Wilson-Fisher without Feynman diagrams

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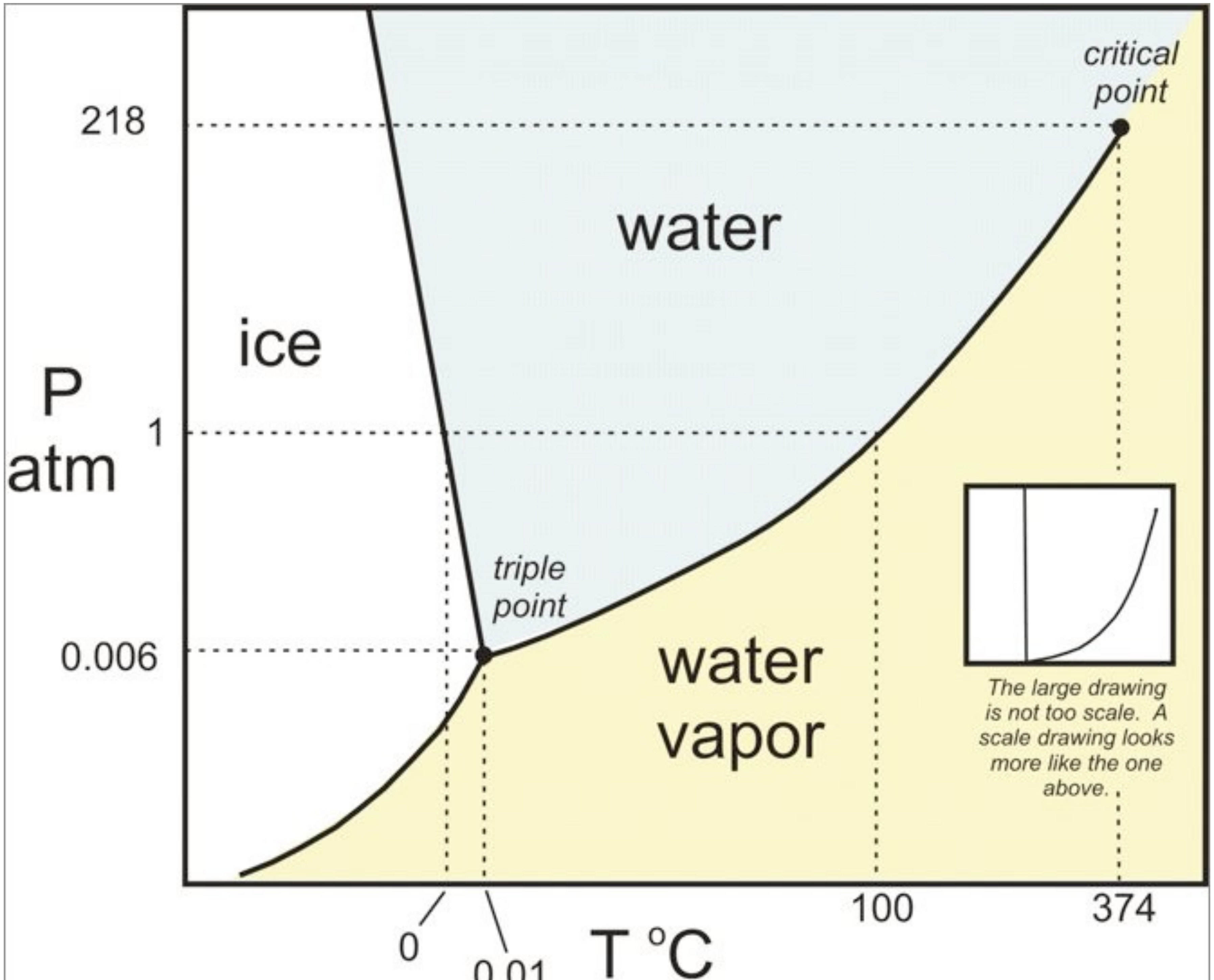
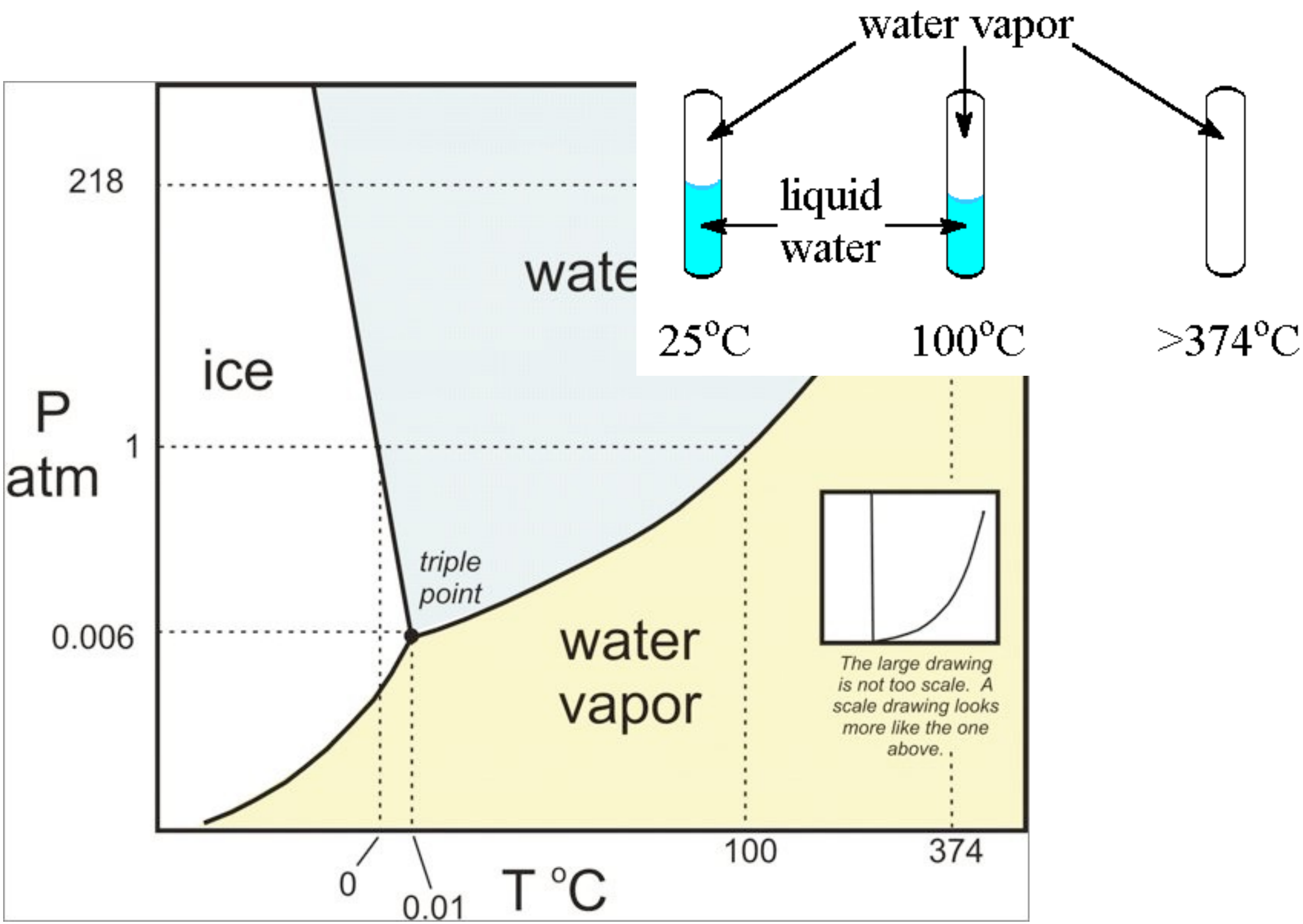
ICTS April 2017



Critical phenomena: Physics



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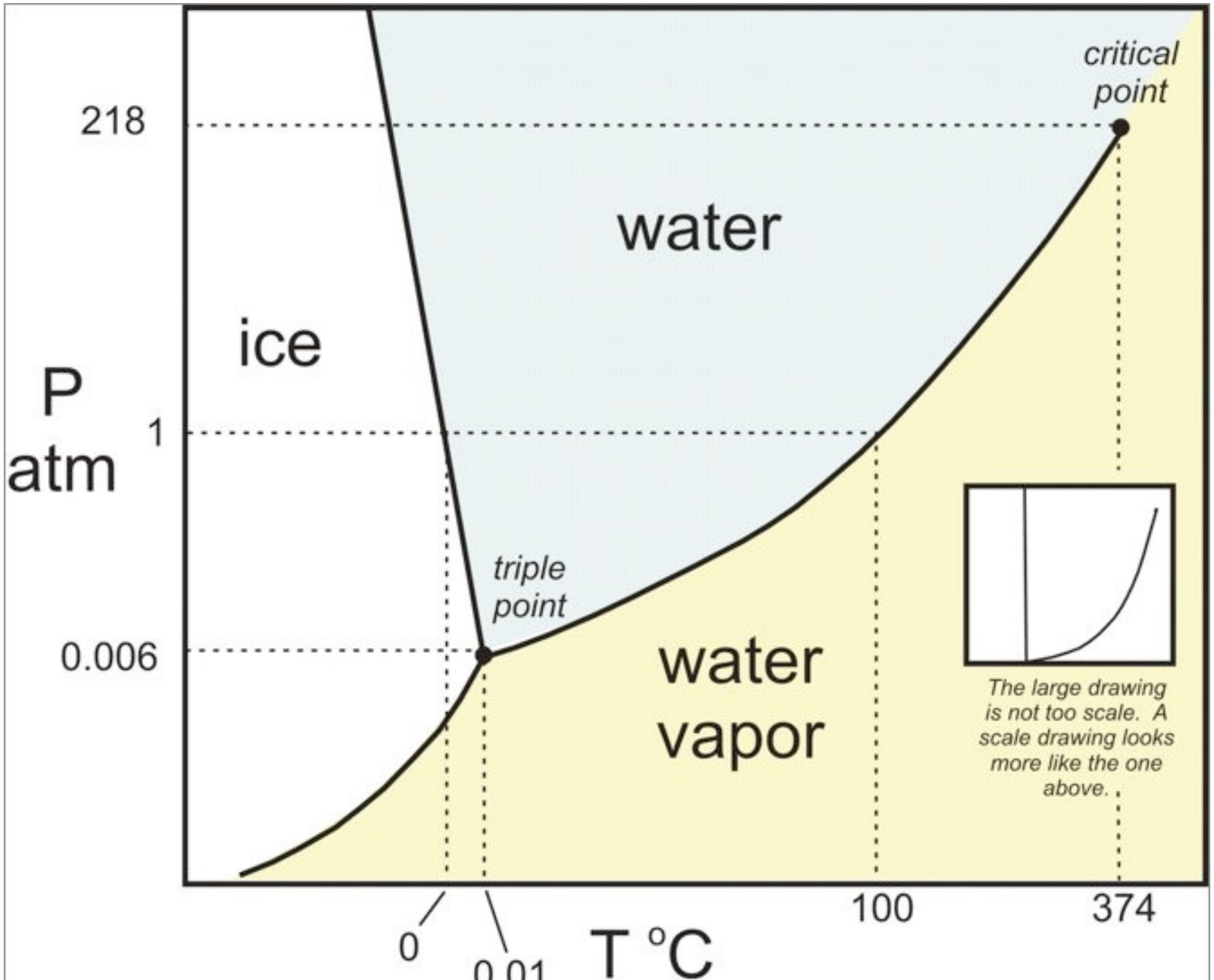

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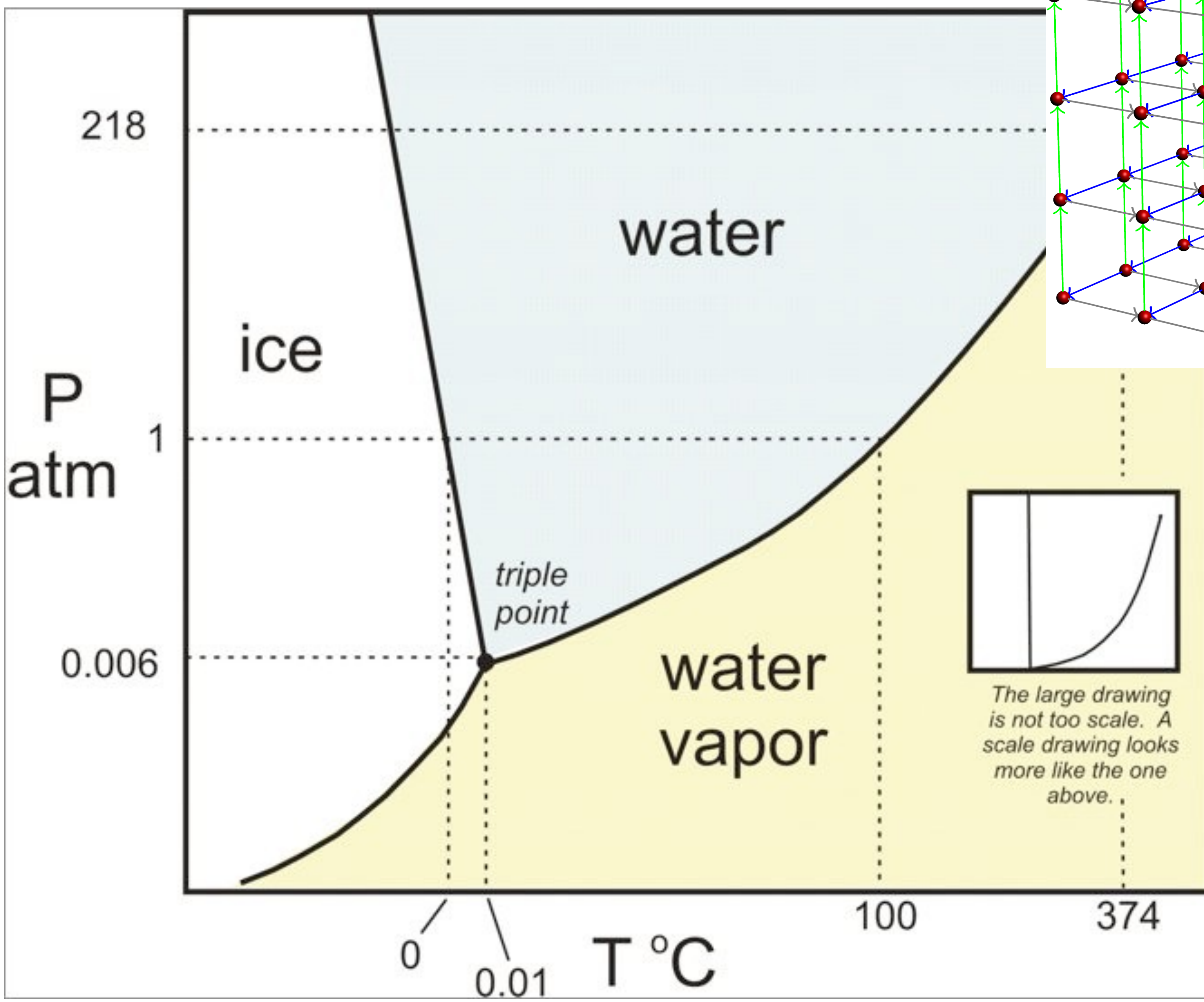

