - \left|\left|(\nabla^{1,0}_{\frac{\partial}{\partial t}} \mathfrak{f})_t\right|\right|^2_t \right) = 0,$$

and from Theorem 11.1.15 we obtain

$$\frac{\partial^2}{\partial \tau \partial \bar{\tau}} \log \frac{1}{||\mathfrak{f}_\tau||^2_t} \bigg|_{\tau = t} \geq \frac{1}{||\mathfrak{f}_t||^2_t} \int_X \left(\frac{\partial^2 \varphi_t}{\partial t \partial \bar{t}} - \left| \bar{\partial}_X \left(\frac{\partial \varphi_t}{\partial t} \right) \right|^2_\omega \right) |\mathfrak{f}_t|^2 e^{-\varphi_t}.$$

Since ξ is a holomorphic section,

$$\tau \mapsto \log \frac{|\langle \xi_\tau, \mathfrak{f}_\tau \rangle|^2}{||\mathfrak{f}_\tau||^2}$$

has non-negative τ-Laplacian at $\tau = t$.

Now, from the definition of the dual norm, the function

$$\Phi : \tau \mapsto \log ||\xi_\tau||^2_{\tau *} - \log \frac{|\langle \xi_\tau, \mathfrak{f}_\tau \rangle|^2}{||\mathfrak{f}_\tau||^2}$$

achieves a local minimum of 0 at $\tau = t$, and therefore

$$\left(\frac{\partial^2 \Phi}{\partial \tau \partial \bar{\tau}} \right)_{\tau = t} \geq 0.$$

But this means

$$\left(\frac{\partial^2}{\partial \tau \partial \bar{\tau}} \log ||\xi_\tau||^2_{\tau *} \right)_{\tau = t} \geq \left(\frac{\partial^2}{\partial \tau \partial \bar{\tau}} \log \frac{|\langle \xi_\tau, \mathfrak{f}_\tau \rangle|^2}{||\mathfrak{f}_\tau||^2} \right)_{\tau = t} \geq 0.$$

Thus we have shown that $t \mapsto \log ||\xi_t||^2_{t*}$ is subharmonic away from the set $\{t \in \mathbb{D}\,;\ \xi_t = 0\}$. Since ξ is holomorphic, this set is a closed discrete subset, and thus $\log ||\xi_\tau||^2_{\tau *}$ continues across the zero set of ξ to a subharmonic function on $\mathbb{D}$. The proof of Theorem 11.1.8 is therefore complete. □

11.2 Application: The Suita Conjecture

Suita's Conjecture was proved by Błocki [B-2013] when X is a domain in $\mathbb{C}$, and in general by Guan and Zhou [GZ-2015]. We shall give a different proof here, that uses Theorem 11.1.8. Our proof below is similar to the proof of Berndtsson and Lempert [BL-2016], but it is slightly different.

11.2.1 Formulation of the Suita Conjecture

Let X be a Riemann surface and assume X admits a non-constant bounded subharmonic function. (Such Riemann surfaces are called *hyperbolic*, or sometimes *potential theoretically-hyperbolic*.) It is well known that such a Riemann surface admits a Green's function $G : X \times X \to [-\infty, 0)$, i.e., a function uniquely characterized by the following properties:

(i) If we write $G_x(y) = G(y, x)$, then for each $x \in X$,

$$\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}G_x = \delta_x,$$

and

(ii) if $H : X \times X \to [-\infty, 0)$ is another function with property (i), then $G \geq H$.

Using the Green's Function, one can construct a conformal metric for X as follows:

$$\omega_F(x) := \lim_{y\to x} \frac{\sqrt{-1}}{2}\partial(e^{G_x})(y) \wedge \bar{\partial}(e^{G_x})(y).$$

The metric ω_F is called the *fundamental metric*.

The metric ω_F can be computed from the Green's Function at a point x as follows. Choose any holomorphic function $f \in \mathcal{O}(X)$ that vanishes to order 1 at x and is non-zero everywhere else. (Since X is an open Riemann surface, and thus Stein, such a function f exists.) From the definition of Green's function, the function

$$h_x := G_x - \log|f|$$

is harmonic. Then

$$\omega_F(x) = e^{2h_x(x)}\frac{\sqrt{-1}}{2}df(x) \wedge d\bar{f}(x).$$

Now, the curvature of ω_F is

$$\Omega_F := -2\partial\bar{\partial}h_x(x).$$

Locally $h_x(y) = G(x, y) - \log|x - y|$ up to a harmonic function, so by the harmonicity of the Green's function we find that

$$\partial\bar{\partial}h_x(y) = \partial\left(\frac{\partial G}{\partial\bar{x}}d\bar{x} + \left(\frac{\partial G}{\partial\bar{y}} + \frac{1}{\bar{x}-\bar{y}}\right)d\bar{y}\right) = \frac{\partial^2 G}{\partial y\partial\bar{x}}dx \wedge d\bar{y} + \frac{\partial^2 G}{\partial y\partial\bar{x}}dy \wedge d\bar{x}.$$

Therefore

$$\Omega_F(x) = 4\left(\lim_{y\to x}\frac{\partial^2 G}{\partial x\partial\bar{y}}\right)dx \wedge d\bar{x}.$$

11.2.1 EXAMPLE. Let X be the unit disk. Then $G(z, \zeta) = \log\left|\frac{\zeta - z}{1-\bar{\zeta}z}\right|$, and we have

$$\omega_F(z) = \frac{\sqrt{-1}dz \wedge d\bar{z}}{2(1-|z|^2)^2}.$$

Thus in the unit disk the fundamental metric agrees with the Poincaré metric (normalized so that its Gaussian curvature is identically -4).

Note that $h_z(\zeta) = -\frac{1}{2}\log|1-\zeta\bar{z}|^2$ and

$$\frac{\partial^2 h_z(\zeta)}{\partial\zeta\partial\bar{z}} = \frac{1}{2(1-\zeta\bar{z})^2}.$$

Consequently $\Omega_F(z) = -4\omega_F(z)$.

Suita's conjecture can be stated as follows:

11.2.2 CONJECTURE. [Su-1971] Let X be a hyperbolic Riemann surface. Then the Gaussian curvature of the fundamental metric of X is at most -4.

11.2.2 Alternate formulation of the Suita Conjecture

Before turning to the proof of the Suita Conjecture, we shall reformulate it in a manner that is more amenable to our methods. To this end, we want to consider Example 11.1.6 in a special setting. On our Riemann surface X, consider the canonical line bundle $K_X \to X$, which in this case is just the cotangent bundle. We can define the Hilbert space

$$\mathscr{H}_X^2 := \left\{\alpha \in \Gamma_{\mathcal{O}}(X, K_X)\ ;\ \int_X \frac{\sqrt{-1}}{2}\alpha\wedge\bar{\alpha} <, +\infty\right\}.$$

This Hilbert space is a closed subspace of the larger Hilbert space $\mathfrak{L}_X^2$ of measurable $(1,0)$-forms θ such that

$$\int_X \frac{\sqrt{-1}}{2}\theta\wedge\bar{\theta} < +\infty,$$

and we have the Bergman projection

$$P_X : \mathfrak{L}_X^2 \to \mathscr{H}_X^2.$$

11.2.1. Prove that the Bergman projection P_X is bounded.

The integral kernel B_X of P_X, defined by the relation

$$P_X\alpha(z) = \int_{w\in X}\alpha(w)\wedge B_X(z,w),$$

is called the Bergman kernel of X. As in Example 11.1.6,

$$B_X \in \Gamma_{\mathcal{O}}(X\times X^\dagger, K_X \boxtimes K_X^\dagger).$$

If the Riemann surface X is a (hyperbolic) plane domain, Schiffer [Sc-1946] observed that the Bergman kernel $B_X(z,\zeta)$ of X is given by the formula

$$B_X(z,\zeta) = \frac{\sqrt{-1}}{\pi}\frac{\partial^2 G(z,\zeta)}{\partial z\partial\bar{\zeta}}dz\otimes d\bar{\zeta}.$$

Schiffer's result extends to other hyperbolic Riemann surfaces.

