Interfacing analytical- and numerical relativity for gravitational-wave astronomy

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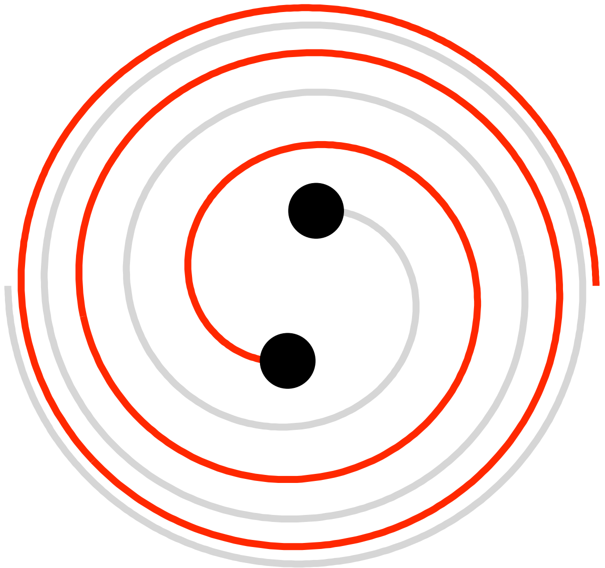
ICTS Seminar 12 June 2012

Overview

- Compact binary coalescences (CBCs): The most promising sources of GWs.
- Modelling different stages of the evolution of CBCs: Quick overview
- Putting them together: Description of the full coalescence process.
- Implications in detection, parameter estimation and tests of GR.

Coalescing compact binaries



Coalescing binaries of compact objects (black holes or neutron stars) are among the most promising sources for the first detection of gravitational waves .

Number of astrophysical channels exist to form such systems.
Once formed, they "inspiral" and coalesce by GW emission.

Coalescing compact binaries: The most promising GW sources

- **Highly "efficient" sources** $\sim 1-15\%$ of the rest mass is radiated as GWs. Can be observed up to large distances.

$$\mathcal{L}_{\mathrm{GW}} \sim \frac{0.1\,Mc^2}{1000\,GM/c^3} \sim 10^{22}\,L_\odot$$

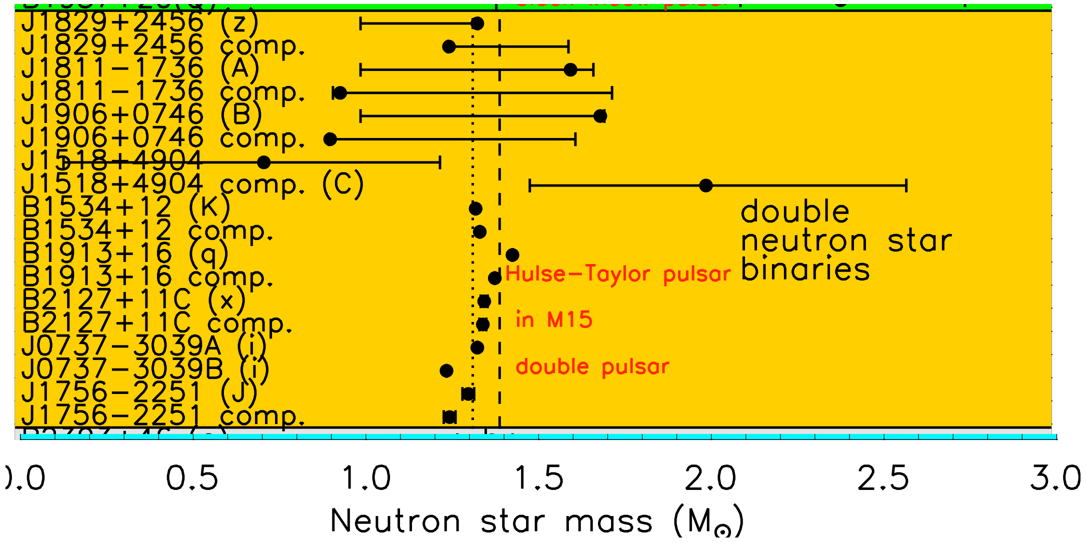
most of the energy is radiated
over the late-inspiral &
merger (time scale ~ 1000 M)

larger than the total
luminosity of the
observable EM universe!

Coalescing compact binaries: The most promising GW sources

- **Highly "efficient" sources** ~1—15% of the rest mass is radiated as GWs. Can be observed up to large distances.

- **Strong observational evidence** A dozen galactic binary neutron stars have been observed. Binary quasars are expected to be supermassive black hole binaries.



[Lattimer & Prakash (2010)]

Coalescing compact binaries: The most promising GW sources

- **Highly "efficient" sources** ~1–15% of the rest mass is radiated as GWs. Can be observed up to large distances.

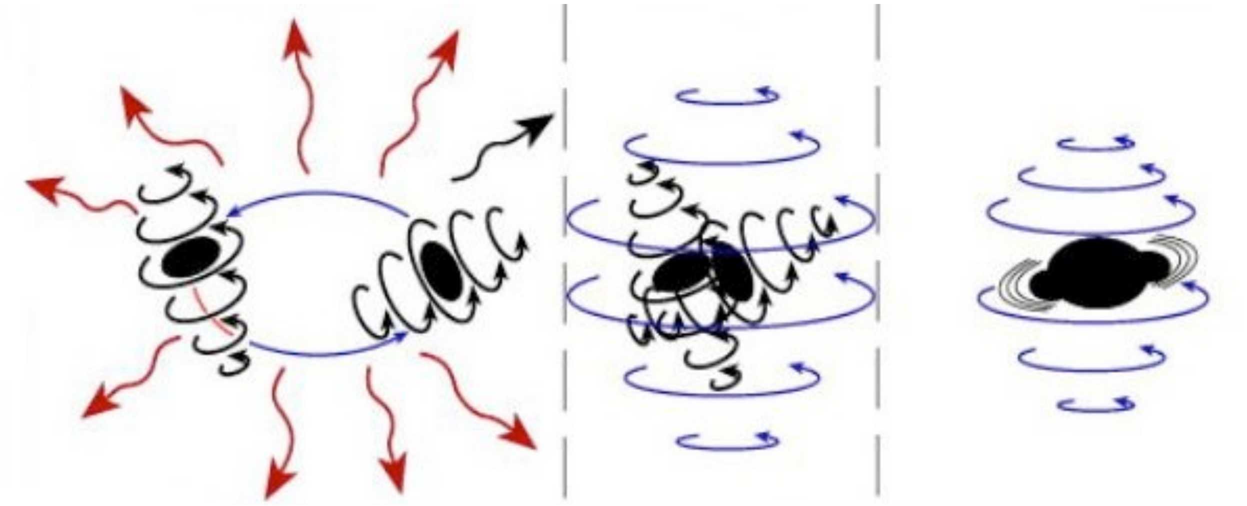
- **Strong observational evidence** A dozen galactic binary neutron stars have been observed. Binary quasars are expected to be supermassive black hole binaries.

- **"Clean systems"** Expected signals can be accurately modelled using GR. Enable us to use the most sensitive search techniques.
  - Matched filter: cross correlate the data with theoretical templates of the expected signals.

Important to have accurate models of the expected signals!

Coalescence of binary black holes

Inspiral Merger Ring down

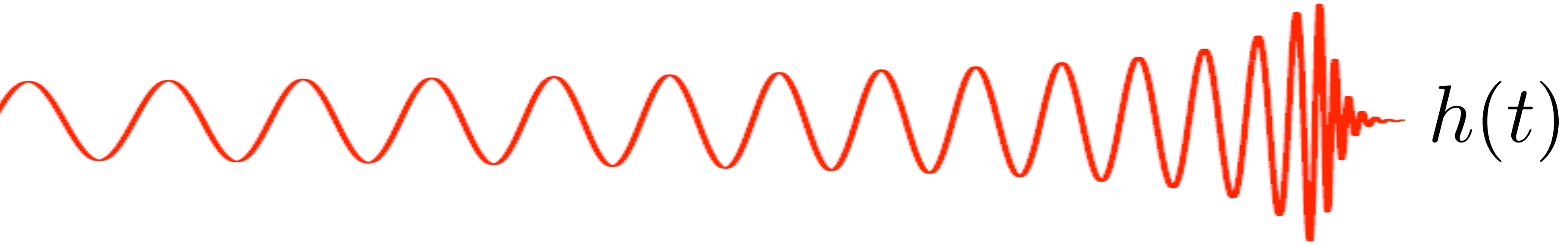


Post-Newtonian theory

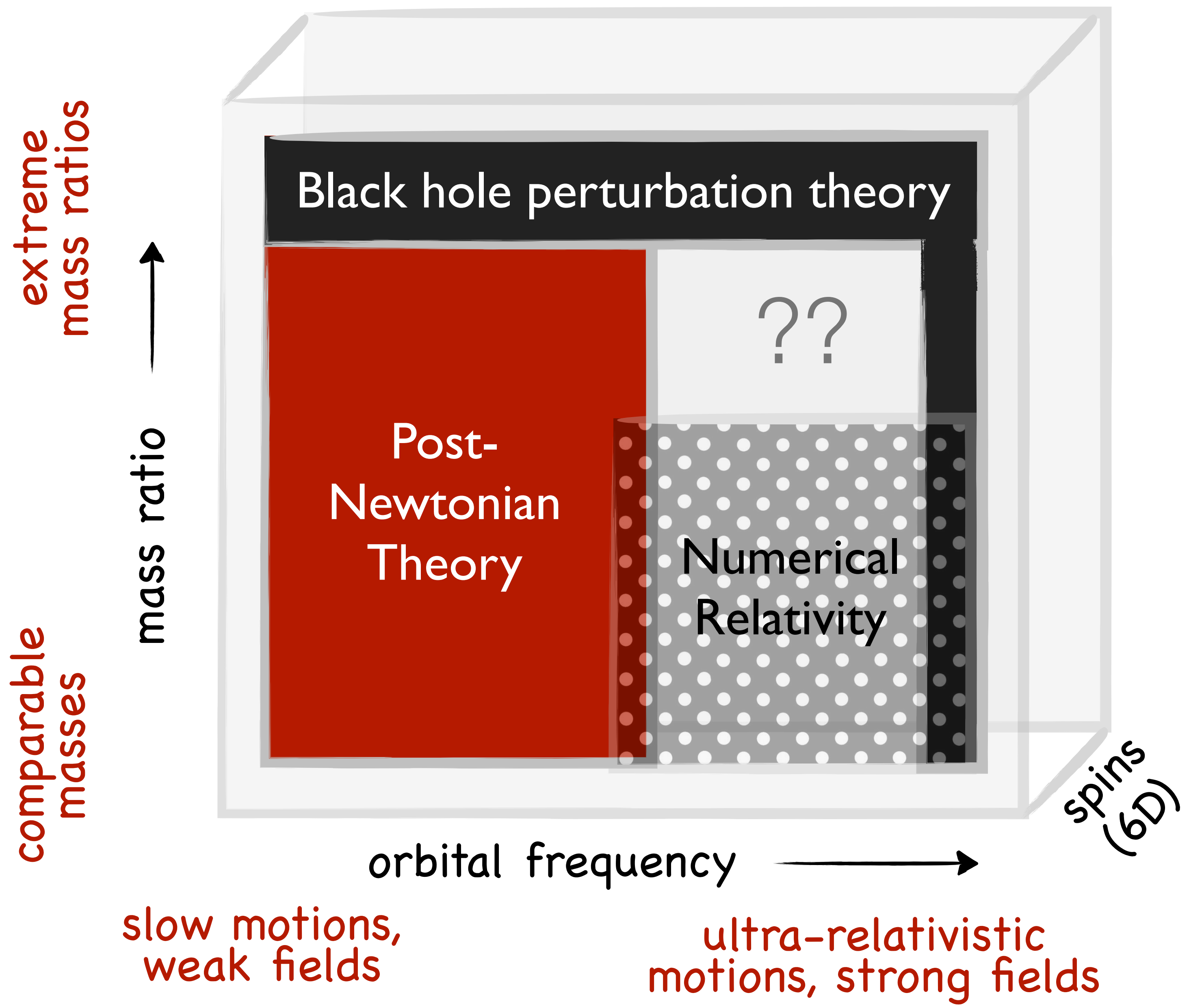
Numerical Relativity

BH perturbation theory

(Pic. K. Thorne)



