Spectral properties of random perturbations of Toeplitz matrices of finite symbols

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Advances in Applied Probability

Based on joint works with Elliot Paquette and Ofer Zeitouni

An example

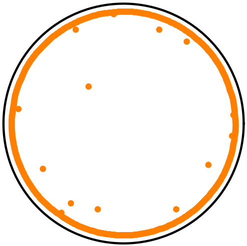
$$T_N := \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 1 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}.$$

Define $\widehat{T}_N := U_N T_N U_N^*$, where U_N is a unitary matrix. The eigenvalues of $\widehat{T}_N$ are same as that of T_N.

Take U_N to be Haar unitary matrix. Compute eigenvalues of $\widehat{T}_N$ using any mathematical software.

An example

Example I. $N = 500$. Eigenvalues of $\widehat{T}_N$ (computed using Mathematica) are in orange. The black circle is the unit circle $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$.

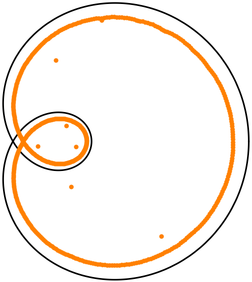


An example

Example II.

$$T_N := \begin{bmatrix} 0 & 1 & 1 & & & \\ & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 1 & 1 \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}$$

Example II. $N = 500$. Eigenvalues of $\widehat{T}_N$ (computed using Mathematica) are in orange. The black Limaçon is $\boldsymbol{a}(\mathbb{S}^1)$, where $\boldsymbol{a}(\lambda) = \lambda + \lambda^2$.



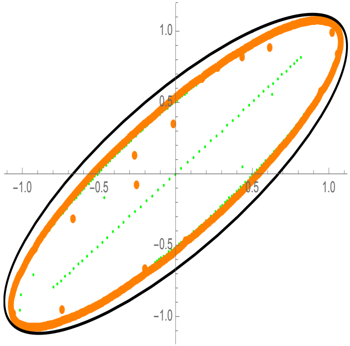
An example

Example III.

$$T_N := \begin{bmatrix} 0 & 0.5 & & & & & \\ i & 0 & 0.5 & & & & \\ & i & 0 & 0.5 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & i & 0 & 0.5 & \\ & & & & i & 0 & 0.5 \\ & & & & & i & 0 \end{bmatrix}$$

An example

Example III. $N = 500$. Eigenvalues of $\widehat{T}_N$ (computed using Mathematica) are in orange. The black ellipse is $\boldsymbol{a}(\mathbb{S}^1)$, where $\boldsymbol{a}(\lambda) = 0.5\lambda + i\lambda^{-1}$. Green dots are the eigenvalues of T_N.



(plausible) explanation

Model: $\widehat{U}_N = U_N + \Delta_N$, where U_N is Haar unitary and Δ_N has a small norm. $\widehat{T}_N := \widehat{U}_N T_N \widehat{U}_N^*$.
The eigenvalues of $\widehat{T}_N$ are same with that of $T_N + \mathcal{E}_N$, where

$$\mathcal{E}_N := U_N^* \Delta_N T_N + T_N \Delta_N^* U_N + U_N^* \Delta_N T_N \Delta_N^* U_N.$$

Spectrum of $T_N + \mathcal{E}_N$

Let E_N be a random matrix with i.i.d. entries of zero mean and finite fourth moment. Then

$$\frac{1}{\sqrt{N}} \|E_N\| \to 2, \qquad \text{almost surely.}$$

Two approaches:

- Eigenvalues of $T_N + \frac{\sigma}{\sqrt{N}} E_N$. First let $N \to \infty$ and then $\sigma \to 0$.
- Eigenvalues of $T_N + N^{-\gamma} E_N$, where $\gamma > 1/2$.