11.2.3 THEOREM (Schiffer[Sc-1946]). *Let X be a hyperbolic Riemann surface and let G be the Green's function of X. Then one has the formula*

$$B_X(z,\zeta) = \frac{1}{\pi}\partial_z\bar{\partial}_\zeta G(z,\zeta). \tag{11.3}$$

Proof. Let $o \in X$. Choose a local coordinate ζ near o and $\varepsilon > 0$ so small that the disk $D_{2\varepsilon} := \{|\zeta| < 2\varepsilon\}$ is a coordinate chart. Fix $\Theta \in \mathscr{H}^2_X$ and write $\Theta = f(\zeta)d\zeta$ in $D_{2\varepsilon}$. Then

$$\begin{aligned}\int_{X-D_\varepsilon} \sqrt{-1}\Theta(\zeta) \wedge \frac{1}{\pi}\partial_z\bar{\partial}_\zeta G(o,\zeta) &= \frac{1}{\sqrt{-1}\pi}\int_{X-D_\varepsilon} d_\zeta\left(\Theta(\zeta)\otimes\partial_z G(o,\zeta)\right) \\ &= \left(\frac{-1}{\sqrt{-1}\pi}\int_{\partial D_\varepsilon} f(\zeta)\frac{\partial G(o,\zeta)}{\partial z}d\zeta\right)dz,\end{aligned}$$

where we have used the fact that $G|_{\partial X} = 0$. (Here ∂X denotes the part of the boundary that has non-zero capacity.) But one has

$$\frac{\partial G(o,\zeta)}{\partial z} = \frac{1}{-2\zeta} + \frac{\partial h(o,\zeta)}{\partial z},$$

and we have

$$\begin{aligned}&\int_{X-D_\varepsilon} \sqrt{-1}\Theta(\zeta)\wedge\frac{1}{\pi}\partial_z\bar{\partial}_\zeta G(o,\zeta) \\ &= \left(\frac{1}{2\sqrt{-1}\pi}\int_{|\zeta|=\varepsilon}\left(\frac{f(\zeta)}{\zeta} - 2\frac{\partial h(z,\zeta)}{\partial z}\right)d\zeta\right)dz \\ &= f(0)dz + O(\varepsilon) = \Theta(o) + O(\varepsilon).\end{aligned}$$

Letting $\varepsilon \to 0$, we see that

$$\Theta(o) = \int_X \sqrt{-1}\Theta(\zeta)\wedge\frac{1}{\pi}\partial_z\bar{\partial}_\zeta G(o,\zeta).$$

Since the Bergman kernel is characterized by its holomorphicity and its reproducing property, the proof is complete. □

Schiffer's Theorem immediately gives the following corollary.

11.2.4 COROLLARY. *If X is a hyperbolic Riemann surface then the curvature of the fundamental metric is $-4\pi K_X(z,z)$.*

11.2.2. Show that the Bergman kernel for the Bergman projection

$$P_{\mathbb{D}} : L^2(\mathbb{D}) \to L^2(\mathbb{D}) \cap \mathcal{O}(\mathbb{D})$$

for the classical Bergman space is

$$K(z,w) = \frac{\sqrt{-1}dz\otimes d\bar{w}}{2\pi(1-z\bar{w})^2}.$$

(Hint: Use Exercise 11.1.4.)

We therefore see that Suita's Conjecture is equivalent to the estimate

$$\pi B_X(x,x) \ge \omega_F(x). \tag{11.4}$$

Note that, in view of Example 11.2.1 and Exercise 11.2.2, equality holds in (11.4) for $X = \mathbb{D}$.

We shall prove the Suita Conjecture by establishing (11.4).

11.2.3 Estimating the Bergman kernel

For the rest of this paragraph we fix the point x at which we will estimate $B_X(x,x)$.

In Theorem 11.1.8 we take our Stein manifold to be the hyperbolic Riemann surface X, and we fix *any* smooth metric ω on X. Note that ω is a Kähler form, and also an area form, so that $dV_\omega = \omega$. Next we let $L = K_X$, and we shall need a family of metrics $e^{-\varphi_t}$.

The family of metrics is constructed as follows. We equip L with the dual metric $e^{-\varphi_o} := \omega^{-1}$. Then

$$\partial\bar\partial\varphi_o = -\mathrm{Ricci}(\omega).$$

Thus if we let $\psi : X \times \mathbb{L} \to (-\infty, 0)$ be a subharmonic function, where $\mathbb{L} := \{\tau \in \mathbb{C}\,;\ \mathrm{Re}\,\tau < 0\}$ is the left half plane, then the family of metrics

$$e^{-\Phi} := e^{-(\varphi(\cdot)+\psi(\cdot,\tau))}$$

satisfies

$$\partial\bar\partial\Phi + \mathbf{p}^*\mathrm{Ricci}(\omega) = \partial\bar\partial\psi \ge 0,$$

where $\mathbf{p} : X \times \mathbb{L} \to X$ is projection to the first factor. Thus for any choice of such ω and ψ we have spaces $\mathscr{H}(\Phi)_t$ and the vector bundle $\mathscr{H}(\Phi) \to \mathbb{D}$ for which Theorem 11.1.8 holds. We fix

$$\omega = \omega_F \quad \text{and} \quad \psi(y,\tau) := \mathrm{Re}\,\tau + p \cdot \max(2G_x(y) - \mathrm{Re}\,\tau, 0)$$

where $p > 0$. At the end of the proof, we will let $p \to \infty$.

Next fix a tangent vector $\mathbf{e} \in T_{X,x} = L^*_x$ such that $|\mathbf{e}|^2_{\omega_F} = 1$, and consider the holomorphic section ξ^x of $\mathscr{H}(\Phi)^* \to \mathbb{D}$ defined in Example 11.1.6:

$$\Gamma_{\mathcal{O}}(X, K_X) \supset \mathscr{H}(\Phi)_t \ni f \mapsto \xi^x f := \langle \mathbf{e}, f(x)\rangle\,.$$

Observe that

$$||\xi^x_t||^2_{t*} = \sup_{f \in \mathscr{H}(\Phi)_t,\ ||f||_t = 1} |\mathbf{e} \otimes f(x)|^2 = \frac{K_t(x,x)}{\omega_F(x,x)}.$$

In particular, when $t = 0$, and thus $\psi = 0$, we see that $K_o = B_X$, and thus we wish to prove that

$$||\xi^x_o||^2_{o*} \ge \frac{1}{\pi}.$$

Since ψ depends only on $\mathrm{Re}\,\tau$, Berndtsson's Theorem 11.1.8 yields the following theorem.

11.2.5 THEOREM. *Let ξ^x be the section of $\mathscr{H}(\Phi)^* \to \mathbb{L}$ defined in Example 11.1.6. Then the function*

$$\mathbb{L} \ni \tau \mapsto \log ||\xi^x_\tau||^2_{\tau*}$$

is convex.

Let us set

$$X_t := \{y \in X \ ; \ 2G_x(y) < t\}.$$

Notice that

(a) if $t << 0$ then X_t is simply connected. In the limit, X_t converges to a point, so once we get past the critical points of G_x, we have a simply connected Riemann surface, which is biholomorphic to the disk by the Uniformization Theorem.

(b) For $\tau \in \mathbb{L}$ the function $\psi(\cdot, \tau)$ vanishes on $X_{\mathrm{Re}\,\tau}$, and equals $\mathrm{Re}\,\tau + p \cdot (2G_x(y) - \mathrm{Re}\,\tau)$ on $X - X_{\mathrm{Re}\,\tau}$. In particular, for $\tau = o$ we see that K_o, the Bergman kernel for $\mathscr{H}(\Phi)_o$, is just B_X.

11.2.6 PROPOSITION. *The function*

$$(-\infty, 0) \ni t \mapsto \log ||\xi^x_t||^2_{t*} \tag{11.5}$$

is bounded, and therefore it is increasing.

Proof. Observe that for $f \in \mathscr{H}(\Phi)_t$ satisfying $||f||_t = 1$, Bergman's inequality implies that

$$|\langle \xi^x_t, f\rangle|^2 = \frac{|f(x)|^2}{\omega_F(x)} \le Ce^{-t} \int_{X_t} |f|^2 \le C||f||^2_t = C,$$

where C is independent of t and f. Thus the function (11.5) is bounded. Since the function (11.5) is also convex and defined in $(-\infty, 0)$, it must be increasing. □

We have therefore proved that

$$||\xi^x_o||^2_{o*} \ge \lim_{t \to -\infty} ||\xi^x_t||^2_{t*}. \tag{11.6}$$

(Note that the limit on the right hand side exists.) To complete the proof of (11.4), we need to show that the limit on the right hand side is $\frac{1}{\pi}$.

To motivate why the latter might be true, observe that since $G_x(y) = \log|x - y| + h_x(y)$ for some harmonic a function h_x, which in particular is smooth across x. If $t << 0$ then X_t is very close to the disk of area e^t centered at x, and on $X - X_t$ the weight $e^{-\varphi_t}$ is equal to $e^{(p-1)t}e^{-\varphi_o - 2pG_x}$, which should be very small when $p >> 0$ because $2G_x > t$. We therefore expect K_t to be very close to $e^{-t}B_{X_t}$.