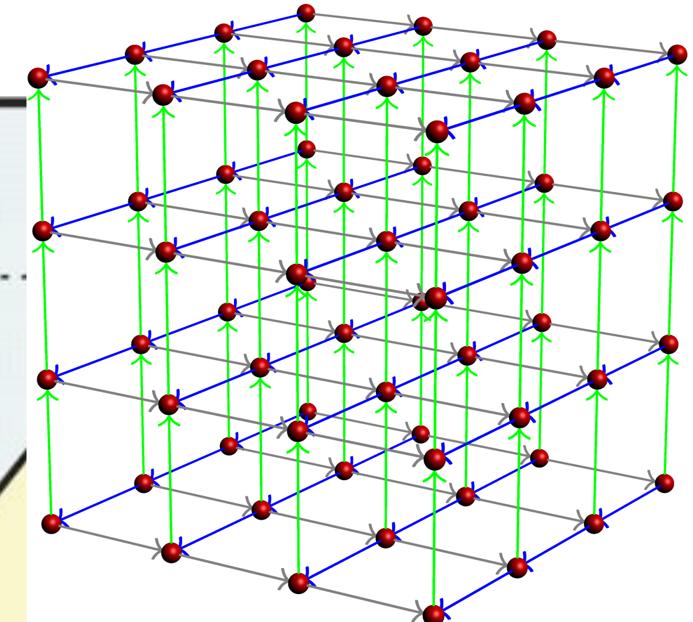

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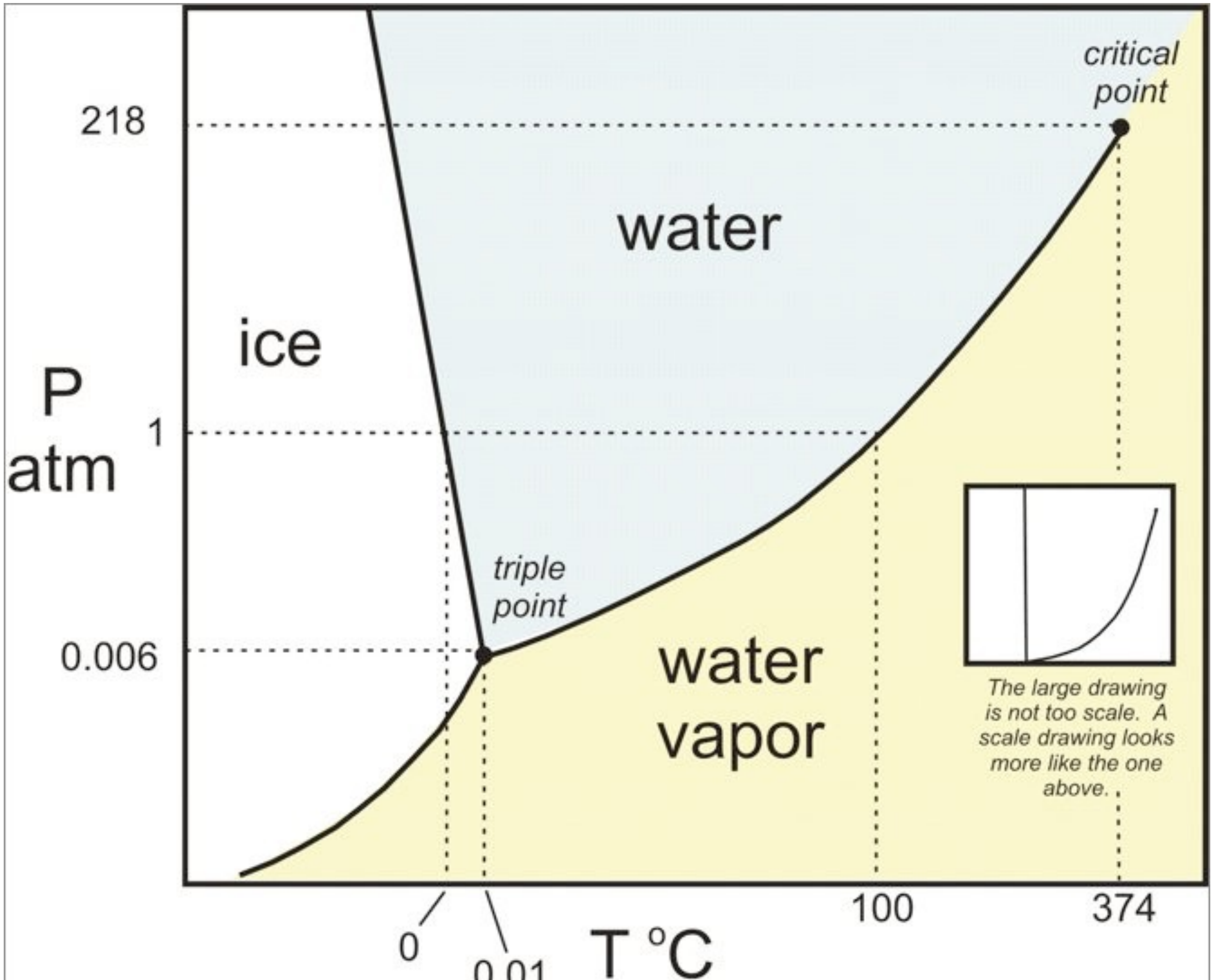

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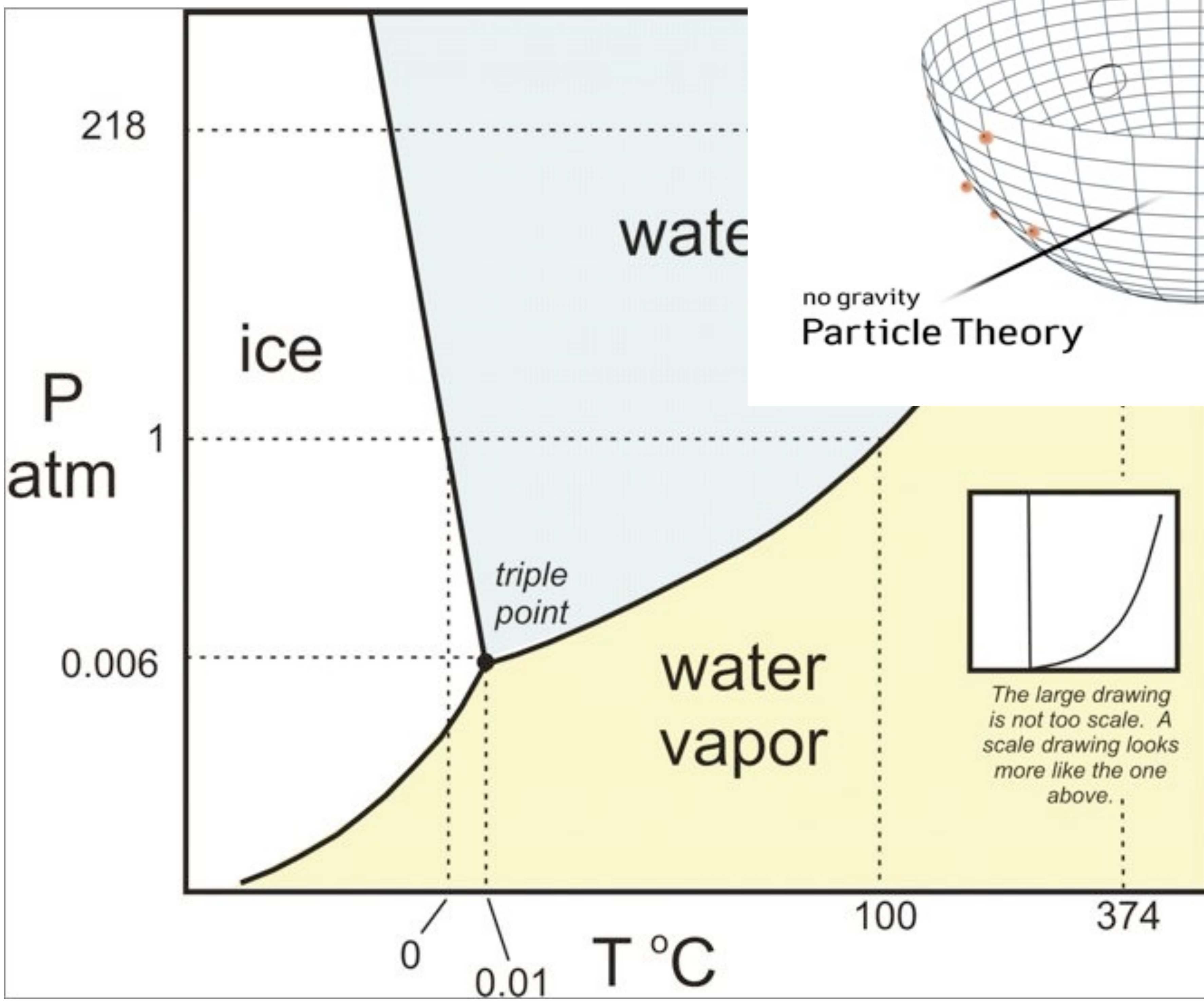

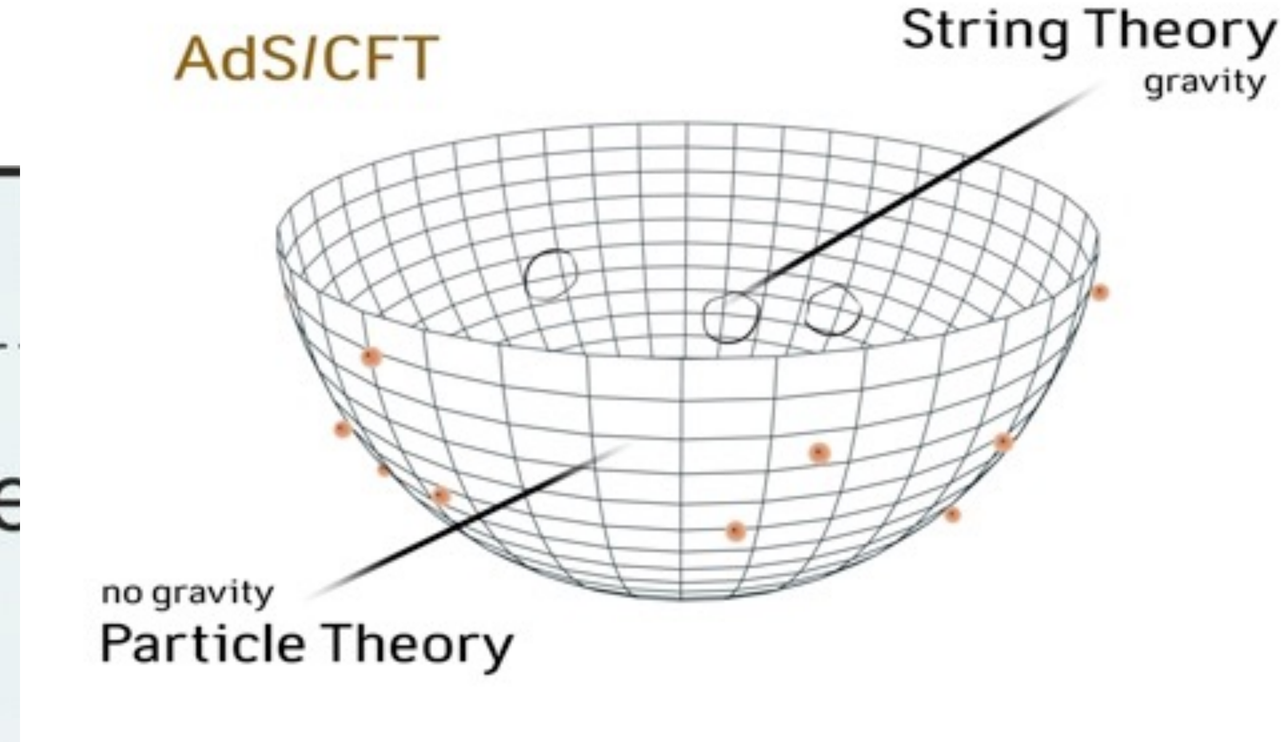

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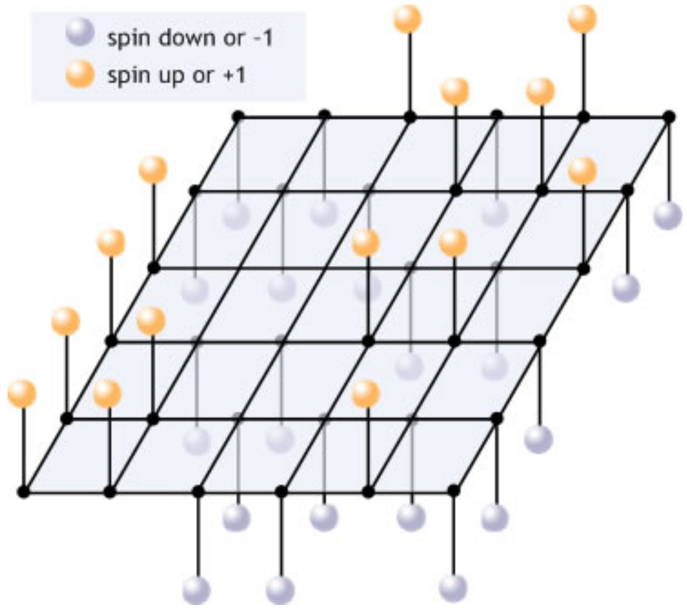
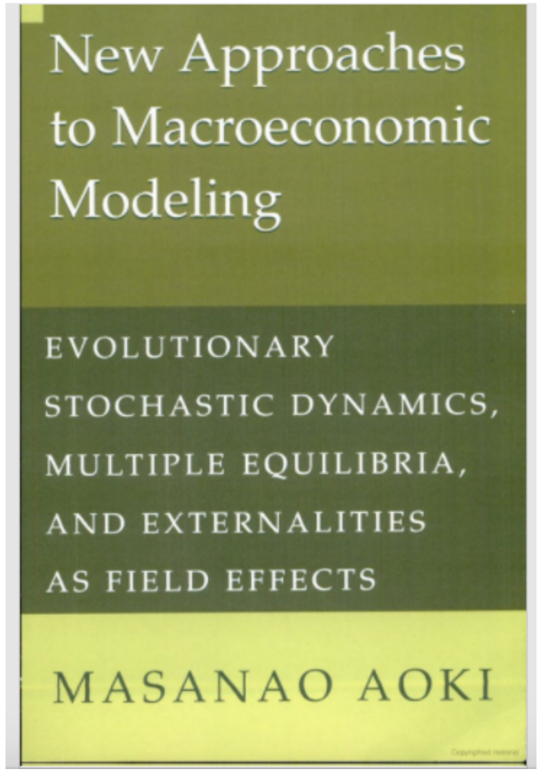


Applications [edit]

Applications arise in physics and chemistry, but also in fields such as sociology. For example, it is natural to describe a system of two political parties by an Ising model. Thereby, at a transition between one majority to the other one the above-mentioned critical phenomena may appear. [2]

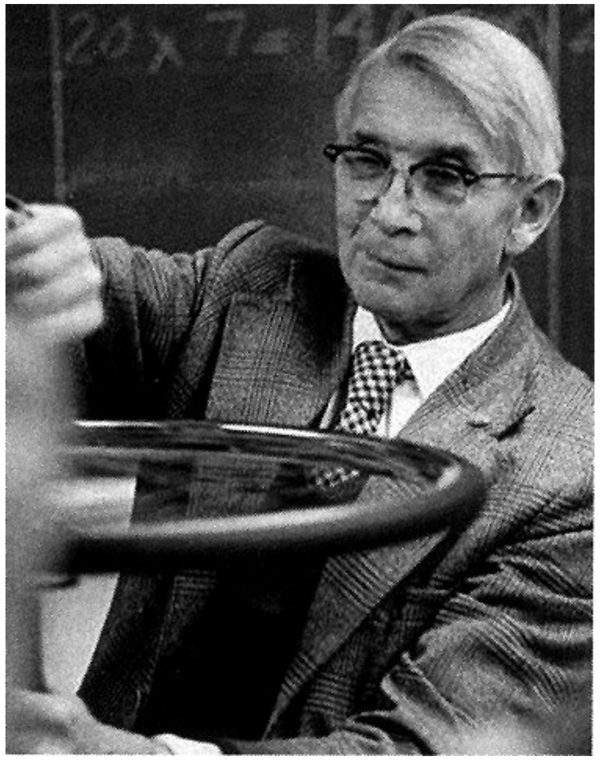
2. ^ W. Weidlich, *Sociodynamics*, reprinted by Dover Publications, London 2006, ISBN 0-486-45027-9

In economics, the Ising model has been adapted to explain diffusion of technical innovations by modeling the process of adoption of new technology as the result of interaction with neighboring firms. The notion of neighborhood is somewhat forced or artificial in this and some other economic applications, however. Föllmer (1973) introduced the Ising model into general equilibrium economics, using the framework introduced by Hildenbrand (1971). In this setup, he needed a certain ergodicity result. As we see in Chapter 5, however, this ergodicity property does not hold in more interesting economic problems that have multiple equilibria, because they call for the framework of ergodic decompositions. Since Föllmer, there have been a few economic models patterned after the Ising model, or its generalized version called the Curie–Weiss model. See Brock (1993) or Durlauf (1991), among others.



Critical phenomena: Sociology, economics

1d



2d

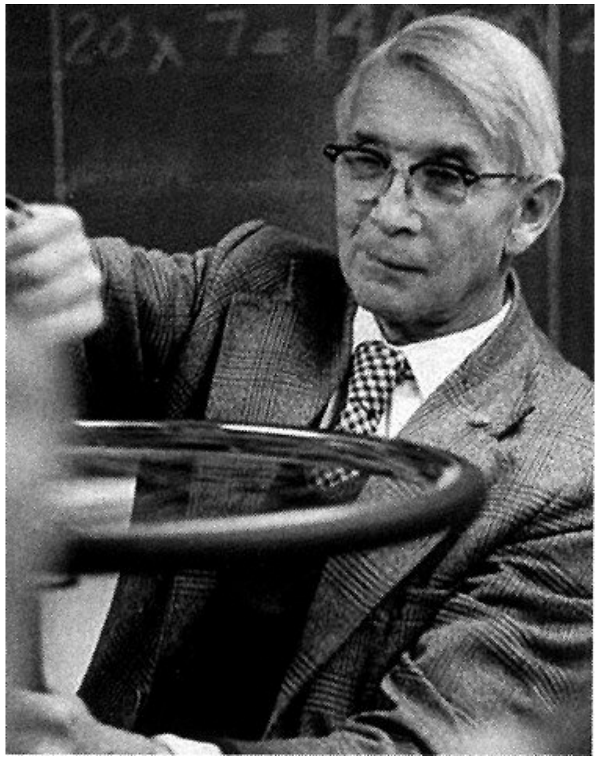
3d

???

