Linear Alexander quandle colorings and finite-fold cyclic covers of S^3 branched over knots

Kanako Oshiro (Sophia Univ.)

Today's talk

Fox p- colorings are related to homomorphisms from the fundamental group of the double branched cover of S^3 branched over knots to $\mathbb{Z}_p$

$$\{\text{Fox } p\text{-colorings for } K\} \xleftrightarrow{p:1} \{\pi_K^{(2)}/(x_i^2) \to \mathbb{Z}_p : \text{hom}\}$$

For linear Alexander quandle colorings, is it satisfied ?

$$\{\text{linear Alexander quandle colorings for } K\}$$

$$\xleftrightarrow{p:1} \{\pi_K^{(\boxed{?})}/(x_i^{\boxed{?}}) \to \mathbb{Z}_p : \text{hom}\} \quad ??$$

§ Preliminaries

A <u>quandle</u> is a set X $(\neq \phi)$ with a binary operation $*$

s.t.

(I) $\forall a \in X$, $a * a = a$

(II) $\forall a, b \in X$, $\exists! c \in X$ s.t. $a = c * b$

(III) $\forall a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$

The <u>symmetry of $a \in X$</u> :

$$\xi_a : X \to X, \quad x \mapsto x * a =: (x)\xi_a$$

<u>Note</u>

ξ_a is a quandle automorphism

(II) $\rightsquigarrow$ bijective

(III) $\rightsquigarrow$ homomorphic $(x * y)\xi_a = (x)\xi_a * (y)\xi_a$

The <u>inner automorphism group of X</u> :

$$\mathrm{Inn}(X) = \{ \text{symmetries of } X \}$$

X is <u>connected</u> $:\Leftrightarrow$ $\mathrm{Inn}(X)$ acts transitively on X

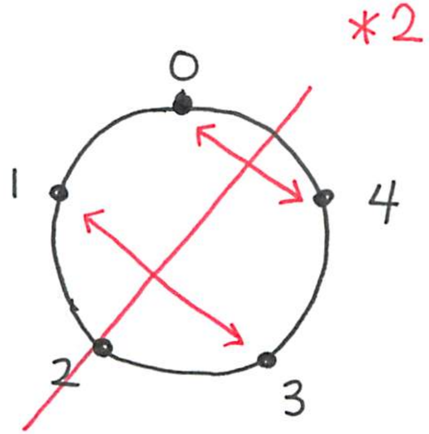
Examples

(1) Trivial quandle :

$$\begin{cases} S : \text{a set} \\ a * b = a \end{cases}$$

(2) Dihedral quandle of order $p\ (\geq 3)$:

$$\begin{cases} \mathbb{Z}/p\mathbb{Z} \\ a * b = 2b - a \end{cases} \quad \leftarrow\!\!\sim\!\sim\!\sim \quad R_p$$



Alexander quandle:

$$\begin{cases} M : \mathbb{Z}[t, t^{-1}]\text{-module} \\ a * b = ta + (1-t)b \end{cases}$$

Linear Alexander quandle

$$M = \mathbb{Z}_p[t, t^{-1}] / (t - \mu)$$

$$p \in \mathbb{N}$$

$$\mu \in \{1, 2, \cdots, p-1\} \quad \text{s.t.} \quad \text{g.c.d.}(p, \mu) = 1$$

Note

- $\mu = p-1 \Rightarrow$ It is R_p $(a*b = 2b - a)$
- $\mu = 1 \Rightarrow$ It is a trivial quandle $(a*b = a)$

Lem (Nelson)

$\mathbb{Z}_p[t, t^{-1}] / (t-M)$: connected

$\Leftrightarrow$ g.c.d. $(p, 1-M) = 1$

$\hookrightarrow$ In this talk, we study in the case that

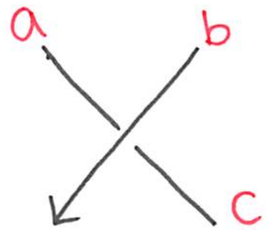
$\mathbb{Z}_p[t, t^{-1}] / (t-M)$ is connected.

i.e., g.c.d $(p, 1-M) = 1$

X : a quandle

D : an ori. knot diagram

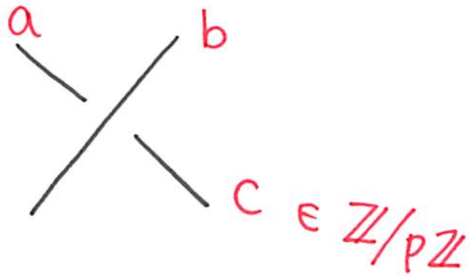
<u>An X-coloring for D</u> : $C : \{\text{arcs of } D\} \to X$ s.t.