Questions

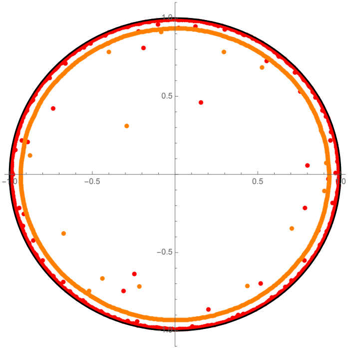
- Limit of the bulk of the spectrum.

$$L_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}.$$

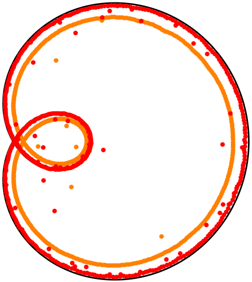
- Limit of the random point process induced by the outliers.

Spectrum of additive random perturbation of T_N

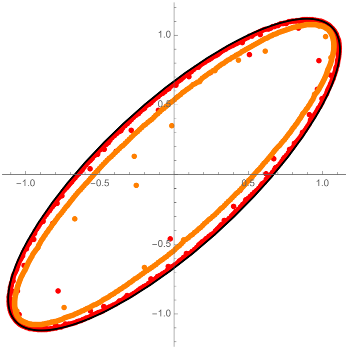
Example I. $N = 500$, $E_N = N^{-1}G_N$, G_N is a complex Ginibre. Eigenvalues of $T_N + E_N$ is in red. Eigenvalues of $\widehat{T}_N = U_N T_N U_N^*$ (computed using Mathematica) is in orange.



Example II. $N = 500$, $E_N = N^{-1}G_N$, G_N is a complex Ginibre. Eigenvalues of $T_N + E_N$ is in red. Eigenvalues of $\widehat{T}_N = U_N T_N U_N^*$ (computed using Mathematica) is in orange.



Example III. $N = 500$, $E_N = N^{-1}G_N$, G_N is a complex Ginibre. Eigenvalues of $T_N + E_N$ is in red. Eigenvalues of $\widehat{T}_N = U_N T_N U_N^*$ (computed using Mathematica) is in orange.



Toeplitz matrix

$$T_N = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_{N-1} \\ a_{-1} & a_0 & a_1 & \ddots & & \vdots \\ a_{-2} & a_{-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_1 & a_2 \\ \vdots & & \ddots & a_{-1} & a_0 & a_1 \\ a_{-(N-1)} & \cdots & \cdots & a_{-2} & a_{-1} & a_0 \end{bmatrix}, \ a_i \in \mathbb{C}.$$

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T_N finitely banded if $a_i = 0$ for $i \geq d_1 + 1$ and $i \leq -(d_2 + 1)$ for some $d_1, d_2 \geq 0$.

Toeplitz matrix

$$T_N = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_{N-1} \\ a_{-1} & a_0 & a_1 & \ddots & & \vdots \\ a_{-2} & a_{-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_1 & a_2 \\ \vdots & & \ddots & a_{-1} & a_0 & a_1 \\ a_{-(N-1)} & \cdots & \cdots & a_{-2} & a_{-1} & a_0 \end{bmatrix}, \ a_i \in \mathbb{C}.$$

- T_N can be viewed as a finite dimensional version of an infinite dimensional matrix/operator T.
- The symbol associated with T is $\boldsymbol{a}$.

$$\boldsymbol{a}(\lambda) := \sum_{k=-\infty}^{\infty} a_k \lambda^k.$$

Toeplitz matrix

- If T (or equivalently T_N) if finitely banded then $\boldsymbol{a}$ is a Laurent polynomial.

$$\boldsymbol{a}(\lambda) = \sum_{k=-d_2}^{d_1} a_k \lambda^k.$$

  - Example I: $\boldsymbol{a}(\lambda) = \lambda$.
  - Example II: $\boldsymbol{a}(\lambda) = \lambda + \lambda^2$.
  - Example III: $\boldsymbol{a}(\lambda) = 0.5\lambda + i\lambda^{-1}$.