Binary-black-hole parameter space

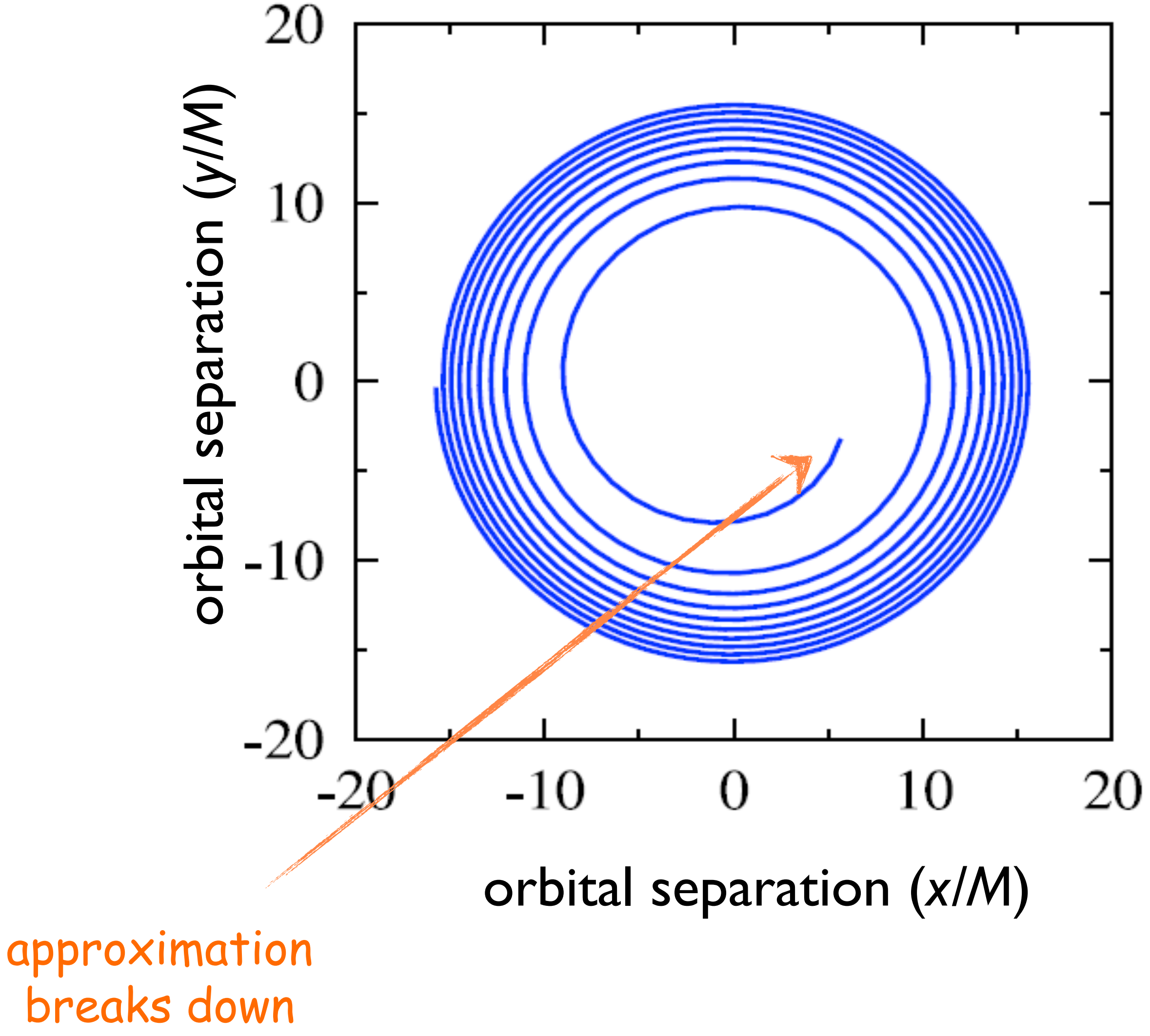


Modeling of compact binaries

- **Post-Newtonian theory** Model the dynamics and waveforms using perturbative expansions in v/c. Suitable for binaries at large separations. Breaks down at close separations.
  - Currently known up to $(v/c)^7$ order. Spin effects up to $(v/c)^6$.

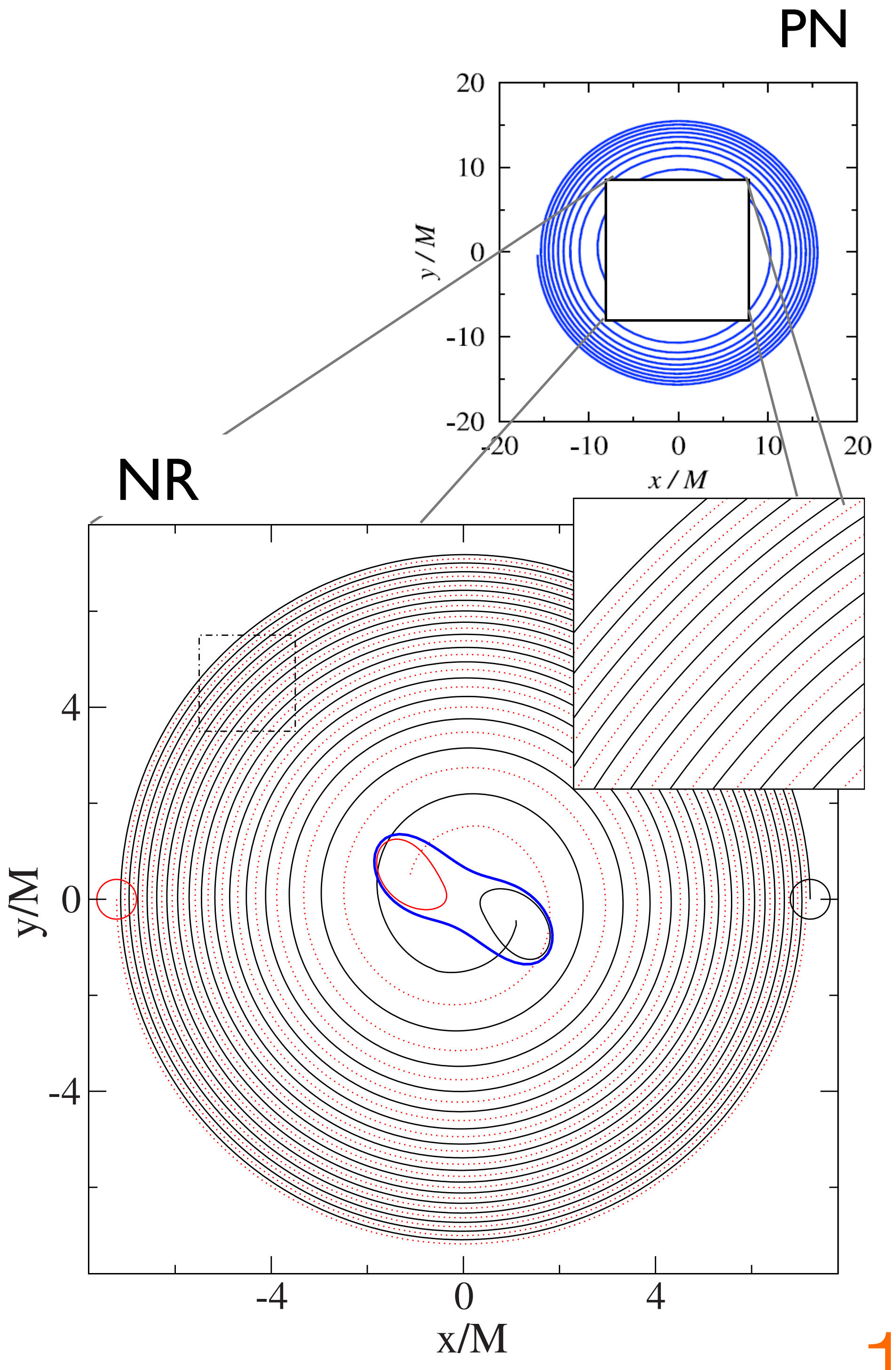
[Major Indian contribution (Iyer's group)]

[Pic. BCV (2002)]



Modeling of compact binaries

- **Post-Newtonian theory** Model the dynamics and waveforms using perturbative expansions in v/c. Suitable for binaries at large separations. Breaks down at close separations.
- **Numerical relativity** Solve Einstein's equations exactly using numerical methods.
  - Great breakthroughs in recent years. "Binary black-hole problem" is solved. The field is going through an "explosive" stage.
  - Can simulate binaries at arbitrary separations. Computational cost prohibitive at large separations.



[Scheel et al (2009)]

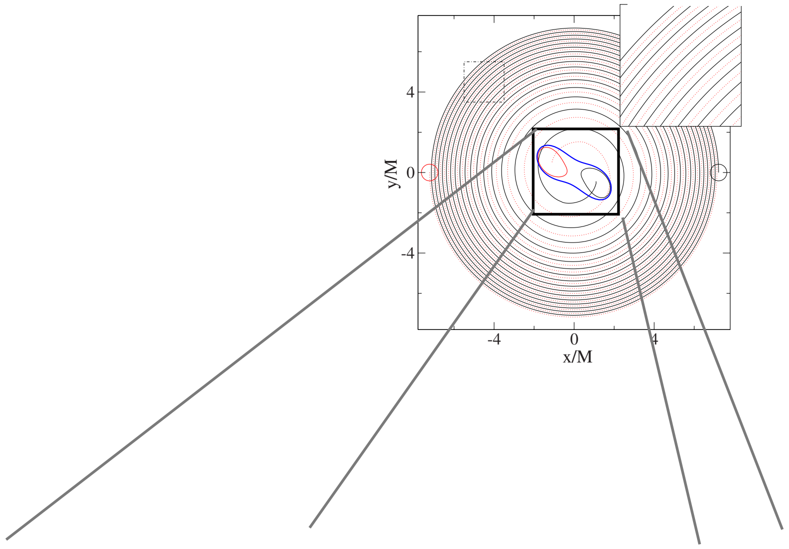
Modeling of compact binaries

- **Post-Newtonian theory** Model the dynamics and waveforms using perturbative expansions in v/c. Suitable for binaries at large separations. Breaks down at close separations.

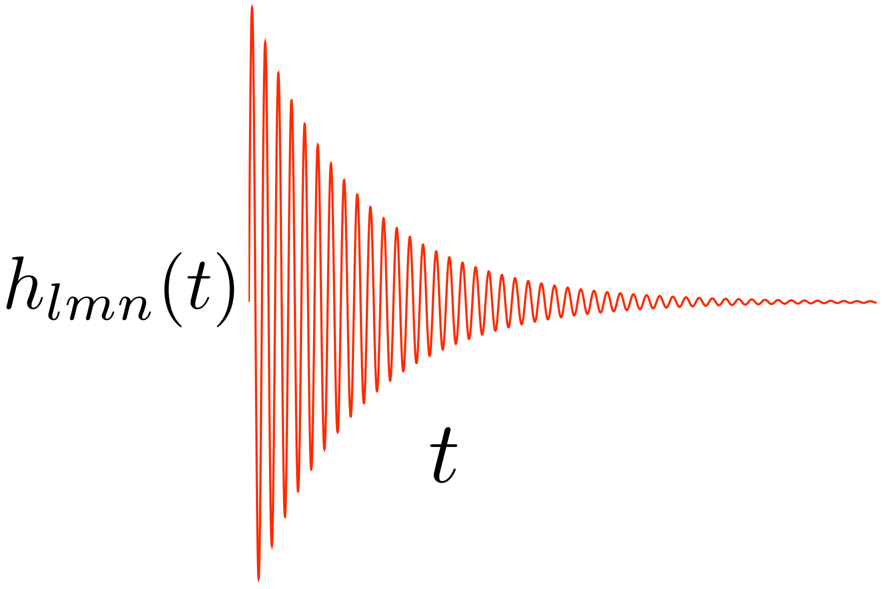
- **Numerical relativity** Solve Einstein's equations exactly using numerical methods.

- **Black hole perturbation theory** Perturbations to black holes die away as exponentially damped sinusoids (quasi-normal modes, QNMs).
  - Frequencies & damping times of QNMs uniquely determined by BH mass & spin.
  - Extreme-mass ratio inspirals known to $(v/c)^{28}$.

NR



BH perturbation



This seminar

Coherent description of the inspiral, merger and ring-down of binary black holes

- Produce analytical waveform templates describing the inspiral, merger and ringdown by combining analytical- and numerical relativity.

P. Ajith et al, Class Quantum Grav **24** S689 (2007).
P. Ajith et al, Phys. Rev. D **77** 104017 (2008).
P. Ajith, Class. Quantum Grav. **27** 114033 (2008).
P. Ajith et al, Phys. Rev. Lett. **106** 241101 (2011).
P. Ajith, Phys. Rev. D **84** 084037 (2011).

Why analytical templates?

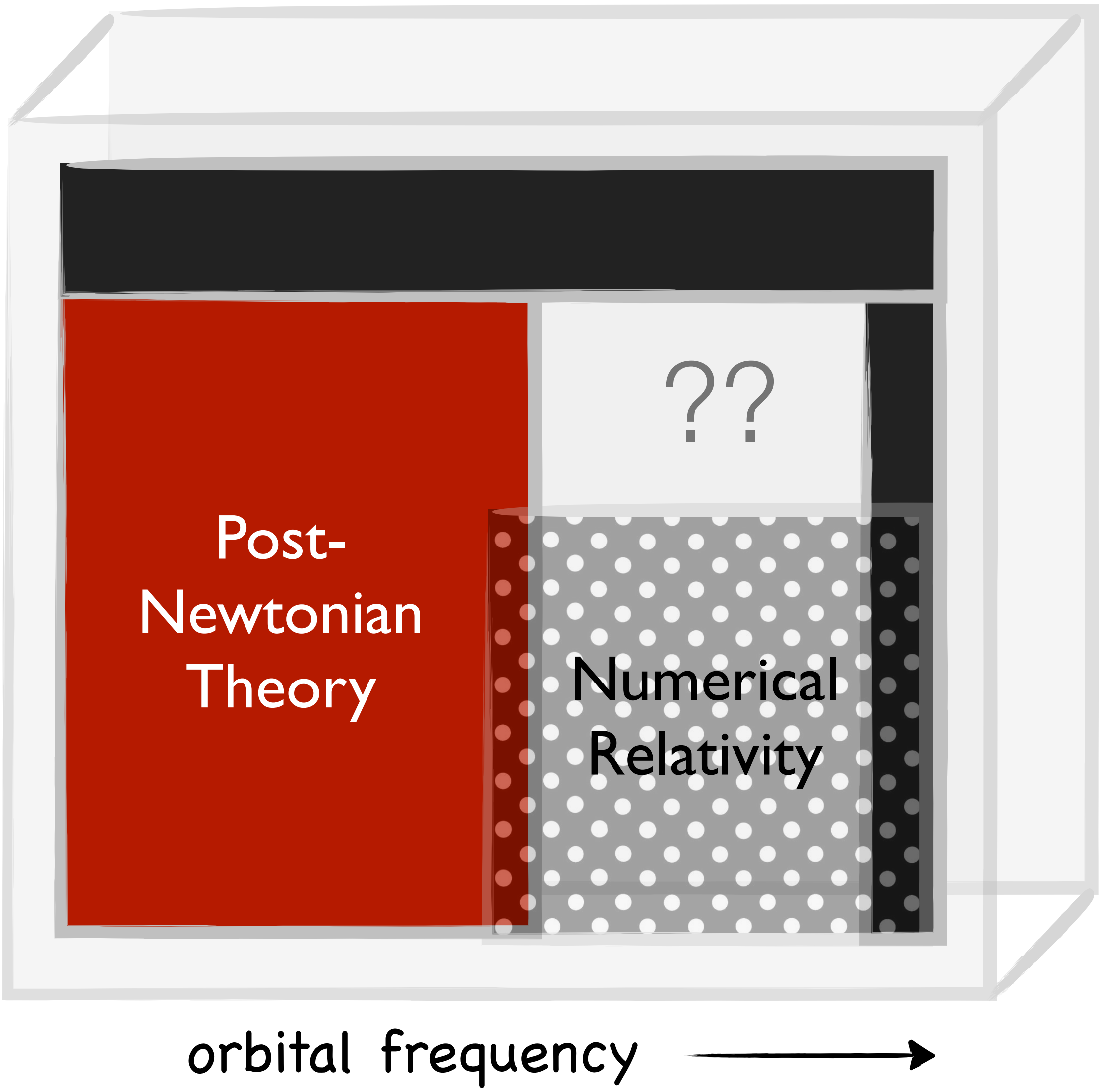
Practically impossible to perform NR simulations over all relevant regions in the parameter space

Computational cost scales steeply with orbital separation, mass ratio and spin

Currently available simulations are with moderate orbital separations, mass ratios and spins

Solution: combine NR with perturbative calculations to produce analytical waveforms describing the full coalescence.

