# Statistics of Geometric Phase Curvature

Pragya Shukla

Department of Physics,
Indian Institute of Technology, Kharagpur, Kanpur (*visiting*)



in collaboration with Professor Sir Michael Berry
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# Geometric Curvature



- Geometric phase defined for a closed loop
  when the Hamiltonian goes around a closed loop $\mathbf{R}(T) = \mathbf{R}(0)$ in parameter space,
- circuit dependent in general
- need for a closed loop usually leads to technical issues
- often better to use "local phase": **geometric curvature**

$$\gamma_n(T) = i \oint_{\mathbf{R}(0)}^{\mathbf{R}(T)} \langle n, \mathbf{R}(\tau)|\, \nabla_R |n, \mathbf{R}(\tau)\rangle \,.\mathrm{d}\mathbf{R} = \iint \mathbf{C}_n .\, \mathrm{d}\mathbf{S}$$



**Geometric curvature** $\quad \mathbf{C}_n \equiv \nabla_{\mathbf{R}} \times i\langle n, \mathbf{R}|\nabla_R|n, \mathbf{R}\rangle$

- Geometric phase for a loop $\rightarrow$ flux of the Geometric curvature

# Curvature is better than phase !

- Geometric phase always associated with a closed path
  Geometric curvature a local quantity
  provides a local description of the geometric properties of the parametric space.

- both gauge-invariant, can act as observables

- Geometric curvature: a geometric magnetic field
  Just as a magnetic field can affect the dynamic of an electron,
  curvature directly affects the dynamic of parameters (adiabatic).
  In this sense, curvature is more fundamental quantity than phase.



- no closed loop needed for curvature analysis

- curvature has many applications: example: essential in a proper description of thr Bloch electrons which has various effects on transport and thermodynamics properties

# in terms of eigenvalues and eigenfunctions



$$\mathbf{C_n(R) = 2\ \mathrm{Im} \sum_{m \neq n} \frac{\langle n|\nabla_R H|m\rangle \times \langle m|\nabla_R H|n\rangle}{(E_n - E_m)^2}}$$

**Hamiltonian $H(\mathbf{R})$ in 3-dimensional parameter space $\mathbf{R} = (x, y, z)$**

$$\mathbf{C_n = (C_{nx}, C_{ny}, C_{nz})} \qquad \text{(in vector form)}$$

$$\mathbf{C_{nz}(x,y) = 2\ \mathrm{Im} \sum_{m \neq n} \frac{\langle n|\partial_x H|m\rangle\langle m|\partial_y H|n\rangle}{(E_n - E_m)^2}}$$



**what is geometric curvature for a complex system?**

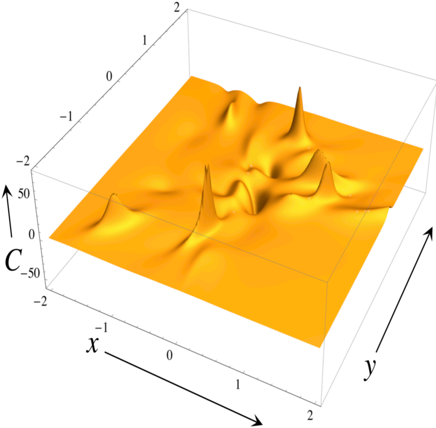
# Curvature of a typical random state:

$H(x, y) = H_0 + xH_x + yH_y$





**for the state $n = 10$ of a $20 \times 20$ Hamitonian with $H_0, H_x, H_y$ sampled from the GUE.**







# Role of Complexity

- Lack of detailed knowledge about interactions manifests itself by randomization (partial / full) of various operators of the system.
- Determination of some/ all matrix elements $\Rightarrow$ within an uncertainty $\Rightarrow$ matrix elements are at best described by a probability distribution.
- Uncertainty associated with various matrix element can be of different types $\Rightarrow$ matrix elements can have different distributions.
- Hamiltonian $\Rightarrow$ a random matrix (**some/ all matrix elements randomly distributed**)
- Distributions of the eigenfunctions and eigenvalues
- What is the distribution of geometric curvature $\mathbf{P_c(C)}$ ?