Connection to pseudospectrum

$$\mathrm{spec}^{\varepsilon}(A)(\varepsilon\text{-pseudospectrum}) := \{z \in \mathbb{C} : \|(A - zI)^{-1}\| \geq 1/\varepsilon\}$$
$$= \bigcup_{\|E\| \leq \varepsilon} \mathrm{spec}(A + E).$$

[Varah '79], [Trefethen, Embree '05]

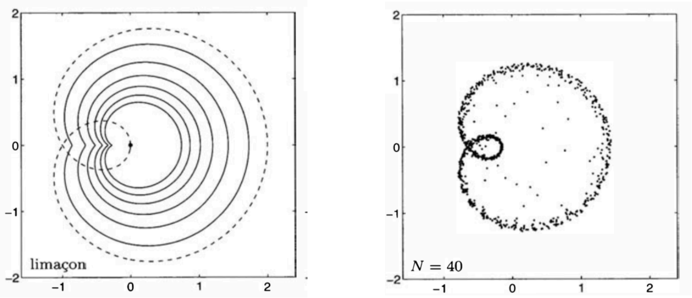

Figure: `Left panel`: ε-pseudospectrum for $\varepsilon = 10^{-2}, 10^{-4}, \ldots, 10^{-12}$ for Toeplitz matrix with symbol $\boldsymbol{a}(\lambda) = \lambda + \lambda^2$; $\boldsymbol{a}(\mathbb{S}^1)$ marked by dashed points. `Right panel`: eigenvalues of 20 random perturbations of norm 10^{-3}. Pictures from [Trefethen, Embree '05].

Limit of the bulk of the spectrum

Theorem 1 (B., Paquette, Zeitouni '18)

Let T_N a Toeplitz matrix of dimension N with symbol $\boldsymbol{a}$, where $\boldsymbol{a}$ is a Laurent polynomial. Let E_N a be random matrix satisfying Assumption (A). Then, for any $\gamma > \frac{1}{2}$, the empirical distribution of the eigenvalues of $T_N + N^{-\gamma} E_N$ converges weakly, in probability, to the law of $\boldsymbol{a}(U)$ where $U \sim \mathrm{Unif}(\mathbb{S}^1)$.

Assumption (A)

(1)

$$\mathbb{E}\left[\|E_N\|_{\mathrm{HS}}^2\right] = \mathbb{E}\left[\sum_{i,j} |e_{i,j}|^2\right] = O(N^2).$$

(2) For every $\alpha > 0$ $\exists\, \beta \in (0, \infty)$, such that for any M_N with $\|M_N\| = O(N^\alpha)$,

$$\mathbb{P}\left(s_{\min}(M_N + E_N) \le N^{-\beta}\right) = o(1).$$

Limit of the bulk of the spectrum

- Example I: $\boldsymbol{a}(\lambda) = \lambda$. $L_N \Rightarrow$ law of U, where $U \sim \mathrm{Unif}(\mathbb{S}^1)$.
- Example II: $\boldsymbol{a}(\lambda) = \lambda + \lambda^2$. $L_N \Rightarrow$ law of $U + U^2$.
- Example III: $\boldsymbol{a}(\lambda) = 0.5\lambda + i\lambda^{-1}$. $L_N \Rightarrow$ law of $0.5U + iU^{-1}$.

Limit of the bulk of the spectrum

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(2) For every $\alpha > 0 \ \exists \beta \in (0, \infty)$, such that for any M_N with $\|M_N\| = O(N^\alpha)$,

$$\mathbb{P}\left(s_{\min}(M_N + E_N) \le N^{-\beta}\right) = o(1).$$

Matrices satisfying Assumption (A)

- The entries of E_N are i.i.d. with finite second moment.

  follows from [Tao-Vu '08]

- $E_N = \sqrt{N} U_N$, where U_N is Haar Unitary.

  follows from [Rudelson-Vershynin '14]

- The entries of E_N are independent, satisfy a uniform anti-concentration bound near zero, and have uniform lower bound on the truncated variance.

  [Bordenave-Chafaï '12]

- The entries of E_N have an inhomogeneous variance profile satisfying some appropriate assumptions.

  [Cook '16]

- E_N can also be sparse random matrix.