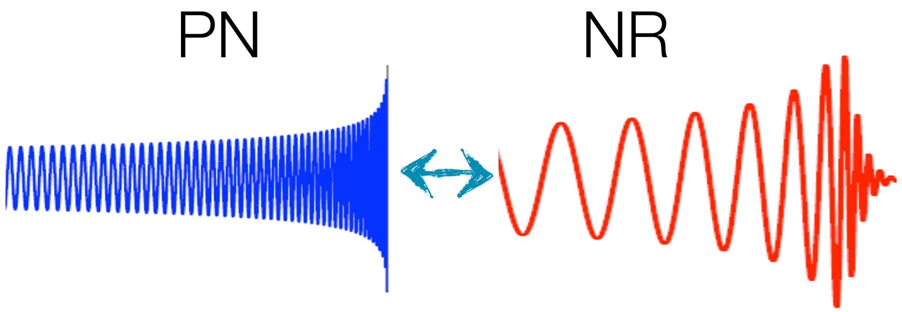
Matching post-Newtonian and numerical relativity



slow motions, weak fields ⟷ ultra-relativistic motions, strong fields

Matching post-Newtonian and numerical relativity

- Construct "hybrid PN-NR waveforms" by matching PN and NR in a region where both calculations are expected to be valid.

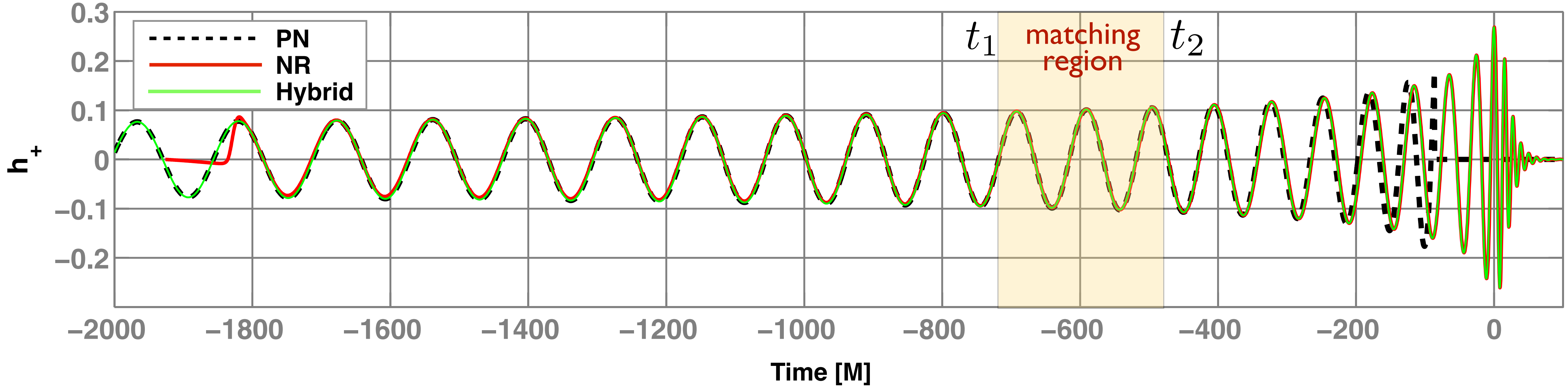


Matching post-Newtonian and numerical relativity

- Construct "hybrid PN-NR waveforms" by matching PN and NR in a region where both calculations are expected to be valid.

excellent agreement
between PN and NR!

[P. Ajith et al (2008)]



Least square fitting: $\min_{\Delta t_0, \Delta\varphi_0} \int_{t_1}^{t_2} \left| \mathsf{h}(t)^{\mathrm{PN}} - \mathsf{h}(t)^{\mathrm{NR}} \right| dt,$ $\mathsf{h} = h_+ - i h_\times$

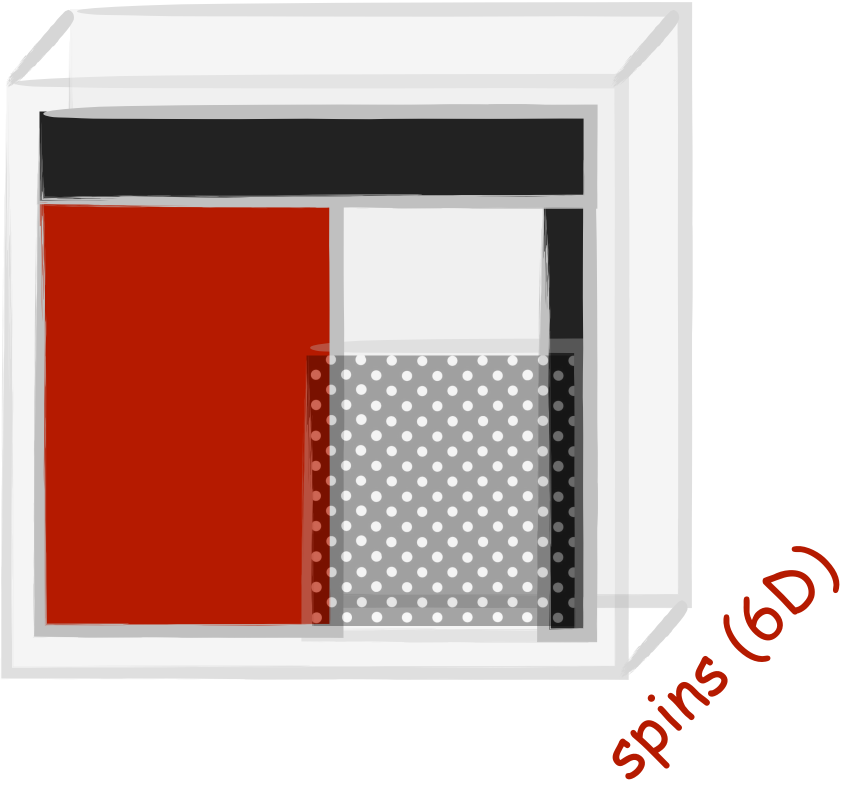
GW polarizations

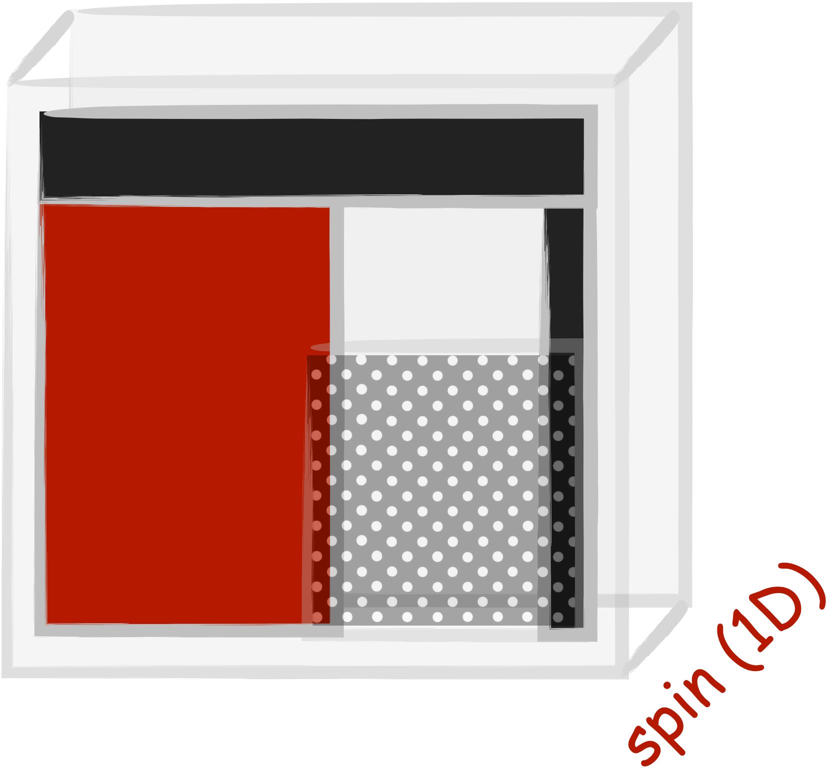
Matching post-Newtonian and numerical relativity

- Construct "hybrid PN-NR waveforms" by matching PN and NR in a region where both calculations are expected to be valid.
- Such "hybrid" waveforms can be constructed for each mass ratio and spin, where NR simulations are available. [Ajith et al (2007), Ajith et al (2008), Ajith (2008)]

fully rescalable with the total mass of the binary

Reducing the spin parameter space





Spin precession in compact binaries

- If spins are misaligned, spin-orbit, spin-spin interactions (frame dragging by the spins) will cause precession of the spins & the orbital plane.

Time scales

$$t_{\text{orbit}} \ll t_{\text{precession}} \ll t_{\text{inspiral}}$$

Precession of spins

$$\frac{d\mathbf{S}_i}{dt} = \mathbf{\Omega}_i \times \mathbf{S}_i\,, \quad i = 1, 2,$$

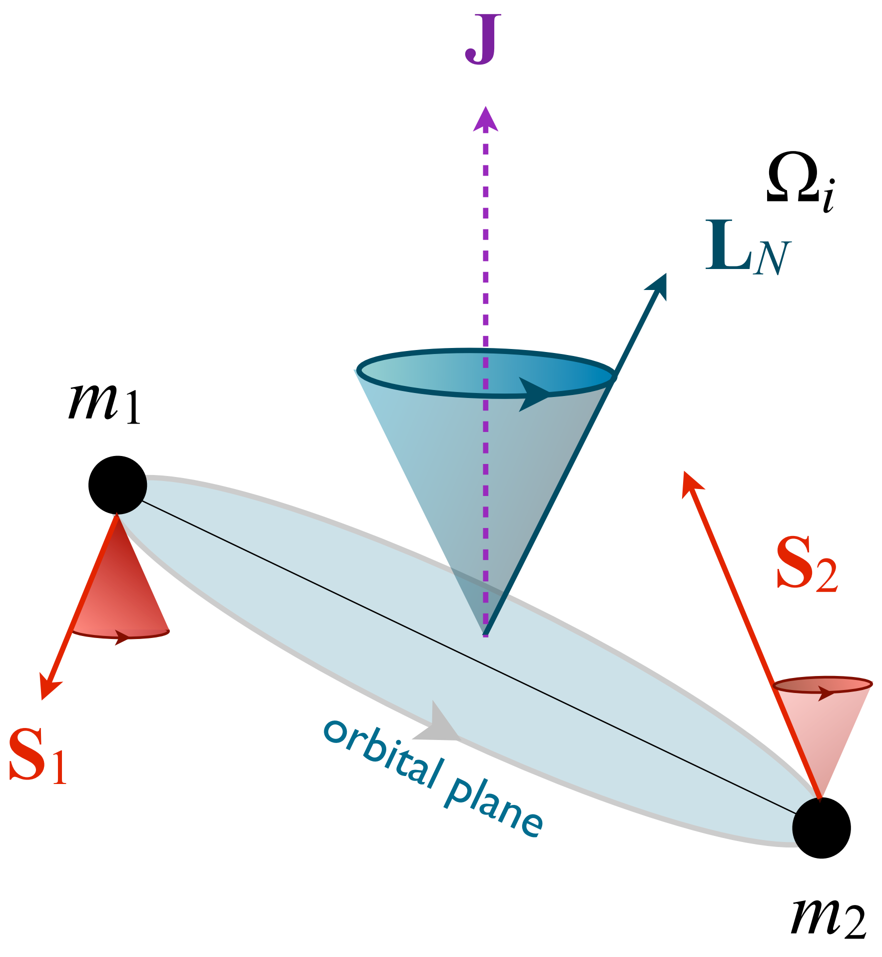
precession frequency

spin vector

Precession of the orbital plane

$$\frac{d\mathbf{L}}{dt} = -\left(\frac{d\mathbf{S}_1}{dt} + \frac{d\mathbf{S}_2}{dt}\right)$$