Let us fix $f \in \mathscr{H}(\Phi)_t)$ such that

$$||f||_t = 1 \quad \text{and} \quad \frac{\sqrt{-1}}{2} f(x) \wedge \overline{f(x)} = K_t(x,x).$$

(C.f. Exercise 11.1.4.) We compute that

$$1 = \int_X |f|^2 e^{-\Phi_t}\omega_F = e^{-t}\int_{X_t}|f|^2 + e^{(p-1)t}\int_{X-X_t}|f|^2 e^{-2G_x}.$$

Now, if s is sufficiently close to 0 then

$$\begin{aligned} e^{(p-1)t}\int_{X-X_t}|f|^2 e^{-2pG_x} &= e^{(p-1)t}\int_{X-X_s}|f|^2 e^{-2pG_x} + e^{(p-1)t}\int_{X_s-X_t}|f|^2 e^{-2pG_x} \\ &\le \varepsilon + \sup_{X_s}\frac{|f|^2}{\omega_F}\cdot e^{(p-1)t}\int_{X_s-X_t} e^{2pG_x}\omega_F. \end{aligned}$$

Letting $\mathcal{V}(t) := \int_{X_s-X_t}\omega_F \le Ce^t$, we find that

$$e^{(p-1)t}\int_{X_s-X_t} e^{2pG_x}\omega_F = e^{-t}\int_t^s e^{-p(u-t)}d\mathcal{V}(u).$$

11.2.7 LEMMA. *Let $\nu : (-\infty, 0) \to \mathbb{R}_+$ be an increasing function such that $\nu(t) \le e^t$ for all $t < 0$. Then for $p > 1$,*

$$\liminf_{t\to-\infty} e^{-t}\int_t^0 e^{-p(s-t)}d\nu(s) \le \frac{2}{p-1}. \tag{11.7}$$

Proof. Let

$$f(t) := e^{-t}\int_t^0 e^{-p(s-t)}d\nu(s).$$

Observe that for $R \ge 1$

$$\begin{aligned} \frac{1}{R}\int_{-R}^0 f(t)dt &= \frac{1}{R}\int_{-R}^0\int_t^0 e^{(p-1)t}e^{-ps}d\nu(s)dt = \frac{1}{R}\int_{-R}^0\int_{-R}^s e^{(p-1)t}e^{-ps}dtd\nu(s) \\ &\le \frac{1}{R(p-1)}\int_{-R}^0 e^{-s}d\nu(s) = \frac{1}{R(p-1)}\left(\nu(0) - e^{-R}\nu(R) + \int_{-R}^0 \nu(s)e^{-s}ds\right) \\ &\le \frac{R+1}{R(p-1)} \le \frac{2}{p-1}. \end{aligned}$$

But if (11.7) fails then there exists $R_o > 0$ such that $f(t) \ge (2+\delta)/(p-1)$ for all $t < R_o$, and then we have

$$\frac{1}{R}\int_{-R}^0 f(t)dt \ge \frac{R-R_o}{R}\frac{2+\delta}{p-1},$$

and we get a contradiction as soon as R is sufficiently large. □

We have thus shown that for any $\varepsilon > 0$ there exists p sufficiently large and $t << 0$ to guarantee that

$$1 - \varepsilon \le e^{-t}\int_{X_t}|f|^2 \le 1.$$

Consequently

$$e^{-t}B_{X_t} \le (1-\varepsilon)K_t.$$

But since X_t is smoothly close to the disk of radius $e^{t/2}$ centered at x, we see from Example 11.2.1 and Exercise 11.2.2 that for $t << 0$,

$$\frac{e^{-t}B_{X_t}(x)}{\omega_F(x)} \ge \frac{1-\varepsilon}{\pi}.$$

Therefore we have shown that for any $\varepsilon > 0$ there exists $p >> 0$ such that ,

$$\lim_{t\to-\infty} ||\xi_t^x||^2_{t^*} \ge \frac{(1-\varepsilon)^2}{\pi}.$$

Since the right hand side of (11.6) does not depend on p, we have proved that

$$\frac{\pi B_X(x,x)}{\omega_F(x)} \ge 1,$$

and thus the Suita conjecture is confirmed. □

11.3 Application: Sharp Estimates for $\mathbf{L}^2$ extension

In this section we are going to give another proof of the L^2 extension theorem. We will focus on the special case of Theorem 9.2.2, in which the line bundle L_Z associated to Z is trivial. In fact, the method we present here, which is due to Berndtsson and Lempert [BL-2016], can be extended to prove a more general version than this case, but so far it is not known how to obtain the full statement of Theorem 9.2.2 from this approach.

11.3.1 Statement of the Sharp $\mathbf{L}^2$ Extension Theorem

11.3.1 THEOREM. *Let (X,ω) be a Stein manifold and let $T : X \to \mathbb{D}$ be a holomorphic function such that $dT(x) \neq 0$ for all $x \in Z := T^{-1}(0)$. Let $L \to X$ be a holomorphic line bundle with singular Hermitian metric $e^{-\varphi}$ such that*

$$\partial\bar\partial\varphi + \mathrm{Ricci}(\omega) \ge 0.$$

Then for any $f \in \Gamma_{\mathcal{O}}(Z,L)$ such that

$$\int_Z |f|^2 e^{-\varphi}\frac{dA_\omega}{|dT|^2_\omega} < +\infty$$

there exists $F \in \Gamma_{\mathcal{O}}(X,L)$ such that

$$F|_Z = f \quad \textit{and} \quad \int_X |F|^2 e^{-\varphi} dV_\omega \le \pi \int_Z |f|^2 e^{-\varphi}\frac{dA_\omega}{|dT|^2_\omega}.$$

Theorem 11.3.1 looks very much like Theorem 10.2.10 (the latter is the version for L-valued holomorphic). The major difference is the constant π, which is better than the constant 24π that we previously established. As we will show in the next paragraph, the constant π is sharp.

11.3.2 REMARK. For many applications (for example, the invariance of plurigenera) this sharp constant does not matter, but there are some interesting applications where the sharpness is important. Interestingly, in the applications known to the author, the fact that the constant is sharp is less important than the fact that it is the area of the unit disk. $\diamond$

11.3.2 Sharpness in Theorem 11.3.1

The constant π in Theorem 11.3.1 is sharp in the sense that there is no smaller constant for which Theorem 11.3.1 holds in its full generality. There are, however, specific cases (and in fact, many of them) in which the constant is far better than π.

11.3.3 EXAMPLE. Let $X = \mathbb{D}$ be the unit disk and let $Z = \{p\}$ be a point in $\mathbb{D}$. We take L to be the trivial bundle with the trivial metric, so that

$$\mathscr{A}(\varphi) = L^2(\mathbb{D}) \cap \mathcal{O}(\mathbb{D}) =: \mathfrak{B}$$

is the classical Bergman space. Letting

$$T := \frac{z - p}{|p| + 1} \in \mathcal{O}(\mathbb{D}),$$

we see that

$$\sup_{\mathbb{D}} |T| = 1 \quad \text{and} \quad |dT(p)|^2 = \frac{1}{(|p| + 1)^2}.$$

Thus we want to know the best constant C such that for each $a \in \mathbb{C}$ there exists $F \in \mathcal{O}(\mathbb{D})$ such that

$$F(p) = a \quad \text{and} \quad \int_{\mathbb{D}} |F|^2 \le C \cdot |a|^2 (|p| + 1)^2$$

Clearly the problem of finding such a function is linear, so we may as well assume that $a = 1$, and then it's clear that the optimal constant is

$$C_{\min}(p) = \min \left\{ ||F||^2 \, ; \; F \in \mathfrak{B} \text{ and } F(p) = 1 \right\}.$$

11.3.1. Show that if $F_o \in \mathfrak{B}$ satisfies $||F_o||^2 = C_{\min}(p)$ then

$$\int_{\mathbb{D}} F_o \overline{G} dA = 0$$

for every $G \in \mathfrak{B}$ satisfying $G(p) = 0$. Use Exercise 11.1.4 to show that consequently

$$C_{\min}(p) = \frac{1}{K(p, p)}$$

where K is the Bergman kernel of the Bergman projection $P : L^2(\mathbb{D}) \to \mathfrak{B}$.

Exercises 11.3.1 and 11.2.2 show that

$$C_{\min}(p) = \pi(1 - |p|)^2.$$

Thus we see, on the one hand, that the optimal constant in Theorem 11.3.1 cannot be less than π. On the other hand, we also see that even though the optimal constant is π, there are certain spaces X with hypersurfaces Z for which the norm of the operator of minimal extension is much smaller than π, and might even go to 0.