$a * b = c$

<u>Rem</u>

Fox p-coloring = R_p - coloring

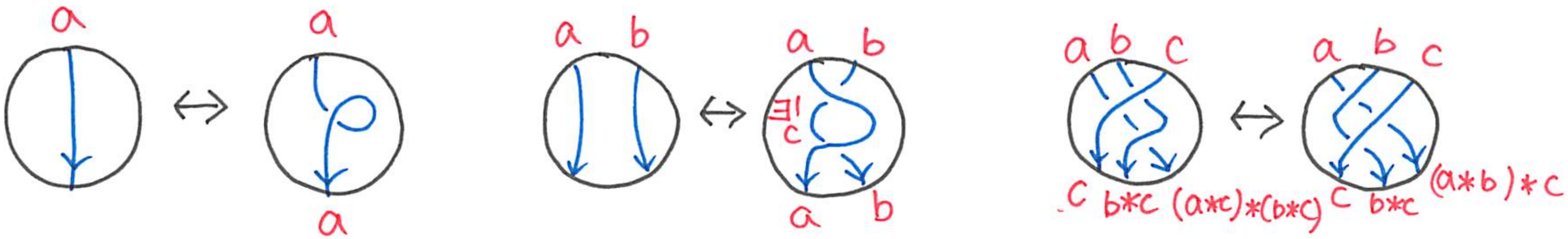


$a + c = 2b \iff 2b - a = c \iff a * b = c$

↑
Operation of R_p

$D \cong D'$

$\Rightarrow \{X\text{-colorings for } D\} \xleftrightarrow{1:1} \{X\text{-colorings for } D'\}$



$\{X\text{-colorings for } K\} = \{X\text{-colorings for } D\}$ for some D

K : a knot

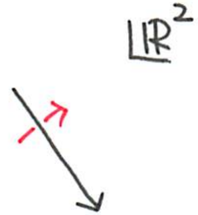
D : a diagram of K (oriented)

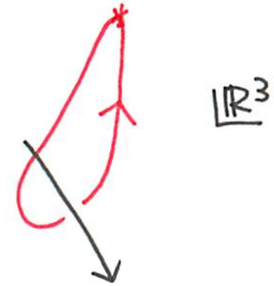
The knot group $\pi_K^{(1)}$ is

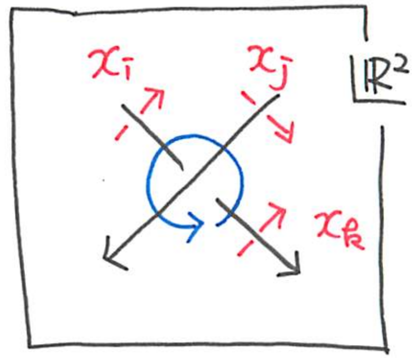
$$\langle \underbrace{x_1, \cdots, x_n}_{\text{arcs}} \mid \underbrace{r_1, \cdots, r_{n-1}}_{\text{relations obtained from crossings}} \rangle$$

$\uparrow$

<u>Wirtinger presentation</u>







$$r_i = x_k x_j^{-1} x_i^{-1} x_j$$

$\pi_K^{(s)}$: the fundamental group of the s-fold cyclic cover of $S^3 - K$

<u>Rem</u>

$$\begin{array}{rcl} \alpha : \pi_K^{(1)} & \twoheadrightarrow & \mathbb{Z}_s \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x_i & \longmapsto & 1 \end{array}$$

$$\operatorname{Ker}\alpha \cong \pi_K^{(s)}$$

Hence, $\pi_K^{(s)} < \pi_K^{(1)}$

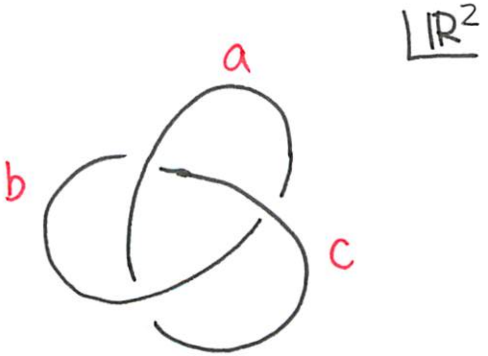
$\pi_K^{(s)}/(x_i^s)$: the fundamental group of the s-fold cyclic cover of S^3 branched over K

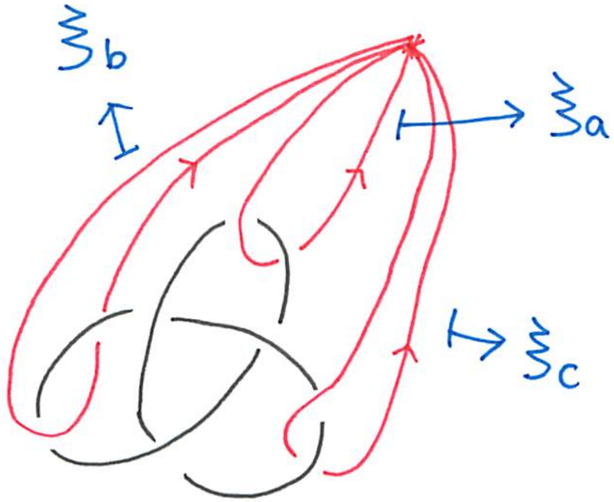
§ A property for Fox colorings

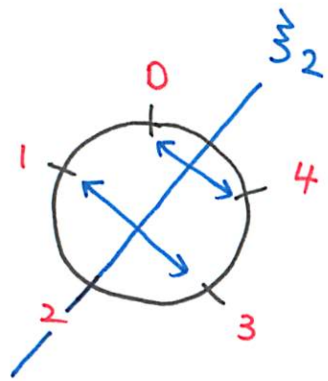
p : an odd prime number

$\{\text{Fox } p\text{-colorings for } K\} \xleftrightarrow{p:1} \{\pi_K^{(2)}/(x_i^2) \to \mathbb{Z}_p : \text{hom}\}$