orbital ang. momentum



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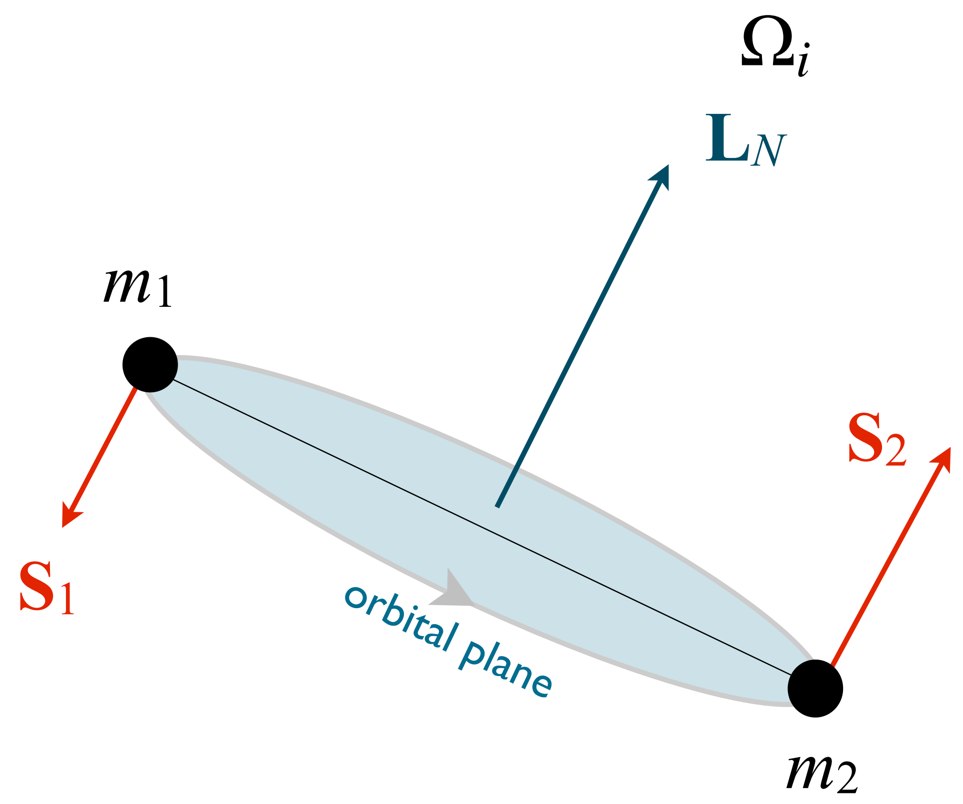
Precession of spins

$$\frac{d\mathbf{S}_i}{dt} = \boldsymbol{\Omega}_i \times \mathbf{S}_i\,, \quad i = 1, 2,$$

if the spins are (anti) aligned with L, no precession



Precession of the orbital plane

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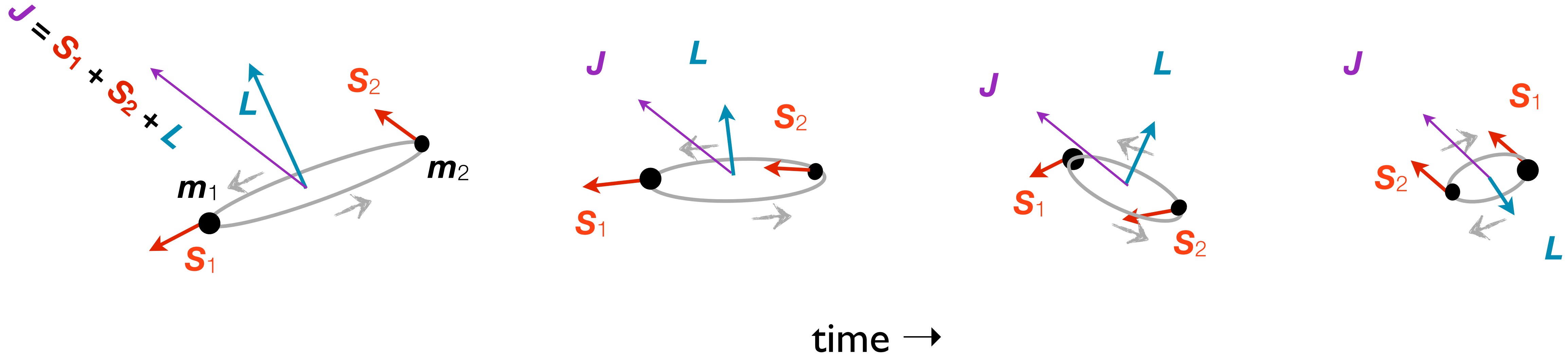
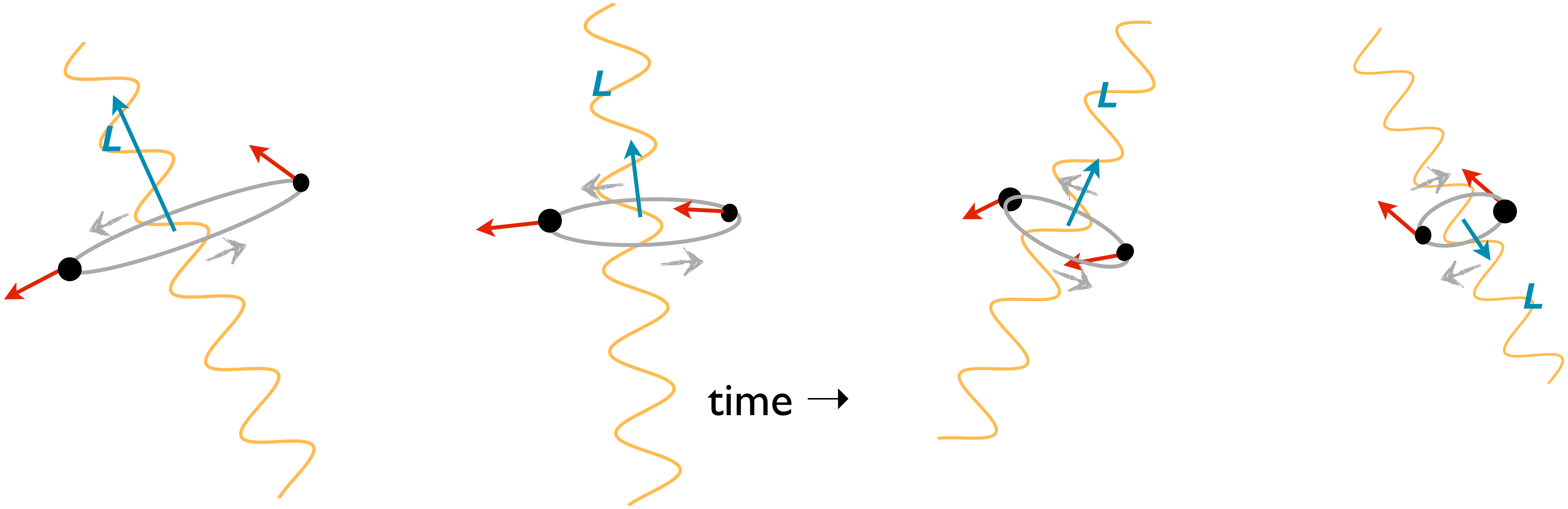


Spin precession in compact binaries

- Waveform observed by a fixed detector will contain amplitude & phase modulations.



Spin precession in compact binaries

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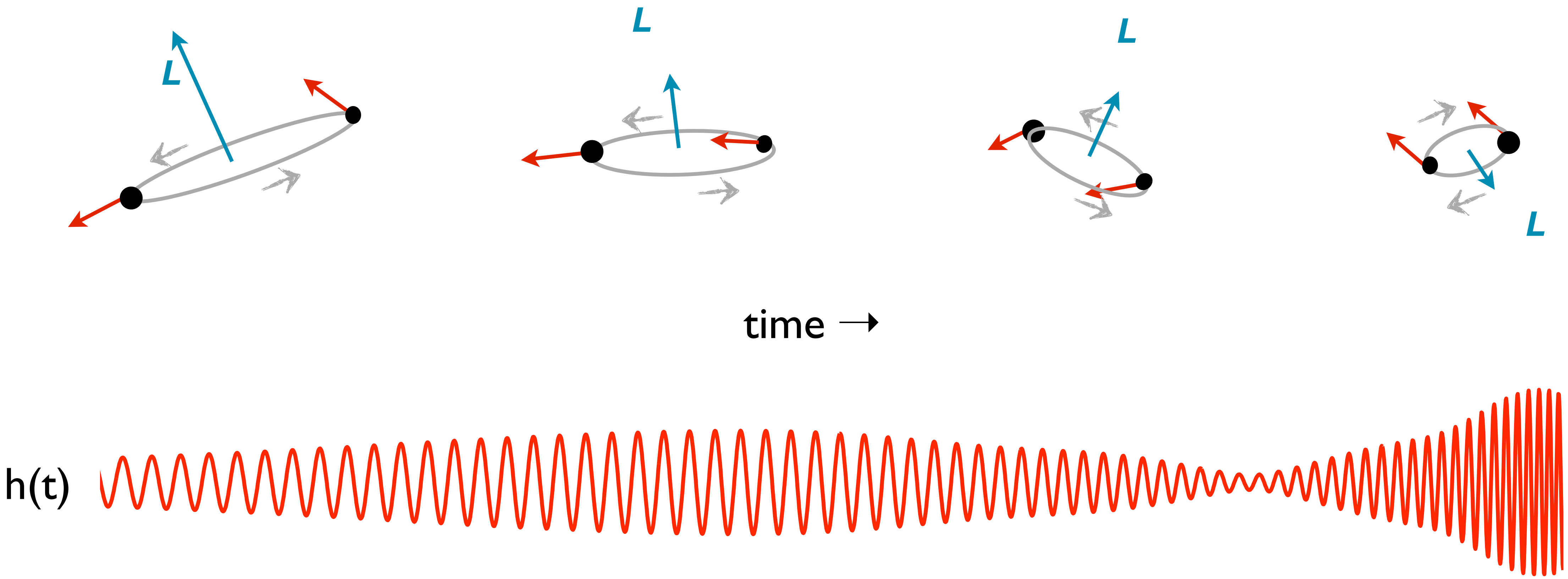


Spin precession in compact binaries

Full spin-parameter space of NR simulations is 6-dimensional

$$\{\mathbf{S}_1 , \mathbf{S}_2\}$$

But, currently possible NR simulations are restricted to a small region in the parameter space

$$\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 \simeq \pm 1 \, , \quad \hat{\mathbf{S}}_i \cdot \hat{\mathbf{L}} \simeq \pm 1, \quad ||\mathbf{S}_i||/m_i^2 \lesssim 0.75$$

(Large spins require larger resolution in the numerical grid,
precessing binaries have less symmetries ...)

Possible way out: Several different spin configurations are nearly "degenerate" (very similar dynamics and waveforms). Make use of this to reduce the dimensionality

Reducing dimensionality: Take motivation from PN theory

$$\varphi(v) = -\frac{1}{32 v^5 \eta}\left\{1 + v^2\,[1\mathrm{PN}] + v^3\,[1.5\mathrm{PN} + \mathrm{SO}] + v^4\,[2\mathrm{PN} + \mathrm{SS}] + \ldots\right\}$$

orbital
phase

PN expansion parameter

$$v = (m\,\omega_{\mathrm{orb}})^{1/3} \simeq \sqrt{m/R}$$

total mass

$(G = c = 1)$

binary
separation

Motivation: post-Newtonian waveforms for spinning binaries

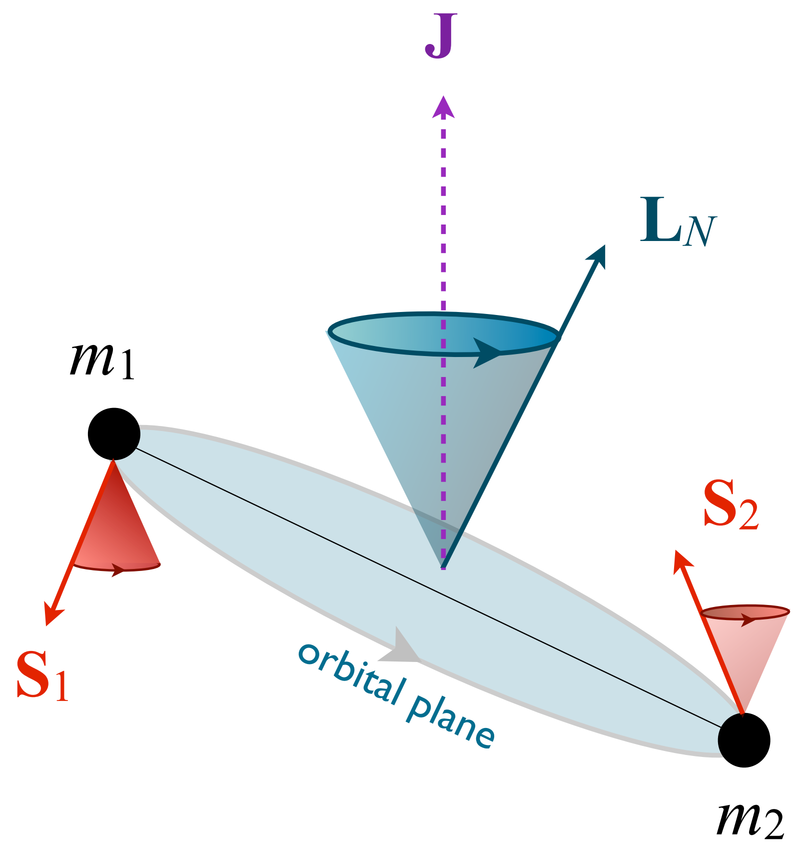
$$\varphi(v) = -\frac{1}{32v^5\eta}\left\{1 + v^2\,[\text{1PN}] + v^3\,[\text{1.5PN} + \text{SO}] + v^4\,[\text{2PN} + \text{SS}] + \dots\right\}$$

leading order spin-orbit coupling term

$$\propto \left(1 - \frac{76\eta}{113}\right)\chi_s \cdot \hat{\mathbf{L}}_N + \delta\chi_a \cdot \hat{\mathbf{L}}_N$$

scalar products change
over the evolution
(oscillatory)

$$2\chi_s = \mathbf{S}_1/m_1^2 + \mathbf{S}_2/m_2^2$$
$$2\chi_a = \mathbf{S}_1/m_1^2 - \mathbf{S}_2/m_2^2$$
$$m = m_1 + m_2$$
$$\eta = \mu/m$$
$$\delta = (m_1 - m_2)/m$$



Motivation: post-Newtonian waveforms for spinning binaries

[P. Ajith (2011)]

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leading order spin-orbit coupling term

$$\propto \underbrace{\left(1 - \frac{76\eta}{113}\right)\boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N + \delta\boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N}_{\chi}$$

“reduced spin”

Motivation: post-Newtonian waveforms for spinning binaries

[P. Ajith (2011)]

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leading order spin-orbit coupling term

$\propto$ χ ("reduced spin")

- χ is a slowly oscillating quantity: $d\chi/dt \sim \mathcal{O}(v^6)$ (while $d\mathbf{S}/dt \sim \mathcal{O}(v^5)$).
- χ is constant (leading order) in several regions in the parameter space:

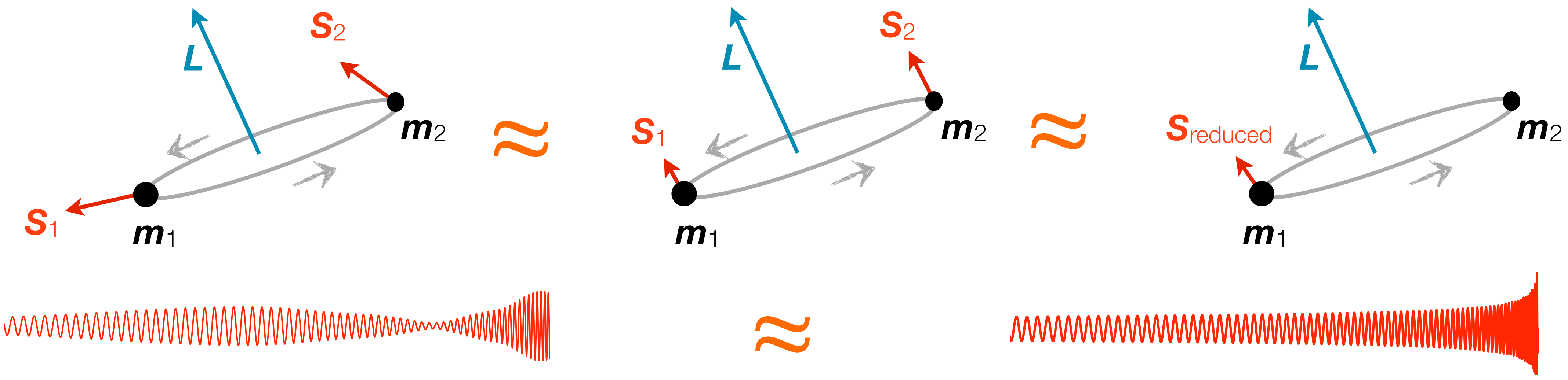
$$\mathbf{S}_i \to 0$$

or $\mathbf{S}_2 \cdot (\mathbf{S}_1 \times \mathbf{L}_N) \to 0$

or $m_1 \to m_2$

Reducing the spin parameter space

[P. Ajith (2011)]



Dominant spin-dependent term in the phasing is nearly secular.

Possible to (approximately) describe the system as a non-precessing binary described by one “reduced-spin” parameter.

Reducing the spin parameter space

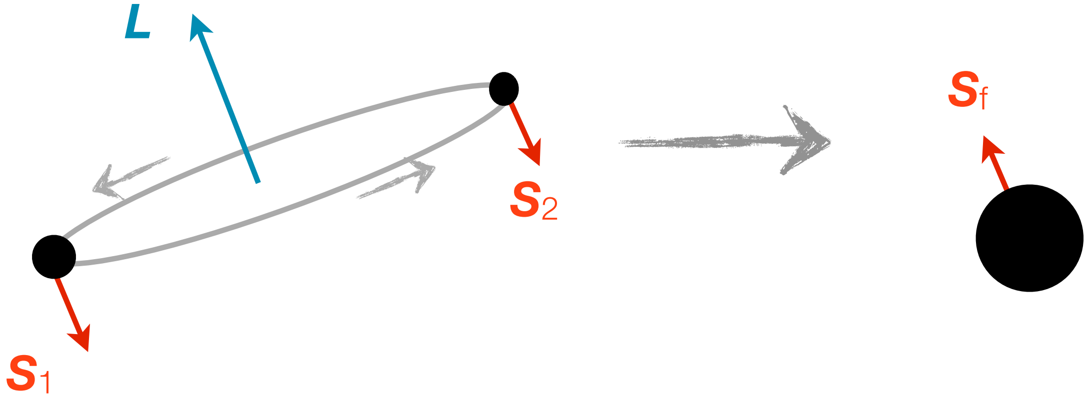
[P. Ajith (2011)]

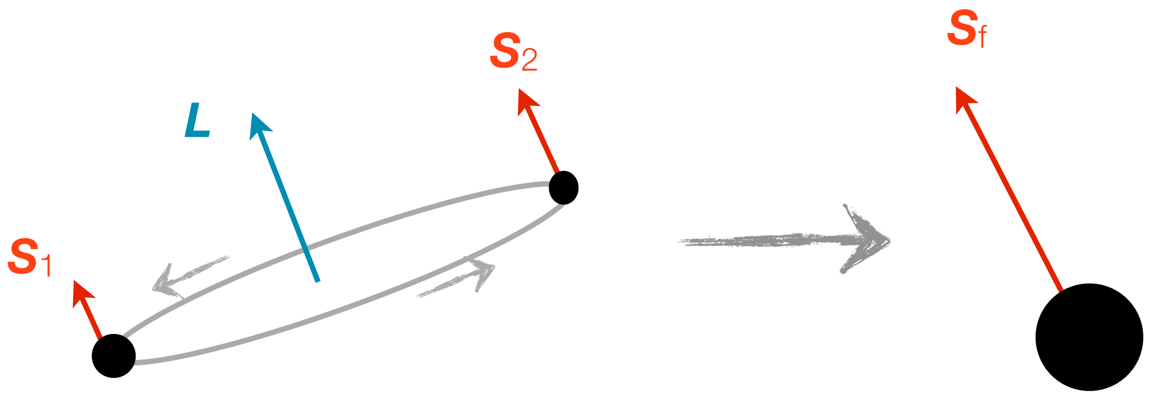
$$\varphi(v) = -\frac{1}{32 v^5 \eta}\left\{1 + v^2\,[\text{1PN}] + v^3\,[\text{1.5PN} + \overset{\overset{= f(\chi)}{\uparrow}}{\text{SO}}] + v^4\,[\text{2PN} + \overset{\overset{\simeq f(\chi)}{\uparrow}}{\text{SS}}] + \overset{\overset{\simeq f(\chi)}{\uparrow}}{\ldots}\right\}$$

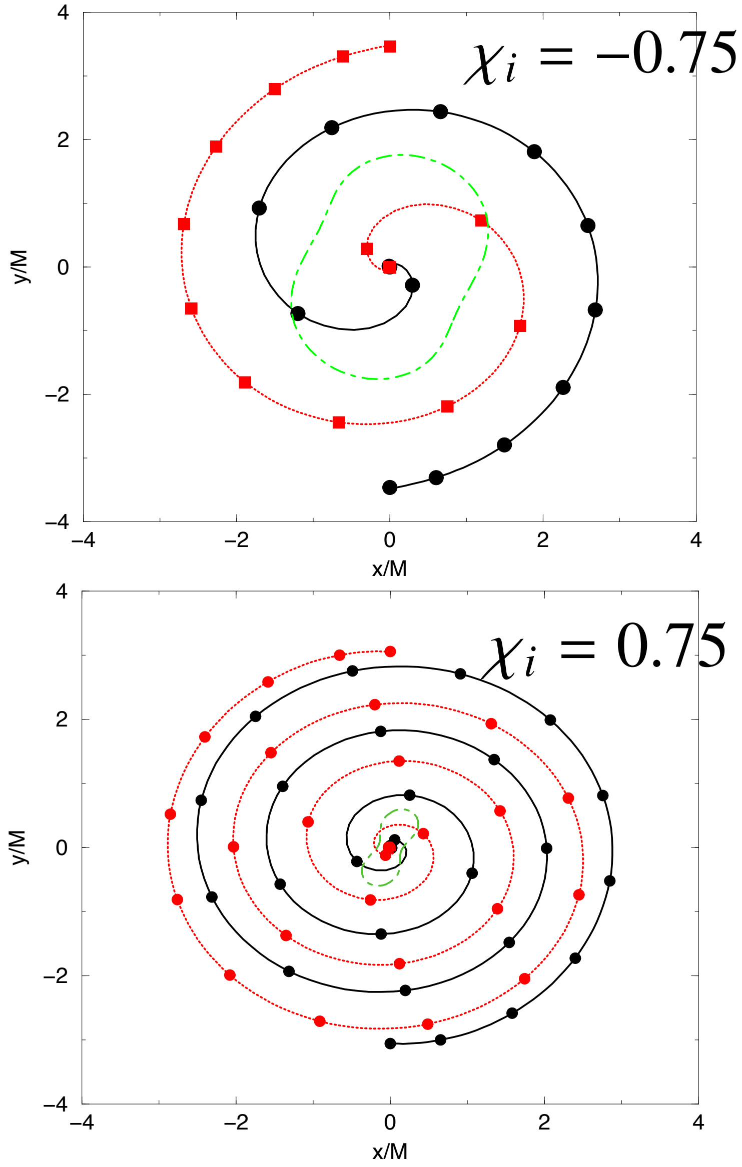
Dominant spin-dependent term in the phasing is nearly secular.

Possible to (approximately) describe the system as a non-precessing binary described by one "reduced-spin" parameter.

Spins produce effects other than precession also: Orbital hangup

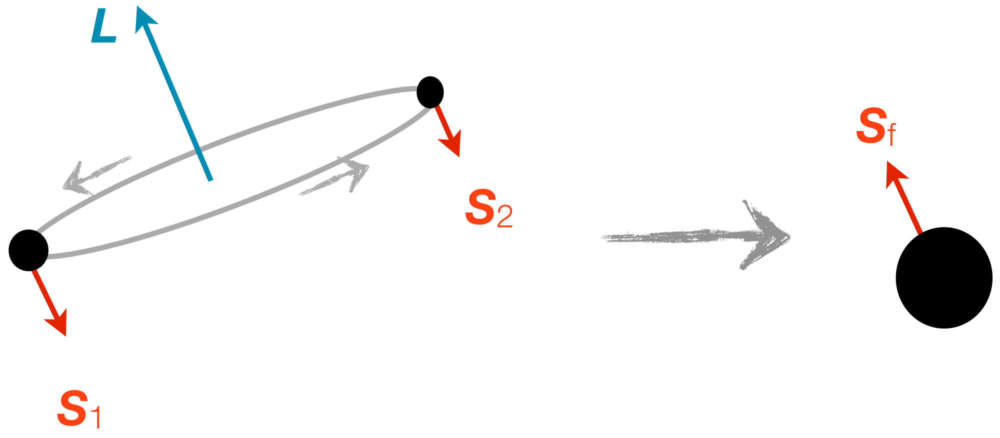


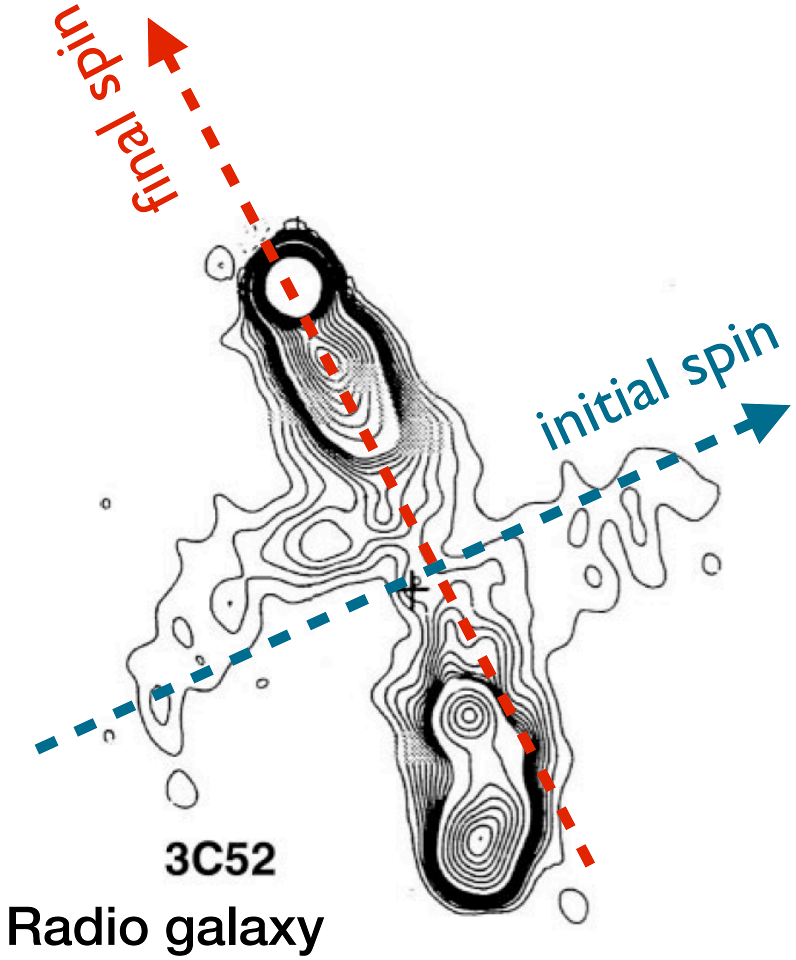




[Campanelli et al 2006]