11.3.3 Beginning of the proof of Theorem 11.3.1

The proof of Theorem 11.3.1 begins with a couple of reductions. First, we may assume that X is actually a bounded pseudoconvex domain in some larger Stein manifold and that Z meets the boundary of X transversely. When objects extend to the larger manifold, we shall avoid referring to this manifold by saying that the objects extend *up to the boundary* of X. Next we can assume that metric ω and the line bundle $L \to X$ extends holomorphically to this larger domain, and that the metric $e^{-\varphi}$ for this extension of L is smooth. If the result is proved under these two assumptions, then it follows in general by weak-$*$ compactness theorems and techniques of approximating singular Hermitian metrics on Stein manifolds.

Let us fix the section $f \in \Gamma_{\mathscr{O}}(Z, L)$ to be extended; we may assume, perhaps after shrinking X, that f is holomorphic up to the boundary of Z. By Theorem 9.1.2 there exists a section $F \in \Gamma_{\mathscr{O}}(X, L)$ such that

$$F|_Z = f \quad \text{and} \quad \int_X |F|^2 e^{-\varphi} dV_\omega < +\infty.$$

Indeed, one extends f to F on a larger ambient manifold, and compactness implies that $F|_X$ is square-integrable.

Since there exists some extension of f with finite L^2 norm, there is an extension of minimal L^2 norm; let us call the minimal extension F_o.

11.3.2. Show that the extension of minimal norm is unique.

Thus our goal is to prove that

$$\int_X |F_o|^2 e^{-\varphi} dV_\omega \le \pi \int_Z |f|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega}.$$

Since the minimal extension is unique (Exercise 11.3.2), its norm can be thought of as a norm of the section f to be extended. This observation suggests that there is a way to describe this norm without reference to the minimal extension itself, and indeed this is the case.

11.3.4 Dual formulation of the norm of the minimal extension

We introduce the notation

$$\mathscr{A}(\varphi) := \left\{ F \in \Gamma_{\mathscr{O}}(X, L) \; ; \; ||F||^2 := \int_X |F|^2 e^{-\varphi} dV_\omega < +\infty \right\}.$$

(The Hilbert space $\mathscr{A}(\varphi)$ looks a lot like the spaces $\mathscr{H}(\varphi)$, but we use different notation to emphasize the fact that the metric $e^{-\varphi}$, and hence the space $\mathscr{A}(\varphi)$, is not varying.)

11.3.4 PROPOSITION. *Let*

$$\mathfrak{I}_\varphi(Z) := \{g \in \mathscr{A}(\varphi)\ ;\ g|_Z \equiv 0\}$$

and let

$$\mathrm{Ann}(\mathfrak{I}_\varphi(Z)) := \{\xi \in \mathscr{A}(\varphi)^*\ ;\ \langle \xi, g\rangle = 0 \textit{ for all } g \in \mathfrak{I}_\varphi(Z)\}.$$

Then for each $f \in \Gamma_{\mathscr{O}}(Z, L)$ that is smooth up to the boundary (or more generally has some extension in $\mathscr{A}(\varphi)$) the minimal extension F_o of f satisfies

$$\int_X |F_o|^2 e^{-\varphi} dV_\omega = \sup\left\{\frac{|\langle \xi, F\rangle|^2}{||\xi||_*^2}\ ;\ \xi \in \mathrm{Ann}(\mathfrak{I}_\varphi(Z))\right\}, \tag{11.8}$$

where $F \in \mathscr{A}(\varphi)$ is any extension of f.

Proof. First note that if $\xi \in \mathrm{Ann}(\mathfrak{I}_\varphi(Z))$ and F_1, F_2 are extensions of f then $F_2 - F_1 \in \mathfrak{I}_\varphi(Z)$ and thus

$$\langle \xi, F_2\rangle = \langle \xi, F_1\rangle + \langle \xi, F_2 - F_1\rangle = \langle \xi, F_1\rangle.$$

Thus the right hand side of (11.8) is independent of the choice of extension. Next observe that $F_o \perp \mathfrak{I}_\varphi(Z)$. Indeed, if $G \in \mathfrak{I}_\varphi(Z)$ then $F_o + \varepsilon G$ is an extension of f for every $\varepsilon \in \mathbb{R}$, and we have

$$||F_o + \varepsilon G||^2 = ||F_o||^2 + \varepsilon 2\mathrm{Re}\,(F_o, G) + O(|\varepsilon|^2),$$

and since the right hand side achieves its minimum at $\varepsilon = 0$, we have

$$2\mathrm{Re}\,(F_o, G) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} ||F_o + \varepsilon G||^2 = 0.$$

Similarly $0 = 2\mathrm{Re}\,(F_o, -\sqrt{-1}G) = 2\mathrm{Im}\,(F_o, G) = 0$, so $(F_o, G) = 0$ as claimed.

It follows that the linear functional $\xi_o \in \mathscr{A}(\varphi)^*$ defined by

$$\langle \xi_o, G\rangle := (G, F_o)$$

lies in $\mathrm{Ann}(\mathfrak{I}_\varphi(Z))$, and therefore its norm is

$$\sup\left\{\frac{|\langle \xi, F_o\rangle|}{||\xi||_*}\ ;\ \xi \in \mathrm{Ann}(\mathfrak{I}_\varphi(Z))\right\}.$$

The proof is complete. □

To estimate the right hand side of (11.8) we may obviously work with a dense subspace that is well-suited for estimation.

11.3.5 LEMMA. *The set of linear functionals in $\mathscr{A}(\varphi)^*$ of the form*

$$\xi_g : \mathscr{A}(\varphi) \ni F \mapsto \int_Z F\bar{g}e^{-\varphi}\frac{dA_\omega}{|dT|^2_\omega}, \quad g \in \Gamma_o(Z, L),$$

(where Γ_o means smooth sections with compact support) forms a dense subspace of $\mathrm{Ann}(\mathfrak{J}_\varphi(Z))$.

Proof. Clearly $\xi_g \in \mathrm{Ann}(\mathfrak{J}_\varphi(Z))$. Moreover, if $\langle \xi_g, F\rangle = 0$ for all $g \in \Gamma_o(Z, L)$ then $F|_Z \equiv 0$. Thus the lemma follows from Exercise 11.3.3. □

11.3.3. Let A be a subspace of a topological vector space H. Then $\overline{A} = H$ if and only if every continuous linear functional on H that vanishes on A is zero.

Using (11.8), it is evident that to get an estimate for the extension of minimal norm, we have to obtain estimates for

$$\frac{|\langle \xi_g, F_o\rangle|^2}{||\xi_g||^2_*}$$

for all $g \in \Gamma_o(Z, L)$. Let us begin with the numerator. We introduce the notation

$$\mathscr{L}^2_Z := L^2(e^{-\varphi}/|dT|^2_\omega) \quad \text{and} \quad \mathscr{E}(\varphi) := \mathscr{L}^2_Z \cap \Gamma_{\mathcal{O}}(Z, L).$$

Letting

$$P : \mathscr{L}^2_Z \to \mathscr{E}(\varphi)$$

denote the Bergman projection, we have the following proposition.

11.3.6 PROPOSITION. *For each $g \in \Gamma_o(Z, L)$*

$$|\langle \xi_g, F_o\rangle|^2 \le \left(\int_Z |f|^2 e^{-\varphi}\frac{dA_\omega}{|dT|^2_\omega}\right)\left(\int_Z |Pg|^2 e^{-\varphi}\frac{dA_\omega}{|dT|^2_\omega}\right).$$

Proof. We calculate that

$$\langle \xi_g, f\rangle = \int_Z f\bar{g}e^{-\varphi}\frac{dA_\omega}{|dT|^2_\omega} = \int_Z f\overline{Pg}e^{-\varphi}\frac{dA_\omega}{|dT|^2_\omega},$$

and the result follows from the Cauchy-Schwarz Inequality. □

Consequently, to prove Theorem 11.3.1, it suffices to show that

$$\int_Z |Pg|^2 e^{-\varphi} \le \pi||\xi_g||^2_* \quad \text{for all } g \in \Gamma_o(Z, L). \tag{11.9}$$

To achieve (11.9), the strategy is to degenerate the domain X to Z through a *good* family of domains X_t parameterized by negative real numbers t, i.e.,

$$X_o = X \quad \text{and} \lim_{t\to-\infty} X_t = Z.$$

We now define this *good* degeneration.