- $\{\text{Fox } p\text{-colorings}\} \xleftrightarrow{1:1} \left\{ \begin{array}{ccc} \pi_K^{(1)} & \longrightarrow & D_{2p} \\ \Psi & & \Psi \\ x_i & \longmapsto & \text{a reflection} \end{array} : \text{hom} \right\}$







uniquely

- Each ϕ_c (= Fox p-coloring) lifts to a homomorphism

$$\tilde{\phi}_c : \pi_K^{(2)} \to \mathbb{Z}_p \quad \text{s.t. } x_i^2 \longmapsto 0$$

$$\begin{tikzcd}
\pi_K^{(2)} \arrow[r, "\tilde{\phi}_c"] \arrow[d] \arrow[dr, phantom, "\text{lifts}"] & \mathbb{Z}_p \arrow[d] \\
\pi_K^{(1)} \arrow[r, "\phi_c"] \arrow[d] & D_{2p} \arrow[d] \\
\mathbb{Z}_2 \arrow[r, equal] & \mathbb{Z}_2
\end{tikzcd}$$

- Any hom. $\tilde{\phi} : \pi_K^{(2)} \to \mathbb{Z}_p$ s.t. $x_i^2 \mapsto 0$ is a lift of exactly p hom.s (Fox p-colorings) $\phi_c : \pi_K^{(1)} \to D_{2p}$
$$\begin{array}{ccc} \pi_K^{(1)} & \longrightarrow & D_{2p} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x_i & \longmapsto & \text{a reflection} \end{array}$$

$$\begin{tikzcd}
\pi_K^{(2)} \arrow[r, "\tilde{\phi}\ \text{s.t.}\ x_i^2 \mapsto 0"] \arrow[d] \arrow[dr, phantom, "\Uparrow\ \text{lift}"] & \mathbb{Z}_p \arrow[d] \\
\pi_K^{(1)} \arrow[r, "\exists\,\text{exactly}\ p\ \phi_c\text{ s}"] \arrow[d] & D_{2p} \arrow[d] \\
\mathbb{Z}_2 \arrow[r, equal] & \mathbb{Z}_2
\end{tikzcd}$$

$$\{\text{Fox } p\text{-colorings for } K\} \xleftrightarrow{p:1} \{\pi_K^{(2)} \longrightarrow \mathbb{Z}_p : \text{hom s.t. } x_i^2 \mapsto 0\}$$

$$\Big\updownarrow \ 1:1$$

$$\{\pi_K^{(2)}/(x_i^2) \longrightarrow \mathbb{Z}_p : \text{hom}\}$$

$$\begin{tikzcd}
\pi_K^{(2)} \arrow[rr, "\tilde{\phi}_c"] \arrow[d] & & \mathbb{Z}_p \arrow[d] \\
\pi_K^{(1)} \arrow[rr, "p\ \ \phi_c\, s"] \arrow[d] \arrow[u, phantom, shift left=8ex, "\textcolor{red}{\text{lift}}"] & & D_{2p} \arrow[d] \\
\mathbb{Z}_2 \arrow[rr, equal] & & \mathbb{Z}_2
\end{tikzcd}$$

$$\{\text{Fox } p\text{-colorings for } K\} \xleftrightarrow{p:1} \{\pi_K^{(2)} \longrightarrow \mathbb{Z}_p : \text{hom s.t. } x_i^2 \mapsto 0\}$$

$$\updownarrow \; 1:1$$

$$\{\pi_K^{(2)}/(x_i^2) \longrightarrow \mathbb{Z}_p : \text{hom}\}$$

$$\begin{tikzcd}
\pi_K^{(2)} \arrow[r, "\tilde{\phi}_c"] \arrow[d] \arrow[dr, phantom, "\uparrow\ \text{lift}"] & \mathbb{Z}_p \arrow[d] \\
\pi_K^{(1)} \arrow[r, "p\ \phi_c s"] \arrow[d] & D_{2p} \cong \mathrm{Inn}(R_p) \arrow[d] \\
\mathbb{Z}_2 \arrow[r, equal] & \mathbb{Z}_2
\end{tikzcd}$$

$$\pi_K^{(1)} \ni x_i \longmapsto \xi_a \in \mathrm{Inn}(R_p)$$

$\underbrace{\mathbb{Z}_2}_{}$: the type of R_p

$$\boxed{\begin{array}{l} \text{※ Fox } p\text{-coloring} \\ \quad = R_p \text{ coloring} \\ \qquad (a*b = 2b-a) \end{array}}$$