# distribution of curvature

Complexity leads to sample to sample fluctuations in eigenvalues and eigenfunctions $\rightarrow$ fluctuations of geometric curvature

- **distribution of curvature components $(\mathbf{C_x}, \mathbf{C_y}, \mathbf{C_z})$ of an arbitrary eigenstate**

$$\mathbf{P}(\mathbf{C_x}, \mathbf{C_y}, \mathbf{C_z}) = \langle \delta(\mathbf{C_x} - \mathbf{C_{nx}}) \delta(\mathbf{C_y} - \mathbf{C_{ny}}) \delta(\mathbf{C_z} - \mathbf{C_{nz}}) \rangle$$

with

$$\mathbf{C_{nz}} = \mathbf{2}\ \mathrm{Im} \sum_{\mathbf{m} \neq \mathbf{n}} \frac{\langle \mathbf{n} | \partial_{\mathbf{x}} \mathbf{H} | \mathbf{m} \rangle \langle \mathbf{m} | \partial_{\mathbf{y}} \mathbf{H} | \mathbf{n} \rangle}{(\mathbf{E_n} - \mathbf{E_m})^{\mathbf{2}}}$$



similarly $\mathbf{C_{nx}}, \mathbf{C_{ny}}$

**needed: distribution of the eigenvalues and eigenfunctions of $H$**

# begin with a simple question





**Hamiltonian $H(\mathbf{R})$ in 2-dimensional parameter space $\mathbf{R} = (\mathbf{x}, \mathbf{y})$**

$$\textbf{For} \qquad \mathbf{H(x,y) = H_0 + x\, H_x + y\, H_y}$$

$$\mathbf{C \equiv C_n(x,y) = 2}\ \mathrm{Im} \sum_{\mathbf{m \neq n}} \frac{\langle \mathbf{n|H_x|m} \rangle \langle \mathbf{m|H_y|n} \rangle}{\mathbf{(E_n - E_m)^2}}$$

# $2 \times 2$ Hamiltonian



- only two states with curvature $\mathbf{C_1, C_2}$

$$\mathbf{C_1 = -C_2 = 2}\ \mathrm{Im}\frac{\langle \mathbf{1}|\mathbf{H_x}|\mathbf{2}\rangle\ \langle \mathbf{2}|\mathbf{H_y}|\mathbf{1}\rangle}{(\mathbf{E_1 - E_2})^\mathbf{2}} = \frac{\mathbf{-i(F - F^*)}}{\mathbf{S^2}}$$

- $\mathbf{E_2 - E_1 \equiv S}$: spacing between eigenvalues $\mathbf{E_1, E_2}$

- **Perturbation $\mathbf{H_x, H_y}$ measured in eigenfunction basis of $\mathbf{H}$**
  $\mathbf{F \equiv F_{12} \equiv \langle 1|H_x|2\rangle\ \langle 2|H_y|1\rangle = F^*_{21}}$



- **Desired Probability distribution**

$$\begin{aligned}\mathbf{P_c(C)} &= \langle \delta\,[\mathbf{C - C_1}]\rangle \\ &= \frac{\mathbf{1}}{\mathbf{2}\pi}\int_{-\infty}^{\infty} \mathrm{dt}\ \exp[\mathbf{iCt}]\ \langle \exp\,[\mathbf{-iC_1t}]\rangle\end{aligned}$$

$\langle . \rangle$ : ensemble average

# $2 \times 2$ Hamiltonians





$$\langle \exp\left[-\mathbf{i C_1 t}\right]\rangle = \int \exp\left[\frac{(\mathbf{F} - \mathbf{F}^*)}{\mathbf{S^2}}\right] \mathbf{P}(\mathbf{F}, \mathbf{F}^*, \mathbf{S})\, \mathrm{d}\mathbf{F}\, \mathrm{d}\mathbf{F}^*\, \mathrm{d}\mathbf{S}$$

- matrix elements of $\mathbf{H_x}, \mathbf{H_y}, \mathbf{H_0}$ correlated in eigenfunction basis of $\mathbf{H}$

$$\mathbf{0} = \langle \mathbf{1}|\mathbf{H}|\mathbf{2}\rangle = \langle \mathbf{1}|\mathbf{H_0}|\mathbf{2}\rangle + \mathbf{x}\,\langle \mathbf{1}|\mathbf{H_x}|\mathbf{2}\rangle + \mathbf{y}\langle \mathbf{1}|\mathbf{H_y}|\mathbf{2}\rangle$$

$\longrightarrow$ F, F$^*$ are correlated



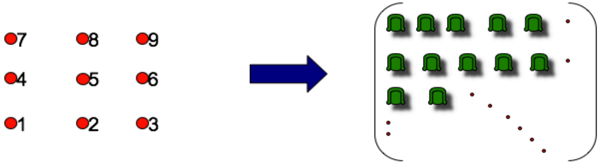
- is $\mathbf{S}$ uncorrelated with $\mathbf{F}, \mathbf{F}^*$?
  $\longrightarrow$**eigenvalues and eigenfunctions of $H$ uncorrelated?**

