Risk-sensitive control for diffusions with jumps

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this talk is based on a joint work with Ari Arapostathis

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Problem setting

We consider controlled jump-diffusion in $\mathbb{R}^d$ given by

$$\mathrm{d}X_t = b(X_t, Z_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t + \int_{\mathbb{R}^m\setminus\{0\}} g(X_{t-}, \xi)\,\mathcal{N}(\mathrm{d}t, \mathrm{d}\xi)\,,$$

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Let $c : \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}_+$ be a non-negative, continuous function .

For $U \in \mathfrak{U}$ (set of admissible controls), we define the risk-sensitive ergodic cost as

$$\mathcal{E}(x, U) = \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}_x \left[e^{\int_0^T c(X_s, U_s)\mathrm{d}s} \right].$$

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The optimal value is given by

$$\inf_{x\in\mathbb{R}^d} \inf_{U\in\mathfrak{U}} \mathcal{E}(x, U) = \Lambda^*.$$

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- Characterization of minimizing Markov controls if it exists.
- Solution $V \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $V > 0$, of the HJB equation ($a := \sigma\sigma^T$)

$$\mathcal{I}V := \frac{1}{2}\mathrm{trace}(a\nabla^2 V) + I[V,x] + \min_{u\in\mathbb{U}}\{b(x,u)\cdot\nabla V + c(x,u)V\} = \Lambda^* V,$$

where

$$I[V,x] = \int_{\mathbb{R}^d} (V(x+z) - V(x))\nu(x,\mathrm{d}z).$$

- Is the above solution (V, Λ^*) unique?

What is the difficulty

To understand the difficulty, let us look back to the situation where $\nu = 0$ i.e. there is no non-local interaction. In this case we would be looking for the principal eigenvalue of the operator

$$\mathcal{L}f := \frac{1}{2}\mathrm{trace}(a\nabla^2 f) + \min_{u\in\mathbb{U}}\{b(x,u)\cdot\nabla f + c(x,u)f\}, \quad f \in \mathcal{C}^2(\mathbb{R}^d).$$

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The general strategy is the following

- Let $\mathcal{B}_n$ be ball of radius of n around 0. Consider the principal eigenpair (Ψ_n, λ_n) solving

$$\mathcal{L}\Psi_n = \lambda_n\Psi_n \quad \text{in } \mathcal{B}_n, \quad \text{and} \quad \Psi_n = 0 \text{ on } \partial\mathcal{B}_n = 0.$$

- Then, under appropriate assumptions, one can then pass the limit, as $n \to \infty$, to obtain the eigenpair (V, Λ) in $\mathbb{R}^d$. See for instance, B.'2011, Arapostathis-B.'2018, Arapostathis-B.-Saha'2019

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- The eigenpair (V, Λ) would be the (generalized) principal eigenpair of the operator $\mathcal{L}$ in $\mathbb{R}^d$.
- One has to make some effort to justify $\Lambda = \Lambda^*$ i.e., the optimal value. In general, $\Lambda \leq \Lambda^*$.
- The uniqueness of V is also not obvious since $\mathbb{R}^d$ is unbounded. One may see Berestycki-Rossi'2015 for further discussion.

Recently, in Arapostathis-B.-Saha'2019 we study monotonicity property of the principal eigenvalue with respect to the potential and show that this is closely related to the uniqueness issue of the principal eigenfunction. One of the key findings in this is stochastic representation of eigenfunction...

Principal eigenvalue

The principal eigenvalue of the $\mathcal{L}$ is defined as (inspired from Berestycki-Nirenberg-Varadhan'1994)

$$\lambda^*(f) = \inf \left\{ \lambda \in \mathbb{R} : \exists \varphi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d),\ \varphi > 0,\ \mathcal{L}\varphi + (f - \lambda)\varphi \le 0, \text{ a.e. in } \mathbb{R}^d \right\}.$$

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Theorem (Berestycki-Rossi'2015)

There exists a positive $\Psi \in \mathcal{W}^{2,d}_{\rm loc}(\mathbb{R}^d)$ satisfying

$$\mathcal{L}\Psi + f\,\Psi = \lambda\Psi \quad \textit{a.e. on } \mathbb{R}^d\,,$$

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We say the principal eigenfunction Ψ^* has a stochastic representation if

$$\Psi^*(x) = \mathbb{E}_x\left[\mathrm{e}^{\int_0^{\breve{\tau}_r}[f(X_s)-\lambda^*(f)]\,\mathrm{d}s}\,\Psi^*(X_{\breve{\tau}})\,\mathbf{1}_{\{\breve{\tau}_r<\infty\}}\right]\ \forall\, x \in \bar{B}_r^c.$$

Coming back to our problem...