§ A generalization for linear Alexander quandle colorings

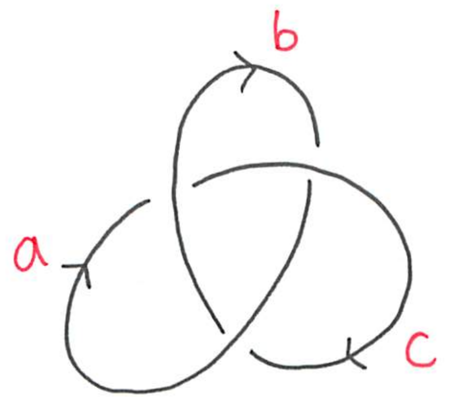
$p \in \mathbb{N}$

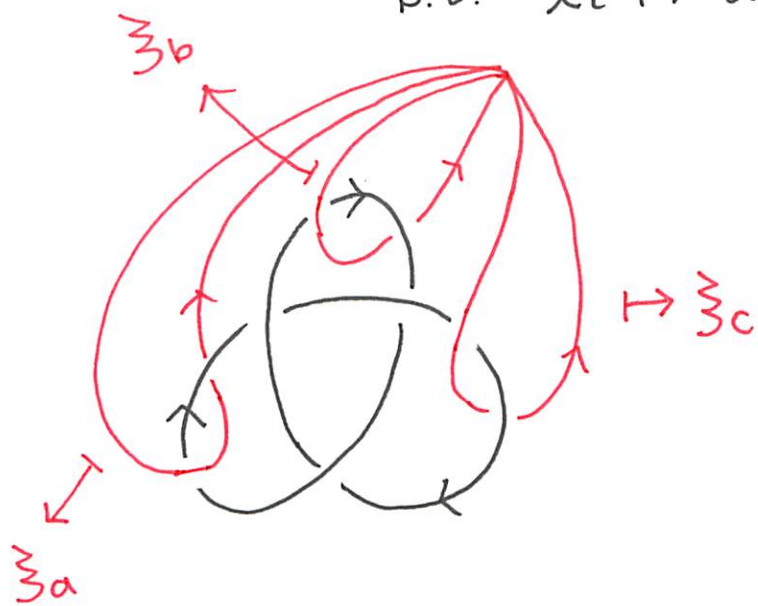
$\mu \in \{1, \cdots, p-1\}$ s.t. $g.c.d(p, 1-\mu) = 1$

$X = \mathbb{Z}_p[t, t^{-1}] / (t-\mu)$

(1) $\{X\text{-colorings for } K\} \xleftrightarrow{1:1} \{\pi_K^{(1)} \longrightarrow \mathrm{Inn}(X) : \text{hom}$

s.t. $x_i \mapsto$ a symmetry$\}$





(2) Each X-coloring ϕ_C lifts uniquely to a hom

$$\phi_C^{(s)} : \pi_K^{(s)} \longrightarrow \mathbb{Z}_p$$

s.t. (i) $x_i^s \longmapsto 0$ ($\forall x_i$: meridian generator)

(ii) $x_i^{-1} w x_i w^{-\mu} \longmapsto 0$ ($\forall x_i$, $\forall w \in \pi_K^{(s)}$)

where $s = \min \{ s \in \mathbb{N} \mid \mu^s \equiv 1 \mod p \}$

<u>Rem</u> $s =$ type X

$$= \min \{ n \in \mathbb{N} \mid \forall a, b \in X,\ a \underbrace{* b * \cdots * b}_{n} = a \}$$

$$
\begin{tikzcd}
\pi_K^{(s)} \arrow[r, "\phi_c^{(s)}"] \arrow[d] \arrow[dr, phantom, "\Uparrow \text{lift}"] & \mathbb{Z}_p \arrow[d] \\
\pi_K^{(1)} \arrow[r, "\phi_c"] \arrow[d] & \mathrm{Inn}(X) \arrow[d] \\
\mathbb{Z}_s \arrow[r, equal] & \mathbb{Z}_s
\end{tikzcd}
$$

(i) $x_i^s \longmapsto 0$ ($\forall x_i$: meridian generator)

(ii) $x_i^{-1} w x_i w^{-\mu} \mapsto 0$ ($\forall x_i$, $\forall w \in \pi_K^{(s)}$)

(3) $\forall \phi^{(s)} : \pi_K^{(s)} \longrightarrow \mathbb{Z}_p$ s.t.

(i) $x_i^s \mapsto 0$

(ii) $x_i^{-1} w x_i w^{-\mu} \mapsto 0$

is a lift of exactly p X-colorings

$$\begin{tikzcd}
\pi_K^{(s)} \arrow[r, "\phi^{(s)}"] \arrow[d] & \mathbb{Z}_p \arrow[d] \\
\pi_K^{(1)} \arrow[r, "\exists \text{exactly } p \ \phi_c \text{ s}"] \arrow[d] \arrow[ur, phantom, "\text{lift}"] & \mathrm{Inn}(X) \arrow[d] \\
\mathbb{Z}_s \arrow[r, equal] & \mathbb{Z}_s
\end{tikzcd}$$

$$\{X\text{-colorings for } K\} \xleftrightarrow{p:1} \{\pi_K^{(s)} \longrightarrow \mathbb{Z}/p : \text{hom. s.t. (i), (ii)}\}$$

$$\updownarrow 1:1$$

$$\{\pi_K^{(s)}/(x_i^s) \longrightarrow \mathbb{Z}/p : \text{hom s.t. (ii)}\}$$

Theorem

$p \in \mathbb{N}$

$\mu \in \{1, \cdots, p-1\}$ s.t. g.c.d. $(p, 1-\mu) = 1$

$X = \mathbb{Z}/p[t, t^{-1}] / (t-\mu)$

We have

$$\{X\text{-colorings for } K\} \xleftrightarrow{p:1} \{\pi_K^{(s)}/(x_i^s) \to \mathbb{Z}/p : \text{hom s.t. } x^{-1} w x w^{-\mu} \mapsto 0\}$$