?

????

1d



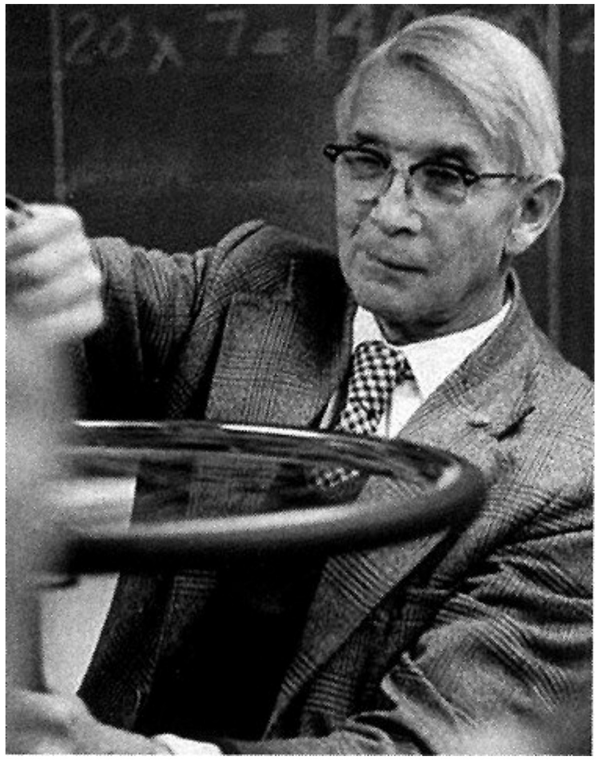
Ernst Ising
1925

2d

3d

???
?
????

1d



Ernst Ising
1925

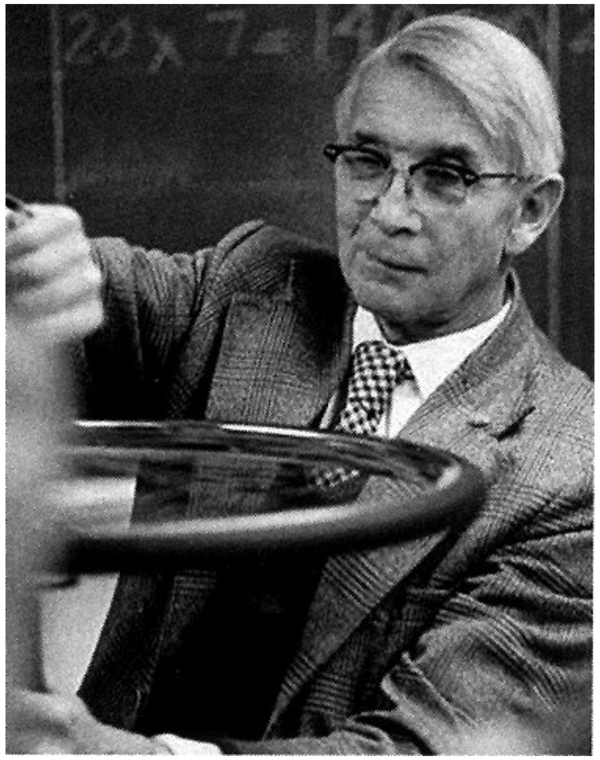
2d

Lars
Onsager
1944, nobel
prize 1968

3d

???
?
????

1d



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1925

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1944, nobel
prize 1968

3d

???
?
????

No analytic
solution to
date, even at
critical point

- Quantum field theory methods typically need us to deal with UV divergences at the critical point.

- I will describe a new approach that enables us to extract physical quantities (in agreement with the QFT methods and yielding new results) in a manifestly finite manner.

Some basic defs

$$\langle O(0)O(x)\rangle \propto \frac{1}{(x^2)^\Delta}$$

Correlation fall-off

$$\boxed{\Delta \rightarrow Conformal - dimension}$$

$$\boxed{\langle O(0)O(x)O(y)\rangle \propto OPE - coefficient}$$

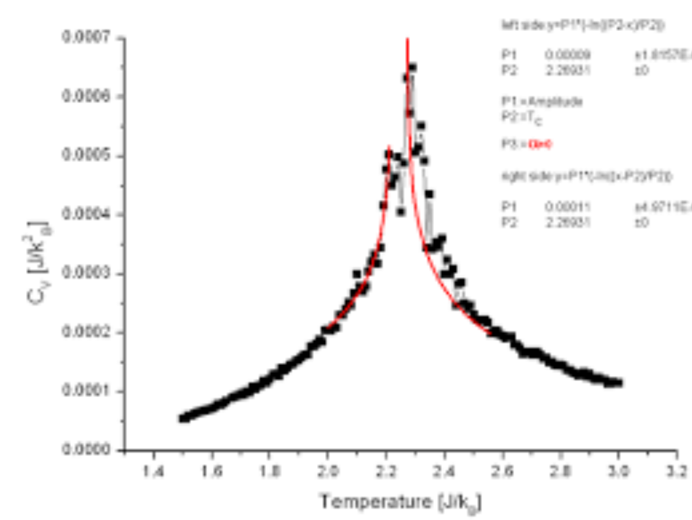
"interaction strength"

Critical exponents and conformal dimensions

$$\Delta_\phi = \frac{\eta + 1}{2}$$

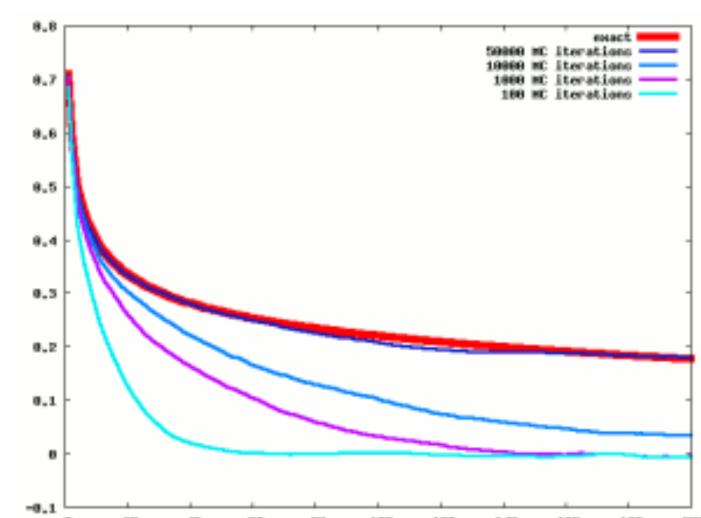
$$\Delta_{\phi^2} = 3\frac{\alpha - 1}{\alpha - 2}$$

$$specific - heat \propto \tau^{-\alpha}$$

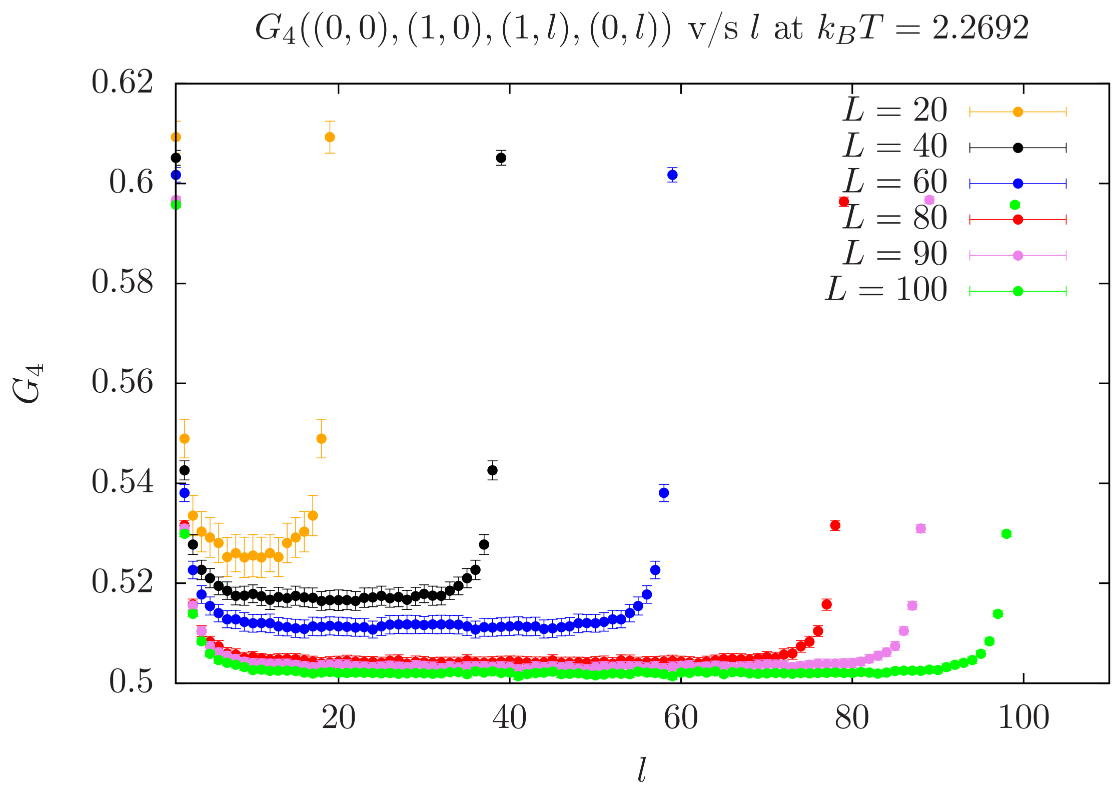


2d

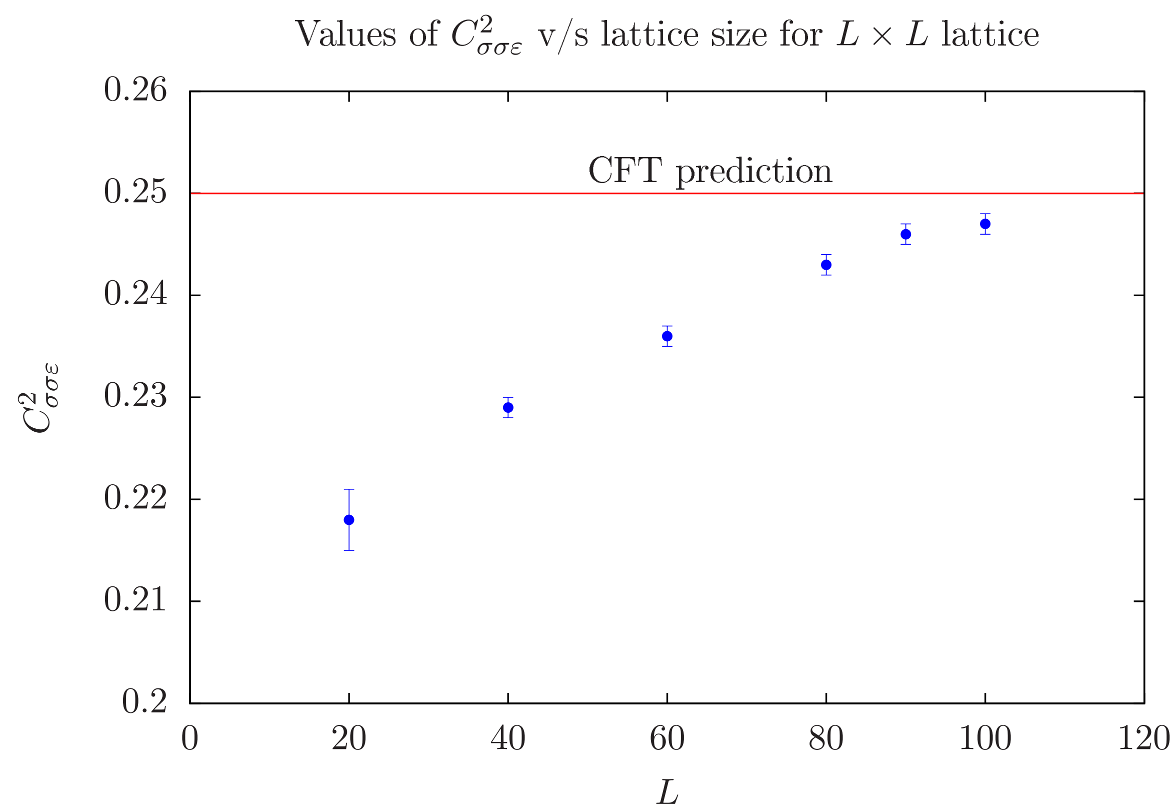
$$\langle \psi(0)\psi(r) \rangle \propto r^{-1-\eta}$$



OPE determination from lattice?



2d



P. Hegde, A. Sharma in progress

One of the many legacies of Ken Wilson
(Nobel prize 1982)

PHYSICAL REVIEW D VOLUME 7, NUMBER 10 15 MAY 1973

Quantum Field-Theory Models in Less Than 4 Dimensions*

Kenneth G. Wilson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
(Received 9 November 1972)

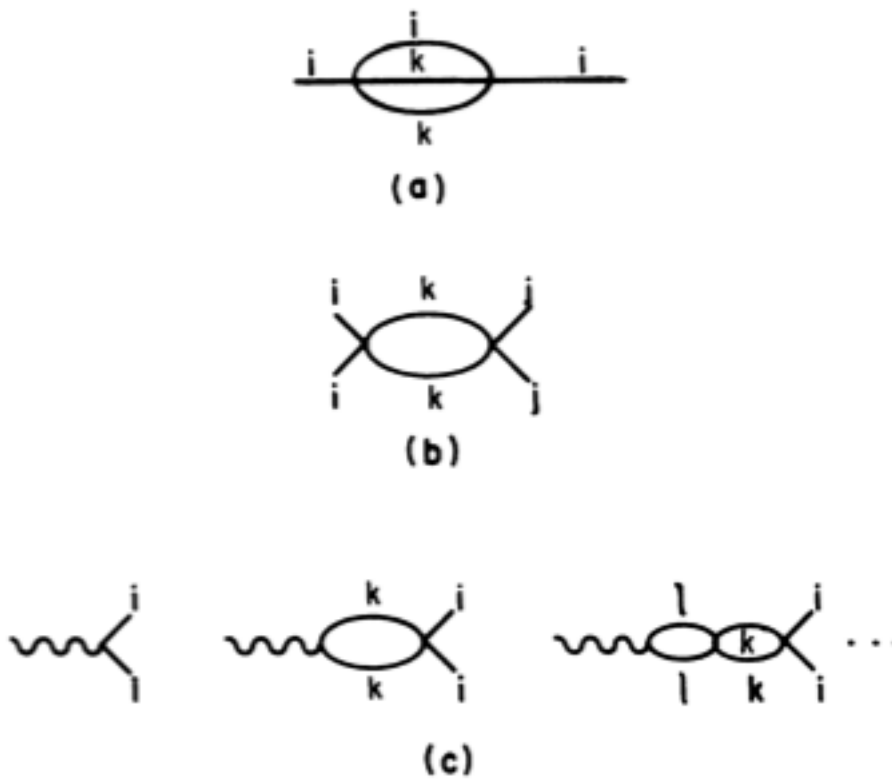


FIG. 1. (a) Diagram giving the lowest-order correction to the propagator. The two lines with internal index k form a loop; the sum over k gives a factor of N. (b) Diagram giving the leading correction to the four-point function (k is summed over). (c) Bubble graphs for vertex function involving ϕ^2 or $\bar{\psi}\psi$. The wavy line represents ϕ^2 or $\bar{\psi}\psi$; the straight lines refer to the elementary fields ϕ, ψ, or $\bar{\psi}$. The indices k and l are summed over.

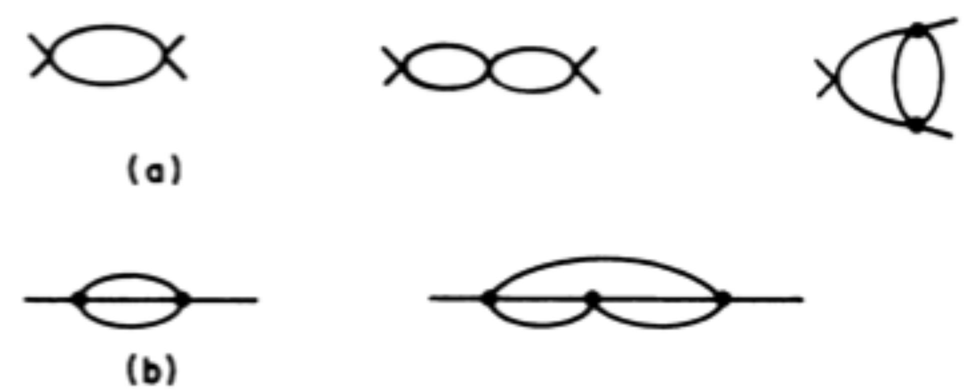


FIG. 2. (a) Diagrams determining u_R to order u_0^3. (b) Diagrams determining $D(q)$ to order u_0^2.