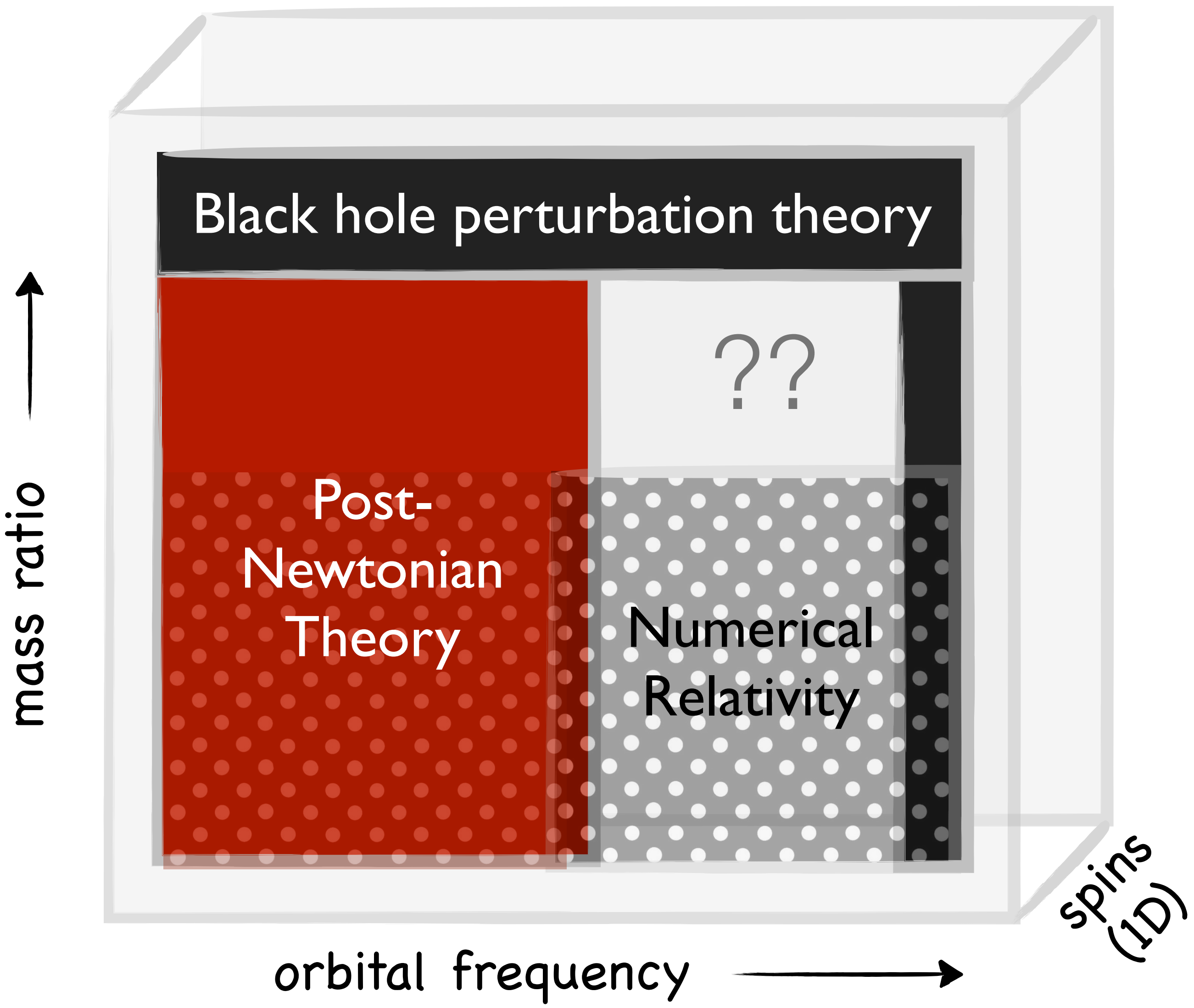
Spins produce effects other than precession also: Spin flips





[Merritt & Ekers 2002]

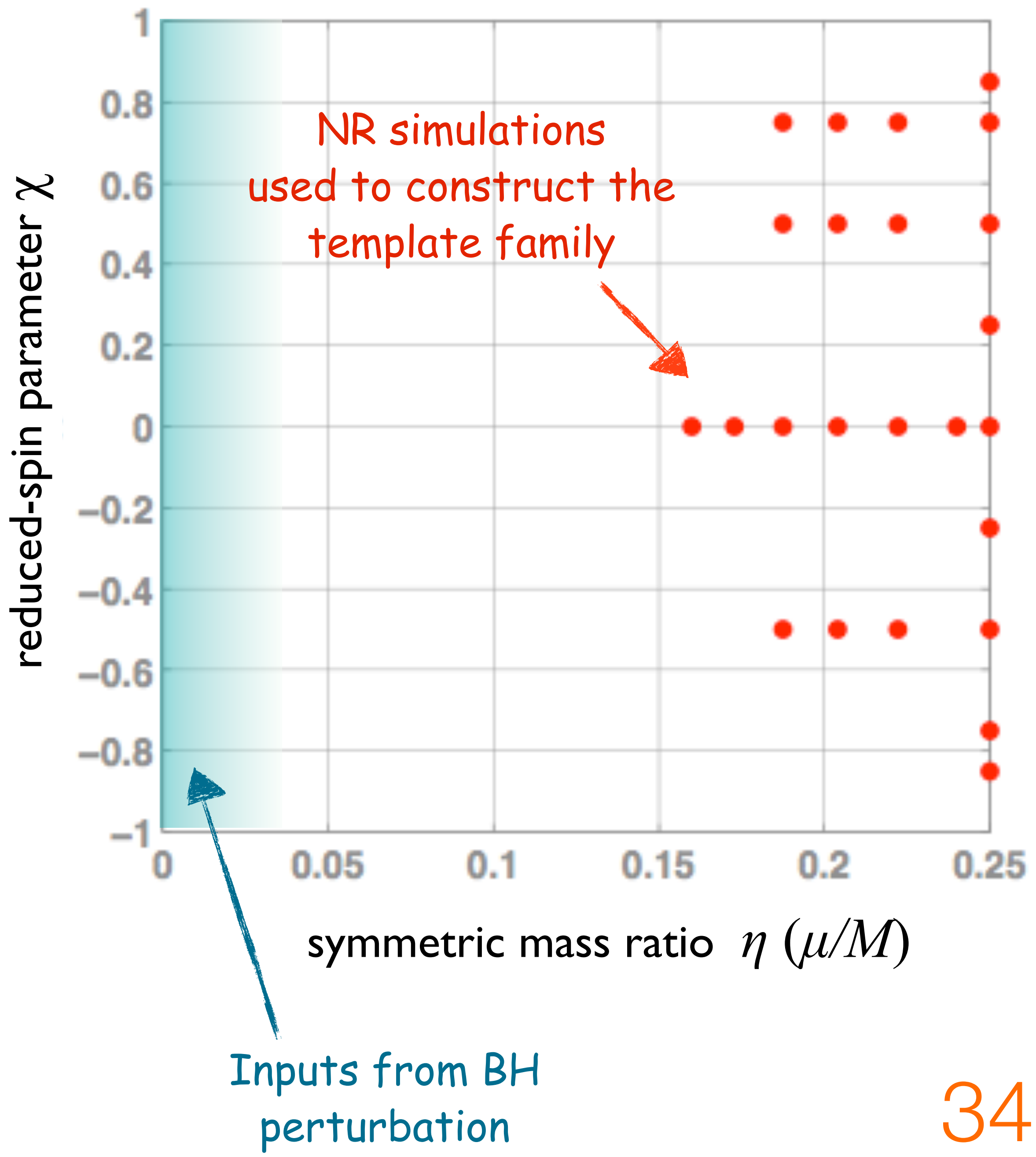
Interpolating the parameter space



Interpolating the parameter space

- Interpolate the non-precessing-spin hybrid waveforms over the parameter space.
    - ... Using appropriate interpolation functions, motivated from perturbative approaches.
- Test the interpolated (analytical) waveform family against a different set of NR simulations (covering more regions in the parameter space).
    - ... a measure of the validity of the approximations used.

[P. Ajith et al (2011)]



NR simulations used to construct the template family

Inputs from BH perturbation

Phenomenological parameterization of hybrid waveforms

Inspiral-merger-ringdown waveform
in the Fourier domain

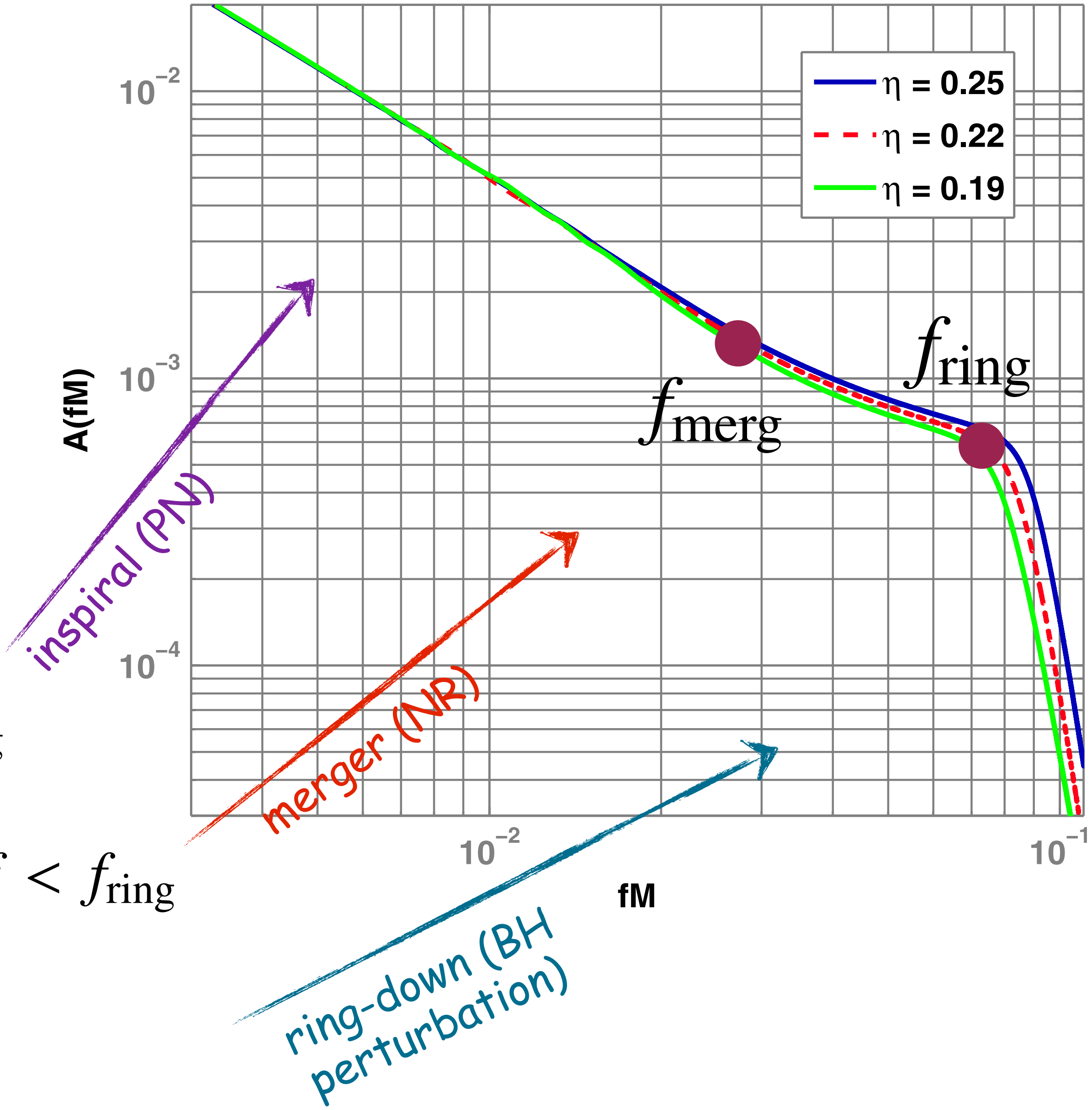
$$h(f) \equiv A(f)\, e^{-\mathrm{i}\Psi(f)}$$

- Modelling the amplitude:

inspiral amplitude
with PN corrections

$$A(f) \propto \begin{cases} f'^{-7/6}\,(1 + \sum_{i=2}^{3} \alpha_i\, v^i) & \text{if } f < f_{\text{merg}} \\ w_m\, f'^{-2/3}\,(1 + \sum_{i=1}^{2} \epsilon_i\, v^i) & \text{if } f_{\text{merg}} \le f < f_{\text{ring}} \\ w_r\, \mathcal{L}(f, f_{\text{ring}}, \sigma) & \text{if } f \ge f_{\text{ring}} \end{cases}$$

Lorentzian (amplitude
of the dominant QNM)



[P. Ajith et al (2008)]

Phenomenological parameterization of hybrid waveforms

Inspiral-merger-ringdown waveform
in the Fourier domain

$$h(f) \equiv A(f)\, e^{-\mathrm{i}\Psi(f)}$$

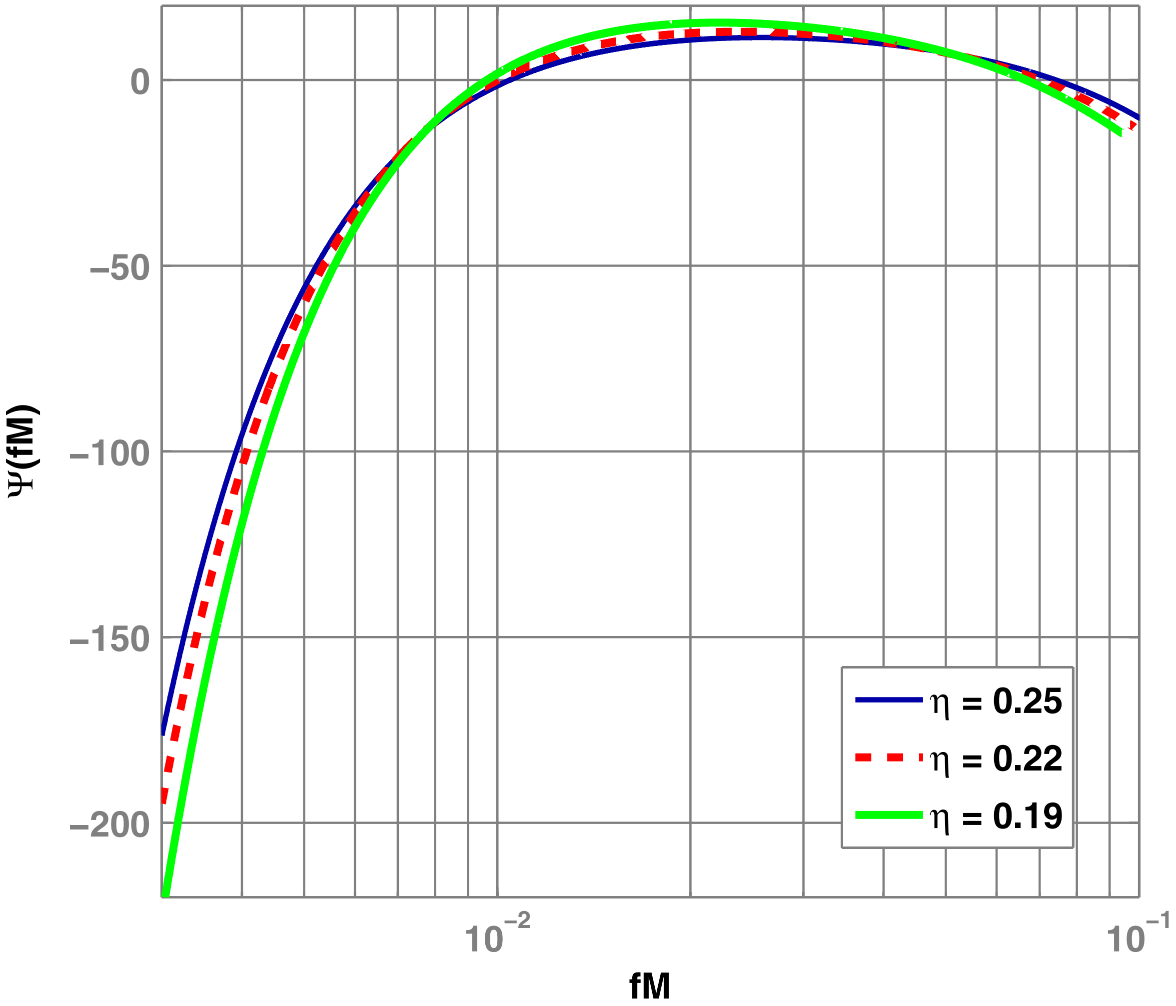
- Modelling the phase:

$$\Psi(f) \equiv \frac{3}{128\,\eta\,v^5}\Big(1 + \sum_{k=2}^{8} v^k\,\psi_k\Big)$$