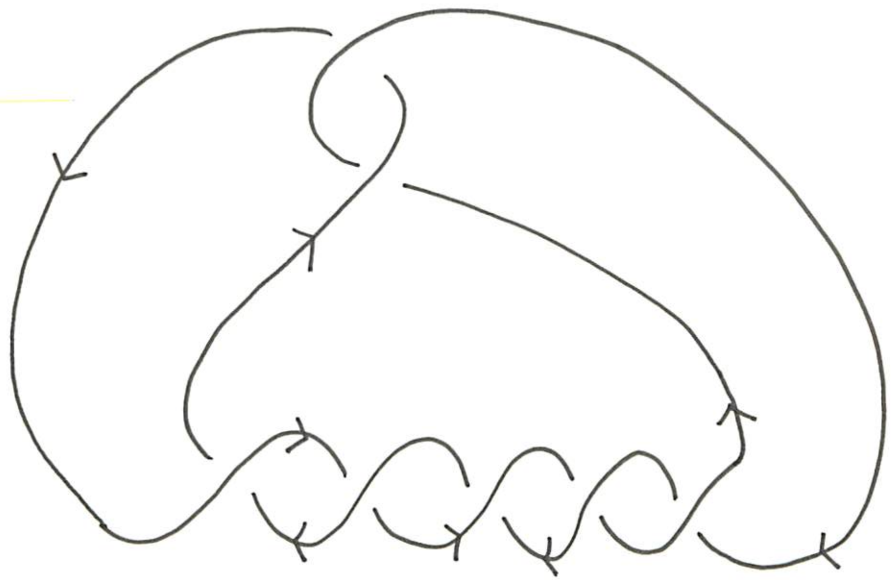
where $s = \text{type } X$

Q. Can we remove the condition (ii) ?

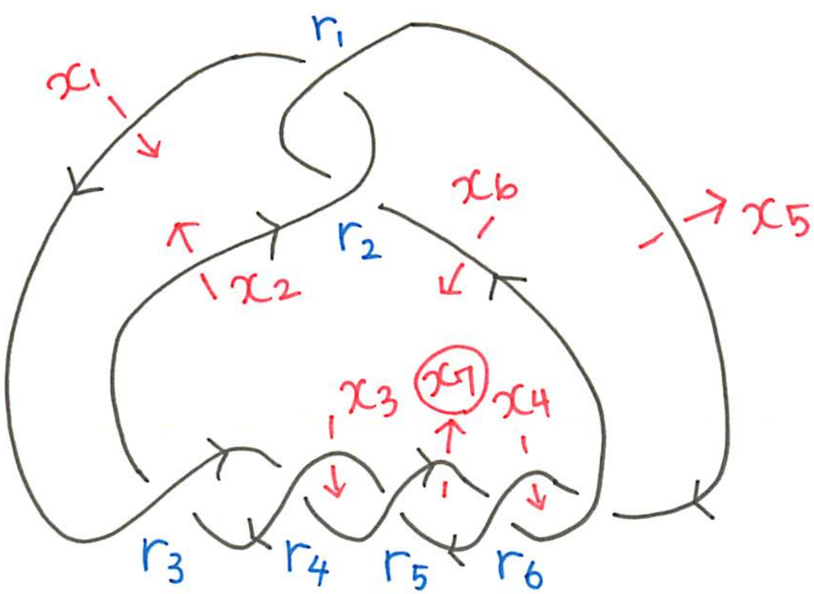
⇝ In the case of Fox colorings, we don't need it.

$X = \mathbb{Z}_5[t, t^{-1}]/(t-2)$

⇝ $S =$ type $X = 4$ $\quad (2^4 \equiv 1 \mod 5)$



⇝ # of X-colorings $= 5^2$



$x = x_7$

$$\left.\begin{aligned} m &:= x^4 \\ m_{ij} &:= x^{-j} x_i x^{-1} x^j \end{aligned}\right\} \text{ are generators of } \pi_K^{(4)}$$

$$r_1 = x_1 x_5^{-1} x_2^{-1} x_5 = (x_1 x^{-1})(x_5 x^{-1})^{-1}(x^{-1} x_2 x^{-1} x)^{-1}(x^{-1} x_5 x^{-1} x)$$

$$= m_{10}\, m_{50}^{-1}\, m_{21}^{-1}\, m_{51}$$

$\vdots$

$$\pi_K^{(4)} = \langle m,\ m_{ij}\ (i=1,\cdots,6,\ j=0,\cdots,3) \mid r_1',\ \cdots,\ r_6' \rangle$$

Case 1 condition (i) (ii)

(i) $m^4 \mapsto 0$

$x_i^4 = m_{i0} \, m \, m_{i3} \, m_{i2} \, m_{i1} \mapsto 0$

(ii) $x^{-1} w x w^{-2} \longmapsto 0$

$\# \{ \pi_K^{(4)} \to \mathbb{Z}_5 : \text{hom. s.t. (i) (ii)} \} = 5$

Case 2   condition $(\bar{i})$

(i) $m^4 \mapsto 0$

$$x_{\bar{i}}^4 = m_{\bar{i}0}\, m\, m_{\bar{i}3}\, m_{\bar{i}2}\, m_{\bar{i}1} \mapsto 0$$

$$\# \{ \pi_K^{(4)} \to \mathbb{Z}_5 : \text{hom s.t. (i)} \} > 5$$

$\rightsquigarrow$ We can't eliminate the condition $(\bar{i}\bar{i})$

§ An interpretation from the viewpoint of $H_1^{(s)} \longrightarrow \mathbb{Z}/p$