Epsilon expansion

Epsilon-expansion; Wilson; Wilson-Fisher; Wilson-Kogut; Polyakov;.Mack…..Rychkov, Tan

O(N) model

$$\int d^{4-\epsilon}x \ \left[(\partial_\mu \phi^i)^2 + \lambda(\phi^i\phi^i)^2\right]$$

- Wilson-Fisher fixed point.
- 3d Ising model (critical point of water)

$$N=1, \epsilon=1$$

- 2d Ising model

$$N=1, \epsilon=2$$

- XY model

$$N=2, \epsilon=1$$

$$\Delta_\phi = \frac{d-2}{2} + \frac{N+2}{4(N+8)^2}\epsilon^2 + \frac{N+2}{4(N+8)^2}\left[6\frac{3N+14}{(N+8)^2} - \frac{1}{4}\right]\epsilon^3$$

Ising

N=1

$$d = 2 \to 0.11$$

$$\text{actual} = \frac{1}{8} = 0.125$$

$$d = 3 \to 0.519$$

$$\text{numerics} \approx 0.518$$

$$\text{experiments} \approx 0.521$$

XY

N=2

$$d = 3 \to 0.52,$$

$$\text{expt} \approx 0.506$$

Numerical results are from bootstrap based on methods pioneered by Rattazzi, Rychkov, Vichi and Tonni [2008] and used by El Showk et al [2012, 2014]. Often quoted as most accurate numerical estimates for the 3d Ising model at criticality.

$$\Delta_{\phi^2} = d - 2 + \frac{N+2}{N+8}\epsilon + \frac{N+2}{2(N+8)^3}(13N+44)\epsilon^2$$

Ising

$d = 2 \to 1.136$ $\quad$ $\text{actual} = 1$

$d = 3 \to 1.45$ $\quad$ $\text{numerics} \approx 1.41$

$\text{expts} \approx 1.41$

XY

$d = 3 \to 1.54$ $\quad$ $\text{expt} \approx 1.51$

International space station superfluid He experiment

$$\Delta_{\phi^2} = d - 2 + \frac{N+2}{N+8}\epsilon + \frac{N+2}{2(N+8)^3}(13N+44)\epsilon^2$$

Ising

$$d = 2 \to 1.136$$

$$\text{actual} = 1$$

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$$\text{expts} \approx 1.41$$

XY

$$d = 3 \to 1.54$$

$$\text{expt} \approx 1.51$$

seems good but!!

$$\alpha = 2 - \frac{d}{d - \Delta_{\phi^2}}$$

$$\to -0.055$$

$$\text{expt} \approx -0.013$$

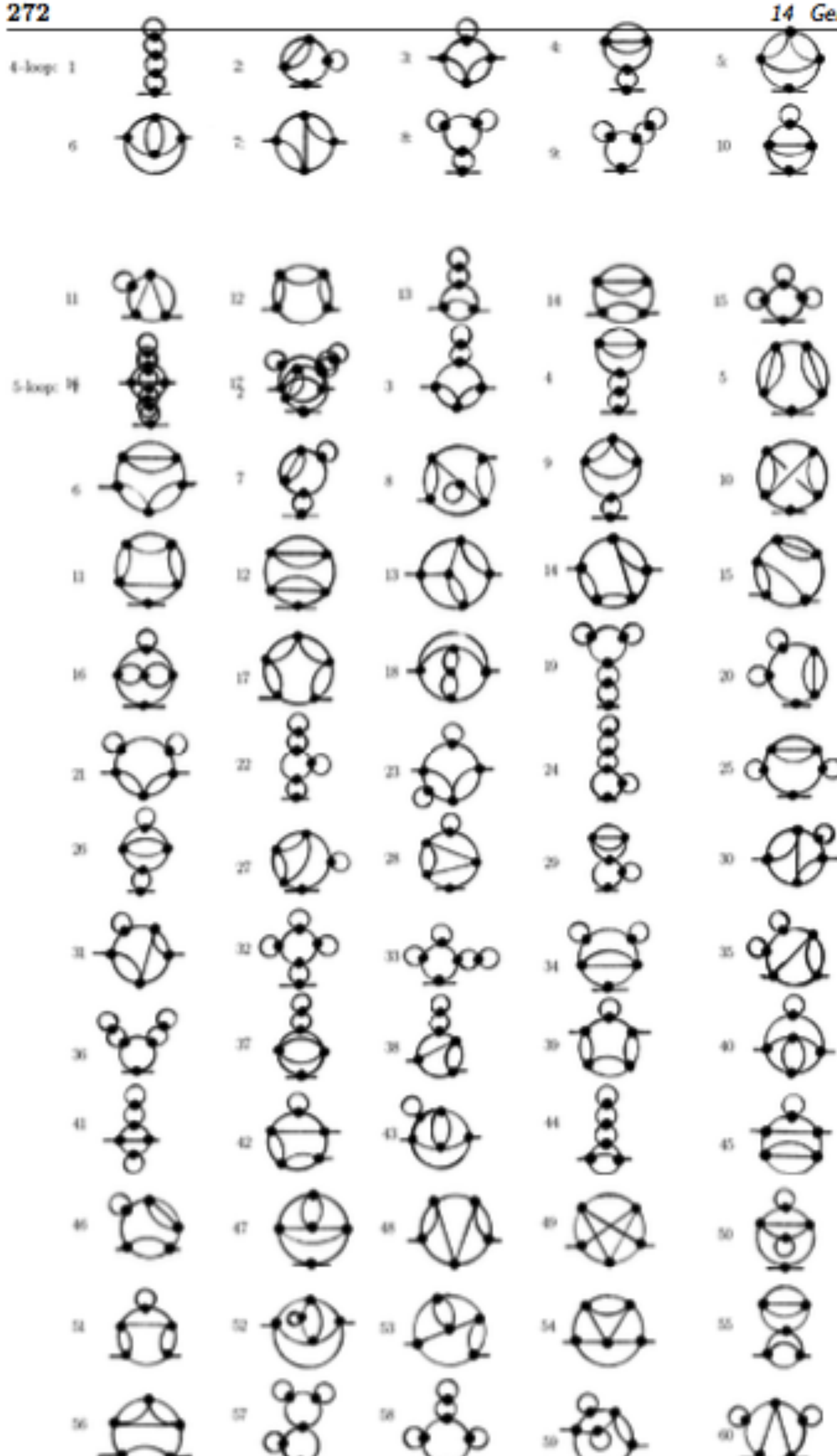
International space station superfluid He experiment

$$O(\epsilon^5) \to -0.004$$

discrepancy!

- Regularize, renormalize: remove infinities.
- Locate fixed point of beta function so that we have scale invariance. This is a special value of the coupling in terms of epsilon.
- Use Callan-Symanzik to determine scaling dimensions.
- Bottom line: quite technical and tedious although framework is well understood.

- State of the art is 5-loops (recent 6 loops) for scalar operators, 4-loops for “double field” operators ϕ, ϕ^2


4-loop:
5-loop:

The five-loop anomalous dimension of the field ϕ and the associated critical exponent η to order ϵ^5 were determined analytically in [11]. The five-loop β-function and the anomalous dimension of the mass were given in Ref. [12]. However, three of the 124 four-point diagrams contributing to the β-function at the five-loop level could be evaluated only numerically. The analytic calculation of the β-function was finally completed in [13]. The ensuing ϵ-expansions for the critical exponents were obtained up to order ϵ^5 in [14].

Intending further applications, the Berlin group of the authors undertook an independent recalculation of the perturbation series of Refs. [11, 12], using the same techniques, and discovered errors in six of the 135 diagrams. This meant that the subsequent results of [13, 14] were also incorrect. When visiting the Moscow group the errors were confirmed and we can now jointly report all expansions in the correct form.

hep-th/9503230

cf: Critical properties of phi4 theories by Kleinert et al. 539 pages.

- Virtually no results for OPE coeffs.
- Needs lots of diagrams. Possibility of mistakes. Even 3 loops requires ~10 diagrams,
- Any way the series is asymptotic.
- QUESTION: Can the 3d Ising model at the critical point yield an analytic solution?

- For this question the Feynman diagram approach is inefficient.
- Needs regularization, renormalization.
- Does not make use of the conformal symmetry at the fixed point.
- Hard to justify epsilon=1, get OPE coefficients, higher order spectrum …….

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Scale invariance implies conformal invariance for the three-dimensional Ising model

Bertrand Delamotte (LPTMC), Matthieu Tissier (LPTMC), Nicolás Wschebor (LPTMC)

(Submitted on 8 Jan 2015 (v1), last revised 27 Jan 2016 (this version, v4))

Using Wilson renormalization group, we show that if no integrated vector operator of scaling dimension -1 exists, then scale invariance implies conformal invariance. By using the Lebowitz inequalities, we prove that this necessary condition is fulfilled in all dimensions for the Ising universality class. This shows, in particular, that scale invariance implies conformal invariance for the three-dimensional Ising model.

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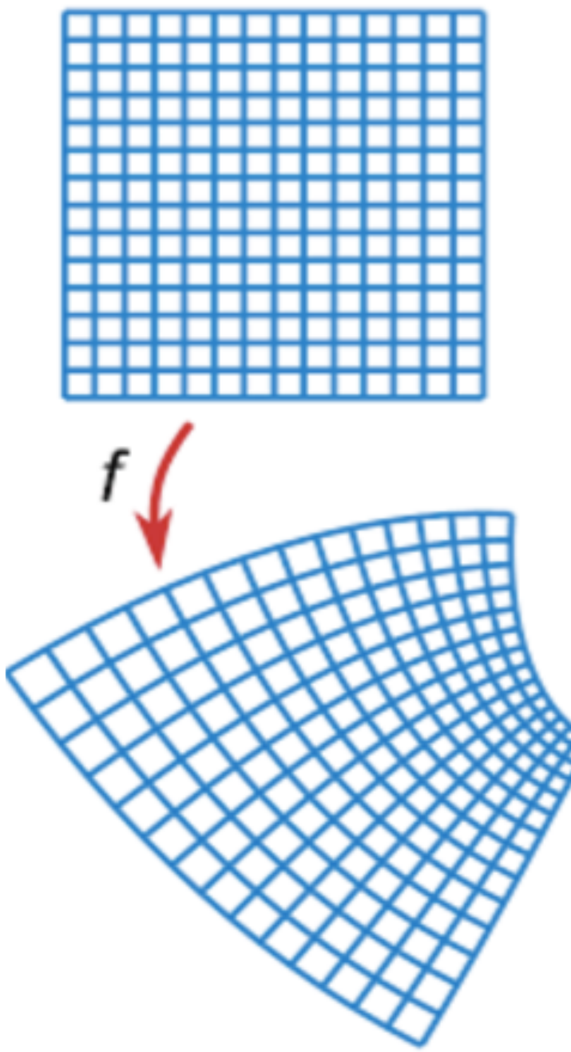
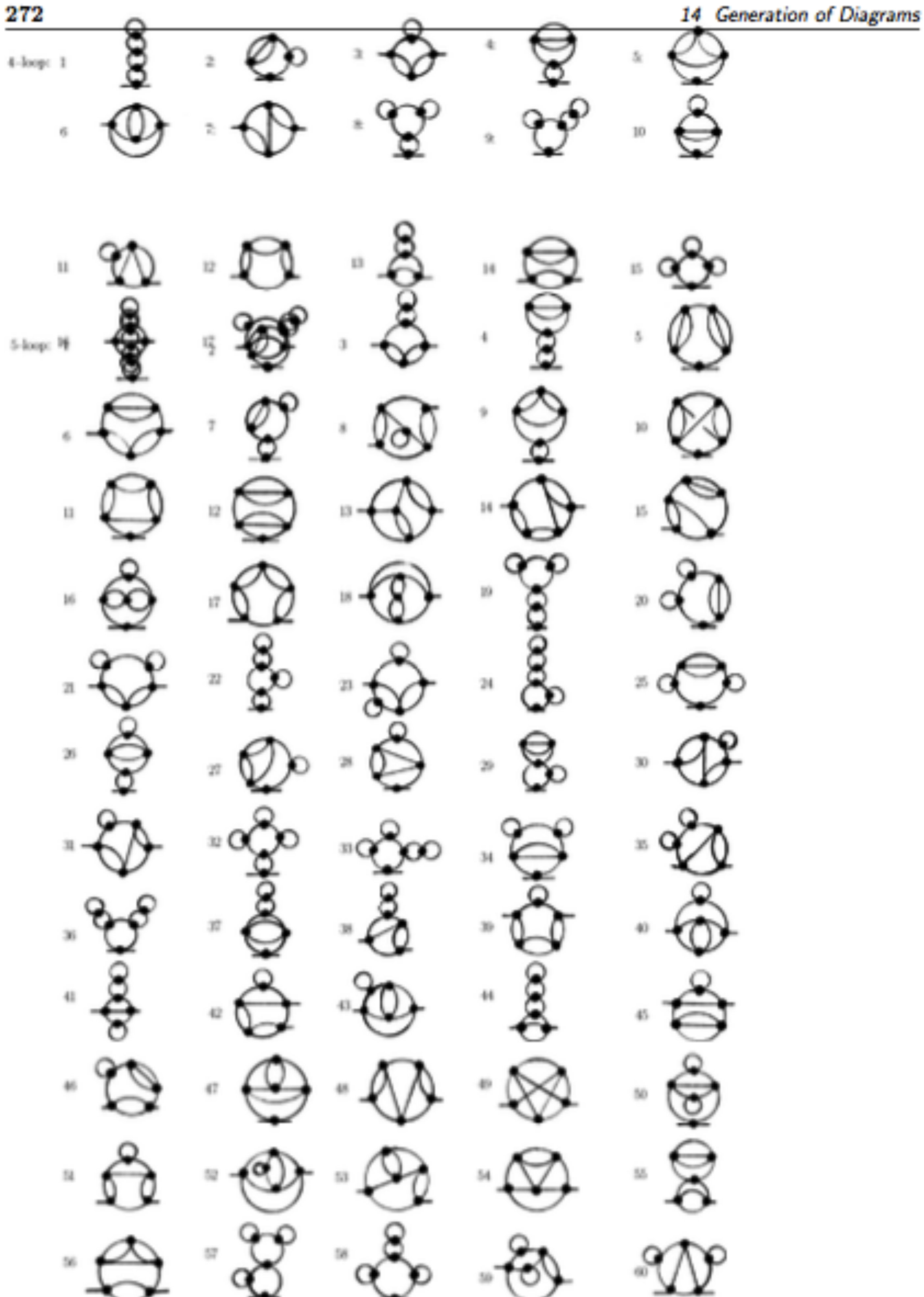
Subjects:	**Statistical Mechanics (cond-mat.stat-mech)**; High Energy Physics - Theory (hep-th)
Journal reference:	Phys. Rev. E 93, 012144 (2016)
DOI:	10.1103/PhysRevE.93.012144
Cite as:	arXiv:1501.01776 **[cond-mat.stat-mech]**
	(or **arXiv:1501.01776v4** **[cond-mat.stat-mech]** for this version)

CFT

- In addition to Lorentz generators (d(d-1)/2) and Translations (d), we have Special Conformal Transformations (d) and Dilatations.
- Conformal dimension is eigen-value of Dilatation. By spin, we will refer to the spin of the symmetric traceless operator entering the OPE.



Eternal Scream



Grad student's reaction at being told to verify this !!

- The technique that makes use of the full conformal symmetry is called the conformal bootstrap.
- This is an old idea that originated in the 1970s.
- However very few results in $d>2$ were obtained using this approach until the work of Rattazzi, Rychkov, Tonni and Vichi in 2008.

Conformal bootstrap: philosophy

- No Lagrangian
- No Feynman diagrams
- No regularization, RG etc.
- Only conformal symmetry, crossing symmetry and OPE.
- Algebraic equations: steps can be made manifestly finite.

dictionary meaning

noun

1. a loop of leather or cloth sewn at the top rear, or sometimes on each side, of a boot to facilitate pulling it on.

2. a means of advancing oneself or accomplishing something:
He used his business experience as a bootstrap to win voters.

adjective

3. relying entirely on one's efforts and resources:
The business was a bootstrap operation for the first ten years.

