Spacetime Without Scattering

(Or "The Bare Bones of AdS-CFT")

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R. G., "What is the Simplest Gauge-String Duality?", 1104.2386

R. G. and Roji Pius, To Appear

ICTS, Bangalore

Sep. 27, 2012

The Goal, The Motivation

- ✦ The Goal
  - ✧ Try to derive the string dual ("spacetime") to the 0-dimensional ("without scattering") Gaussian Matrix Model (in the large N 't Hooft limit).

$$Z[t_k] = \int [dM]_{N\times N} e^{-N\mathrm{Tr}M^2 + \sum_k t_k N \mathrm{Tr} M^{2k}}$$

- ✦ The Motivation
  - ✧ Understand the "inner workings" of AdS/CFT in a model which keeps enough structural complexity as well as analytic tractability.
  - ✧ Could describe a simple subsector of the string theory on $AdS_5 \times S^5$ describing half BPS Wilson Loops in $\mathcal{N} = 4$ Super Yang-Mills theory.

The Thread of Logic

- Describe the suggestive combinatorics of correlators $\langle\prod_{i=1}^{n} \mathrm{Tr} M^{2k_i}\rangle_{g=0}$ in a Gaussian matrix model - in terms of counting branched holomorphic covers from $\mathbb{P}^1 \to \mathbb{P}^1$ with three branching points (Belyi Maps).

- Realisation of this picture in terms of gluing together Feynman diagrams ("open-closed string duality") and explicit Belyi maps.

- Identify target $\mathbb{P}^1$ with the Riemann Surface associated to the Gaussian matrix model $(y^2 = x^2 - 4)$ and see a skeletal version of AdS/CFT.

- Connect this string description to the conventional A-model topological string theory on $\mathbb{P}^1$, by comparing with correlators $\langle\prod_i \sigma_{2k_i}(Q)\rangle_{g=0}$:
  - Obtain the right selection rule for the degree $= k = \sum_i k_i$.
  - Matching of explicit answers on both sides for arbitrary k_i and any $n-$point function (at genus zero).

1 Belyi Maps

The Importance of Three Permutations

- Gaussian Matrix model correlators can be expressed in terms of *three* sets of permutations. (Itzykson et.al., de Mello Koch and Ramgoolam)

- $N^n \langle \prod_i^n \mathrm{Tr} M^{2k_i} \rangle_g = \sum_{\alpha,\gamma \in S_{2k}} \delta(\alpha \cdot \beta \cdot \gamma) N^{2-2g(\alpha,\beta,\gamma)}$.

- (α, β, γ) are conjugacy classes in S_{2k} where $2k = 2\sum_i k_i$ is the total number of half-edges in the diagram. Related to vertices, edges and faces.

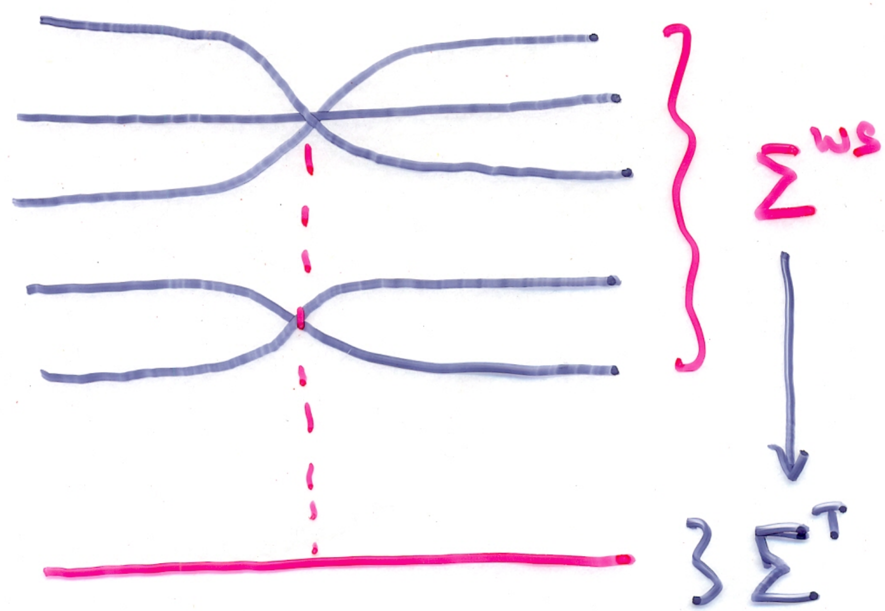
- VERTICES: $\beta \in (2k_1)(2k_2)\ldots(2k_n)$ captures the cyclic information in the half edges at each vertex.