We recall our non-local operator

$$\mathcal{I} V := \frac{1}{2}\mathrm{trace}(a\nabla^2 V) + I[V, x] + \min_{u\in\mathbb{U}}\{b(x, u)\cdot\nabla V + c(x, u)V\}.$$

• **First hurdle**

There is nothing known for the Dirichlet eigenvalue problem for $\mathcal{I}$. Also, there are very few works available in this direction but for fractional Laplacian type kernel.

We prove..

Theorem

There exists a unique $\varphi_D \in \mathcal{C}_{b,+}(\mathbb{R}^d) \cap \mathcal{W}^{2,p}_{\mathrm{loc}}(D)$, $p > d$, *satisfying*

$$\begin{aligned} \mathcal{I}\varphi_D &= \lambda_D \varphi_D \quad \text{in } D\,, \\ \varphi_D &= 0 \quad \text{in } D^c\,, \\ \varphi_D &> 0 \quad \text{in } D\,, \quad \varphi_D(0) = 1\,. \end{aligned}$$

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Corollary

Suppose that $D \subsetneq D'$. *Then we have* $\lambda_D < \lambda_{D'}$.

For linear operators

Theorem

Suppose $c \leq c'$, and $c' > c$ on subset of D with positive Lebesgue measure. Then $\lambda_D(c) < \lambda_D(c')$.

Theorem

Let c and c' be two potentials. Then

$$\lambda_D\big(\theta c + (1-\theta)c'\big) \;\leq\; \theta\lambda_D(c) + (1-\theta)\lambda_D(c') \quad \text{for all } \theta \in [0,1]\,.$$

Controlled eigenvalue problem

The above developments help us to generalize a result on controlled eigenvalue problem. Fix a smooth domain D and let τ be the exit time from D. Also, assume that $c = 0$ and consider the operator

$$\tilde{\mathcal{I}} f(x) \,=\, \mathrm{trace}(a\nabla^2 f) + I[f, x] + \inf_{\zeta \in \mathbb{U}} b(x, \zeta) \cdot \nabla f(x) \,.$$

Suppose we are interested in

$$\Theta \,=\, \sup_{U \in \mathfrak{U}} \limsup_{T \to \infty} \frac{1}{T} \log \mathbb{P}^U_x(\tau > T) \,.$$

We could prove the following

- There exists a unique $\psi_D \in \mathcal{C}_b(\mathbb{R}^d) \cap \mathcal{W}^{2,p}_{\text{loc}}(D)$, $p > d$, and $\Theta_D < 0$ satisfying

$$\begin{aligned} \tilde{\mathcal{I}}\psi_D &= \Theta_D \, \psi_D \quad \text{in } D\,, \\ \psi_D &= 0 \quad \text{in } D^c\,, \\ \psi_D &< 0 \quad \text{in } D, \quad \psi_D(0) = -1\,. \end{aligned}$$

- $\Theta_D = \Theta$.
- Any minimizing selector of above equation is an optimal Markov control.
- Any optimal stationary Markov control is a measurable selector of the above equation.

Back to risk-sensitive

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• What happens if we enlarge the domains to $\mathbb{R}^d$? Can we justify the passage of limits in eigenpair (φ_D, λ_D)?

Because of the monotonicity property of eigenvalues the limit $\lim_{D\to\mathbb{R}^d} \lambda_D$ would exists. But passage of limit in φ_D is not clear since we do not have Harnack's property.

Stability hypothesis

We impose the following Lyapunov criterion

Assumption

There exists a positive function $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$, $\mathcal{V} \geq 1$, and an inf-compact function ℓ such that

$$\operatorname{trace}\big(a(x)\nabla^2\mathcal{V}(x)\big) + I[\mathcal{V}, x] + \max_{\zeta\in\mathbb{U}} b(x,\zeta)\cdot\nabla\mathcal{V}(x) \;\leq\; \theta_1 \mathbf{1}_{\mathcal{K}}(x) - \ell(x)\mathcal{V}(x)$$

for some constant θ_1 and compact set $\mathcal{K}$. In addition,

$$x \mapsto \int_{\mathbb{R}^d} \mathcal{V}(x+z)\nu(x, \mathrm{d}z)$$

is locally bounded, and for some $\beta \in (0,1)$, we have

$$\limsup_{|x|\to\infty} \sup_{\zeta\in\mathbb{U}} \frac{c(x,\zeta)}{\ell(x)} \;<\; \beta\,.$$

How does it help..