4. self-generating or self-sustaining:
a bootstrap process.

verb (used with object), **bootstrapped, bootstrapping.**

5. *Computers.* boot[1](def 24).

6. to help (oneself) without the aid of others:
She spent years bootstrapping herself through college.

Idioms

7. **pull oneself up by one's bootstraps,** to help oneself without the aid of others; use one's resources:
I admire him for pulling himself up by his own bootstraps.

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In the late '60's, Migdal and Polyakov[64] developed a "bootstrap" formulation

of critical phenomena based on a skeleton Feynman graph expansion, in which all parameters including the expansion parameter inself would be determined self-consistently. They were unable to solve the bootstrap equations because of their complexity, although after the ε expansion about four dimensions was discovered, Mack showed that the bootstrap could be solved to lowest order in ε. If the 1971 renormalization group ideas had not been developed, the Migdal-Polyakov bootstrap would have been the most promising framework of its time for trying to further understand critical phenomena. However, the renormalization group methods have proved both easier to use and more versatile, and the bootstrap receives very little attention today.

Wilson-Nobel lecture 1982

I would like to quote from a 2003 interview with A.M. Polyakov [12], where the desire for a better 3D theory has been stated with clarity:

"Let me tell you what I think of the renormalization group. I think there are two types of useful equations. One type is human-made, they are invented by people. The other type reflects some 'pre-established harmony.' They can be discovered (uncovered) and not invented. Renormalization

5

group is clearly a human made thing. It's clearly a smart way of calculating things but it doesn't have a breathtaking quality of, say, the Dirac equation.

The example of the second kind is operator product expansions. They form some beautiful mathematical relations and I was dreaming in the 1970s to have some classification of fixed points based on the possible operator product expansions. The program was a little like classifying Lie algebras. In that case you start with the commutator relations which define the Lie algebra and then you classify all possible semi-simple algebras. You arrive at a stunningly beautiful theory (which was clearly discovered and not invented). I was working on that project in the 1970s and I still think it might have a chance. It was successful in two dimensions. We can classify possible fixed points in two dimensions using operator product expansions. That's what conformal field theories are about. And I think it's not excluded, that in 3 dimensions something like that is still possible. I was working for a while on this without much success in the 1970s and then I switched to other things.

I think the epsilon expansion ended the subject in the practical sense. You can calculate more or less what you want with good accuracy but aesthetically the subject is not closed yet. It's possible that there will be classification of fixed points in three dimensions, based on string theory, similar to what we have in two dimensions. But that's just dreams."

Polyakov interview 2003. Source: Rychkov, 2011

OPE

Operator Product Expansion

$$\phi(0)\phi(x) \sim \sum c_{\Delta,\ell}(x^2)^{\Delta_O/2-\Delta_\phi-\ell/2}x^{a_1}\cdots x^{a_\ell}O_{a_1\cdots a_\ell}(0)$$

- Convergent power series expansion.
- Radius of convergence set by closest operator insertion.
- Operator relation.

OPE

Operator Product Expansion

unknowns

$$\phi(0)\phi(x) \sim \sum \boxed{c_{\Delta,\ell}} (x^2)^{\boxed{\Delta_O}/2-\boxed{\Delta_\phi}-\ell/2} x^{a_1} \cdots x^{a_\ell} O_{a_1\cdots a_\ell}(0)$$

- Convergent power series expansion.
- Radius of convergence set by closest operator insertion.
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Quick review of modern bootstrap

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle\phi(x_1)\phi(x_4)\phi(x_3)\phi(x_2)\rangle$$

"direct-channel"

"crossed-channel"

Crossing

Early work: Ferrara, Gatto, Grillo; Polyakov; Modern revival: Rattazzi, Rychkov, Tonni, Vichi; Many, many others

Quick review of modern bootstrap

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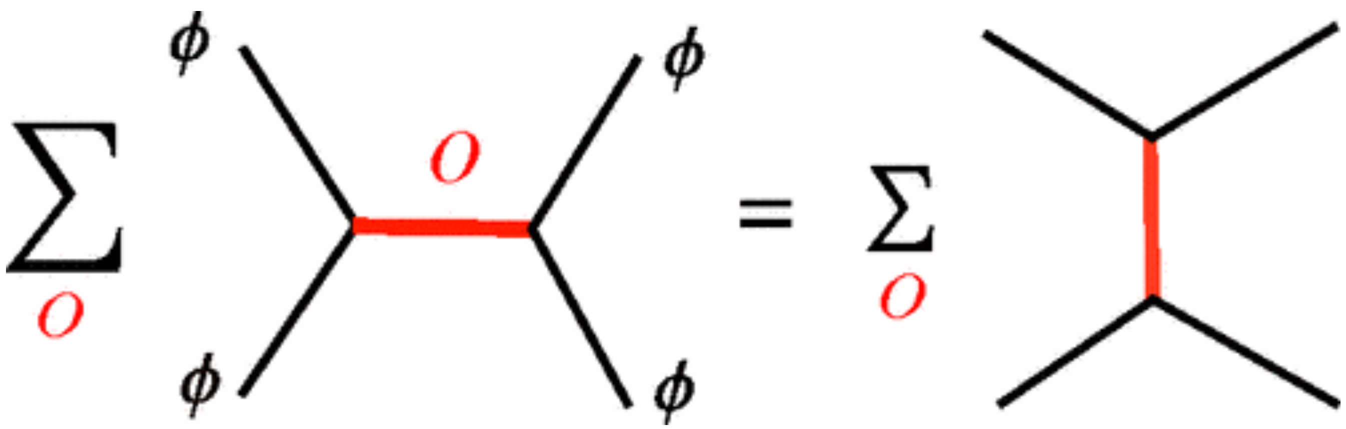

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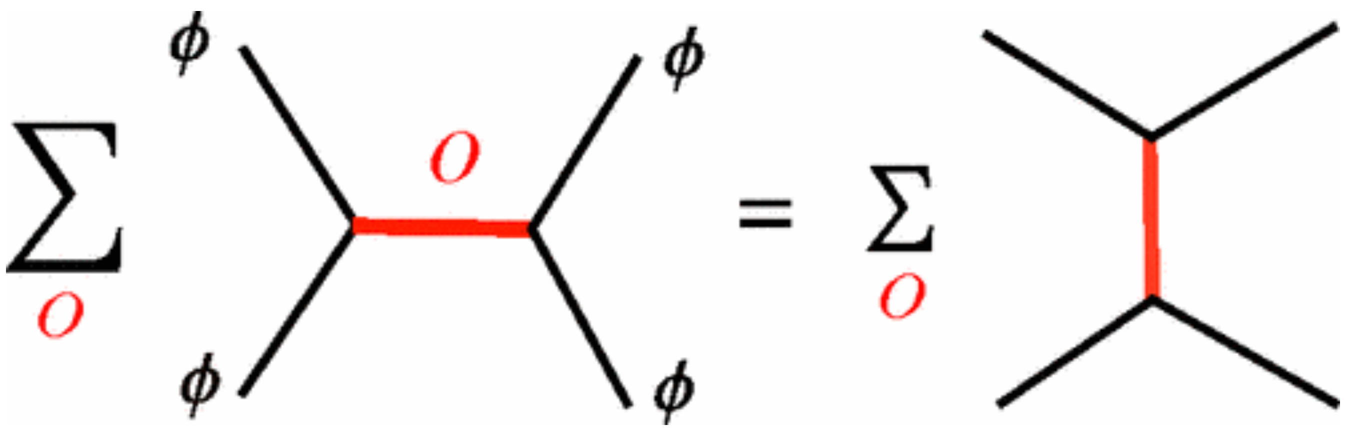


even spin

"direct-channel"

"crossed-channel"

Crossing

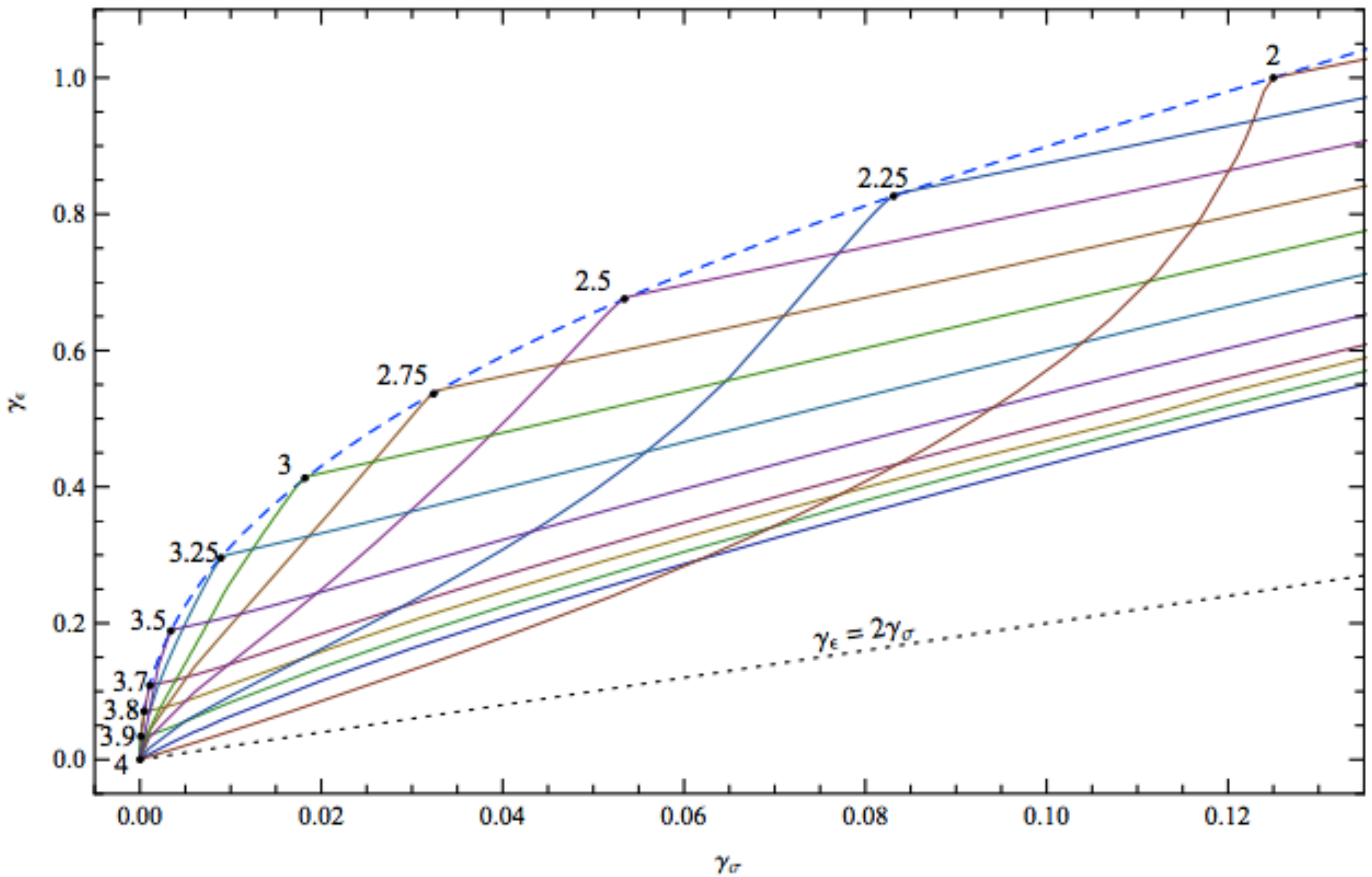
Early work: Ferrara, Gatto, Grillo; Polyakov; Modern revival: Rattazzi, Rychkov, Tonni, Vichi; Many, many others

- Modern numerical methods can numerically extract anomalous dimensions of certain operators by exploiting a “kink”.

- Difficult to do analytically, get information for general operators, get OPE coefficients…..large spin approx/ resummation methods needed.

Alday-Zhiboedov; Komargodski-Zhiboedov; Fitzpatrick et al; Sen, Kaviraj, AS; Alday et al.

Numerical bounds in fractional dimensions

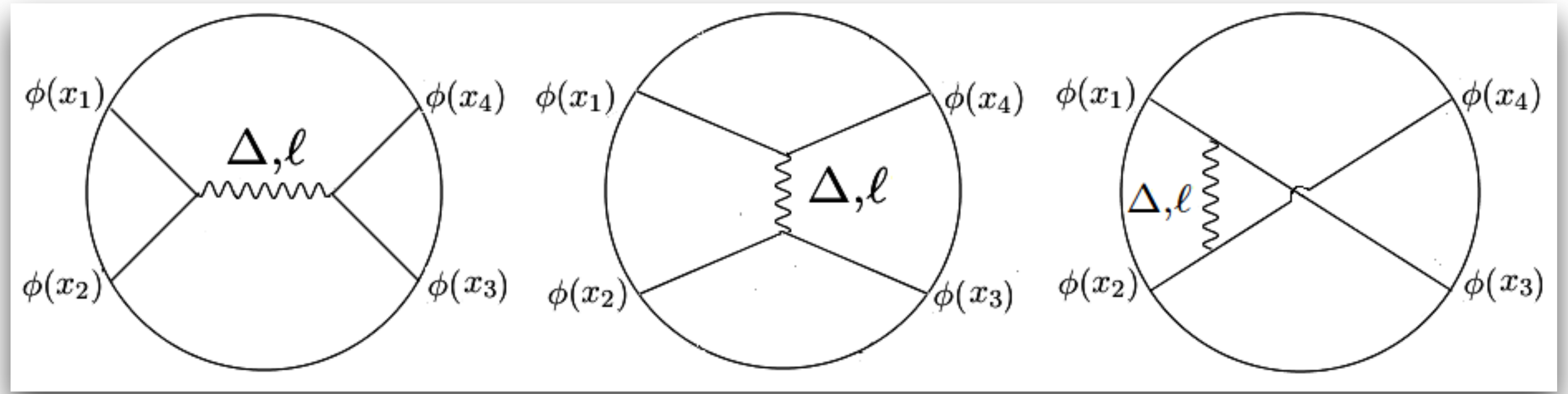


1309.5089, El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi

New approach

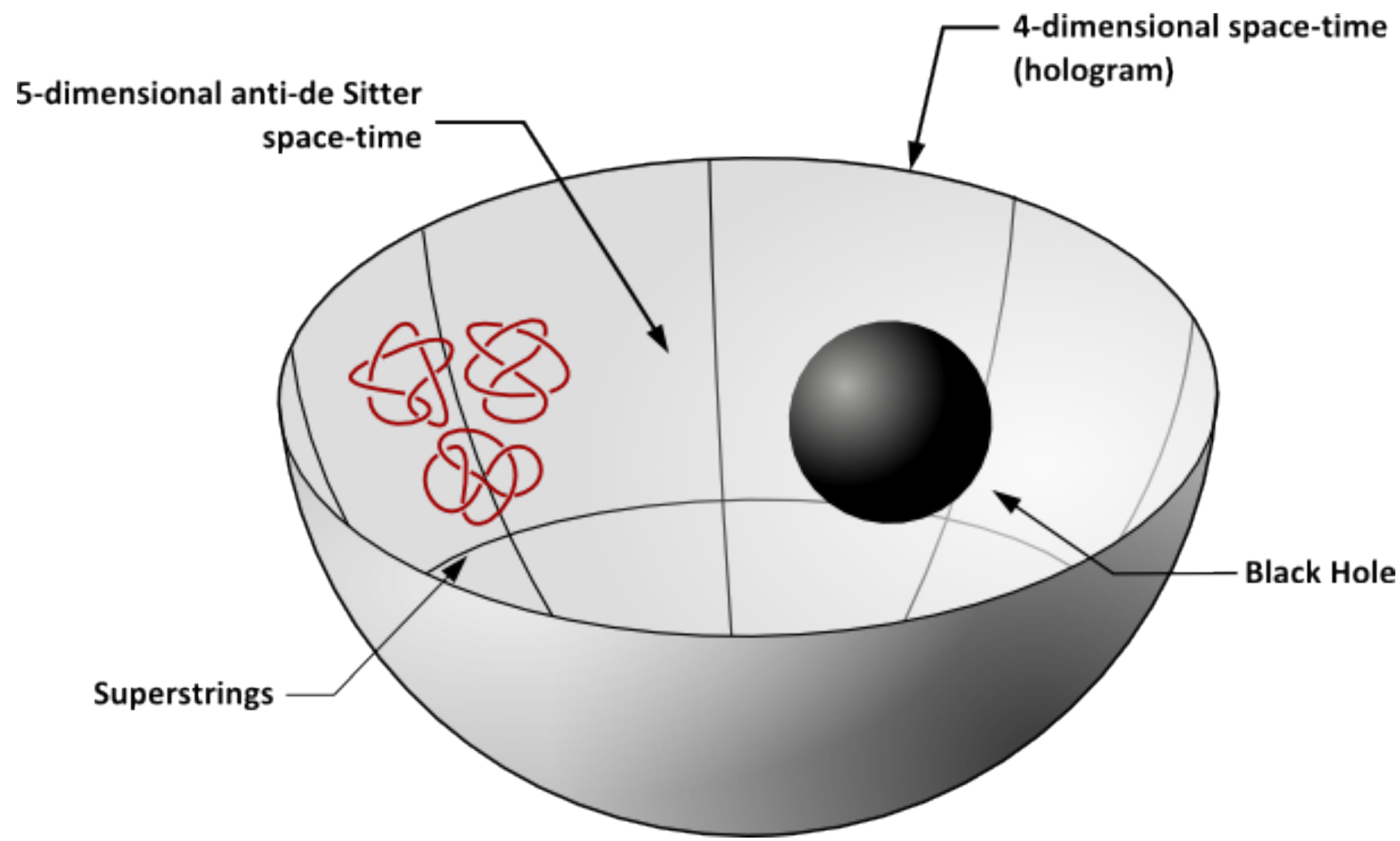
- 1510.07770, JPhysA (w K. Sen)
- 1609.00572, Phys. Rev. Lett. (w R. Gopakumar, A. Kaviraj and K. Sen), longer version 1611.08407
- 1612.05032 (w A. Kaviraj and P. Dey)
- 17xx.xxxx (w R. Gopakumar)
- …….. (a few others which are preliminary)

- Partly motivated by the 1974 seminal paper by Polyakov.
- New ingredients are Mellin space and connection with Witten diagrams.