- EDGES: $\alpha \in [2^k]$ consists only of two cycles (transpositions) and captures the various possible Wick contractions amongst the half-edges (edge information).

- FACES: $\gamma = \beta^{-1}\alpha^{-1}$ captures the arrangement of half-edges around a face.

Permutations and Branched Coverings

✦ Permutations can be naturally associated with branching structure of holomorphic maps of Riemann surfaces $\Sigma_g^{(WS)} \to \Sigma_G^{(T)}$. (Riemann)



✦ Cycle structure $(l_1)(l_2)\ldots(l_n) \in S_d$ associated to each branch point. Branching number $b = \sum_i (l_i - 1) = d - n$. ($d$ is the degree of the covering).

✦ Riemann-Hurwitz relation: $2 - 2g = d(2 - 2G) - B$. $\quad (B = \sum_{br.pts.} b)$

Three Permutations and Belyi Maps

- Associate $(\alpha, \beta, \gamma) \in S_{2k}$ with three branchings in a degree $2k$ cover of $\mathbb{P}^1$ by $\Sigma_g^{(WS)}$.

- $b_\beta = 2k - n; \quad b_\alpha = 2k - k; \quad b_\gamma = 2k - F \Rightarrow B = 5k - n - F.$

- $2d(2 - 2 \times 0) - B = n - k + F = 2 - 2g.$

- Satisfies Riemann-Hurwitz relation with $G = 0$. Moreover, need $\alpha \cdot \beta \cdot \gamma = 1$ for three branch points on a sphere.

- Thus, combinatorics of the matrix model correlators suggests one is simply counting a sum over holomorphic maps (of degree $2k$) from $\Sigma^{(WS)} \to \mathbb{P}^1$ with branching over $(0, 1, \infty)$ - **Belyi Maps**. (Itzykson et.al., de Mello Koch-Ramgoolam)

- Can we see this as some kind of realization of AdS/CFT? Relate it to known facts about this matrix model? Is there a conventional description of this dual string theory?

2 Belyi Maps and Open-Closed String Duality

Belyi Maps and Arithmetic Surfaces

- Belyi's Theorem: Belyi maps are admissible at special points in the moduli space of riemann surfaces - *arithmetic* Riemann surfaces.

- An arbitrary Riemann surface can be obtained by gluing together strips of varying width (Strebel).

- Underlying mechanism for open-closed string duality converting Feynman-'tHooft diagrams (strips) to world sheets (R.G. 2003-05).

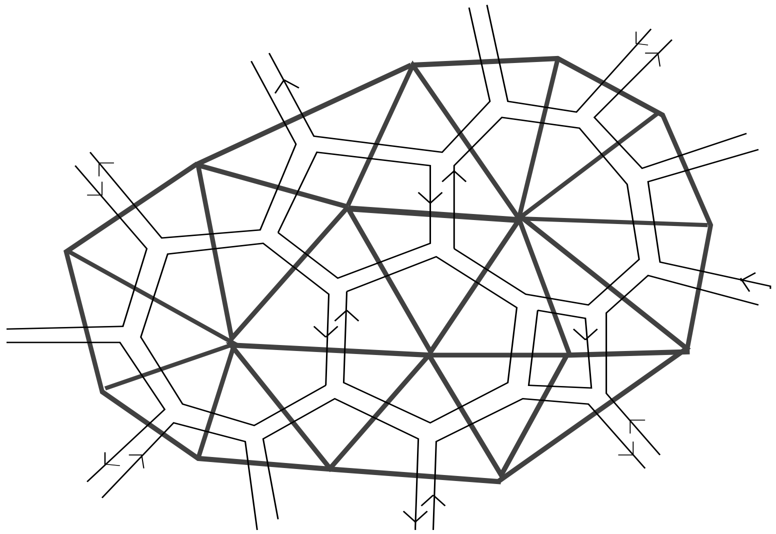
- Arithmetic surfaces are those where the width is a *positive integer* (Mulase-Penkava).