$\phi_c^{(s)} : \pi_K^{(s)} \longrightarrow \mathbb{Z}/p$ goes through $\underline{\text{the abelianization of } \pi_K^{(s)}}$

$$\underset{\displaystyle H_1^{(s)}}{\shortparallel}$$

$$\pi_K^{(1)} = \langle x_1, \cdots, \underset{\underset{\displaystyle x}{\shortparallel}}{x_n} \mid r_1, \cdots, r_{n-1} \rangle$$

$\downarrow$

$$\pi_K^{(s)} = \langle x^s ,\ x^{-j}(x_i x^{-1}) x^j \mid r_1', \cdots, r_{n-1}' \rangle$$

$$\begin{pmatrix} i = 1, \cdots, n-1 \\ j = 0, \cdots, s-1 \end{pmatrix}$$

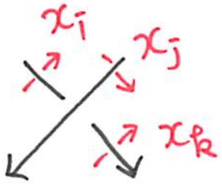
$$x^{-j}(x_i x^{-1}) x^j \underset{\text{abel}}{\longmapsto} t^j a_i , \quad x^s \underset{\text{abel}}{\longmapsto} a$$

$H_1^{(s)} = \langle a, a_1, \cdots, a_{n-1} \mid r_1'', \cdots, r_{n-1}'' \rangle$ as $\mathbb{Z}/p[t.t^{-1}]$-module

$$\#\{\pi_K^{(s)} \to \mathbb{Z}_p : \text{hom s.t. (i) (ii)}\}$$

$$= \#\{H_1^{(s)} \xrightarrow[\psi]{} \mathbb{Z}_p : \text{hom s.t. (i)' (ii)'}\}$$

$\boxed{r_i''}$



$$r = x_k x_j^{-1} x_i^{-1} x_j$$

$$r' = x_k x^{-1} (x_j x^{-1})(x^{-1} x_i x^{-1} x)^{-1} (x^{-1} x_j x^{-1} x)$$

$$r'' = a_k - a_j - t a_i + t a_j$$

$$= -t a_i - (1-t) a_j + a_k$$

$\hookrightarrow$ $t a_i + (1-t) a_j - a_k = 0$

$\boxed{(\mathrm{i})'}$ (i) $x_i^s \xrightarrow{\phi^{(s)}} 0 \quad (\forall i \in \{1, \cdots, n\})$

$$x_i^s = (x_i x^{-1}) x^s (x^{-s+1} x_i x^{-1} x^{s-1}) \cdots (x^{-1} x_i x^{-1} x)$$

$$\xmapsto{\text{abel}} a_i + a + t^{s-1} a_i + t^{s-2} a_i + \cdots + t a_i$$

$$= (t^{s-1} + t^{s-2} + \cdots + 1) a_i$$

↳ $(t^{s-1} + t^{s-2} + \cdots + 1) a_i = 0$

$x^s \xrightarrow{\phi^{(s)}} 0$

↳ $a = 0$

$\boxed{(\mathrm{ii})'}$ (ii) $x_i^{-1} w x_i w^{-\mu} \xrightarrow{\phi^{(s)}} 0$

$$x^{-1}(x_i x^{-1}) x (x_i x^{-1})^{-\mu} \xmapsto{\text{abel}} t a_i - \mu a_i = (t - \mu) a_i$$

↳ $(t - \mu) a_i = 0$

We have a simultaneous equations

$$\left\{\begin{array}{ll} \vdots & \\ ta_i + (1-t)a_j - a_k = 0 & \\ \vdots & \\ a = 0 & \\ \vdots & \\ (t^{s-1} + \cdots + 1)\, a_i = 0 & \\ \vdots & \\ (t-\mu)\, a_i = 0 & \\ \vdots & \end{array}\right.$$

crossing ← Alexander quandle coloring conditions s.t. $x \mapsto 0$

condition(i)

condition(ii)

When we have a non-trivial solution, $t = \mu$ (∵ (ii)).

$$\left\{\begin{array}{l} \vdots \\ \mu a_i + (1-\mu) a_j - a_k = 0 \\ \vdots \end{array}\right.$$

← $X = \mathbb{Z}_p[t, t^{-1}]/(t-\mu)$ coloring conditions s.t. $x \mapsto 0$

$\therefore$ $\#\{X\text{-colorings}\} = p \times \#\{H_1^{(s)} \to \mathbb{Z}_p$ s.t. $\underline{(i)'}$, $(ii)'\}$

we can eliminate

In the case where $\mu = p-1$ (i.e., $X = R_p$)

$$\left\{\begin{array}{rcll}
 & \vdots & & \\
t a_i + (1-t) a_j - a_k & = & 0 & \\
 & \vdots & & \\
a & = & 0 & \\
 & \vdots & & \text{condition (i)} \\
(t+1)\, a_i & = & 0 & \\
 & \vdots & & \\
(t+1)\, a_i & = & 0 & \text{condition (ii)}
\end{array}\right.$$

We can eliminate the condition (ii)

Thank you very much !!