11.3.5 Degeneration to the infinitesimal neighborhood

Degeneration to the infinitesimal neighborhood: first version

Letting

$$\mathbb{L} := \{\tau \in \mathbb{C}\, ;\ \mathrm{Re}\,\tau < 0\}$$

denote the left half plane, define

$$X_t := \{x \in X\ ;\ \log |T(x)|^2 < t\}, \qquad t \le 0.$$

Then $X_o = X$ and, for $t << 0$, X_t is a neighborhood of Z. (Again, recall that we have assumed T to be holomorphic up to the boundary of X.) One can define the Hilbert spaces

$$\mathbf{H}(\varphi)_\tau := \left\{ F \in \Gamma_{\mathcal{O}}(X_{\mathrm{Re}\,\tau}, L)\ ;\ e^{-\mathrm{Re}\,\tau} \int_{X_{\mathrm{Re}\,\tau}} |F|^2 e^{-\varphi} dV_\omega < +\infty \right\}.$$

If we could apply Berndtsson's Theorem to the bundle of Hilbert spaces $\mathbf{H}(\varphi) \to \mathbb{L}$ then we could conclude that $\log ||\xi_g||^2_{\tau *}$ is subharmonic in τ. As we shall see in a moment, this piece of information is particularly useful in the proof of Theorem 11.3.1.

Unfortunately, it is not possible to apply Berndtsson's Theorem here, because the Hilbert spaces $\mathbf{H}(\varphi)_\tau$ might not have the same underlying vector space; in fact, each of them is defined over a different complex manifold, so it is hard to decide if $\mathbf{H}(\varphi) \to \mathbb{L}$ is a vector bundle.

To get around this problem, we now modify the Hilbert spaces slightly.

Degeneration to the infinitesimal neighborhood: better version

Define the plurisubharmonic function

$$\psi_\tau : X \times \mathbb{L} \ni (x, \tau) \mapsto \max(\log |T(x)|^2 - \mathrm{Re}\,\tau, 0).$$

Note that (i) ψ_τ depends only on $\mathrm{Re}\,\tau$, and (ii) the support of ψ_τ is $X - X_{\mathrm{Re}\,\tau}$.

For a positive number $p >> 0$, consider the family of positively curved Hermitian metrics

$$e^{-\varphi_\tau} := e^{-(\varphi + p\psi_\tau)}, \qquad \tau \in \mathbb{L}.$$

We define the Hilbert spaces

$$\mathscr{H}(\varphi)_\tau := \Gamma_{\mathcal{O}}(X, L) \cap L^2(e^{-\varphi_\tau - \mathrm{Re}\,\tau}).$$

Note that for each $F \in \mathscr{H}(\varphi)_\tau$ one has

$$\lim_{p \to \infty} \int_X |F|^2 e^{-\varphi_\tau} dV_\omega = e^{-\mathrm{Re}\,\tau} \int_{X_{\mathrm{Re}\,\tau}} |F|^2 e^{-\varphi} dV_\omega,$$

so that the Hilbert spaces $\mathscr{H}(\varphi)_\tau$ are in some sense approximations of $\mathbf{H}(\varphi)_\tau$. However, since all metrics extend up to the boundary, the Hilbert spaces $\mathscr{H}(\varphi)_\tau$ all have the same underlying vector space (seen as a subset of $\Gamma_{\mathcal{O}}(X, L)$) and only their norms vary. Thus Berndtsson's Theorem 11.1.8 applies to $\mathscr{H}(\varphi) \to \mathbb{L}$.

11.3.6 The proof of Theorem 11.3.1

11.3.7 LEMMA. *Let $g \in \Gamma_o(Z, L)$. Then*

$$\sup_{\tau\in\mathbb{L}} ||\xi_g||^2_{\tau*} < +\infty.$$

Proof. Let $H \in \mathscr{H}(\varphi)_\tau$ with $||H||^2_\tau = 1$. By the sub-mean value property and the smoothness of all metrics up to the boundary, for each $x \in \mathrm{Support}(g) \subset Z$ we have

$$|H(x)|^2 e^{-\varphi(x)} \le C_g e^{-\mathrm{Re}\,\tau} \int_{X_{\mathrm{Re}\,\tau}} |H|^2 e^{-\varphi} \le C_g e^{-\mathrm{Re}\,\tau} \int_X |H|^2 e^{-\varphi_\tau} = C_g.$$

Thus

$$||\xi_g||^2_{\tau*} = \sup_{||H||_\tau=1} \left| \int_Z H\bar{g} e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega} \right|^2 \le C_g,$$

as desired. □

11.3.8 LEMMA. *Let $g \in \Gamma_o(Z, L)$. Then the function*

$$\lambda_g : (-\infty, 0] \ni t \mapsto \log ||\xi_g||^2_{t*}$$

is non-decreasing. In particular,

$$||\xi_g||^2_{o*} \ge ||\xi_g||^2_{t*}$$

for all $t < 0$.

Proof. By Berndtsson's Theorem, the function $\tau \mapsto \log ||\xi_g||^2_{\tau*}$ is subharmonic on $\mathbb{L}$. On the other hand, $||\xi_g||^2_{\tau*}$ depends only on $\mathrm{Re}\,\tau$, and thus λ_g is convex on $(-\infty, 0)$. If λ_g decreases anywhere on $(-\infty, 0)$ then by convexity $\lim_{t\to-\infty} \lambda_g = +\infty$. But by Lemma 11.3.7 λ_g is bounded above. □

11.3.9 THEOREM. *Let $g \in \Gamma_o(Z, L)$. Then for each $\delta > 0$ there exists p sufficiently large so that*

$$\lim_{t\to-\infty} ||\xi_g||^2_{t*} \ge \frac{1}{2\pi} \int_Z |Pg|^2 e^{-\varphi} - \delta.$$

Theorem 11.3.9 will be proved using Lemma 11.2.7 and the following two lemmas.

11.3.10 LEMMA. *Let $\mathcal{F}$ be a smooth function on $\overline{X}$. Then*

$$\limsup_{t\to-\infty} e^{-t} \int_{X_t} \mathcal{F} dV_\omega = \pi \int_Z \mathcal{F} \frac{dA_\omega}{|dT|^2_\omega}.$$

The proof is left to Exercise 11.3.4.

11.3.4. Prove Lemma 11.3.10.

11.3.11 LEMMA. *Let $g \in \Gamma_o(Z, L)$. Then for any $\varepsilon > 0$ there exists $G \in \Gamma_{\mathcal{O}}(X, L)$ that is holomorphic up to the boundary, such that*

$$\int_Z |G - Pg|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega} < \varepsilon^2.$$

Proof. By L^2 approximation theorems (see, e.g., [H-1990, Theorem 5.6.2]) we can find a section that is holomorphic on a neighborhood of $\overline{Z}$ and approximate Pg uniformly on any compact subset of Z. If we take the compact subset to be sufficiently large then we also approximation in L^2. Finally, since the ambient manifold is Stein, the approximation on a neighborhood of $\overline{Z}$ can be extended to a neighborhood of $\overline{X}$, and therefore the extension will have finite L^2 norm on X. □

Proof of Theorem 11.3.9. Fix $\varepsilon > 0$ and let G be as in Lemma 11.3.11. Then

$$||\xi_g||^2_{t^*} \geq \frac{1}{||G||_t} \left| \int_Z G\overline{Pg} e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega} \right|^2 \geq (1-\varepsilon)^2 \frac{||Pg||^4}{||G||^2_t}.$$

Now,

$$||G||^2_t = e^{-t} \int_{X_t} |G|^2 e^{-\varphi} dV_\omega + e^{(p-1)t} \int_{X - X_t} |G|^2 e^{-(\varphi + p\log|T|^2)} dV_\omega.$$

By Lemma 11.3.10,

$$e^{-t} \int_{X_t} |G|^2 e^{-\varphi} dV_\omega \leq (1+\varepsilon)||Pg||^2$$

for $t << 0$. Turning to the second integral, we have

$$e^{(p-1)t} \int_{X-X_t} |G|^2 e^{-(\varphi + p\log|T|^2)} dV_\omega \leq \left(\sup_X |G|^2 e^{-\varphi} \right) e^{(p-1)t} \int_{X-X_t} e^{-p(\log|T|^2)} dV_\omega.$$