- In fact, natural in the open-closed string duality of matrix models to consider integer width strips (Razamat).

- Use this to give an explicit realization of Belyi maps from the world sheet of the matrix model to a target $\mathbb{P}^1$.

Triangulated Riemann Surfaces

- Gaussian matrix model generates *equilaterally* triangulated Riemann surfaces - dual edges (i.e. width of double lines) are of unit length.



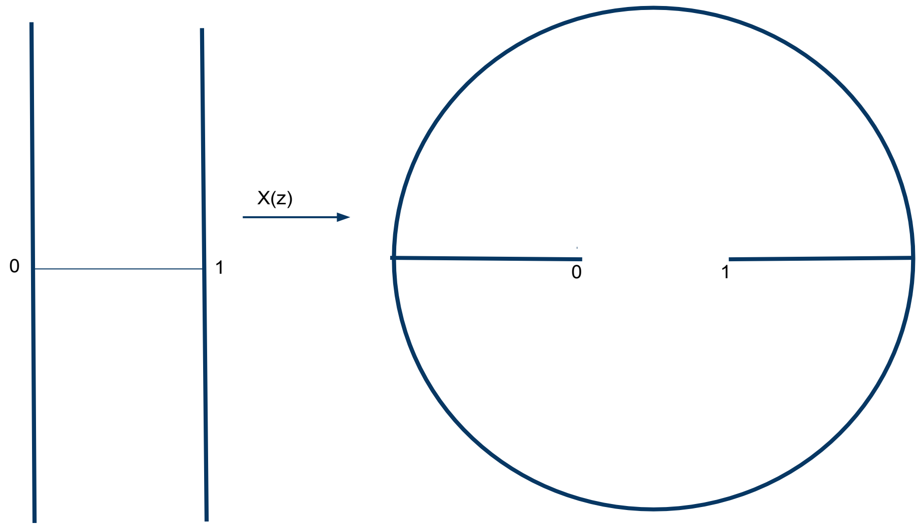
- Earlier Intuition: A discrete approximation to a Riemann surface.

- NOW: They are special (interior) points in the moduli space of Riemann surfaces.

- Can explicitly define Belyi maps from these special Riemann surfaces to $\mathbb{P}^1$.

Explicit Belyi Map

- ✦ First, give the Belyi Map for each strip (Feynman-'t Hooft fat graph edge) and then show how this is compatible with the gluing of strips.