Convenient Kinematical Basis

- Witten diagrams are building blocks in the AdS/CFT correspondence which was postulated in 1997!



How did Polyakov know about Witten diagrams in 1974?

Non-Hamiltonian approach to conformal quantum field theory

A. M. Polyakov

L. D. Landau Theoretical Physics Institute, USSR Academy of Sciences
(Submitted July 9, 1973)
Zh. Eksp. Teor. Fiz. 66, 23–42 (January 1974)

The completeness requirement for the set of operators appearing in field theory at short distances is formulated, and replaces the S-matrix unitarity condition in the usual theory. Explicit expressions are obtained for the contribution of an intermediate state with given symmetry in the Wightman function. Together with the "locality" condition, the completeness condition leads to a system of algebraic equations for the anomalous dimensions and coupling constants; these equations can be regarded as sum rules for these quantities. The approximate solutions found for these equations in a space of $4-\epsilon$ dimensions give results equivalent to those of the Hamiltonian approach.

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I will come back to this in the end.

ℓ	Δ_{numerics}	$\Delta_{\epsilon\text{-expansion}}$	% variation
4	5.02267	5.02495	0.0454992
6	7.02849	7.03091	0.0344653
8	9.03192	9.03332	0.0154195
10	11.0324	11.0345	0.0192799
12	13.0333	13.0353	0.015114
14	15.0338	15.0357	0.0124619
16	17.0343	17.036	0.0103496

Table 1: Comparison of Δ from ϵ expansion with numerical estimates

Simmons-Duffin: 1612.08471

Spin ℓ	$f_{\phi\phi J_\ell}\|_{DSD}$	$f_{\phi\phi J_\ell}\|_{\epsilon=1}$	Percentage Deviation
$\ell = 2$	0.326	0.328	0.33
$\ell = 4$	0.069	0.070	0.86
$\ell = 6$	1.57×10^{-2}	1.61×10^{-2}	2.55
$\ell = 8$	3.69×10^{-3}	3.76×10^{-3}	2.04
$\ell = 10$	8.76×10^{-4}	8.82×10^{-4}	0.79
$\ell = 12$	2.10×10^{-4}	2.06×10^{-4}	1.86
$\ell = 14$	5.06×10^{-5}	4.79×10^{-5}	5.52
$\ell = 16$	1.22×10^{-5}	1.10×10^{-5}	10.6

Table 2 : Comparison of OPE

$O(\epsilon^3)$

- In the traditional approach we expand in terms of partial waves which are consistent with OPE.
- Impose crossing symmetry as constraint.

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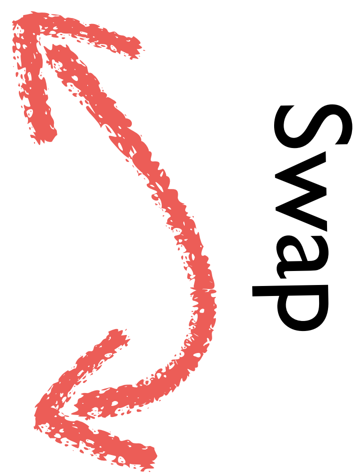
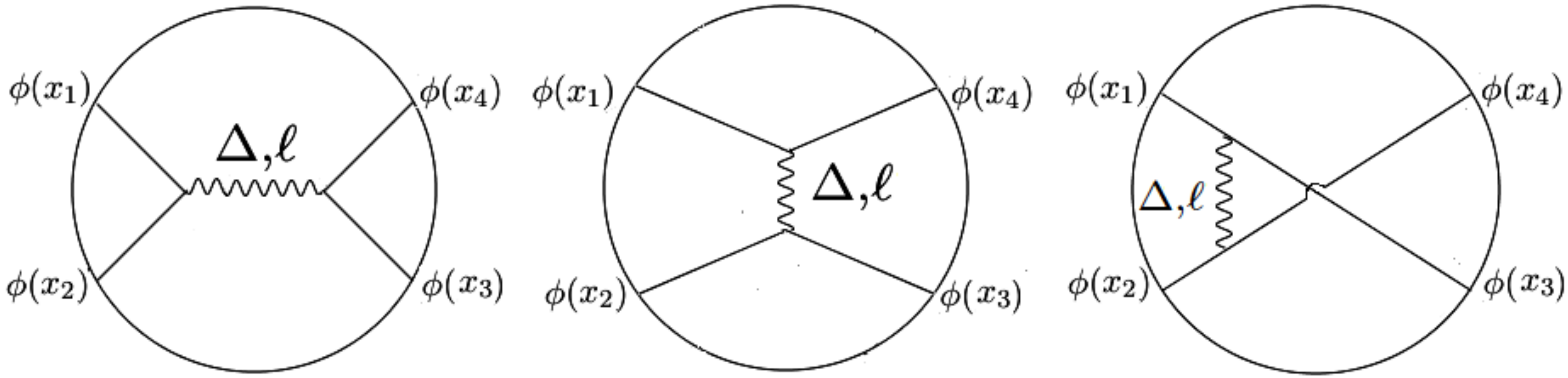


- Impose crossing symmetry as constraint.

New approach: key point

- In the new approach we expand in terms of crossing symmetric partial waves.
- Impose OPE consistency as constraint.



- New partial wave basis: Tree level Witten diagrams (better to call Witten blocks).
- Use this to sum over all physical states (including double trace operators)—a bit different from AdS/CFT at infinite N.
- Has in-built crossing symmetry.

"Cross-ratios"

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \mathcal{A}(u,v)$$

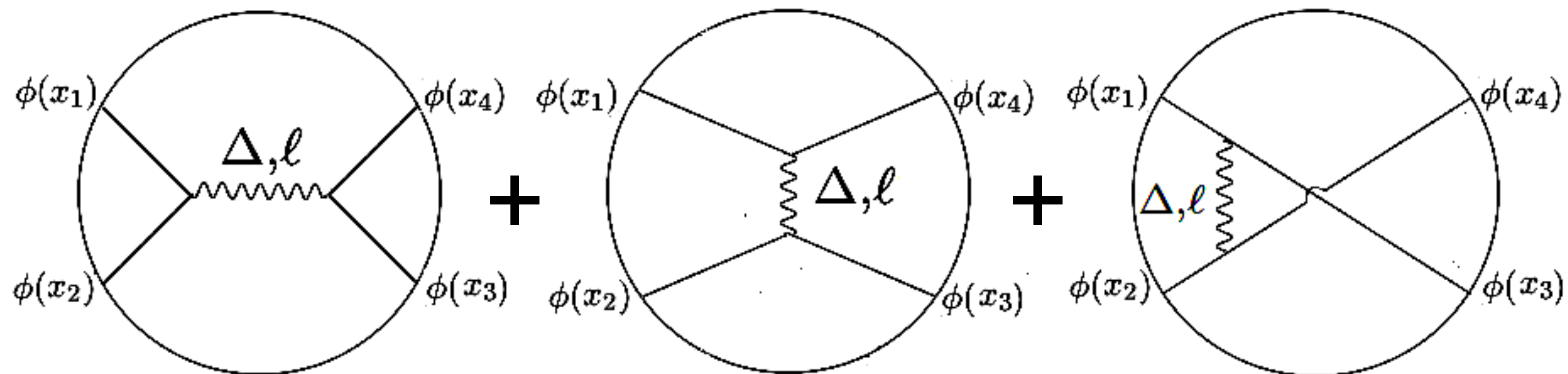
$$\mathcal{A}(u,v) = \int \frac{ds dt}{(2\pi i)^2} \Gamma(-t)^2 \Gamma(s+t)^2 \Gamma(\Delta_\phi - s)^2 \mathcal{M}(s,t) u^s v^t$$

$\mathcal{M}(s,t)$: Mellin amplitude

Hjalmar Mellin (Finnish mathematician)

$$f(x) = \int_{c-i\infty}^{c+i\infty} \tilde{f}(s) x^{-s} \, ds$$

$$\mathcal{M}(s,t) = \sum_{\Delta,\ell} c_{\Delta,\ell}$$

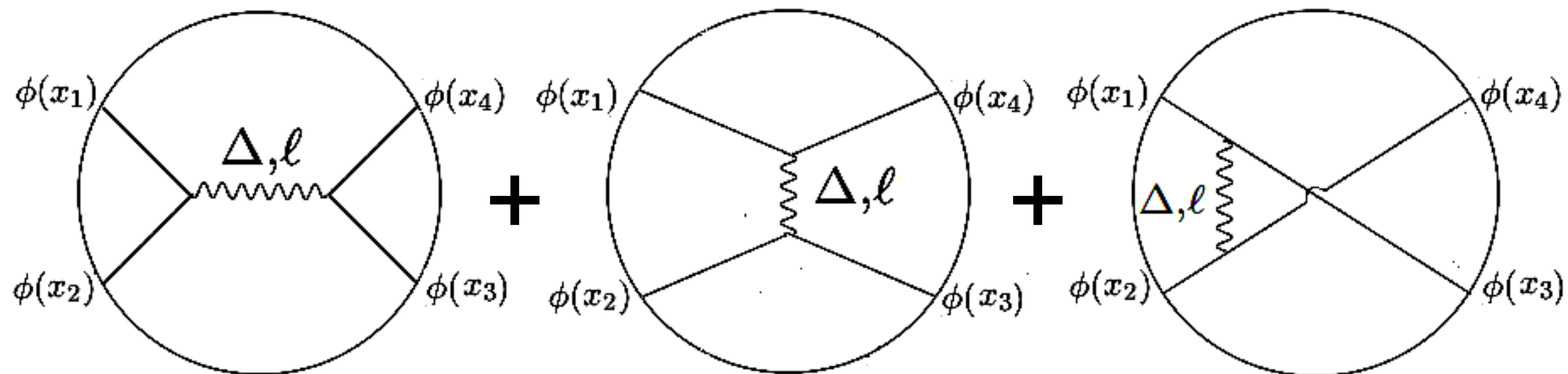


Mellin amplitudes should factorize on physical poles.
m: descendant label, ℓ: spin
Q: polynomials in t

$$\sum_m \frac{Q^{\Delta}_{\ell,m}(t)}{s - \frac{\Delta-\ell}{2} - m} + \boxed{\ldots\ldots}$$

Modern Mellin amplitude literature: Mack; Penedones; Paulos;……….

$$\mathcal{M}(s,t) = \sum_{\Delta,\ell} c_{\Delta,\ell}$$



Residue fixed by conformal invariance; Mack polynomials

Mellin amplitudes should factorize on physical poles.
m: descendant label, ℓ: spin
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$$\sum_m \frac{Q^{\Delta}_{\ell,m}(t)}{s - \frac{\Delta-\ell}{2} - m} + \boxed{\cdots\cdots}$$

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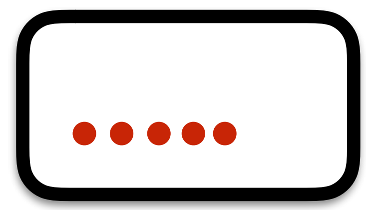
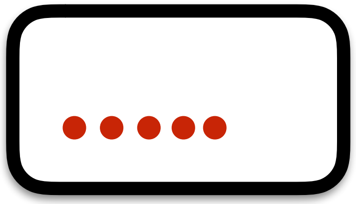
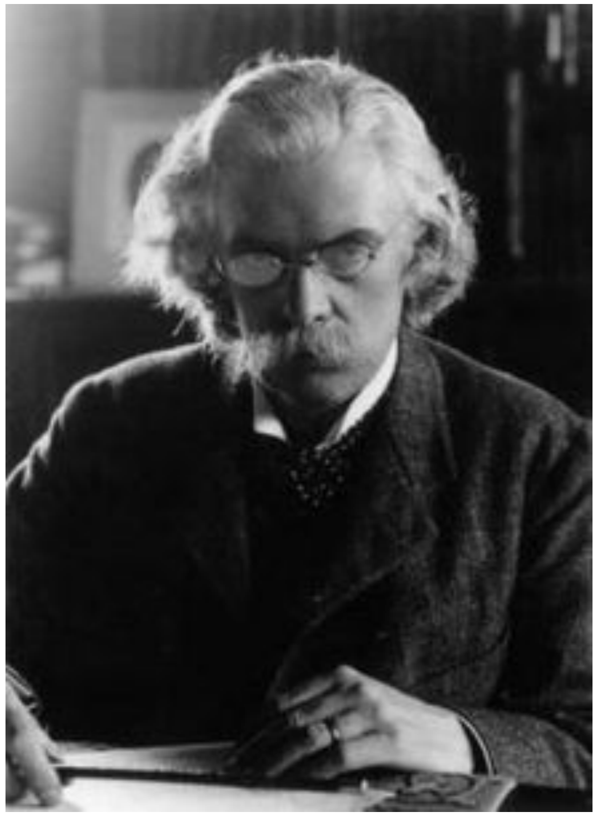




Difference between usual conformal block expansion and Witten diagram expansion lies in the …. pieces. The regular piece for usual conformal block is exponential at infinity while for the Witten block it is polynomial.

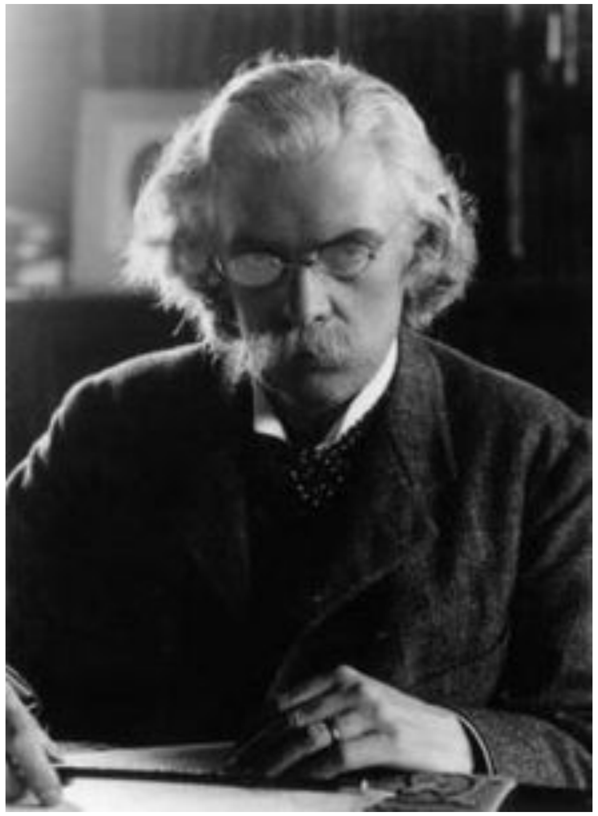
Strictly



Gosta Mittag-Leffler
(Swedish mathematician)

Thm: Existence of meromorphic functions with prescribed poles

Strictly



Mellin's
advisor!