Newtonian phasing
(Quadrupole formula)

PN-like series. Coefficients determined from the hybrid waveforms

[P. Ajith et al (2008)]



Phenomenological parameterization of hybrid waveforms

- Map the phenomenological parameters in to the physical parameter space:

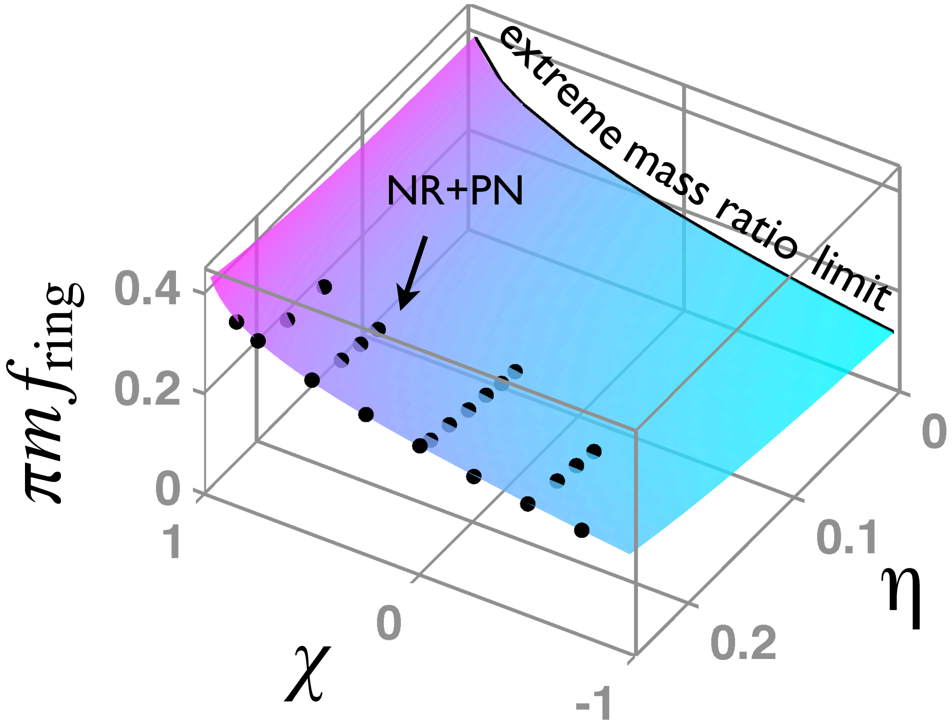
phenomenological params

symm. mass ratio

$$\left\{f_{\rm merg}, f_{\rm ring}, \sigma, \psi_k\right\} = f(m, \eta, \chi)$$

total mass

reduced spin



- such that, in the extreme-mass-ratio limit:

$$f_{\rm merg} \to f^0_{\rm ISCO}, \quad f_{\rm ring} \to f^0_{\rm QNM}, \quad \sigma \to \tau^0, \quad \psi_k \to \psi^0_k$$

ISCO frequency of a Kerr BH with appropr. mass & spin

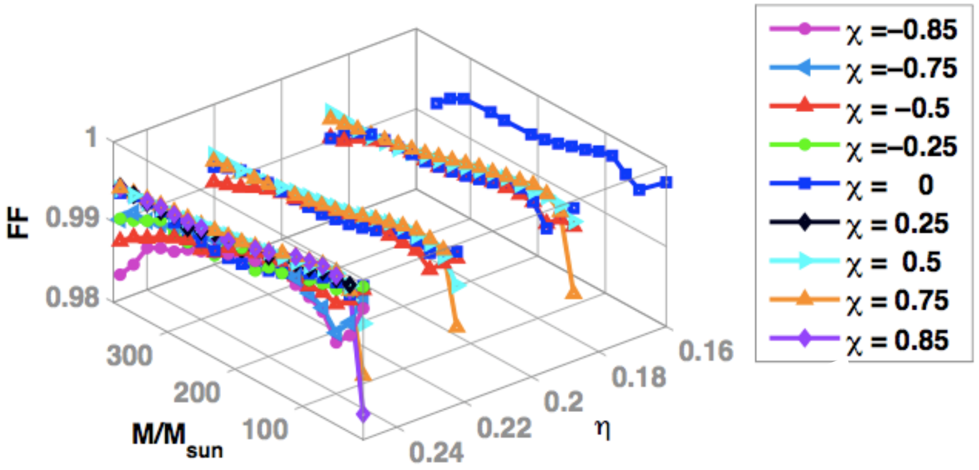
Freq & damping time of the dominant QNM

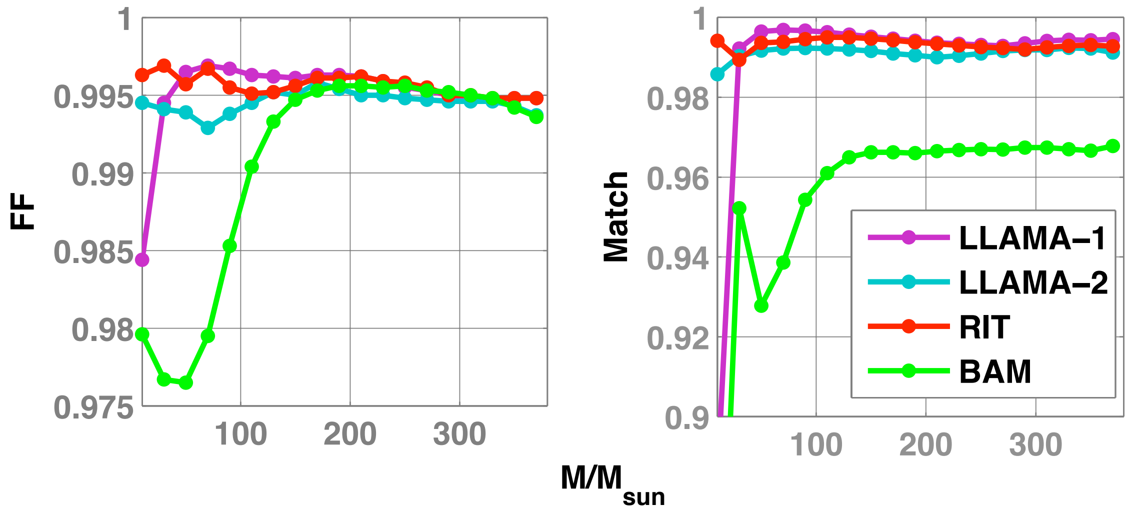
PN phasing coefficients in the test-mass limit

Summary

Analytical description of the GWs from inspiral-merger-ringdown of binary black holes.

Combines NR, PN and BH perturbation.



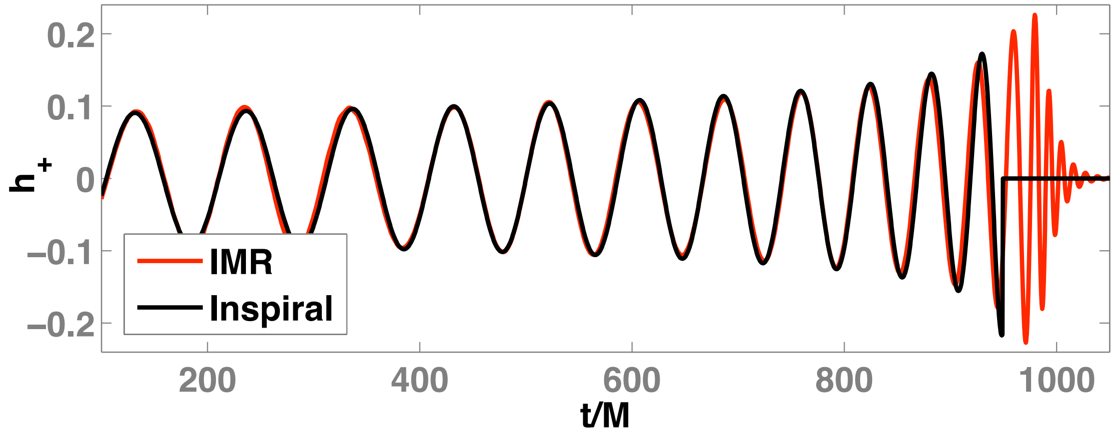


[P. Ajith et al (2011)]

Coherent description of the inspiral, merger and ring-down

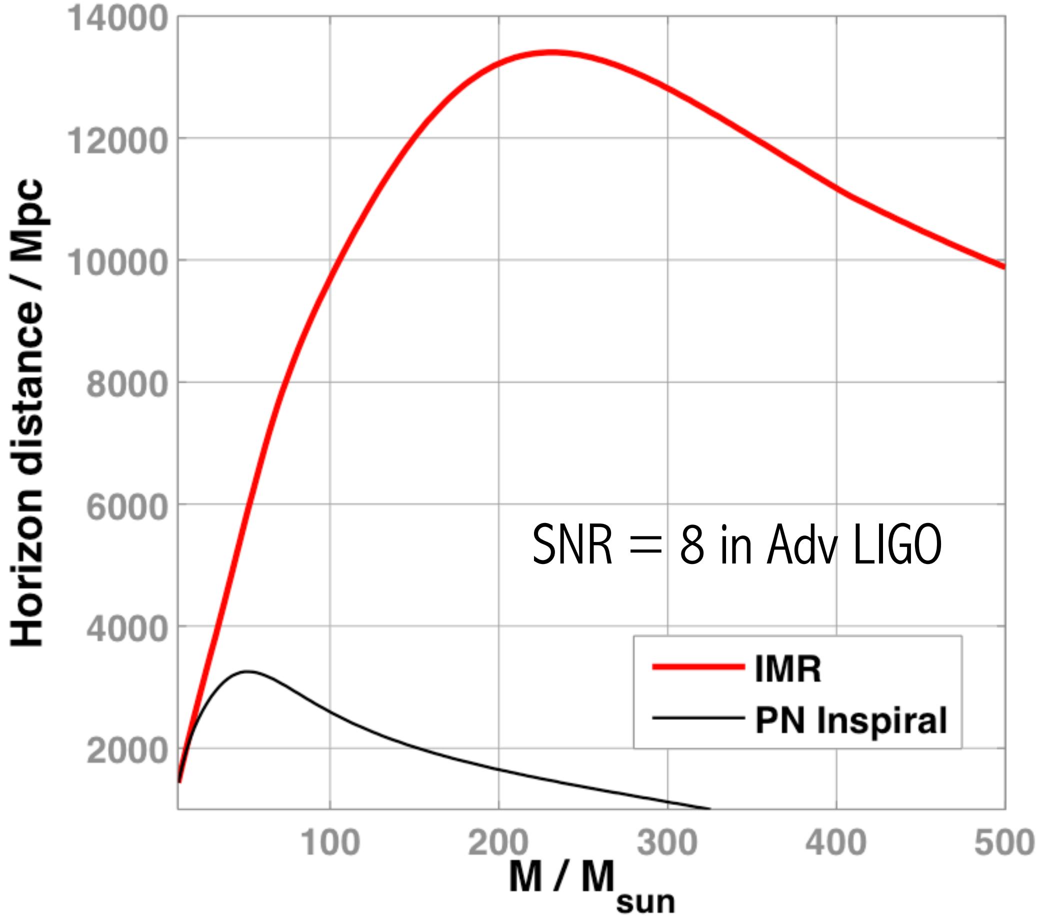
Implications in GW astronomy

IMPLICATIONS Improved distance reach for massive BH binaries



Significant improvement in the "distance reach" of "high-mass" black hole binaries.