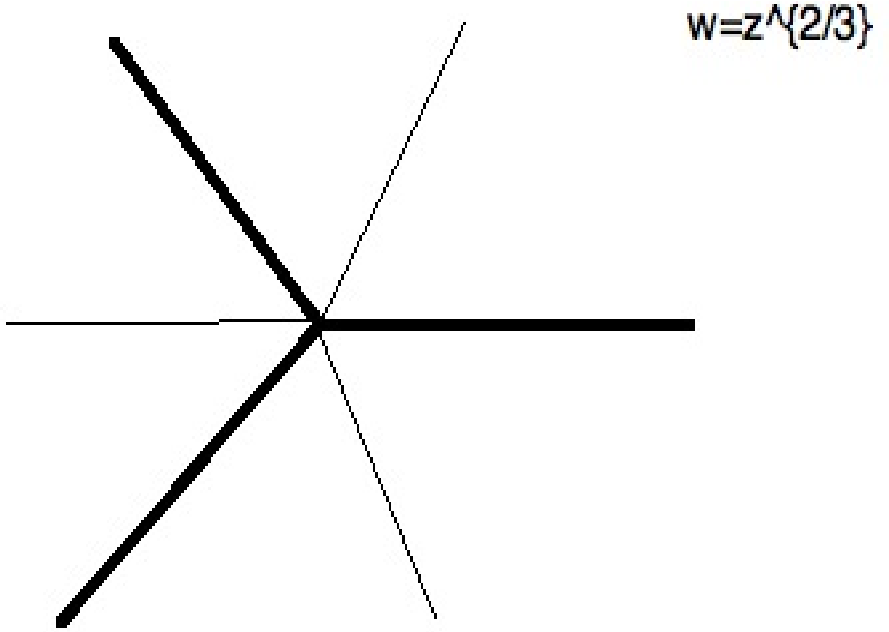


$$X(z) = \sin^2 \frac{\pi z}{2}.$$

Each strip of width one is mapped once onto a $\mathbb{P}^1$. $\quad (0,1) \to (0,1)$.

Gluing at the Faces

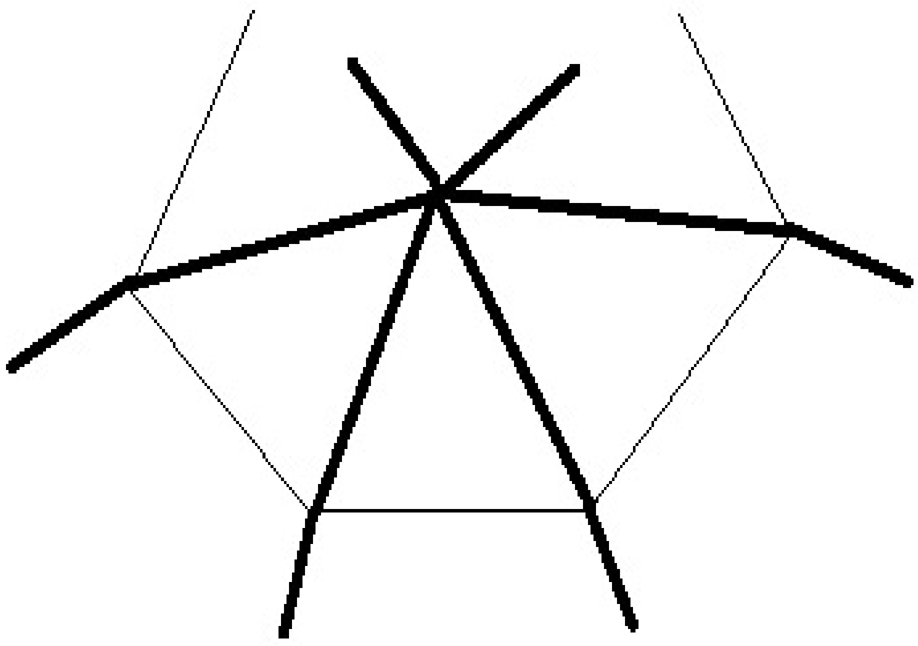
- At each p-gonal face of Feynman graph, (i.e. dual vertex of order p) the gluing is through $w = z^{\frac{2}{p}}$ ("closing up holes").



- Composing with $X(z) = \sin^2 \frac{\pi z}{2} \to X(w) \propto w^p$.
- There is a branch point of order p at the centre of each face.

Gluing at the Vertices

- At each $2k$-fold vertex of the Feynman graph ($2k$-gonal dual face) the gluing is through $u \propto e^{\frac{2\pi i z}{2k}}$. Here the vertex is at $z = i\infty$.



- Composed with $X(z) = \sin^2 \frac{\pi z}{2} \to X(u) \propto u^{-k}$.
- There is a branch point of order k at each vertex (insertion of $\mathrm{Tr} M^{2k}$).

Relation to the Combinatorics

- Therefore, we have a branched cover of $\mathbb{P}^1$ with a branching $(k_1)(k_2)\ldots(k_n)$ at ∞ of degree $k = \sum_i k_i$.

- We have branching at $(0,1)$ as well determined by the faces. Only three branch points on the target $\mathbb{P}^1$.

- Consider now the further map $Y^2 = 4X(1-X)$ - a double cover of the above $\mathbb{P}^1$.

- $X = 0, 1 \to Y = 0$ and $X = \frac{1}{2} \to Y = 1$.