Gosta Mittag-Leffler
(Swedish mathematician)

Thm: Existence of meromorphic functions with prescribed poles

Roughly

$$\Gamma(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(x+n)} \qquad + \Gamma_1(x)$$

Entire function

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(x+n)} \frac{\Gamma(y)}{\Gamma(y-n)}$$

no extra entire
function piece

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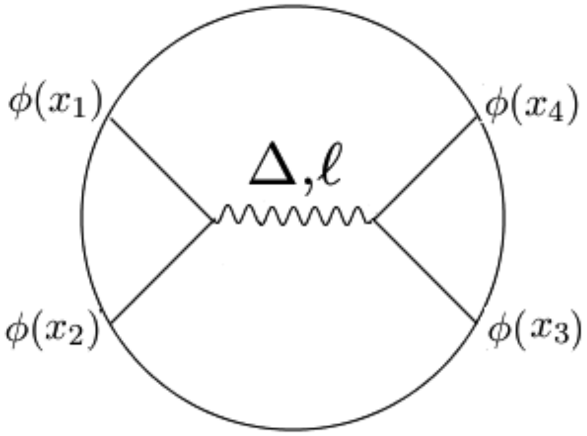
$$\Gamma(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(x+n)} \qquad + \Gamma_1(x)$$

Entire function

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(x+n)} \frac{\Gamma(y)}{\Gamma(y-n)}$$

famous formula responsible for the birth of string theory!

no extra entire function piece



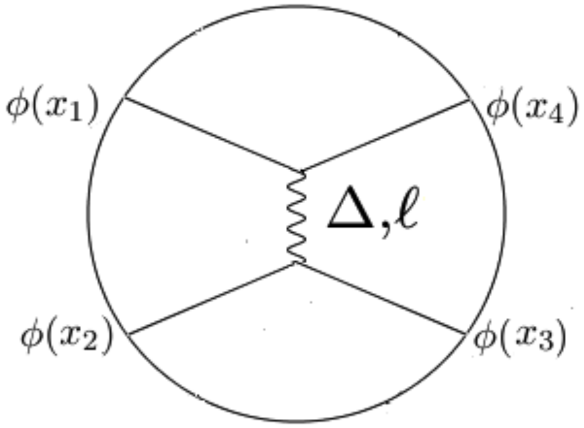
$$\sim \int \frac{ds dt}{(2\pi i)^2} \Gamma(-t)^2 \Gamma(s+t)^2 \Gamma(\Delta_\phi - s)^2 \frac{\sum_m Q^{\Delta}_{\ell,m}(t)}{s - \frac{\Delta-\ell}{2} - m} u^s v^t$$

$$\rightarrow u^{\Delta_\phi + r} \log u,\, u^{\Delta_\phi + r}$$

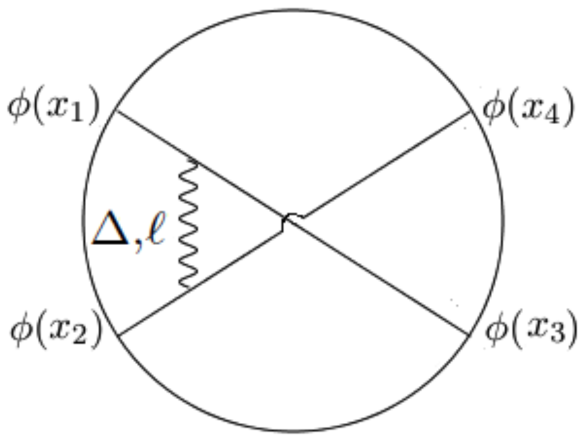
$$\rightarrow \text{incompatible with s} - \text{channel OPE}$$

$$\rightarrow \text{conditions needed to cancel}$$

NB: log u is genuinely spurious as it comes singly and not due to expanding some u-power



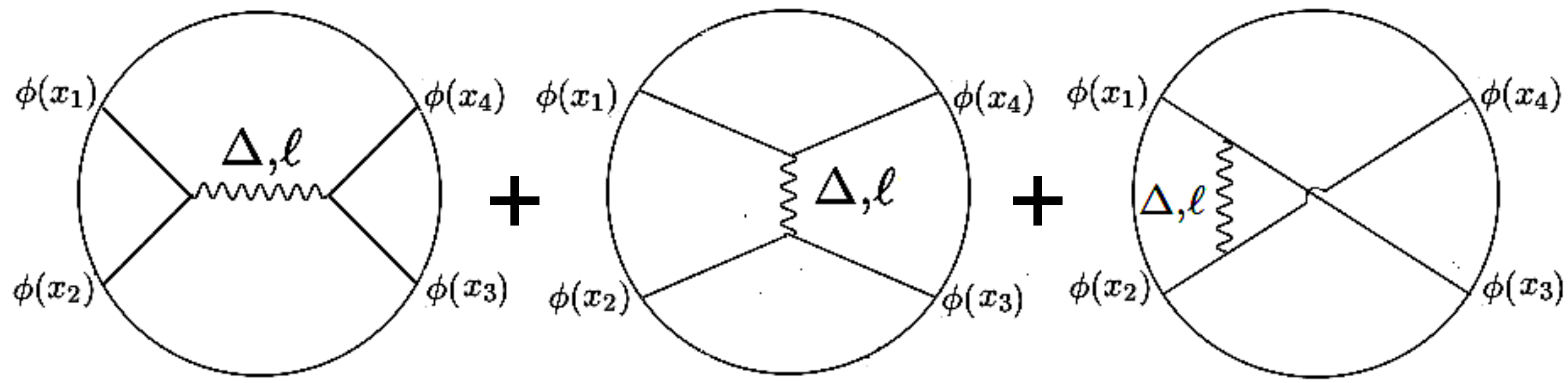
$$\sim \int \frac{ds dt}{(2\pi i)^2} \Gamma(-t)^2 \Gamma(s+t)^2 \Gamma(\Delta_\phi - s)^2 \frac{\sum_m Q^{\Delta}_{\ell,m}(s-\Delta_\phi)}{t+\Delta_\phi - \frac{\Delta-\ell}{2} - m} u^s v^t$$



$$\sim \int \frac{ds dt}{(2\pi i)^2} \Gamma(-t)^2 \Gamma(s+t)^2 \Gamma(\Delta_\phi - s)^2 \frac{\sum_m Q^{\Delta}_{\ell,m}(t)}{\Delta_\phi - s - t - \frac{\Delta-\ell}{2} - m} u^s v^t$$

$$\to u^{\Delta_\phi + r} \log u, u^{\Delta_\phi + r}$$

Demand that the spurious poles in s+t+u cancel to have compatibility with OPE

$$\mathcal{M}(s,t) = \sum_{\Delta,\ell} c_{\Delta,\ell}$$ 

should not have $(s-\Delta_\phi - r)^0, (s-\Delta_\phi - r)^1$ terms

$$\forall r \in \mathbb{Z}_+$$

This should hold for all t.

Lots of conditions!!

Nuts and bolts

Mack polynomials

Mack; Dolan-Osborn

$$P^{(s)}_{\nu,\ell}(s,t) = \widetilde{\sum} \frac{\Gamma^2(\lambda_1)\Gamma^2(\bar\lambda_1)(\lambda_2-s)_k(\bar\lambda_2-s)_k(s+t)_\beta(s+t)_\alpha(-t)_{m-\alpha}(-t)_{\ell-2k-m-\beta}}{\prod_i \Gamma(l_i)},$$

$$\text{where}\quad \widetilde{\sum} \equiv \frac{\ell!}{2^\ell(h-1)_\ell}\sum_{k=0}^{[\frac{\ell}{2}]}\sum_{m=0}^{\ell-2k}\sum_{\alpha=0}^{m}\sum_{\beta=0}^{\ell-2k-m}\frac{(-1)^{\ell-k-\alpha-\beta}\Gamma(\ell-k+h-1)}{\Gamma(h-1)k!(\ell-2k)!}\binom{\ell-2k}{m}\binom{m}{\alpha}$$
$$\times\binom{\ell-2k-m}{\beta}.$$

$$h = d/2$$

$$\lambda_1 = \frac{h+\nu+\ell}{2},\quad \bar\lambda_1 = \frac{h-\nu+\ell}{2},\quad \lambda_2 = \frac{h+\nu-\ell}{2}\quad\text{and}\quad \bar\lambda_2 = \frac{h-\nu-\ell}{2},$$

$$l_1 = \lambda_2+\ell-k-m+\alpha-\beta,\ l_2 = \lambda_2+k+m-\alpha+\beta,\ l_3 = \bar\lambda_2+k+m,\ l_4 = \bar\lambda_2+\ell-k-m.$$

$$P_{\Delta-h,\ell}(s=\frac{\Delta-\ell}{2}+m,t) \;=\; 4^{-\ell}(\Delta-1)_\ell(2h-\Delta-1)_\ell Q^{\Delta}_{\ell,m}(t)$$

Nuts and bolts

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$\times \binom{\ell-2k-m}{\beta}.$

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$$\lambda_1 = \frac{h+\nu+\ell}{2}, \quad \bar\lambda_1 = \frac{h-\nu+\ell}{2}, \quad \lambda_2 = \frac{h+\nu-\ell}{2} \quad \text{and} \quad \bar\lambda_2 = \frac{h-\nu-\ell}{2},$$

$$l_1 = \lambda_2 + \ell - k - m + \alpha - \beta\,,\ l_2 = \lambda_2 + k + m - \alpha + \beta\,,\ l_3 = \bar\lambda_2 + k + m\,,\ l_4 = \bar\lambda_2 + \ell - k - m\,.$$

$$P_{\Delta-h,\ell}\left(s = \frac{\Delta-\ell}{2} + m, t\right) = 4^{-\ell}(\Delta-1)_\ell(2h-\Delta-1)_\ell Q^{\Delta}_{\ell,m}(t)$$

(you are nuts)

(but I have not lost it....yet)

$$P_{\Delta-h,\ell}(s=\frac{\Delta-\ell}{2}+m,t) \;=\; 4^{-\ell}(\Delta-1)_\ell(2h-\Delta-1)_\ell Q^{\Delta}_{\ell,m}(t)$$

- One extra ingredient we will make use of.
- It turns out m=0 case ⇒ orthogonal polynomials in t called the continuous Hahn polynomials which play a role in integrability.
- We will expand the t dependence in terms of these polynomials.

Continuous Hahn polynomials

$$H_\ell^\tau(t) = 2^\ell \frac{(\tau)_\ell^2}{(2\tau+\ell-1)_\ell}\, {}_3F_2[-\ell, 2\tau+\ell-1, \tau+t; \tau, \tau; 1]$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

Play a key role in repackaging the equations. Without this we would need to take clever linear combinations to recover our results.

$$M^{(s)}(s,t) = \sum_{\ell'} q^{(s)}_{\ell'}(s) Q^{2s+\ell'}_{\ell',0}(t)$$

$$M^{(t)}(s,t) = M^{(u)}(s,t) = \sum_{\ell'} q^{(t)}_{\ell'}(s) Q^{2s+\ell'}_{\ell',0}(t)$$

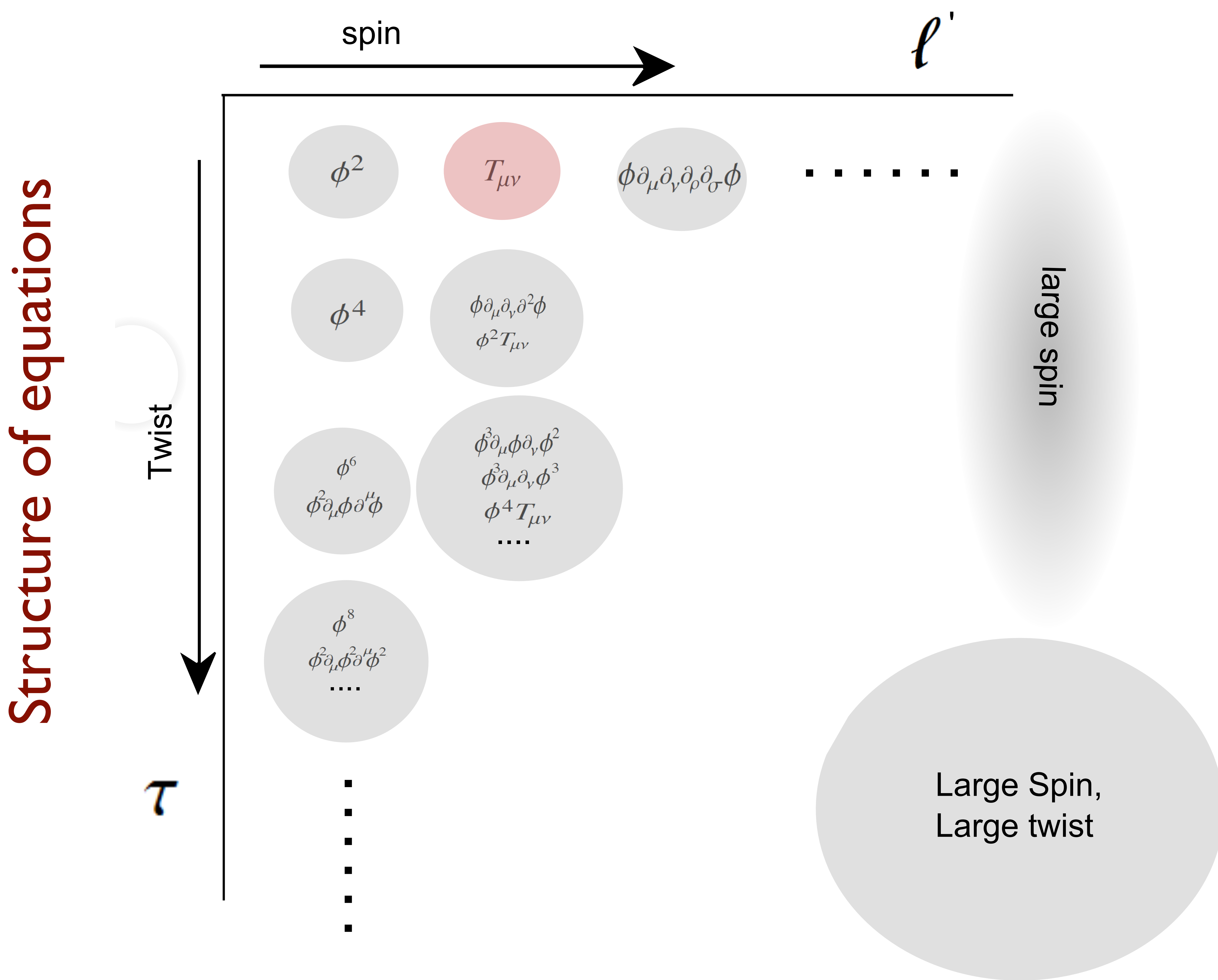
$$\left(q^{(s)}_{\ell'}(s) + 2q^{(t)}_{\ell'}(s)\right)|_{s=\Delta_\phi+r} = 0$$

$$\forall\ \ell', r$$

$$\partial_s \left(q^{(s)}_{\ell'}(s) + 2q^{(t)}_{\ell'}(s)\right)|_{s=\Delta_\phi+r} = 0$$

$$q^{(s,t,u)}_{\ell'}(s) \sim \sum_{\Delta,\ell}^{\infty} c_{\Delta,\ell} \text{ lots of Pochhammers} \times {}_4F_3[]{}_3F_2[]{}_3F_2[]$$

R.Gopakumar, AS; to appear



Structure of equations
spin
ℓ'
Twist
τ
ϕ^2
$T_{\mu\nu}$
$\phi\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi$
ϕ^4
$\phi\partial_\mu\partial_\nu\partial^2\phi$
$\phi^2 T_{\mu\nu}$
ϕ^6
$\phi^2\partial_\mu\phi\partial^\mu\phi$
$\phi^3\partial_\mu\phi\partial_\nu\phi^2$
$\phi^3\partial_\mu\partial_\nu\phi^3$
$\phi^4 T_{\mu\nu}$
ϕ^8
$\phi^2\partial_\mu\phi^2\partial^\mu\phi^2$
large spin
Large Spin,
Large twist

- Demanding consistency we recover a whole bunch of known results upto 3-loop order in the epsilon expansion quite easily. We also recover large spin results quite easily and more transparently.

- We also get new results for OPE which have never been computed using the Feynman diagram approach.