# H-matrix in site basis: ergodic limit





Delocalized Wave Dynamics: waves extended in entire system




In the basis associated with the space

$\langle k|\mathrm{H}|l\rangle = \mathrm{H}_{kl}$ : Similar Distribution about most probable value

$\langle \mathrm{H}_{kl}^2\rangle$ : Indicator of determinstic uncertainty

If available information : $\langle \mathrm{H}_{kl}^2\rangle = \sigma^2, \quad \langle \mathrm{H}_{kl}\rangle = 0$

$$\rho(H) = \exp\left[-\frac{1}{2\sigma^2}\sum_{k\le l}\mathrm{H}_{kl}^2\right] = \exp\left[-\frac{1}{2\sigma^2}\mathrm{Tr}\,\mathrm{H}^2\right]$$

Wigner-Dyson Ensembles

# extended eigenstates: $\mathbf{H(x,y) = H_0 + x\, H_x + y\, H_y}$



**what is $P(C)$ when dynamics of the system represented by $H$ is ergodic?**

**eigenfunctions delocalized in basis of interest, H: metallic, chaotic**

$H$ : represented by Gaussian unitary ensembles

$$\rho(\mathbf{H}) = \mathcal{N} \exp\left[-\mathrm{Tr}\, \mathbf{H^2}\right]$$

$$\begin{aligned} \mathbf{P_{e,u}(E,U)} &= \mathbf{J(E,U|H)}\, \rho(\mathbf{H}) \\ &= |\mathbf{E_1} - \mathbf{E_2}|^{\mathbf{2}} \exp\left[-(\mathbf{E_1^2} + \mathbf{E_2^2})\right] \delta(\mathbf{U^\dagger U - I}) \end{aligned}$$



eigenvalues and eigenfunctions of $\mathrm{H}$ not correlated

$\longrightarrow$ lack of correlations between $\mathrm{S} = |\mathrm{E_1} - \mathrm{E_2}|$ and $\mathrm{F}$

$$\mathbf{F = H_{x;12}\, H_{y;21} = \sum_{k,l} U^*_{k1}\, U_{l2}\, U^*_{j2}\, U_{r1}\, H_{x;kl}\, H_{y;jr}}$$

$$\mathbf{P(F,F^*,S) = P_f(F,F^*)\, P_s(S)}$$



# $2 \times 2$ Hamiltonians

- For mutually independent, Gaussian distributed (zero mean, unit variance) matrix elements of $\mathrm{H_0, H_x, H_y}$ in an arbitrary basis, distribution of $\mathrm{F} \longrightarrow$ correlated Gaussian in eigenfunction basis

$$\mathbf{P_f(F_x, F_y)} = \mathcal{N} \exp\left[-\frac{\mathbf{1}}{\mathbf{2(1-K^2)}}\left(\frac{\mathbf{F_x^2}}{\mathbf{V_x}} - \frac{\mathbf{2KF_xF_y}}{\sqrt{\mathbf{V_xV_y}}} + \frac{\mathbf{F_y^2}}{\mathbf{V_y}}\right)\right]$$

- $\mathrm{V_x, V_y}$: variances of $\mathrm{F_x, F_y}$, varainces of real and imaginary parts of $\mathrm{F} \equiv \mathrm{F_{12}}$
- normalization $\mathcal{N} = \frac{1}{2\pi\sqrt{\mathrm{V_xV_y(1-K^2)}}}$



$$\begin{aligned}\langle \exp[-\mathbf{iC_1t}]\rangle &= \int \exp\left[\frac{\mathbf{(F-F^*)}}{\mathbf{S^2}}\right] \mathbf{P(F,F^*)\,P_s(S)\,dF\,dF^*\,dS} \\ &= \int \frac{\mathbf{S^4}}{\mathbf{S^4 + 4t^2V_xV_y(1-K^2)}}\,\mathbf{P_s(S)\,dS}\end{aligned}$$

# $2 \times 2$ Hamiltonians





$$\mathbf{P_c(C)} = \frac{\mathbf{1}}{\mathbf{2}\pi} \int_{-\infty}^{\infty} \mathrm{dt}\ \exp[\mathbf{iCt}]\ \langle \exp\left[-\mathbf{iC_1 t}\right] \rangle$$