- Now we have a branched cover of degree $2k$ with branching $\beta = (2k_1)(2k_2)\ldots(2k_n)$ at $Y = \infty$, $\alpha \in [2^k]$ at $Y = 1$ and γ determined by the faces at $Y = 0$.

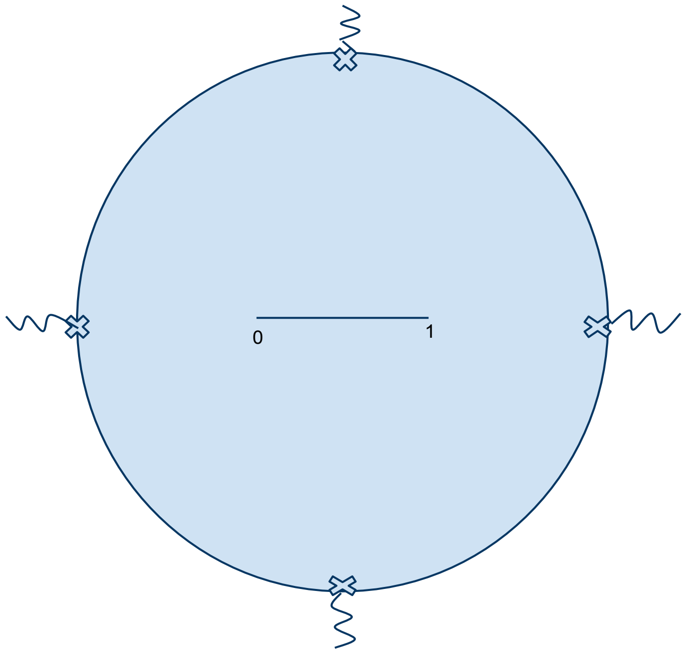
- Thus open-closed string duality as implemented by the above gluing, gives a complete realization of the combinatoric picture of Belyi maps.

3 Skeletal AdS/CFT

The Skeleton *continued*

- The target space $\mathbb{P}^1$ given by $Y^2 = 4X(1-X)$ (which is covered k times by the worldsheet) is nothing but the Riemann surface associated to the Gaussian matrix model.

- The branch cut at $(0, 1)$ is the usual cut corresponding to the Wigner semi-circle law.



The Skeleton *continued*

- External Vertex operator insertions $\mathcal{O}_{k_i} \leftrightarrow \mathrm{Tr}M^{2k_i}$ at infinity ("UV") each have a branching number k_i.
- Different Feynman graphs $\rightarrow$ sum over holomorphic maps with fixed branching data at infinity: $(k_1)\ldots(k_n)$.
- Leads to additional branching at 0 and 1 ("IR"), the two endpoints of the holographically emergent "space" of eigenvalues.
- The spatial slices of the worldsheet (the interval $[0,1]$ on each strip) mapped onto the eigenvalue cut $[0,1]$ (the "emergent space").
- Basically the strings coming in from infinity split and join at these specific points (cf. Gross-Mende, Eynard-Orantin).
- These holomorphic maps *are* stringy Witten diagrams in the target space $\mathbb{P}^1$.

4 Relation to the A-Model Topological String

A Conventional Worldsheet Description ?

- Many features of the dual string seem to suggest a topological character.
- In particular, there is a conventional string theory which "counts" holomorphic maps to a target space - the A-Model topological string theory.
- The theory with target $\mathbb{P}^1$ has basically two sets of observables $\sigma_m(P), \sigma_m(Q)$, $(m = 0, 1, \ldots)$ - "gravitational descendants" of the two nontrivial cohomology classes.
- We make the natural identification $\sigma_{2k_i}(Q) \leftrightarrow \mathrm{Tr}M^{2k_i}$ and explicitly compare correlators.
- Consider correlators $\langle \sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\prod_{i=3}^{n}\sigma_{2k_i}(Q)\rangle_{g=0}$. They obey a selection rule for the degree:

$$\sum_{i=1}^{n} 2k_i - 2 = 2(G-1) + c_1(\Sigma_G)d = -2 + 2d \Rightarrow d = \sum_{i=1}^{n} k_i = k.$$

Comparing Correlators

- Thus the topological string gets a non-zero contribution only from holomorphic maps of the same degree as in the Belyi map picture. Important check.