  [Tao-Vu '08]

Related results

- $\boldsymbol{a}(\lambda) = \lambda$; $\gamma > \frac{1}{2}$. Gaussian perturbation.

  Corollary 6 of [Guionnet, Wood, Zeitouni '14]

- T_N as above. Entries of E_N are i.i.d., and $\gamma > \frac{3}{2}$.

  [Wood '16]

- $\boldsymbol{a}(\lambda) = a_1\lambda + a_{-1}\lambda^{-1}$. Gaussian perturbation.

  [Sjöstrand, Vogel '16]

- For Toeplitz band matrices under Gaussian perturbations.

  [Sjöstrand, Vogel '19]

Regions of no outliers

Theorem 2 (B., Zeitouni '19)

Let T_N be a Toeplitz matrix with symbol $\boldsymbol{a}$, where $\boldsymbol{a}$ is a Laurent polynomial. Let E_N be a random matrix with independent entries having zero mean and unit variance. Then for any $\gamma > \frac{1}{2}$, with probability tending to one, there are no outliers outside $\mathrm{Spec}\, T(\boldsymbol{a})$, where T is limiting Toeplitz operator.

Spectrum of Toeplitz operator

$$\mathrm{spec}(T) = \boldsymbol{a}(\mathbb{S}^1) \cup \{z \notin \boldsymbol{a}(\mathbb{S}^1) : \mathrm{wind}(\boldsymbol{a} - z) \neq 0\}.$$

[Krein' 58], [Caldéron, Spitzer, Widom '59]

$$\mathbb{S}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$$

$\mathrm{wind}(\cdot)$ denotes the winding number around 0.

Spectrum of Toeplitz operator

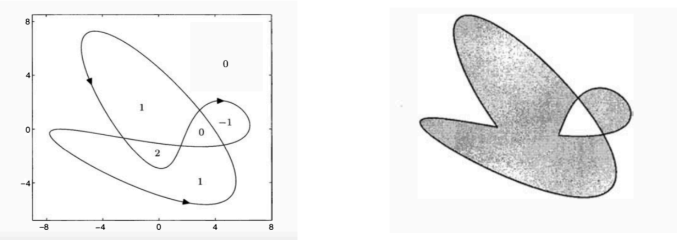


Figure: Spectrum of the Toeplitz operator with symbol
$\boldsymbol{a}(\lambda) = 2\lambda^3 - \lambda^2 + 2\mathrm{i}\lambda - 4\lambda^{-2} - 2\mathrm{i}\lambda^{-3}$ is in grey.

Regions of no outliers

Example I. $\boldsymbol{a}(\lambda) = \lambda$.

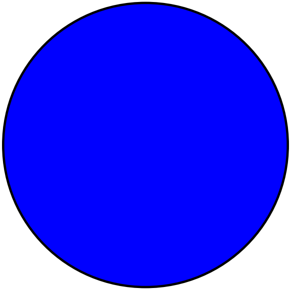
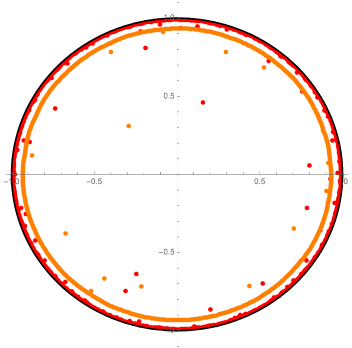


Figure: The spectrum is the unit disk.

Regions of no outliers

Example I. $N = 500$, $E_N = N^{-1}G_N$, G_N is a complex Ginibre. Eigenvalues of $T_N + E_N$ is in red. Eigenvalues of $\widehat{T}_N = U_N T_N U_N^*$ (computed using Mathematica) is in orange.



Example II. $\boldsymbol{a}(\lambda) = \lambda + \lambda^2$.