- $\mathrm{H_0, H_x, H_y}$: **GUE** $\Rightarrow H$ **GUE**
- nearest neighbor spacing distribution for **GUE**

$$\mathbf{P_s(S)} = \frac{\mathbf{32}}{\pi^{\mathbf{2}} \langle \mathbf{S} \rangle^{\mathbf{3}}}\ \mathbf{S^2}\ \exp\left[-\frac{\mathbf{4S^2}}{\pi \langle \mathbf{S} \rangle^{\mathbf{2}}}\right]$$

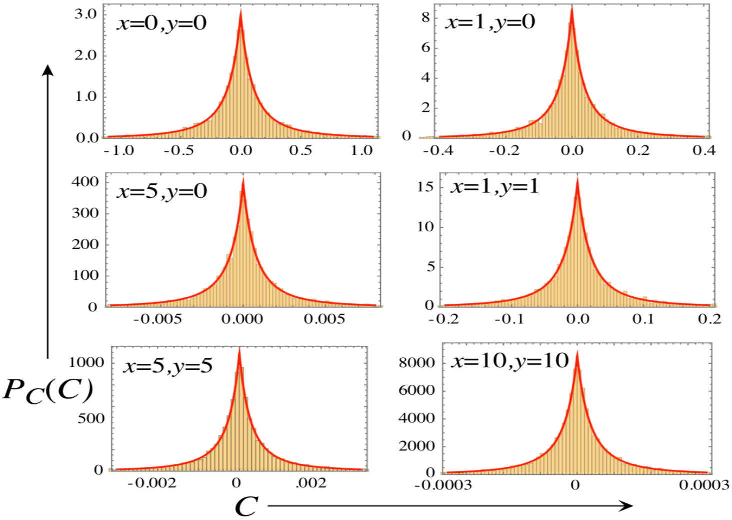


$$\Rightarrow \qquad \mathbf{P_c(C)} = \frac{\mathbf{3}}{\mathbf{4A}} \left(\mathbf{1} + \frac{|\mathbf{C}|}{\mathbf{A}}\right)^{-\mathbf{5/2}}$$

**Rescaling by** $A = \frac{8\sqrt{V_x V_y (1-K^2)}}{\pi \langle S \rangle^2} = \frac{1}{4(1+x^2+y^2)^{5/2}} \rightarrow$ **universal form**

# Case $N = 2$: theory with numerics





# $N \times N$ Hamiltonians





- $(N-1)$ **states contributing to curvature** $C_n$

$$
\begin{aligned}
C_n &= 2\,\mathrm{Im} \sum_{m \neq n} \frac{\langle n|H_x|m\rangle \, \langle m|H_y|n\rangle}{(E_n - E_m)^2} \\
&\approx 2\,\mathrm{Im} \left[ \frac{\langle n|H_x|n+1\rangle \, \langle n+1|H_y|n\rangle}{(E_n - E_{n+1})^2} + \frac{\langle n|H_x|n-1\rangle \, \langle n-1|H_y|n\rangle}{(E_n - E_{n-1})^2} \right]
\end{aligned}
$$

- $\langle n|H_x|n+1\rangle \equiv u_{x+} + i v_{x+}, \quad \langle n+1|H_y|n\rangle \equiv u_{y+} + i v_{y+}, \quad E_{n+1} - E_n \equiv S_+$



**Desired Probability distribution**

$$
\begin{aligned}
\mathbf{P_c(C)} &= \langle \delta\,[\mathbf{C} - \mathbf{C_n}] \rangle \\
&= \frac{\mathbf{1}}{\mathbf{2}\pi} \int_{-\infty}^{\infty} \mathrm{d}\mathbf{t}\, \exp[\mathbf{i}\,\mathbf{Ct}]\, \langle \exp\left[-\mathbf{i}\,\mathbf{C_n t}\right] \rangle
\end{aligned}
$$