- So try to compute the answers explicitly on both matrix model and string theory side and compare.

- $$4\langle\sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\rangle_{g=0} = \frac{2k_1!}{(k_1!)^2}\frac{2k_2!}{(k_2!)^2}\frac{1}{k_1+k_2} = \langle\frac{1}{k_1}\mathrm{Tr}M^{2k_1}\frac{1}{k_2}\mathrm{Tr}M^{2k_2}\rangle_{g=0}.$$

- $$4\langle\sigma_{2k_1-1}\sigma_{2k_2-1}\sigma_{2k_3}\rangle_{g=0} = \frac{2k_1!}{(k_1!)^2}\frac{2k_2!}{(k_2!)^2}\frac{2k_3!}{(k_3!)^2} = \langle\frac{1}{k_1}\mathrm{Tr}M^{2k_1}\frac{1}{k_2}\mathrm{Tr}M^{2k_2}\frac{1}{k_3}\mathrm{Tr}M^{2k_3}\rangle_{g=0}.$$

- So two and three point functions agree very nicely.

Comparing Correlators *continued*

- The general n-point string correlator at genus zero given by:

$$4\langle\sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\prod_{i=3}^{n}\sigma_{2k_i}(Q)\rangle_{g=0}=(\sum_{i=1}^{n}k_i)^{n-3}\prod_{i=1}^{n}\frac{2k_i!}{(k_i!)^2}.$$

- The general matrix model planar connected n-point correlator given by:

$$\langle\prod_{i=1}^{n}\frac{1}{k_i}\mathrm{Tr}M^{2k_i}\rangle_{g=0}=\frac{(\sum_{i=1}^{n}k_i-1)!}{(\sum_{i=1}^{n}k_i-n+2)!}\prod_{i=1}^{n}\frac{2k_i!}{(k_i!)^2}.$$

- There appears to be a slight mismatch for $n>3$. Note that $\frac{(k-1)!}{(k-n+2)!}=k^{n-3}+ck^{n-2}+\ldots$.

- There is agreement in the large k limit (BMN type regime).

- Actually, there appears to be an attractive interpretation of the finite k corrections.

Comparing Correlators *continued*

- In fact, in the string correlator on the LHS: $\langle \sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\prod_{i=3}^{n}\sigma_{2k_i}(Q)\rangle_{g=0}$ we have for (for $n > 3$) contact contributions when two of the operators collide (Dijkgraaf, Verlinde and Verlinde).

- This corresponds in our language to two branch points on the worldsheet (with cycle structure $(k_3)(k_4)$) coming together and giving a single branchpoint with cycle structure $(k_3 + k_4)$.

- So the string correlator also includes contributions from matrix model correlators where $\mathrm{Tr}M^{2k_3}\mathrm{Tr}M^{2k_4} \to \mathrm{Tr}M^{(2k_3+2k_4)}$.

- Use the identity $k^{n-3} = \sum_{m=3}^{n} \tilde{S}_{n-2}^{(m)} \frac{(k-1)!}{(k-m+2)!}$ where $\tilde{S}_{n-2}^{(m)}$ is the number of ways of partitioning a set of $(n-2)$ elements into m non-empty subsets.

- Then one can write the string correlator exactly as a sum of all the matrix model correlators which include bringing together various subsets of operators into collision.

To Conclude....To Continue

- Matrix models can give a hands on picture of gauge-string duality with some unusual features.
- A dual topological string with amplitudes localised at special points on the moduli space which counts holomorphic maps to $\mathbb{P}^1$ with three branchpoints.
- Arises in a precise way from an implementation of open-closed string duality via gluing of Feynman-'tHooft fatgraphs.
- The target space: Riemann surface associated with the matrix model eigenvalues.
- The holomorphic (Belyi) maps describe the stringy Witten diagrams in the target space $\mathbb{P}^1$.
- Seems to give striking agreement with conventional A-model topological string correlators.
- Can we find the A-model topological string by localization of the IIB string theory on $AdS_5 \times S^5$?

"This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections....I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact."

- A. Grothendieck (on Belyi maps and the Belyi theorem)