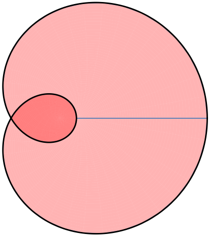
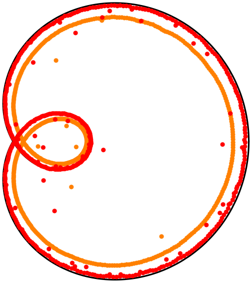


Figure: $\boldsymbol{a}(\mathbb{S}^1)$ is the Limaçon. The spectrum is the image of $\boldsymbol{a}$ over the unit disk.

Regions of no outliers

Example II. $N = 500$, $E_N = N^{-1}G_N$, G_N is a complex Ginibre. Eigenvalues of $T_N + E_N$ is in red. Eigenvalues of $\widehat{T}_N = U_N T_N U_N^*$ (computed using Mathematica) is in orange.



Example III. $\boldsymbol{a}(\lambda) = 0.5\lambda + i\lambda^{-1}$.

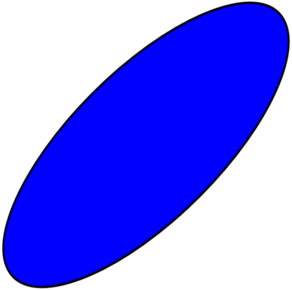
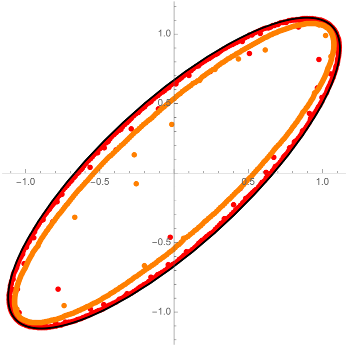


Figure: The spectrum is the solid ellipse and its boundary is $\boldsymbol{a}(\mathbb{S}^1)$.

Example III. $N = 500$, $E_N = N^{-1}G_N$, G_N is a complex Ginibre. Eigenvalues of $T_N + E_N$ is in red. Eigenvalues of $\widehat{T}_N = U_N T_N U_N^*$ (computed using Mathematica) is in orange.



Limit of the random point process induced by outliers

- Recall

$$\boldsymbol{a}(\lambda) = \sum_{\ell=-d_2}^{d_1} a_\ell \lambda^\ell.$$

- Fix $z \in \mathbb{C}$. Let $\lambda_1(z), \ldots, \lambda_d(z)$ be the roots of the polynomial $(\boldsymbol{a}(\lambda) - z) \cdot \lambda^{d_2}$, arranged in non-increasing order of their moduli. Here $d := d_1 + d_2$.
- Denote

$$\mathcal{R}_k := \{z \in \mathbb{C} : |\lambda_k(z)| > 1 > |\lambda_{k+1}(z)|\}.$$

Limit of the random point process induced by outliers

Theorem 3 (B., Zeitouni '19)

Let T_N be a Toeplitz matrix with symbol $\boldsymbol{a}$, where $\boldsymbol{a}$ is a Laurent polynomial and let the entries of E_N be i.i.d. complex valued random variables with bounded density. Then the random point process induced by the outliers of $T_N + N^{-\gamma} E_N$ converge weakly to the random point process induced by the zero set of some (explicit) random analytic function $\mathfrak{F}$.
$\mathfrak{F}$ is non-universal and the description of $\mathfrak{F}$ is differs across the regions $\{\mathcal{R}_k\}$.

Earlier results. $\boldsymbol{a}(\lambda) = \lambda$ or $\boldsymbol{a}(\lambda) = a_1\lambda + a_{-1}\lambda^{-1}$. Under Gaussian perturbations the process induced by the outlier eigenvalues has a limit given by the zero set of some Gaussian analytic function.

[Sjöstrand, Vogel '16, '17]

Example I: $\boldsymbol{a}(\lambda) = \lambda$.

- For z with $|z| < 1$

$$\mathfrak{F}(z) = \sum_{x,y \geq 1} z^{x+y-2}(-1)^{x+y-2} e_{x,y},$$

where $\{e(x,y)\}_{x,y\in\mathbb{N}}$ is an i.i.d. array with distributions same as that of E_N.