-

# $P_c(c)$: full form: $\mathbf{P_C(C) = \frac{1}{A} P_c\left(\frac{C}{A}\right)}$



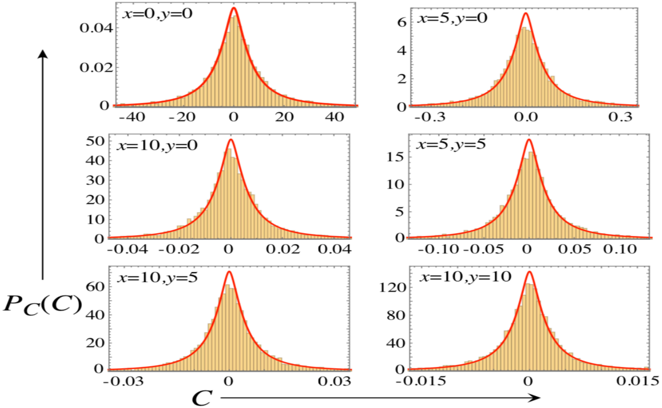
$$P_c(c) = \frac{27\sqrt{3}}{(1+4c^2)^5}\left(\frac{6f_1(|c|)\tan^{-1}\left(\sqrt{3+8|c|}\right)}{\pi(3+2c^2)^5(3+8|c|)^{9/2}} + \frac{3f_2(|c|)\log(1+2|c|)}{\pi(3+2c^2)^5} + \frac{f_3(|c|)}{2\pi(3+2c^2)^4(3+8|c|)^4} + 3\left(1-40c^2+80c^4\right)\right),$$

$$\begin{aligned} f_1(c) = {} & -33777 - 517914c - 2031912c^2 + 8531280c^3 + \\ & 106694400c^4 + 420839296c^5 + 818860032c^6 + 596420096c^7 - \\ & 755948800c^8 - 2317539840c^9 - 2585458688c^{10} - 1558491136c^{11} - \\ & 472834048c^{12} + 4341760c^{13} + 75366400c^{14} + 46137344c^{15} + \\ & 12320768c^{16} + 1572864c^{17}, \end{aligned}$$

$$\begin{aligned} f_2(c) = {} & 1 + 2430c + 8120c^2 - 8640c^3 - 57440c^4 - 76224c^5 - \\ & 30720c^6 + 15360c^7 + 21760c^8 + 7680c^9 + 2048c^{10}, \end{aligned}$$

$$\begin{aligned} f_3(c) = {} & (-41553 - 509328c - 1668384c^2 + 3797712c^3 + 44854416c^4 + \\ & 163496512c^5 + 381128320c^6 + 630179840c^7 + 631230720c^8 + \\ & 166594560c^9 - 372436992c^1 - 457797632c^{11} - 216436736c^{12} - \\ & 44941312c^{13} + 6979584c^{14} + 1179648c^{14}. \end{aligned}$$

# Case $N = 40$: theory with numerics




- **slight deviation from theory near $C = 0$: effect of neglecting non-neighbor levels**

# localized/ partially localized eigenstates



**what happens when wave-dynamics is not ergodic?**

$H$ : represented by a system-dependent random matrix ensemble



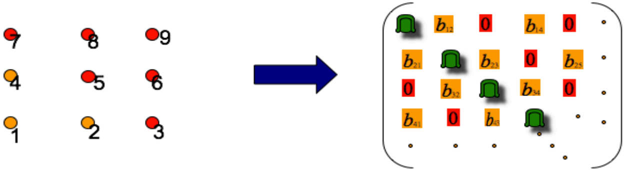
**how the distribution is sensitive to behavior of $\partial_x \mathbf{H}, \partial_y \mathbf{H}$?**

$\partial_x \mathbf{H}, \partial_y \mathbf{H}$: ensembles different from $\mathbf{H}$

# H-matrix in site basis: $\mathbf{H} = \sum_{i,j} \varepsilon_i \, c_i^+ . c_i + \sum_{i,j} V_{ij} \, c_i^+ . c_j$



Localized Wave Dynamics:




In the basis associated with the space

1. $H_{kl} \neq 0$ if $k = l$ or $k = l \pm L^{d-1}$

a. $H_{kk}$ : Randomly Distributed $\rho_{kk} = e^{-H_{kk}^2 / 2 v_{kk}^2}$

b. Nearest Neighbor interactions may be random or non random

If non-random $\rho_{kl} = \delta(H_{kl} - b_{kl})$

If random $\rho_{kl} = e^{-(H_{kl} - b_{kl})^2 / 2 v_{kl}^2}$

Hopping Strength $b_{kl} = 0$ if $k \neq l$ or $k \neq l \pm L^{d-1}$



# system-dependent random matrix ensembles

Complex systems represented by multi-parametric Gaussian ensemble

$$\rho(\mathbf{H}, \mathbf{v}, \mathbf{b}) = \mathbf{C}\exp\left[-\sum_{\mathbf{k}\le\mathbf{l}}\sum_{\mathbf{s}=\mathbf{1}}^{\mathbf{2}} \frac{\mathbf{1}}{\mathbf{2v_{kl;s}}}(\mathbf{H_{kl;s}} - \mathbf{b_{kl;s}})^{\mathbf{2}}\right]$$