Strategy for solving equations

- Helps to have an expansion parameter
- Epsilon expansion/large N expansion
- Or focus on large spin exchange and use 1/spin as the expansion parameter
- Mild assumptions like being close to a free theory which places mild constraints on OPE
- Assume a stress tensor

- Typically a particular spin exchange dominates in the direct channel upto some order in the expansion parameter

- The crossed channel in the examples we consider has the first composite scalar that dominates upto some order

- Enough to fix the first few orders for anomalous dimensions of all higher spin operators quadratic in the fields

Sampling of (new) results

$$O_{\Delta,\ell} \sim \phi\partial_{\mu_1}\cdots\partial_{\mu_\ell}\phi$$

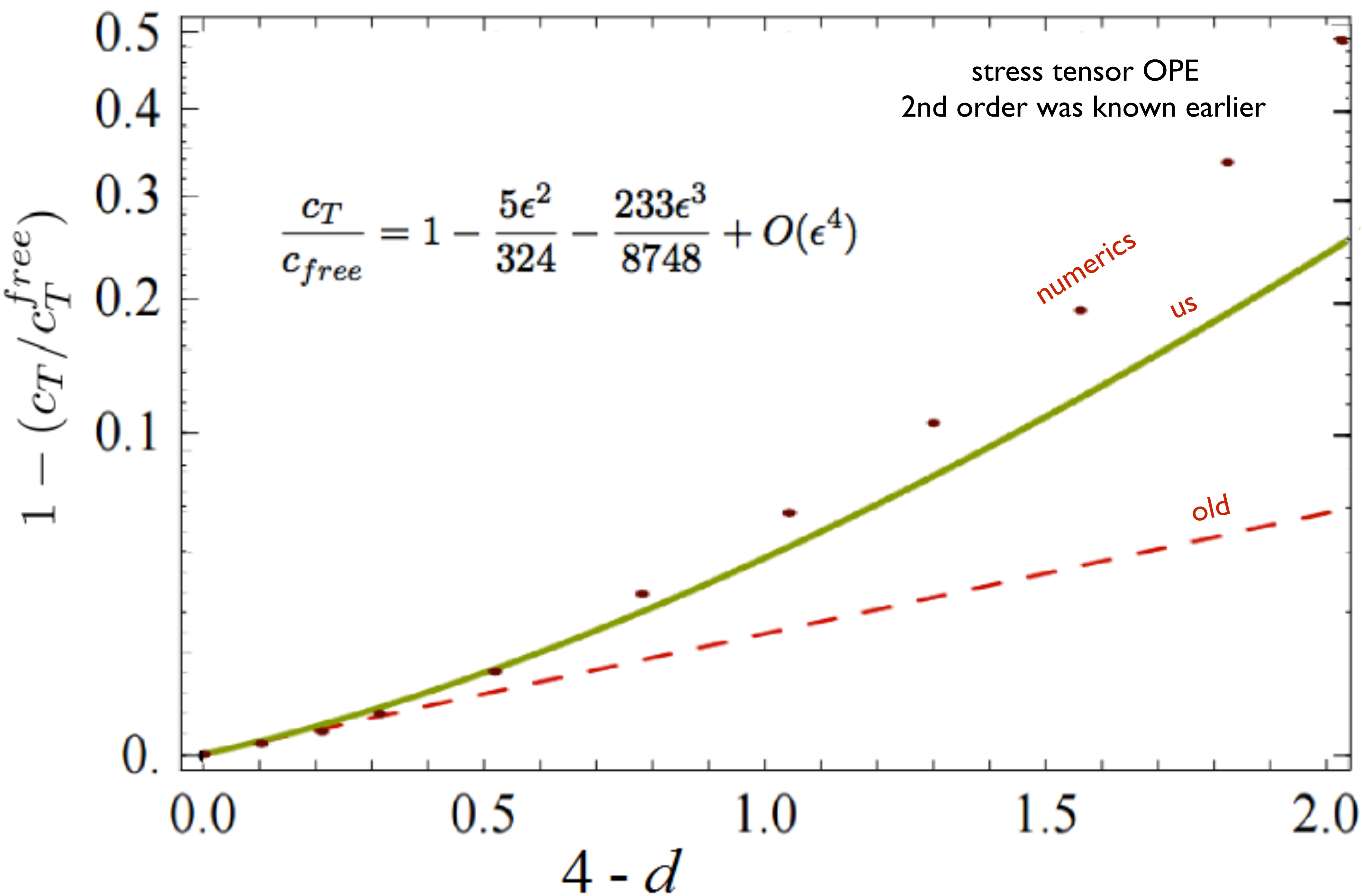
$$\Delta_\ell = d - 2 + \ell + \left(1 - \frac{6}{\ell(\ell+1)}\right)\frac{\epsilon^2}{54} + \frac{373\ell^2 - 384\ell - 324 + 109\ell^3(\ell+2) - 432\ell(\ell+1)H_\ell}{5832\ell^2(\ell+1)^2}\epsilon^3$$

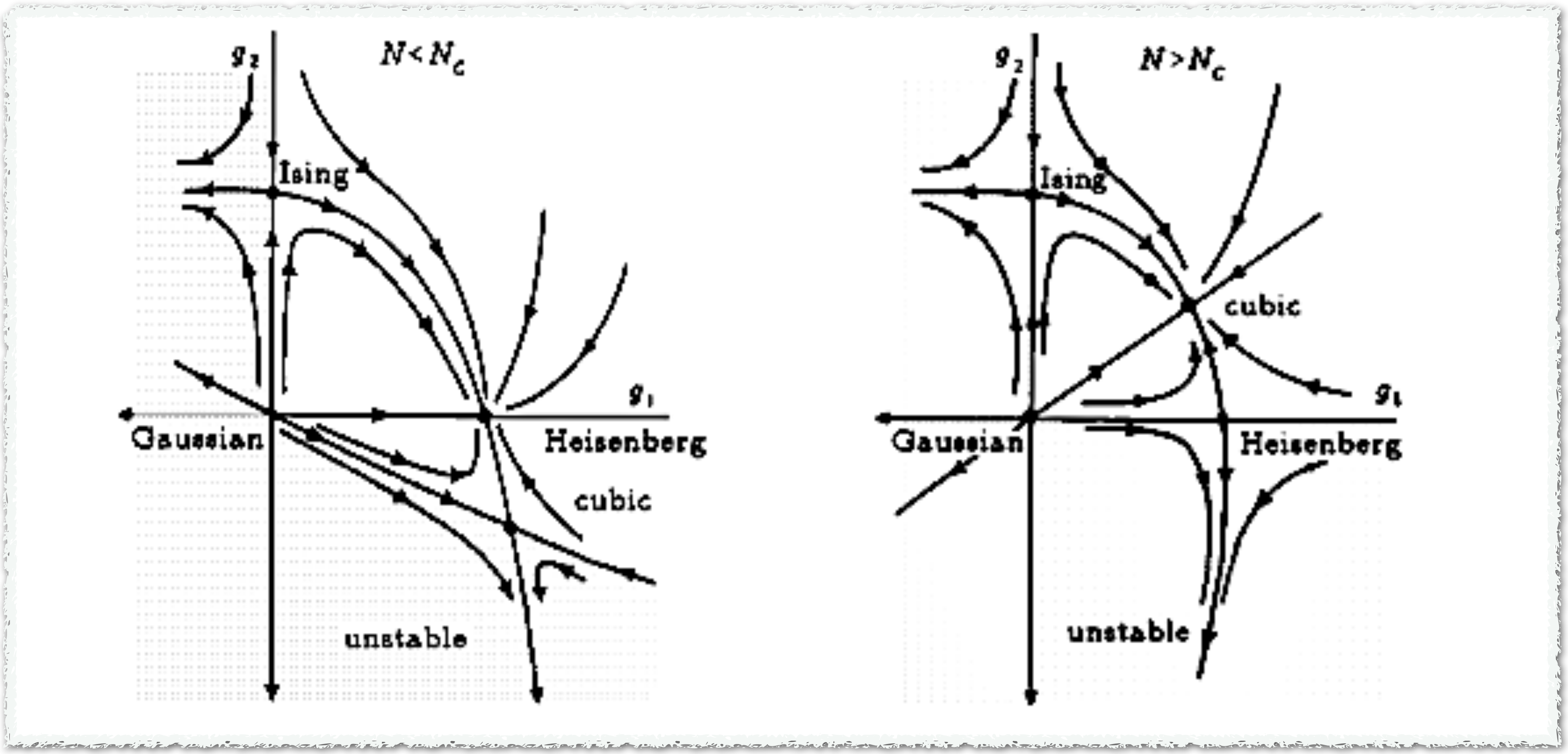
[agrees with Derkachov, Gracey, Manashov's 3-loop Feynman diagram calc!!]

$$\begin{aligned}\frac{C_\ell}{C_\ell^{MFT}} &= 1 + \frac{\epsilon^2\left(\ell(1+\ell)\left(H_{2\ell} - H_{\ell-1}\right) - 1\right)}{9\ell^2(1+\ell)^2} \\ &+ \frac{\epsilon^3}{486\ell^2(1+\ell)^3}\Big[27 + (59 - 22\ell)\ell - 36\ell(1+\ell)^2 H_\ell^2 + (1+\ell)H_\ell\left(22\ell^2 + 4\ell - 27 + 36\ell(1+\ell)H_{2\ell}\right) \\ &+ (1+\ell)\Big(\left(27 + 32\ell - 22\ell^2\right)H_{2\ell} + 18\ell(1+\ell)\big(3H_{2\ell}^{(2)} - 2H_\ell^{(2)}\big)\Big)\Big] \; + \; O(\epsilon^4)\,.\end{aligned}$$

[new]

Sampling of new results





Global symmetry: N scalars

1612.05032; P.Dey, A.Kaviraj, AS

Sampling of new results

Conductivity superfluid-insulator quantum critical point O(2)

$$\frac{\sigma(\infty)}{\sigma_Q} = 0.36$$

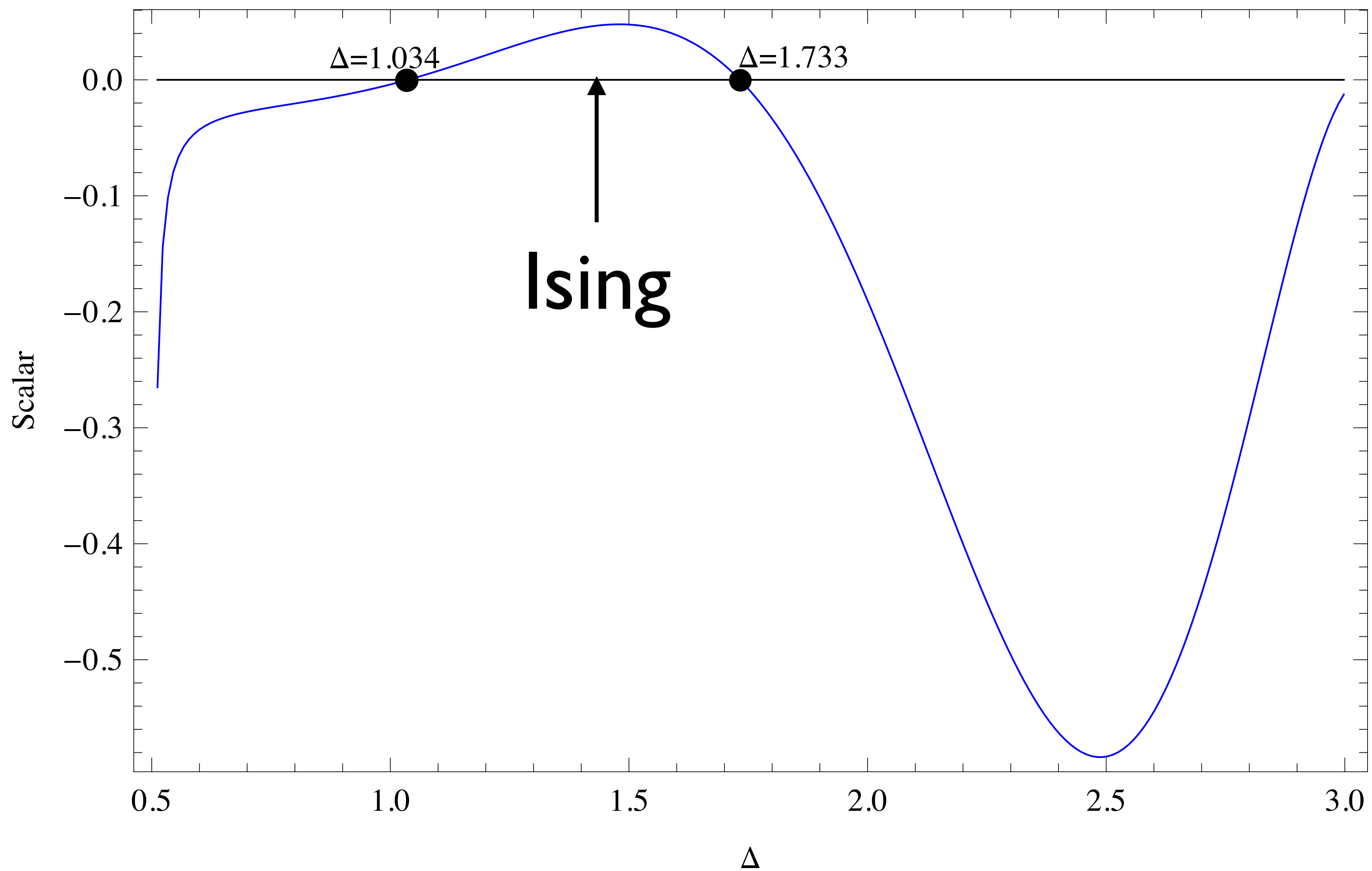
Witzcak-Krempa, Sorensen, Sachdev—Nature Physics 2013

$$\frac{\sigma(\infty)}{\sigma_Q} = \frac{\pi}{8}(1 - \frac{3}{100}\epsilon^2 - \frac{9}{200}\epsilon^3) = 0.363$$

P. Dey, A. Kaviraj, A.S.—1612.05032

FUTURE

Potentially powerful numerics



Loose end to fix: Justify truncation

Epsilon expansion questions

- Can we understand why the first few terms give good agreement?
- Is it guaranteed that we will get the Feynman diagram results order by order?
- I think the answer is no. In the epsilon expansion, there are new operators called evanescent operators in fractional dimensions and affect Feynman diagrams.
- These spoil unitarity. [Hogervorst, Rychkov, van Rees]

- In the bootstrap formalism, in principle we can demand unitarity order by order.
- There are two possibilities: We get a consistent expansion which will not agree with the diagrammatic approach from some order; Or we see evidence of these evanescent operators. [RG, AS, in progress]

- Various generalizations are in progress eg N=4 SYM, external operators etc.

- Hints at a different way to systematize perturbative expansions rather than Feynman diagrams (at least for critical phenomena)................

How did Polyakov know about Witten diagrams in 1974?

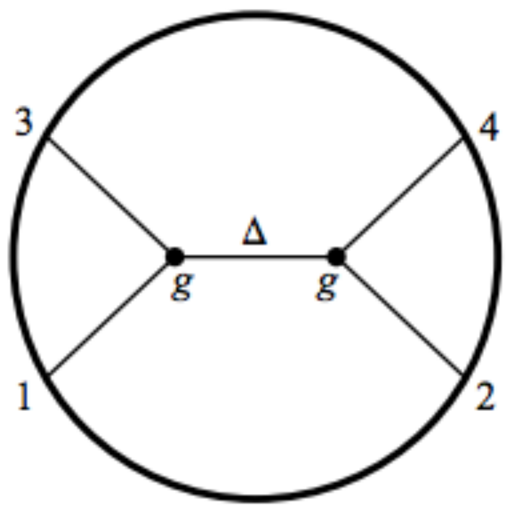
Non-Hamiltonian approach to conformal quantum field theory

A. M. Polyakov

L. D. Landau Theoretical Physics Institute, USSR Academy of Sciences
(Submitted July 9, 1973)
Zh. Eksp. Teor. Fiz. 66, 23–42 (January 1974)

The completeness requirement for the set of operators appearing in field theory at short distances is formulated, and replaces the S-matrix unitarity condition in the usual theory. Explicit expressions are obtained for the contribution of an intermediate state with given symmetry in the Wightman function. Together with the "locality" condition, the completeness condition leads to a system of algebraic equations for the anomalous dimensions and coupling constants; these equations can be regarded as sum rules for these quantities. The approximate solutions found for these equations in a space of $4-\epsilon$ dimensions give results equivalent to those of the Hamiltonian approach.

Reverse argument



$$= \frac{1}{2s-\Delta}\frac{\Gamma^2(\Delta_\phi+\frac{\Delta-2h}{2})}{\Gamma(1+\Delta-h)}\,{}_3F_2\left[\begin{matrix}1-\Delta_\phi+\frac{\Delta}{2}, 1-\Delta_\phi+\frac{\Delta}{2}, \frac{\Delta}{2}-s\\ 1+\frac{\Delta}{2}-s, 1+\Delta-h\end{matrix};1\right]$$

$$= \frac{1}{4\pi i\Gamma(\Delta_\phi-s)^2}\int_{-i\infty}^{i\infty} d\nu q[\nu]q[-\nu]$$

$$q[\nu] = \frac{\Gamma(\frac{h+\nu}{2}-s)\Gamma^2(\frac{2\Delta_\phi-h+\nu}{2})}{(\Delta-h)+\nu}$$

$$\mathcal{M}(\nu) = q[\nu]q[-\nu]$$

Spectral function. Polyakov gave a different physical argument for the double poles. Exactly the same form!! Momentum/position space are not ideal to see the simplification we saw in Mellin space.

Thank you