But

$$e^{(p-1)t} \int_{X-X_t} e^{-p\log|T|^2} dV_\omega \leq Ce^{-t} \int_t^0 e^{-p(s-t)} d\nu(s),$$

where $\nu(t) = \int_{X_t} dV_\omega$. By Exercise 11.3.4, Since $\nu(t) \leq C' e^t$. Lemma 11.2.7 therefore implies that

$$e^{(p-1)t} \int_{X-X_t} |G|^2 e^{-(\varphi + p\log|T|^2)} \leq \frac{C_o}{p-1},$$

for a sequence of 't's tending to $-\infty$. Thus for sufficiently large p and sufficiently negative t we obtain

$$||G||^2_t \leq (1+\varepsilon)||Pg||^2 + \varepsilon.$$

Choosing $\varepsilon > 0$ small enough yields the desired estimate. □

By combining Lemma 11.3.8 and Theorem 11.3.9 we obtain (11.9), and thus the proof of Theorem 11.3.1 is complete. □

11.4 Equivalence of Sharp Extension and Positive Variation

The L^2 extension theorem 11.3.1 has a long history, into which we will not delve very deeply. In brief, the first result was due to Ohsawa and Takegoshi [OT-1987]. Shortly after the appearance of the original result of Ohsawa and Takegoshi, some quite spectacular applications were discovered by Demailly, Diederich, Ohsawa, Siu and others. Manivel [Ma-1993] established a geometric version [Ma-1993], and three new (and mostly rather similar) proofs of the extension theorem appeared almost simultaneously by Berndtsson [B-1996], McNeal [Mc-1996] and Siu [S-1996]. But the true cementing of the L^2 extension theorem as a result lying at the foundation of complex analytic geometry finally occurred around 1998, when Siu established the deformation invariance of plurigenera [S-1998]. (See also [S-2002].) Since that time, there have been and continue to be numerous extensions, applications, and new proofs, of the L^2 extension theorem.

The question of the sharp constant came into focus first in [O-2001], where Ohsawa pointed out that the L^2 extension theorem could be used to establish a conjecture of Suita, on the comparison of the curvature of the hyperbolic metric and of the logarithmic capacity of a Riemann surface. Ohsawa noted that the extension theorem gives a comparison of the two metrics (via the link with the Bergman kernel that we explained in Chapter 11.2), and his proof made it clear that the conjectured comparison holds if and only if the sharp constant is π.

The sharp extension theorem was established first by Błocki [B-2013] in a very particular setting of hyperplanes in domains, though experts knew by that point that this setting already contains all of the key difficulties of the general proof of L^2 extension. Shortly afterwards, a very general version of the sharp extension theorem was proved by Guan and Zhou [GZ-2015]. The method of proof is not fundamentally different from that of Błocki, but the work is much more general, and should still see applications beyond the sizable number of applications appearing in the paper. The method of Błocki-Guan-Zhou can be used to establish the sharp constant

$$\frac{\pi(1+\delta)}{\delta}$$

in Theorem 9.2.2; incidentally this constant works for all $\delta > 0$, and does not require the assumption $\delta \le 1$.

11.4.1 Log plurisubharmonicity of the Bergman kernel

One particular application in [GZ-2015], which is at the heart of the present chapter, concerns a new proof of the following theorem of Berndtsson [B-2006].

11.4.1 THEOREM. *[B-2006] Let $D \subset \mathbb{C}^k \times \mathbb{C}^n$ be a pseudoconvex domain and let $\phi \in \mathrm{PSH}(D)$. For $t \in \mathbb{C}^k$ let*

$$D_t := \{z \in \mathbb{C}^n \ ; \ (t,z) \in D\}, \quad \phi^t := \phi|_{D_t}, \quad \text{and} \quad \mathscr{A}_t := L^2(D_t, e^{-\phi^t}) \cap \mathcal{O}(D_t).$$

Denote by $K_t(z,w)$ the kernel of the Bergman projection $P_t : L^2(D_t, e^{-\phi^t}) \to \mathscr{A}_t$, i.e,

$$P_t f(z) = \int_{D_t} f(w) K_t(z,w) e^{-\phi^t} dV(z).$$

Then the function

$$D \ni (t,z) \mapsto \log K_t(z,z) \tag{11.10}$$

is plurisubharmonic.

A special case of Theorem 11.4.1, in which the domain D is a product $\Omega \times Z$ with $\Omega \subset \mathbb{C}^k$ and $X \subset \mathbb{C}^n$, follows from Berndtsson's Theorem 11.1.8, as we now show.

To prove the plurisubharmonicity of (11.10), it suffices (and is necessary) to show that (11.10) is subharmonic in each variable t_i separately, for any fixed x, because in view of Exercise 11.1.4 (11.10) is already plurisubharmonic in z. Thus we may assume the base Ω is 1-dimensional.

As we showed in Example 11.1.6, the linear functional ξ_t^x of point evaluation at x multiplied by some vector $\mathbf{e} \in L_x^*$, which is given by

$$\langle \xi_t^x, f\rangle = \mathbf{e} \otimes \left(\int_X f(y) K_t(x,y) e^{-\phi_t(y)} dV(y) \right),$$

is a holomorphic section of $\mathscr{H}(\phi) \to \Omega$. On the other hand, by Exercise 11.1.4

$$||\xi_t^x||_{t^*}^2 = \sup_{||f||=1} |\langle \xi_t^x, f\rangle|^2 = \sup_{||f||=1} |f(x)|^2 e^{-\varphi_t(x)} = \langle K_t(x,x), \mathbf{e} \otimes \overline{\mathbf{e}}\rangle,$$

and therefore by Theorem 11.1.8 the function

$$t \mapsto \log ||\xi_t^x||_{t^*}^2 = \log \langle K_t(x,x), \mathbf{e} \otimes \overline{\mathbf{e}}\rangle$$

is subharmonic. Thus Theorem 11.4.1 is proved in the special case of product domains.

Quite remarkably, Theorem 11.4.1 also follows from the L^2 extension theorem with optimal constant, as was shown by Guan and Zhou [GZ-2015]. In fact, there is a somewhat more general result, with the same proof as that of Guan-Zhou.

Before we state the more general theorem, define the volume forms $d\nu_t$ on X_t by the relation

$$\int_X \Phi dV_\omega := \int_{t \in \mathbb{D}} \left(\int_{X_t} \Phi d\nu_t \right) dA(t).$$

Since the definition is clearly local, we compute $d\nu_t$ in terms of ω, in local coordinates, and the submersion $T : X \to \mathbb{D}$. The function T, being a submersion, is locally a coordinate function, and X is locally a product $X = X_o \times D$ where D is a small disk in the complex plane. We use coordinates $x = (x^1, ..., x^n)$ on X_o and use the coordinate T on D. Then

$$\omega = \omega_{o\bar{o}} \tfrac{\sqrt{-1}}{2} dT \wedge d\bar{T} + \omega_{o\bar{j}} \tfrac{\sqrt{-1}}{2} dT \wedge d\bar{x}^j + \omega_{i\bar{o}} \tfrac{\sqrt{-1}}{2} dx^i \wedge d\bar{T} + \omega_{i\bar{j}} \tfrac{\sqrt{-1}}{2} dx^i \wedge d\bar{x}^j,$$

where the indices i and j vary in $\{1, ..., n\}$. Since ω is a metric, the matrix $(\omega_{i\bar{j}})$ is an invertible $n \times n$ Hermitian matrix, and we write its inverse as $(\omega^{\bar{j}i})$. From the formula

$$\det \begin{pmatrix} c & v \\ v^\dagger & A \end{pmatrix} = \big(c - (A^{-1}v, v)\big) \det A,$$

we have $dV_\omega = \det(\omega^{i\bar{j}})dV(x)\left(\omega_{o\bar{o}} - \omega^{\bar{j}i}\omega_{o\bar{j}}\omega_{i\bar{o}}\right)\frac{\sqrt{-1}}{2}dT \wedge d\bar{T}$. Now, $|dT|^2_\omega := \sup_{\xi \neq 0} \frac{|dT(\xi)|^2}{\omega(\xi,\xi)}$. To compute the supremum, we can use the vectors $\xi = \frac{\partial}{\partial T} + \eta^i \frac{\partial}{\partial x^i}$ without loss of generality, so that

$$\frac{|dT(\xi)|^2}{\omega(\xi,\xi)} = \frac{1}{\omega_{o\bar{o}} + \omega_{i\bar{j}}\eta^i\bar{\eta}^j + 2\mathrm{Re}\left(\omega_{i\bar{o}}\eta^i\right)}.$$