System information contained in parameter-sets $\mathbf{v}, \mathbf{b}$:

$$\mathrm{v}^2_{\mathrm{kl;s}}(\mathrm{x},\mathrm{y}) = \langle \mathrm{H_{kl;s}}(\mathrm{x},\mathrm{y})^2\rangle - \langle \mathrm{H_{kl;s}}(\mathrm{x},\mathrm{y})\rangle^2, \qquad \mathrm{b_{kl;s}}(\mathrm{x},\mathrm{y}) = \langle \mathrm{H_{kl;s}}(\mathrm{x},\mathrm{y})\rangle$$



$$\mathbf{P_{e,u}(E, U) = J(H|E, U)}\ \rho(\mathbf{H}) = \prod_{\mathbf{m}\ne\mathbf{n}} |\mathbf{E_m} - \mathbf{E_n}|^{\mathbf{2}}\ \rho(\mathbf{E}, \mathbf{U})$$

**eigenvalues and eigenfunctions of $H$ now correlated !**

# curvature distribution for generic case

$$\mathbf{C_b} = \mathbf{2}\ \mathrm{Im} \sum_{\mathbf{a \neq b}} \frac{\langle \mathbf{b} | \partial_\mathbf{x} \mathbf{H} | \mathbf{a} \rangle \ \langle \mathbf{a} | \partial_\mathbf{y} \mathbf{H} | \mathbf{b} \rangle}{(\mathbf{E_a} - \mathbf{E_b})^\mathbf{2}} = \mathbf{2}\ \mathrm{Im} \sum_{\mathbf{a \neq b}} \frac{\mathbf{F_{ab}}}{(\mathbf{E_a} - \mathbf{E_b})^\mathbf{2}}$$

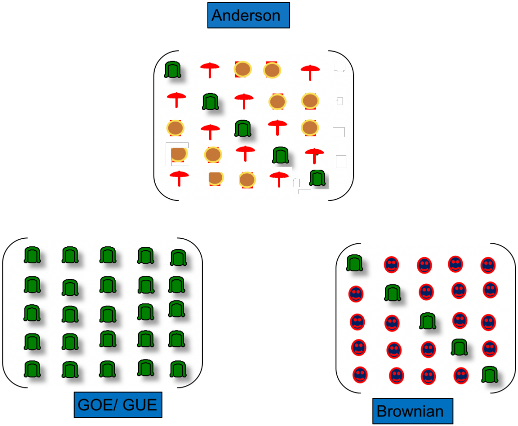
$$\mathrm{F_{ab}} = (\partial_\mathrm{x} \mathrm{H})_\mathrm{ab} \ \partial_\mathrm{y} \mathrm{H}_\mathrm{ba} = \sum_{\mathrm{j,r,k,l}} \mathrm{U^*_{ka} U_{lb} U^*_{jb} U_{ra}} (\partial_\mathrm{x} \mathrm{H})_\mathrm{kl} \ \partial_\mathrm{y} \mathrm{H}_\mathrm{jr}$$

$$\mathbf{P_c(C)} = \int \delta\left(\mathbf{C} - \mathbf{C_b}\right) \ \mathbf{P_{e,u}(U, U^*, E)} \ \rho_\mathbf{x}(\partial_\mathbf{x} \mathbf{H}) \ \rho_\mathbf{y}(\partial_\mathbf{y} \mathbf{H}) \ \mathrm{D}\boldsymbol{\Gamma}$$



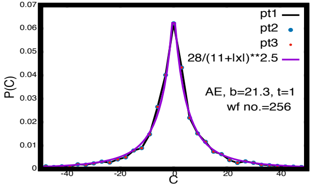
- Integration measure $\mathrm{D}\Gamma \equiv \mathrm{D}U \ \mathrm{D}U^* \ \mathrm{D}E \ \mathrm{D}(\partial_x H) \ \mathrm{D}(\partial_y H)$
- $\mathrm{E} \equiv \{\mathrm{E_1}, \ldots, \mathrm{E_N}\}$ **and** $\mathrm{F_b} \equiv \{\mathrm{F_{1b}}, \ldots, \mathrm{F_{Nb}}\}$ **correlated**