- If the entries of E_N are complex standard Gaussian then

$$\mathfrak{F}(z) = \sum_{k=0}^{\infty} z^k \mathfrak{g}_k \sqrt{k+1},$$

where $\{\mathfrak{g}_k\}$ are i.i.d. standard complex Gaussian.

Limit of the random point process induced by outliers

Generally

$$\mathfrak{F}(z) = \sum_{\mathfrak{x},\mathfrak{y}} \mathfrak{c}(\mathfrak{x},\mathfrak{y}) \cdot \det(E_\infty[\mathfrak{X}(\mathfrak{x},\mathfrak{y}), \mathfrak{Y}(\mathfrak{x},\mathfrak{y})]),$$

where $|\mathfrak{X}| = |\mathfrak{Y}| = |\mathfrak{d}|$, $\mathfrak{d} = d_1 - d_0$, and d_0 is the number of roots greater than one.

Idea of the proof

Outliers are the roots $\det(T_N + N^{-\gamma} E_N - z\mathrm{Id}_N) = 0$ that are in $\mathrm{Spec}\,(T(\boldsymbol{a}))\backslash\boldsymbol{a}(\mathbb{S}^1)$.

Idea:

Expand the determinant

Idea of the proof

Outliers are the roots $\det(T_N + N^{-\gamma}E_N - z\mathrm{Id}_N) = 0$ that are in $\mathrm{Spec}\,(T(\boldsymbol{a}))\backslash\boldsymbol{a}(\mathbb{S}^1)$.

Idea:

Expand the determinant

$$\begin{aligned}&\det(T_N + N^{-\gamma}E_N - z\mathrm{Id}_N)\\ = &\sum_{\substack{X,Y\subset[N]\\ |X|=|Y|}} (\pm)\cdot\det((T_N - z\mathrm{Id}_N)[X^c;Y^c])\cdot\det(N^{-\gamma}E_N[X;Y])\end{aligned}$$

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Idea:

Expand the determinant,

find the dominant term,

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Expand the determinant,

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show that the roots of the dominant term are close to that of the determinant (careful application of Rouché's theorem),

Idea of the proof

Outliers are the roots $\det(T_N + N^{-\gamma} E_N - z\mathrm{Id}_N) = 0$ that are in $\mathrm{Spec}\,(T(\boldsymbol{a}))\backslash\boldsymbol{a}(\mathbb{S}^1)$.

Idea:

Expand the determinant,

find the dominant term,

show that the roots of the dominant term are close to that of the determinant (careful application of Rouché's theorem), and

the dominant term converges weakly to the limiting random analytic function.

Idea of the proof

Formally

- $$\det(T_N + N^{-\gamma}E_N - z\mathrm{Id}_N)$$
$$= \sum_{\substack{X,Y\subset[N] \\ |X|=|Y|}} (\pm)\cdot\det((T_N - z\mathrm{Id}_N)[X^c;Y^c])\cdot\det(N^{-\gamma}E_N[X;Y])$$
$$= \sum_{k=0}^{N} P_k(z),$$

where $P_k(z)$ is the homogeneous polynomial of degree k in the expansion of the determinant in the entries of E_N.

Idea of the proof

Formally

- For every fixed $z \in \mathcal{R}_\ell$

  Step 1.

$$\sum_{k \neq \ell} P_k(z) = o\left(\prod_{i=1}^{\ell} |\lambda_i(z)|\right) \sim P_\ell(z).$$

  Second moment method: To compute the second moment we use some combinatorial arguments.

  - Upper bound follows from second moment method.
  - Lower bound requires some anti-concentration bounds.

Idea of the proof

Step 2.

$$\sum_{k \neq \ell} P_k(z) = o\left(\prod_{i=1}^{\ell} |\lambda_i(z)|\right) \sim P_\ell(z),$$

uniformly over the boundaries of a net (with appropriate mesh size) of $\mathcal{R}_\ell$.

Thank you!