Let (V_n, Λ_n^*) be the principal eigenpair in $\mathcal{B}_n$ (the ball of radius n around 0) i.e.

$$\mathcal{I} V_n = \Lambda_n^* V_n \quad \text{in } \mathcal{B}_n.$$

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Thus in $\mathcal{B}^c \cap \mathcal{B}_n$ we have

$$\operatorname{trace}\big(a(x)\nabla^2(\mathcal{V} - \kappa V_n)(x)\big) + I[\mathcal{V} - \kappa V_n, x] + \min_{\zeta\in\mathbb{U}}\{b(x,\zeta)\cdot\nabla(\mathcal{V} - \kappa V_n)(x)$$
$$+ (c(x,\zeta) - \Lambda_n^*)(\mathcal{V} - \kappa V_n)\} \leq 0.$$

Now choose $\kappa = \kappa_n$ suitablly so that V_n touches $\mathcal{V}$ from below and then using above equation it follows that κV_n must touch $\mathcal{V}$ inside $\mathcal{B}$. This forms a barrier for the solutions.

Normalize V_n to $\kappa_n V_n$ where κ_n chosen as above. Define

$$\mathcal{J}_n(x) = \int_{\mathbb{R}^d} V_n(x+z)\nu(x, \mathrm{d}z)\,.$$

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$$\begin{aligned}\operatorname{trace}\bigl(a(x)\nabla^2 V_n(x)\bigr) + I[V_n, x] + b(x, v_n)\cdot\nabla V_n(x)\\ + \bigl(c(x, v_n(x)) - \Lambda_n^* - \nu(x, \mathbb{R}^d)\bigr) V_n(x) = -\mathcal{J}_n.\end{aligned}$$

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Now applying a result from elliptic PDE we see that for any compact $D_1 \Supset D \Supset \mathcal{B}$ we get

$$\begin{aligned} \sup_D V_n \le \kappa_D \left(\inf_D V_n + \|\mathcal{J}_n\|_{L^d(D_1)}\right) &\le \kappa_D \left(\inf_{\mathcal{B}} V_n + \|\mathcal{J}_n\|_{L^d(D_1)}\right) \\ &\le \kappa_D\big(\inf_{\mathcal{B}} \mathcal{V} + \|\mathcal{J}_n\|_{L^d(D_1)}\big)\,. \end{aligned}$$

Rest is standard.

So we can pass to the limit to arrive at the eigenpair (V, Λ) solving

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In the local case (Arapostathis-B.-Saha'2019) this issue was dealt using the twisted process and stability of the twisted process. This method does not seem promising in the non-local setting. We use stochastic representation method and a perturbation method to establish the result.

Our final theorem

Theorem

Grant the stability hypothesis. Then we have the following.

(a) *There exists a positive* $V \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathcal{O}(\mathcal{V})$, $p > d$, *satisfying*

$$\mathcal{I}V(x) = \Lambda^* V(x) \ \ \textit{a.e. in } \mathbb{R}^d, \quad V(0) = 1\,.$$

Let $\overline{\mathfrak{U}}_{\mathrm{SM}}$ *denote the class of stationary Markov control* v *satisfying*

$$b_v(x)\cdot\nabla V(x) + c_v(x)V(x) = \min_{\zeta\in\mathbb{U}} \bigl\{b(x,\zeta)\cdot\nabla V(x) + c(x,\zeta)V(x)\bigr\} \ \textit{a.e.}$$

(b) *Any member of* $\overline{\mathfrak{U}}_{\mathrm{SM}}$ *is an optimal control.*

(c) *There exists a unique positive* $V \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathcal{O}(\mathcal{V})$ *satisfying above equation and* $V(0) = 1$.

(d) *Every optimal* $v \in \mathfrak{U}_{\mathrm{SM}}$ *belongs to* $\overline{\mathfrak{U}}_{\mathrm{SM}}$.

Maximization problem without stability

We also study a maximization problem where the optimal value is given by

$$\varrho^* = \sup_{x \in \mathbb{R}^d} \sup_{U \in \mathfrak{U}} \limsup_{T \to \infty} \frac{1}{T} \log \mathbb{E}_x \left[e^{\int_0^T c(X_s, U_s) \mathrm{d}s} \right] .$$

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Under a near-monotonicity type condition we produce all the results.

What next..

It would be interesting and necessary to look at the minimization problem with near-monotonicity criterion.

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Thank You!