The minimum of the denominator, as a function of η, is $\omega_{o\bar{o}} - \omega^{\bar{j}i}\omega_{o\bar{j}}\omega_{i\bar{o}}$, and thus

$$d\nu_t = \frac{dA_\omega}{|dT|^2_\omega}.$$

11.4.2 THEOREM. *Let (X,ω) be a Stein manifold. Let $T : X \to \mathbb{D}$ be a holomorphic submersion, and denote by $X_t := T^{-1}(t)$ the fiber, which is itself a Stein manifold. Fix a holomorphic line bundle $L \to X$ with Hermitian metric $e^{-\varphi}$ such that*

$$\partial\bar{\partial}\varphi + \mathrm{Ricci}(\omega) \geq 0.$$

Define

$$\mathfrak{L}_t(\varphi) := \left\{ f \in \Gamma(X_t, L|_{X_t})\ ;\ \int_{X_t} |f|^2 e^{-\varphi}\frac{dA_\omega}{|dT|^2_\omega} < +\infty \right\}$$

and

$$\mathscr{A}_t(\varphi) := \mathfrak{L}_t(\varphi) \cap \Gamma_{\mathcal{O}}(X_t, L|_{X_t}),$$

and let K_t be the Bergman kernel of the Bergman projection $P_t : \mathfrak{L}_t(\varphi) \to \mathscr{A}_t(\varphi)$. Then for any holomorphic frame $\mathbf{e}$ for $L^* \to X$ in a neighborhood U in X, the function

$$U \ni x \mapsto \log\left\langle K_{T(x)}(x,x), \mathbf{e}_x \otimes \mathbf{e}_x \right\rangle$$

is plurisubharmonic.

11.4.1. Show that Theorem 11.4.1 is a special case of Theorem 11.4.2.

Proof of Theorem 11.4.2. As was to be shown in Exercise 11.1.4,

$$K_t(x,x)e^{-\varphi_t(x)} = \sup\{|f(x)|^2 e^{-\varphi_t(x)}\ ;\ f \in \mathscr{A}_t \text{ with } ||f|| = 1\}. \tag{11.11}$$

By Bergman's Inequality the point evaluation functional is bounded, and hence by weak-$*$ compactness there exists a section $g_{x,t} \in \mathscr{A}_t$ that realizes the supremum, in the sense that

$$K_t(x,x)e^{-\varphi_t(x)} = \frac{|g_{x,t}(x)|^2 e^{-\varphi_t(x)}}{\int_{X_t} |g_{x,t}(y)|^2 e^{-\varphi_t(y)}\frac{dA_\omega}{|dT|^2_\omega}}.$$

Now fix $t_o \in \mathbb{D}$ and $\varepsilon > 0$ such that $D_\varepsilon(t_o) := \{t \in \mathbb{C}\ ;\ |t - t_o| < \varepsilon\} \subset\subset \mathbb{D}$ and let

$$X(\varepsilon) := T^{-1}(D_\varepsilon(t_o)) \quad \text{and} \quad \tau := \frac{T - t_o}{\varepsilon} \in \mathcal{O}(X).$$

Then by Theorem 11.3.1 there exists a holomorphic section $G \in \Gamma_{\mathcal{O}}(X(\varepsilon), L)$ such that $G|_{X_t} = g_{x,t}$ and

$$\int_{X(\varepsilon)} |G|^2 e^{-\varphi} dV_\omega \le \pi \int_{X_{t_o}} |g_{x,t}|^2 e^{-\varphi_t} \frac{dA_\omega}{|d\tau|^2_\omega} = \pi\varepsilon^2 \int_{X_{t_o}} |g_{x,t}|^2 e^{-\varphi_t} \frac{dA_\omega}{|dT|^2_\omega}.$$

Thus

$$\begin{aligned}
\log K_{t_o}(x,x) e^{-\varphi_{t_o}(x)} &\le \log |G(x)|^2 e^{-\varphi_{t_o}(x)} - \log \frac{1}{\pi\varepsilon^2} \int_{X(\varepsilon)} |G|^2 e^{-\varphi} dV_\omega \\
&= \log |G(x)|^2 e^{-\varphi_{t_o}(x)} - \log \frac{1}{\pi\varepsilon^2} \int_{t \in D_\varepsilon(t_o)} \left(\int_{X_t} |G|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega} \right) dA(t) \\
&\le \log |G(x)|^2 e^{-\varphi_{t_o}(x)} - \frac{1}{\pi\varepsilon^2} \int_{t \in D_\varepsilon(t_o)} \log \left(\int_{X_t} |G|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega} \right) dA(t),
\end{aligned}$$

where the equality is just Fubini's Theorem and the second inequality follows from the concavity of the logarithm.

Fix a frame $\mathbf{e}$ for L^* over an open set $U \subset X$ of the form $U = T^{-1}(D_\varepsilon(t_o))$ with U so small that it is biholomorphically a product. After multiplying $\mathbf{e}$ by a constant, we may assume without loss of generality that $\mathbf{e}_x \otimes \overline{\mathbf{e}_x} = e^{-\varphi_{t_o}(x)}$. Then the function $\langle G, \mathbf{e} \rangle$ is holomorphic on U, and we have

$$\begin{aligned}
&\log \langle K_{t_o}(x,x), \mathbf{e}_x \otimes \mathbf{e}_x \rangle = \log K_{t_o}(x,x) e^{-\varphi_{t_o}(x)} \\
&\le \log |G(x)|^2 e^{-\varphi_{t_o}(x)} - \frac{1}{\pi\varepsilon^2} \int_{t \in D_\varepsilon(t_o)} \log \left(\int_{X_t} |G|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega} \right) dA(t) \\
&= \log |\langle G, \mathbf{e} \rangle (x)|^2 - \frac{1}{\pi\varepsilon^2} \int_{D_\varepsilon(t_o)} \log \left(\int_{X_t} |G|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega} \right) dA(t) \\
&\le \frac{1}{\pi\varepsilon^2} \int_{D_\varepsilon(t_o)} \left(\log |\langle G, \mathbf{e} \rangle|^2 - \log \left(\int_{X_t} |G|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega} \right) \right) dA(t) \\
&= \frac{1}{\pi\varepsilon^2} \int_{D_\varepsilon(t_o)} \log |\mathbf{e}|^2 e^{\varphi_t} dA(t) \\
&\qquad + \frac{1}{\pi\varepsilon^2} \int_{D_\varepsilon(t_o)} \left(\log(|G|^2 e^{-\varphi_t}) - \log \left(\int_{X_t} |G|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega} \right) \right) dA(t) \\
&\le \frac{1}{\pi\varepsilon^2} \int_{D_\varepsilon(t_o)} \log |\mathbf{e}|^2 e^{\varphi_t} dA(t) + \frac{1}{\pi\varepsilon^2} \int_{D_\varepsilon(t_o)} \log(|K_t(x,x)|^2 e^{-\varphi_t}) dA(t) \\
&= \frac{1}{\pi\varepsilon^2} \int_{D_\varepsilon(t_o)} \log \langle K_t(x,x), \mathbf{e}_x \otimes \mathbf{e}_x \rangle dA(t),
\end{aligned}$$

where the second inequality is an application of the sub-mean value property for the plurisubharmonic function $\log |\langle G, \mathbf{e} \rangle (x)|^2$ and the third and last inequality comes from (11.11). We have therefore shown that the function $\log \langle K_t(x,x), \mathbf{e}_x \otimes \mathbf{e}_x \rangle$ is subharmonic in t, and since it is clearly plurisubharmonic in x, the proof is complete. □

11.4.3 REMARK. In Chapter 11.2 our proof of the Suita conjecture was made longer only because Berndtsson's Theorem implies the log-subharmonicity of the Bergman kernels K_t, rather than $e^{-t}B_{X_t}$. With Theorem 11.4.2 in hand, a much shorter proof is possible. The interested reader should consult [BL-2016], which treats the case of domains. It is a good exercise to adapt the proof of the Suita Conjecture in [BL-2016] to the setting of general Riemann surfaces. ⋄

11.4.2 Plurisubharmonic variation versus sharp L^2 extension

We have seen that Berndtsson's Theorem 11.1.8 on (pluri)subharmonic variation directly implies

(i) Theorem 11.4.1 in the special case of a product domain $D = \Omega \times X$, and

(ii) Theorem 11.3.1 on L^2 extension with sharp constants.

We have also seen that Theorem 11.3.1 implies Theorem 11.4.1 in its full generality. We shall now show that Theorem 11.3.1 implies a much stronger version of Berndtsson's Theorem, in which the product manifold $X \times \mathbb{D}$ is replaced by a more general Stein manifold, and the Hilbert spaces $\mathscr{H}(\varphi)_t$ are no longer required to be quasi-isometric, which is to say, that $\mathscr{H}(\varphi) \to \mathbb{D}$ need not be a vector bundle. We shall now make things more precise.