# numerics: three different ensembles

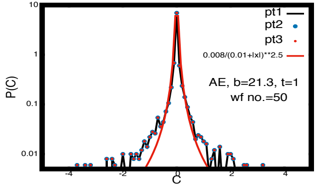


# perturbation ensemble has any role?

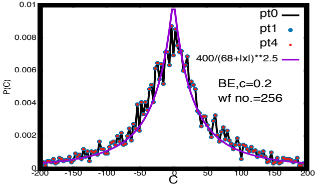
## AE, wf=256



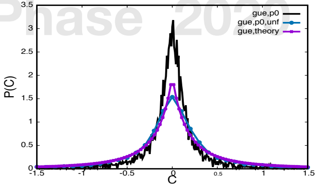

## AE, wf=50




## BE,wf=256




## GUE, wf=256



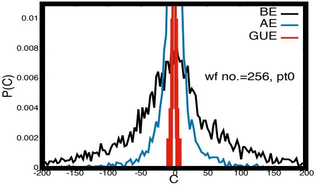




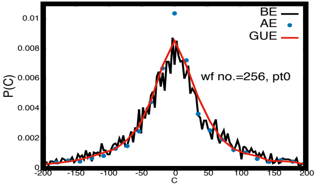


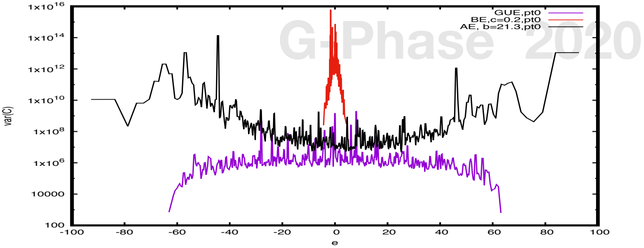

# comparison: Anderson, Brownian, GUE ensembles

## without rescaling




## after rescaling



# Single parametric diffusion of $\rho \propto e^{-\sum \frac{(H_{kl;s}-b_{kl;s})^2}{2v_{kl;s}}}$





Changing complexity i.e. v, b leads to evolution of $\rho(\mathbf{H})$

possible to define a function $\mathbf{Y}$ such that (with $\mathbf{g_{kl} = 1 + \delta_{kl}}$)

$$\frac{\partial \rho}{\partial \mathbf{Y}} = \sum_{\mathbf{k,l;s}} \frac{\partial}{\partial \mathbf{H_{kl;s}}} \left[ \frac{\mathbf{g_{kl}}}{\mathbf{2}} \frac{\partial}{\partial \mathbf{H_{kl;s}}} + \mathbf{H_{kl;s}} \right] \rho$$

$$\text{where} \qquad \mathbf{Y} = -\frac{\mathbf{1}}{\mathbf{N(N+1)}} \ \ln \left[ \prod_{\mathbf{k \le l}} \prod_{\mathbf{s=1}}^{\mathbf{2}} |\mathbf{g_{kl}} - \mathbf{2v_{kl;s}}| \ \ |\mathbf{b_{kl;s}}|^{\mathbf{2}} \right]$$



**multi-parametric diffusion of $\rho(H)$ can be reduced to a single parameter**

**Y= average distribution parameter, a measure of average uncertainty of system**

- moments $\langle \delta \mathbf{H_{kl;s}} \rangle = -\mathbf{H_{kl;s}} \, \delta \mathbf{Y}, \quad \langle |\delta \mathbf{H_{kl;s}}|^{\mathbf{2}} \rangle = \mathbf{g_{kl}} \, \delta \mathbf{Y}$

(P.S.: Phys. Rev. (2001), J. Phys. C (2005), J. Phys. A.(2008)

# 2nd order perturbation theory



$$\delta e_n = \delta H_{nn} + \sum_{m \neq n} \frac{|\delta H_{mn}|^2}{e_n - e_m} + o((\delta H_{mn})^3)$$

$$\delta U_{jn} = \sum_{m \neq n} \left( \frac{\delta H_{mn} U_{jm}}{e_n - e_m} + \sum_{m'}^{N} \frac{\delta H_{mn} \delta H_{m'n} U_{jm}}{(e_n - e_m)(e_n - e_{m'})} - \frac{\delta H_{mn} \delta H_{nn}}{(e_n - e_m)^2} U_{jm} + \frac{\delta H_{mn} \delta U_{nm}}{2\,(e_n - e_m)^2} U \right.$$

$$\langle \delta e_n \rangle = \langle \delta H_{nn} \rangle + \sum_{m \neq n}^{N} \frac{\langle |\delta H_{mn}|^2 \rangle}{e_n - e_m} = \left[ \gamma e_n + \sum_{m \neq n}^{N} \frac{1}{e_n - e_m} \right] \delta Y$$

$$\langle \delta e_n \delta e_m \rangle = \langle \delta H_{nn} \delta H_{mm} \rangle = (2f)^{-1} \delta_{nm} \delta Y$$

$$\langle \delta U_{jn} \rangle = -\frac{1}{2} \sum_{m=1, m \neq n}^{N} \frac{U_{jn}}{(e_n - e_m)^2} \delta Y$$

$$\langle \delta U_{jr} \delta U_{kl} \rangle = -\frac{U_{jl} U_{kr}}{(e_r - e_l)^2} (1 - \delta_{rl})\, \delta Y$$

$$\langle \delta U_{jr} \delta U^*_{kl} \rangle = \sum_{m \neq n}^{N} \frac{U_{jm} U^*_{km}}{(e_r - e_m)^2}\, \delta_{rl} \delta Y$$