[P. Ajith et al (2008)]



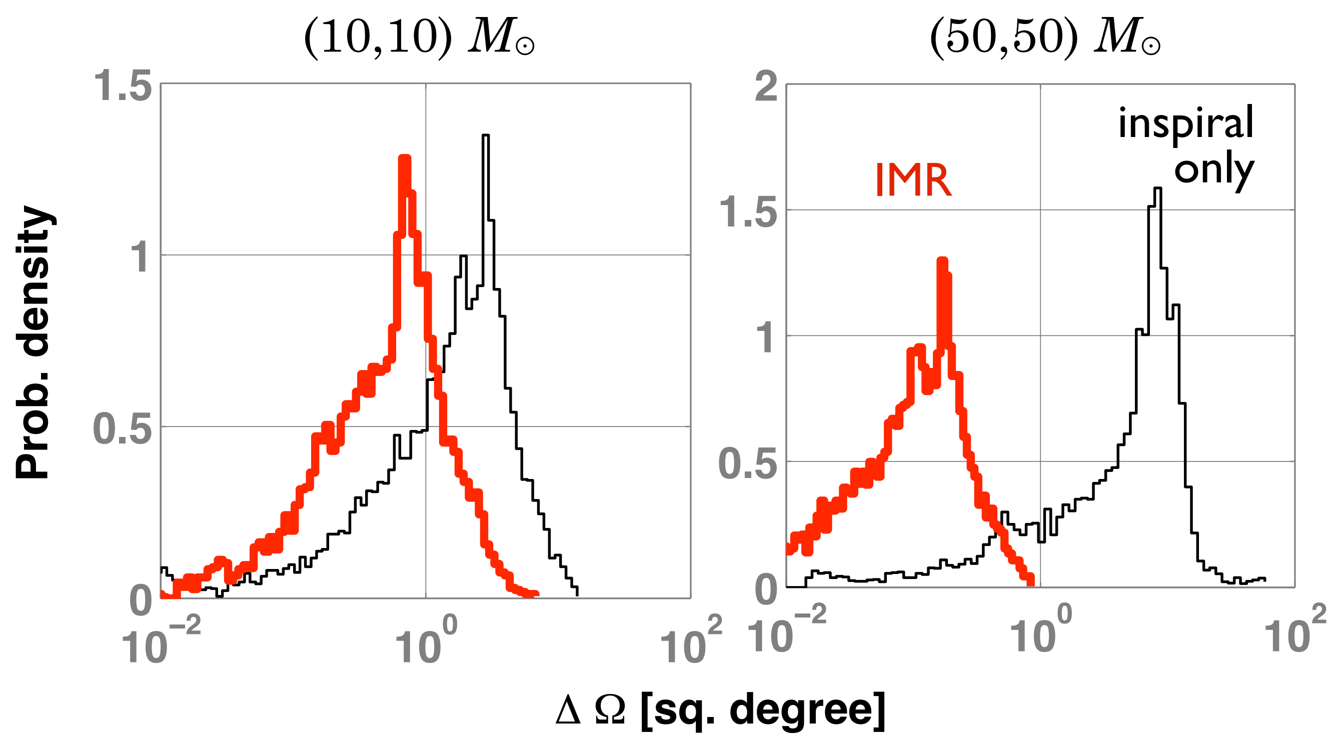
IMPLICATIONS Improved parameter estimation

Helps to disentangle the correlation between different parameters of the binary

(e.g. two component masses)

Improved parameter estimation

[Ajith & Bose (2009)]

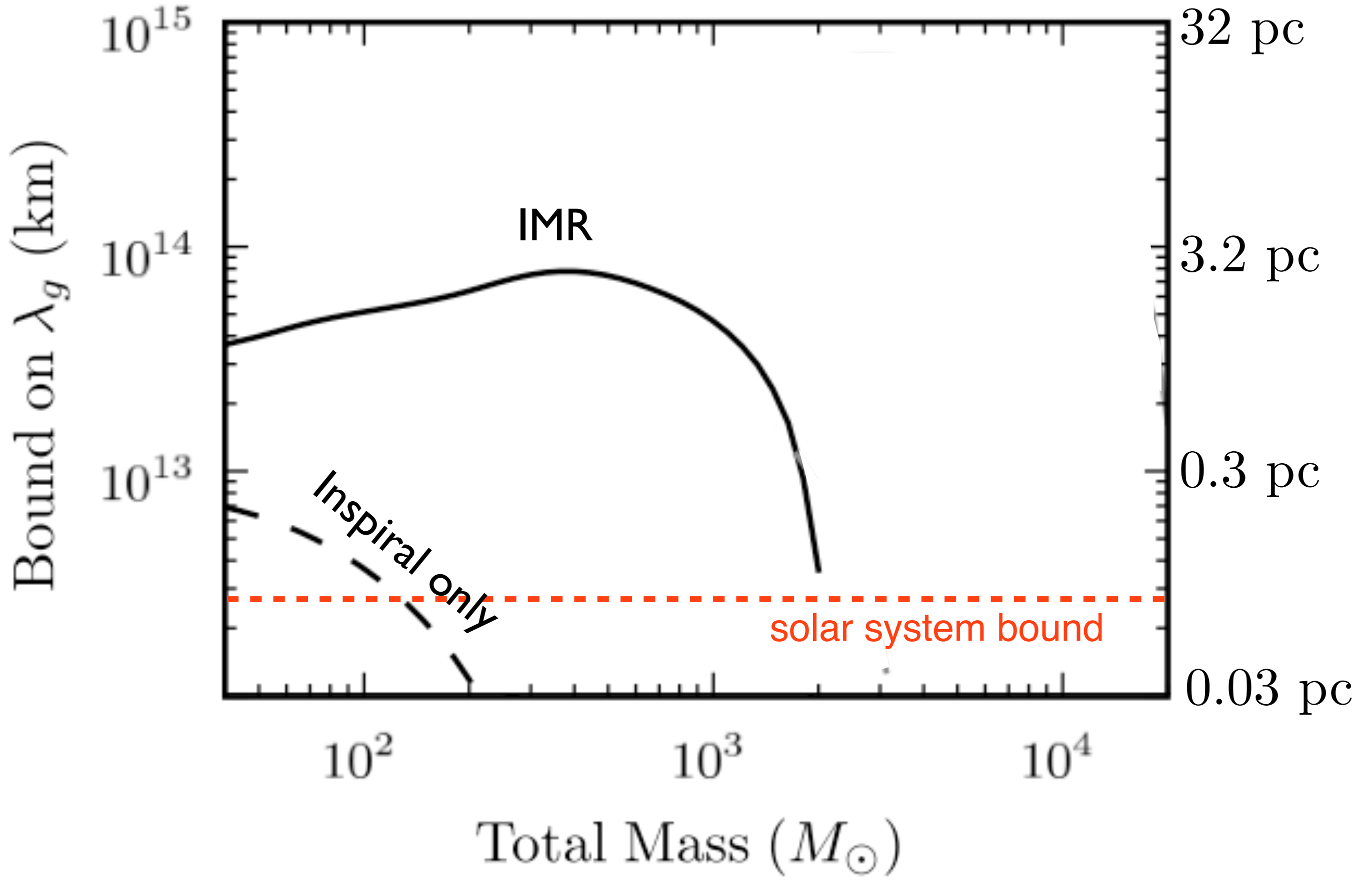


Sky localization errors in Adv LIGO - Adv Virgo Network.
Binary at 1 Gpc

IMPLICATIONS Tests of General Relativity

- **Constraining graviton mass**
  Massive graviton will propagate with (frequency-dependent) speed < c, producing an observable signature in the GW signal.

[Keppel & Ajith (2010)]

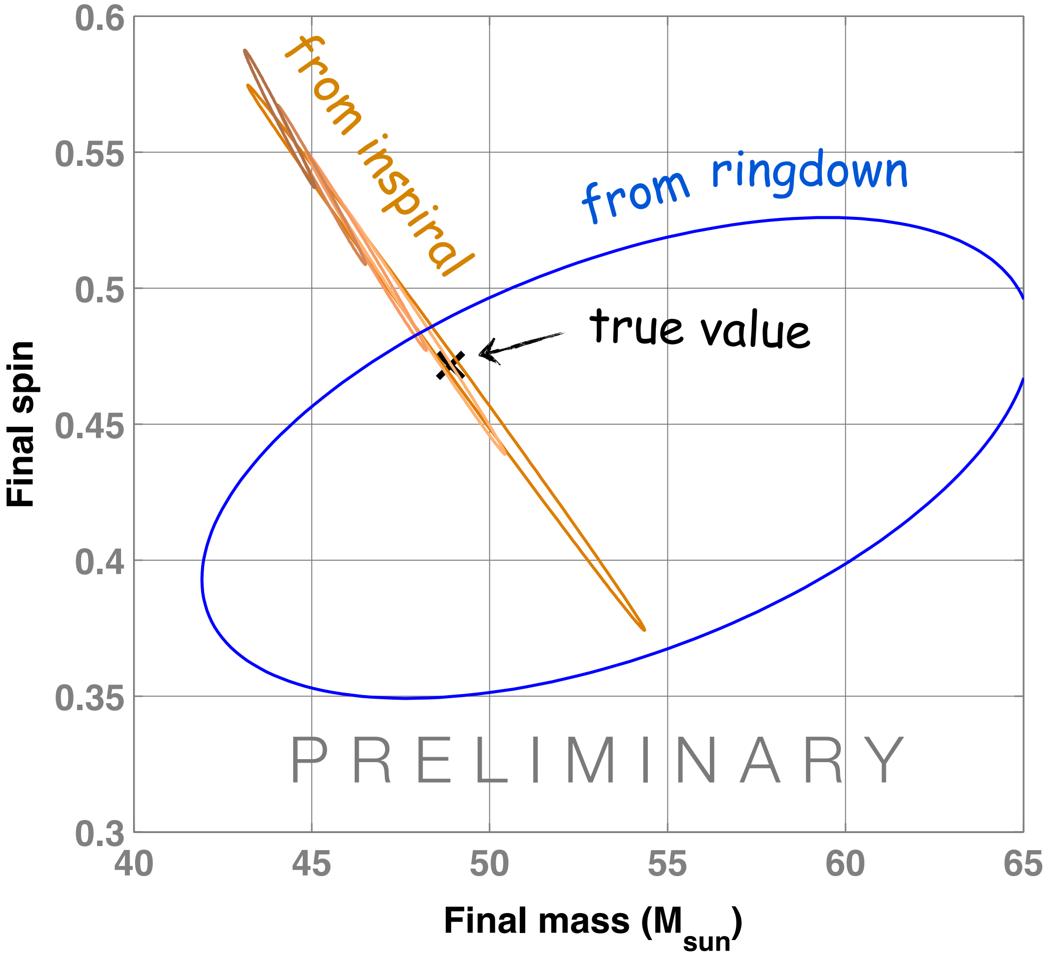


Expected bounds on the graviton mass (Advanced LIGO).
Binary at 1 Gpc

IMPLICATIONS Tests of General Relativity

- **Un-accounted loss of energy & angular momentum from BBHs** Accurate understanding of the energy & angular momentum radiated during the coalescence (from NR) → Can predict the final state of the system if the initial mass & spins known.
- Estimate the initial masses & spins just using the PN inspiral → prediction of the final mass & spin.
- Final mass & spin from the ring-down.

[Ajith, Nichols, Chen (in progress)]



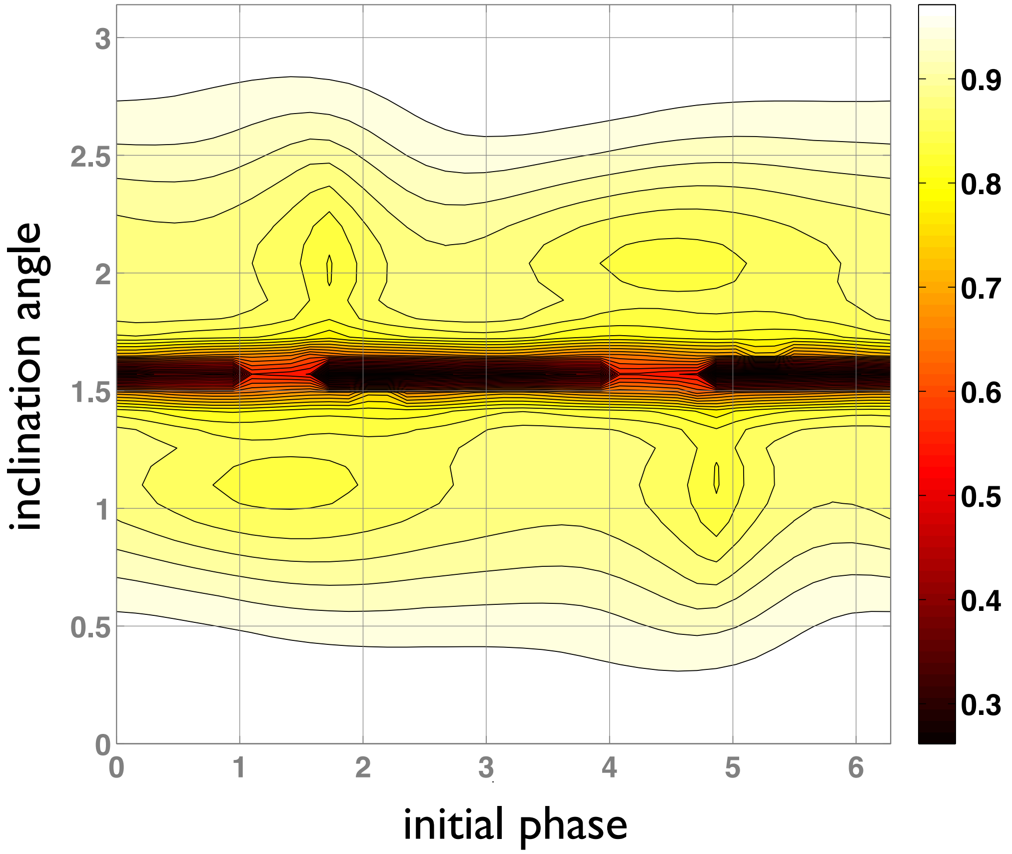
Two estimates has to agree if GR is correct (analogous to binary pulsar tests).

Future work

- Inclusion of the effect of higher multipoles & spin precession: Breaks the degeneracy between the luminosity distance and inclination angle: implication in cosmology using BBH "standard sirens".
- Binaries involving neutron stars: include the effect of equation of state. Implication in estimating the EoS using GW observations.

[P. Ajith et al (prep)]

Effect of neglecting non-quadrupole modes



Fraction of signal-to-noise ratio retrieved using only the quadrupole mode in the template.

Future work

Require thousands of NR simulations + analytical work

Consortium of numerical relativists & analytical relativists formed to address the grand challenge.

Different approaches for combining NR & AR (e.g. EOB)
Enables cross-checking, validation

https://ninja-project.org/nr-ar

Summary

- Great recent breakthroughs in numerical relativity (NR) and analytical relativity (AR) in modeling compact binaries.
- NR+AR enable us to model the coalescence of compact binaries accurately. Important impacts in the detection rates and parameter estimation accuracies, tests of GR.
- NR has a number of applications in non GW-science: understanding black holes, accreting compact objects etc. through EM astronomy, EM transients, high-energy physics.