Let (X, ω) be a Kähler manifold of complex dimension $m+n$ and let $T : X \to \mathbb{D}^m$ a holomorphic submersion of X in the unit polydisk in $\mathbb{C}^m$. Denote by $X_t := T^{-1}(t)$ the (n-dimensional) fibers of T. Fix a holomorphic line bundle $L \to X$ with singular Hermitian metric $e^{-\varphi}$, and define the Hilbert spaces

$$\mathscr{H}(\varphi)_t := \left\{ f \in \Gamma_{\mathcal{O}}(X_t, L)\ ;\ ||f||_t^2 := \int_{X_t} |f|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega} < +\infty \right\}, \quad t \in \mathbb{D}^m.$$

Then we have a map $\mathscr{H}(\varphi) \to \mathbb{D}^m$ whose fibers are Hilbert spaces, and which we call the Hilbert fibration associated to $(X, \omega; L, e^{-\varphi}; T)$. Note that we do not require local triviality, or even equivalence of the fibers, from a Hilbert fibration.

11.4.4 DEFINITION. *(S) A section $\mathfrak{f}$ of a Hilbert fibration $\mathscr{H}(\varphi) \to \mathbb{D}^m$ is a section $F_{\mathfrak{f}} \in \Gamma(X, L)$ such that $F_{\mathfrak{f}}|_{X_t} \in \mathscr{H}(\varphi)_t$ for each $t \in \mathbb{D}^m$. We write*

$$\mathfrak{f}_t := F_{\mathfrak{f}}|_{X_t}.$$

The section $\mathfrak{f}$ of $\mathscr{H}(\varphi) \to \Omega$ is said to be holomorphic if $F_{\mathfrak{f}} \in \Gamma_{\mathcal{O}}(X, L)$. We denote by

$$\Gamma(\Omega, \mathscr{H}(\varphi)) \quad \textit{and} \quad \Gamma_{\mathcal{O}}(\Omega, \mathscr{H}(\varphi))$$

the set of sections, and holomorphic sections, of $\mathscr{H}(\varphi) \to \mathbb{D}^m$ respectively. (Thus all sections are holomorphic on the fibers, and a holomorphic section means it is holomorphic in the total space.)

(B) Let $\mathscr{H}(\varphi)^* \to \mathbb{D}^m$ denote the bundle of dual spaces, i.e., the fiber $\mathscr{H}(\varphi)^*_t$ of this bundle over $t \in \mathbb{D}^m$ is the dual Hilbert space of $\mathscr{H}(\varphi)_t$, with its usual Hilbert norm*

$$||\xi||_{t*} := \sup_{f \in \mathscr{H}(\varphi)_t - \{0\}} \frac{|\langle \xi, f \rangle|}{||f||_t}.$$

(S^*) A section of $\mathscr{H}(\varphi)^* \to \Omega$ is a map $\xi : \mathscr{H}(\varphi) \to \mathbb{C}$ such that

$$\xi_t := \xi|_{\mathscr{H}(\varphi)_t} \in \mathscr{H}(\varphi)^*_t.$$

The section ξ of $\mathscr{H}(\varphi)^* \to \Omega$ is said to be holomorphic if for each $\mathfrak{f} \in \Gamma_{\mathcal{O}}(\mathbb{D}^m, \mathscr{H}(\varphi))$ the function

$$\mathbb{D}^m \ni t \mapsto \langle \xi_t, f \rangle \in \mathbb{C}$$

is holomorphic. The set of holomorphic sections is denoted $\Gamma_{\mathcal{O}}(\Omega, \mathscr{H}(\varphi)^*)$.

Let us start with the case $m = 1$. We can now state the following generalization of Theorem 11.1.8.

11.4.5 THEOREM. *Let (X, ω) be a Stein Kähler manifold and let $T : X \to \mathbb{D}$ be a holomorphic submersion. Let $L \to X$ be a holomorphic line bundle with singular Hermitian metric $e^{-\varphi}$. Let $\mathscr{H}(\varphi) \to \mathbb{D}$ be the Hilbert fibration associated to $(X, \omega; L, e^{-\varphi}; T)$. Assume that*

$$\partial\bar{\partial}\varphi + \mathrm{Ricci}(\omega) \geq 0.$$

Then for any holomorphic section $\xi \in \Gamma_{\mathcal{O}}(\mathbb{D}, \mathscr{H}(\varphi)^*)$ the function

$$\mathbb{D} \ni t \mapsto \log ||\xi_t||^2_{t*}$$

is subharmonic.

Proof. The proof is very similar to that of Theorem 11.4.2. First,

$$||\xi_t||^2_{t*} = \sup_{f \in \mathscr{H}(\varphi)_t} \frac{|\langle \xi_t, f \rangle|^2}{\int_{X_t} |f|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega}} \tag{11.12}$$

Now fix $t_o \in \mathbb{D}$ and $\varepsilon > 0$ such that $D_\varepsilon(t_o) := \{t \in \mathbb{C}\ ;\ |t - t_o| < \varepsilon\} \subset\subset \mathbb{D}$ and let

$$X(\varepsilon) := T^{-1}(D_\varepsilon(t_o)) \quad \text{and} \quad \tau := \frac{T - t_o}{\varepsilon} \in \mathcal{O}(X).$$

Then by Theorem 11.3.1 there exists a holomorphic section $F \in \Gamma_{\mathcal{O}}(X(\varepsilon), L)$ such that $F|_{X_{t_o}} = f$ and

$$\int_{X(\varepsilon)} |F|^2 e^{-\varphi} dV_\omega \leq \pi \int_{X_{t_o}} |f|^2 e^{-\varphi} \frac{dA_\omega}{|d\tau|^2_\omega} = \pi\varepsilon^2 \int_{X_{t_o}} |f|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega}.$$

Fixing $f \in \mathscr{H}(\varphi)_t$, we let $F_f \in \Gamma_{\mathcal{O}}(X(\varepsilon), L)$ denote the unique extension of minimal norm. Then

$$\frac{|\langle \xi_{t_o}, f \rangle|^2}{\int_{X_{t_o}} |f|^2 e^{-\varphi} \frac{dV_\omega}{|dT|^2_\omega}} \leq \frac{1}{\pi\varepsilon^2} \frac{|\langle \xi_{t_o}, f \rangle|^2}{\int_{X(\varepsilon)} |F_f|^2 e^{-\varphi} dV_\omega}.$$

Choosing the section f_o that realizes the supremum (11.12) and writing $F_o := F_{f_o}$, we have

$$\begin{aligned}
\log ||\xi_{t_o}||^2_{t_o *} &= \log |\langle \xi_{t_o}, f_o\rangle|^2 - \log \int_{X_{t_o}} |f_o|^2 e^{-\varphi} \frac{dV_\omega}{|dT|^2_\omega} \\
&= \log |\langle \xi_{t_o}, F_o\rangle|^2 - \log \int_{X_{t_o}} |f_o|^2 e^{-\varphi} \frac{dV_\omega}{|dT|^2_\omega} \\
&\le \log |\langle \xi_{t_o}, F_o\rangle|^2 - \log \left(\frac{1}{\pi\varepsilon^2} \int_{X(\varepsilon)} |F_o|^2 e^{-\varphi} dV_\omega \right) \\
&= \log |\langle \xi_{t_o}, F_o\rangle|^2 - \log \left(\frac{1}{\pi\varepsilon^2} \int_{t \in D_\varepsilon(t_o)} \left(\int_{X_t} |F_o|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega} \right) dA(t) \right) \\
&\le \log |\langle \xi_{t_o}, F_o\rangle|^2 - \frac{1}{\pi\varepsilon^2} \int_{t \in D_\varepsilon(t_o)} \log \left(\int_{X_t} |F_o|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega} \right) dA(t) \\
&\le \frac{1}{\pi\varepsilon^2} \int_{t \in D_\varepsilon(t_o)} \left(\log |\langle \xi_t, F_o\rangle|^2 - \log \left(\int_{X_t} |F_o|^2 e^{-\varphi} \frac{dA_\omega}{|dT|^2_\omega} \right) dA(t) \right) \\
&\le \frac{1}{\pi\varepsilon^2} \int_{D_\varepsilon(t_o)} \log ||\xi_t||^2_{t*} dA(t),
\end{aligned}$$

where the second inequality follows from concavity of the logarithm, the third inequality follows from the holomorphicity of $t \mapsto \langle \xi_t, F_o\rangle$ (i.e., the fact that ξ_t is a holomorphic section), and the last inequality follows from (11.12). The proof of Theorem 11.4.5 is complete. □

EXERCISES

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