(2) Each X-coloring ϕ_c lifts uniquely to a hom

$$\phi_c^{(s)} : \pi_K^{(s)} \longrightarrow \mathbb{Z}_p$$

s.t. (i) $x_i^s \longmapsto 0$ ($\forall x_i$: meridian generator)

(ii) $x_i^{-1} w x_i w^{-\mu} \longmapsto 0$ ($\forall x_i$, $\forall w \in \pi_K^{(s)}$)

where $s = \min\{s \in \mathbb{N} \mid \mu^s \equiv 1 \mod p\}$

<u>Rem</u> $s =$ type X

$$= \min\{n \in \mathbb{N} \mid {}^\forall a, b \in X,\ a \underbrace{* b * \cdots * b}_{n} = a\}$$

$$
\begin{tikzcd}
\pi_K^{(s)} \arrow[rr, "\phi_c^{(s)}"] \arrow[d] \arrow[drr, phantom, "\Uparrow\ \text{lift}"] & & \mathbb{Z}_p \arrow[d] \\
\pi_K^{(1)} \arrow[rr, "\phi_c"] \arrow[d] & & \mathrm{Inn}(X) \arrow[d] \\
\mathbb{Z}_s \arrow[rr, equal] & & \mathbb{Z}_s
\end{tikzcd}
$$

(i) $x_i^s \longmapsto 0$ ($\forall x_i$: meridian generator)

(ii) $x_i^{-1} w x_i w^{-\mu} \longmapsto 0$ ($\forall x_i$, $\forall w \in \pi_K^{(s)}$)

(2)' Each X-coloring ϕ_c lifts uniquely to a hom.

$$\phi_c^{(s)} : \pi_K^{(s)} \longrightarrow \ker\alpha$$

∵

$$\begin{array}{ccccc}
\phi_c^{(s)} : & \pi_K^{(s)} & \longrightarrow & \ker\alpha \\
& \Psi & & \Psi \\
& w & \longmapsto & \phi_c(w)
\end{array}$$

$$\begin{tikzcd}
\pi_K^{(s)} \arrow[r, "\phi_c^{(s)}"] \arrow[d] \arrow[dr, phantom, "\uparrow\ \text{lifts}"] & \ker\alpha \cong \mathbb{Z}_p \arrow[d] \\
\pi_K^{(1)} \arrow[r, "\phi_c"] \arrow[d, "\beta"'] & \mathrm{Inn}(X) \arrow[d, "\alpha"] \\
\mathbb{Z}_s \arrow[r, equal] & \mathbb{Z}_s
\end{tikzcd}$$

※ $\pi_K^{(s)} = \ker\beta < \pi_K^{(1)}$

- $\phi_c^{(s)}$ satisfies (i) $x_i^s \mapsto 0$

  (ii) $x_i^{-1} w x_i w^{-\mu} \mapsto 0$

(3) $\forall \phi^{(s)} : \pi_K^{(s)} \longrightarrow \mathbb{Z}_p$ s.t.

(i) $x_i^s \mapsto 0$

(ii) $x_i^{-1} w x_i w^{-\mu} \longmapsto 0$

is a lift of exactly p X-colorings

$$\begin{tikzcd}
\pi_K^{(s)} \arrow[rr, "\phi^{(s)}"] \arrow[d] & & \mathbb{Z}_p \arrow[d] \\
\pi_K^{(1)} \arrow[rr, "\exists \text{exactly } p \ \phi_c \text{ s}"] \arrow[d] \arrow[r, phantom, shift left=6, "\text{lift}"] & & \mathrm{Inn}(X) \arrow[d] \\
\mathbb{Z}_s \arrow[rr, equal] & & \mathbb{Z}_s
\end{tikzcd}$$

(3)' $\forall \phi^{(s)} : \pi_K^{(s)} \longrightarrow \ker\alpha$ s.t. (i), (ii) is a lift of exactly p X-colorings.

$$
\begin{tikzcd}
\pi_K^{(s)} \arrow[r, "\phi^{(s)}"] \arrow[d] \arrow[dr, phantom, "\Uparrow \text{lift}"] & \ker\alpha \cong \mathbb{Z}_p \arrow[d] \\
\pi_K^{(1)} \arrow[r, "\text{exactly } p \ \phi_c \text{ s}"] \arrow[d] & \mathrm{Inn}(X) \arrow[d, "\alpha"] \\
\mathbb{Z}/s \arrow[r, equal] & \mathbb{Z}/s
\end{tikzcd}
$$

∵

x : a wirtinger generator in $\pi_K^{(1)}$: fix

$\forall a \in X$, defined ϕ_c s.t. $x \mapsto \xi_a$ by

$$\phi_c(x_{k_1}^{\varepsilon_1} \cdots x_{k_i}^{\varepsilon_i})$$

$$= \phi^{(s)}(x_{k_1}^{\varepsilon_1} \cdots x_{k_i}^{\varepsilon_i}\, x^{-\varepsilon_1 - \cdots - \varepsilon_i})\, \xi_a^{\varepsilon_1 + \cdots + \varepsilon_i}$$