# Diffusion Equation for $P_{e,u}(e_1, \ldots, e_N; U_1, \ldots, U_N; Y)$





Using standard second order perturbation theory for Hermitian operators, the second order change due to a small change $\delta Y$

Joint probability distribution of eigenfunction components $P_{e,u}$

$$\frac{\partial \mathbf{P_{e,u}}}{\partial \mathbf{Y}} = (\mathbf{L_e} + \mathbf{L_u} + \mathbf{L_u^*})\mathbf{P_{e,u}}$$

$$\mathbf{L_e}\, \delta \mathbf{Y} = \sum_{\mathbf{n}} \frac{\partial}{\partial \mathbf{e_n}} \left[ \frac{\mathbf{1}}{\mathbf{2}} \frac{\partial}{\partial \mathbf{e_n}} \langle (\delta \mathbf{e_n})^{\mathbf{2}} \rangle - \langle \delta \mathbf{e_n} \rangle \right]$$

$$\mathbf{L_u}\, \delta \mathbf{Y} = \sum_{\mathbf{k,l,j,r}} \frac{\partial}{\partial \mathbf{U_{jr}}} \left[ \frac{\mathbf{1}}{\mathbf{2}} \frac{\partial}{\partial \mathbf{U_{kl}}} \langle \delta \mathbf{U_{jr}} \delta \mathbf{U_{kl}} \rangle + \frac{\mathbf{1}}{\mathbf{2}} \frac{\partial}{\partial \mathbf{U_{kl}^*}} \langle \delta \mathbf{U_{jr}} \delta \mathbf{U_{kl}^*} \rangle - \langle \delta \mathbf{U_{jr}} \rangle \right]$$

# single parametric diffusion of curvature





$$\mathbf{P_c}(\mathbf{C}) = \int \delta\left(\mathbf{C} - \mathbf{C_b}\right)\ \mathbf{P_{e,u}}(\mathbf{U}, \mathbf{U}^*, \mathbf{E})\ \rho_{\mathbf{x}}(\partial_{\mathbf{x}}\mathbf{H})\ \rho_{\mathbf{y}}(\partial_{\mathbf{y}}\mathbf{H})\ \mathrm{D}\boldsymbol{\Gamma}$$

$$\frac{\partial \mathbf{P_c}(\mathbf{C})}{\partial \mathbf{Y}} = \int \delta\left(\mathbf{C} - \mathbf{C_b}\right)\ \frac{\partial \mathbf{P_{e,u}}}{\partial \mathbf{Y}}\ \rho_{\mathbf{x}}(\partial_{\mathbf{x}}\mathbf{H})\ \rho_{\mathbf{y}}(\partial_{\mathbf{y}}\mathbf{H})\ \mathrm{D}\boldsymbol{\Gamma}$$

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# single parametric diffusion of curvature

$$\rho(\mathbf{H}) \propto \mathrm{e}^{-\sum \frac{(\mathbf{H_{kl;s}} - \mathbf{b_{kl;s}})^2}{2\mathbf{v_{kl;s}}}}$$

,

$$\mathbf{Y} = -\frac{\mathbf{1}}{\mathbf{N(N+1)}} \ln \left[ \prod_{\mathbf{k \le l;s}} |\mathbf{g_{kl}} - \mathbf{2v_{kl;s}}| \quad |\mathbf{b_{kl;s}}|^{\mathbf{2}} \right]$$

**For** $2 \times 2$ $H$**-matrix**

$$\frac{\partial \mathbf{P_c(C)}}{\partial \mathbf{Y}} = \frac{\mathbf{1}}{\mathbf{2}} \frac{\partial}{\partial \mathbf{C}} \left[ \alpha \frac{\partial}{\partial \mathbf{C}} + \eta \, \mathbf{C} \right] \mathbf{P_c}$$



**a single parametric formulation of curvature distribution for multi-parametric Gaussian ensemble** $(\alpha, \eta = \text{function(mean level spacing)})$

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- **role of fluctuations in systems where phase is topologically invariant?**

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Thank you