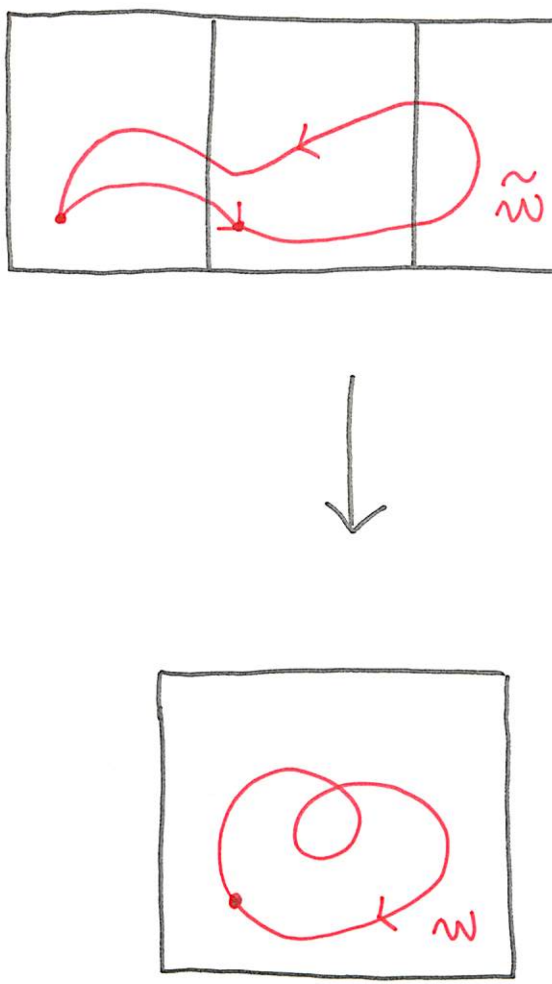
$$\begin{tikzcd}
\pi_K^{(s)} \arrow[r, "\phi^{(s)}"] \arrow[d] & \ker\alpha \arrow[d] \\
\pi_K^{(1)} \arrow[r, "\phi_c"] \arrow[d] & \mathrm{Inn}(X) \arrow[d] \\
\mathbb{Z}/s \arrow[r, equal] & \mathbb{Z}/s
\end{tikzcd}$$

$x \in \pi_K^{(1)} \longmapsto \xi_a \in \mathrm{Inn}(X)$

- ϕ_c : $x_i \longmapsto$ a symmetry

- ϕ_c s.t. $x \mapsto \xi_a$ is uniquely defined

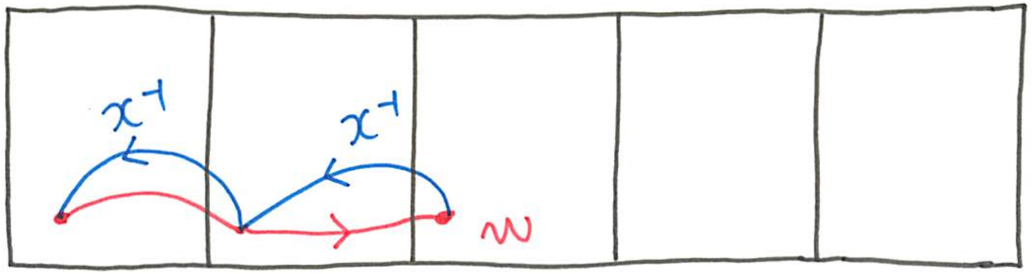
$$\pi_K^{(s)} = \ker\beta = \{ x_{k_1}^{\varepsilon_1} \cdots\cdots x_{k_r}^{\varepsilon_r} \quad \text{s.t. } \varepsilon_1 + \cdots + \varepsilon_r \equiv 0 \bmod s \}$$



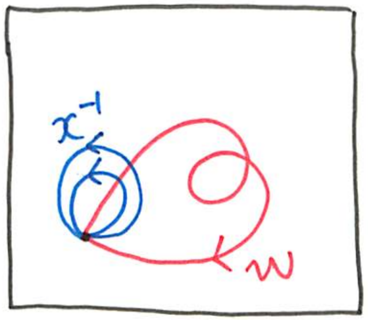
$$\begin{tikzcd}
 & \mathbb{Z}_p \arrow[d, phantom, "\cong" sloped] \\
\pi_K^{(s)} \arrow[r] \arrow[d] & \ker\alpha \arrow[d] \\
\pi_K^{(1)} \arrow[r] \arrow[d, "\beta"'] & \mathrm{Inn}(X) \arrow[d, "\alpha"] \\
\mathbb{Z}_s \arrow[r, equal] & \mathbb{Z}_s
\end{tikzcd}$$

$$\phi_c(w) = \phi_c^{(s)}(w)$$

$$\pi_K^{(s)} = \ker\beta = \{ x_{k_1}^{\varepsilon_1} \cdots\cdots x_{k_i}^{\varepsilon_i} \quad \text{s.t. } \varepsilon_1 + \cdots + \varepsilon_i \equiv 0 \mod s \}$$



$$\begin{tikzcd}
 & \mathbb{Z}_p \\
\pi_K^{(s)} \arrow[r] \arrow[d] & \ker\alpha \arrow[d] \arrow[u, phantom, "\cong" sloped] \\
\pi_K^{(1)} \arrow[r] \arrow[d, "\beta"'] & \mathrm{Inn}(X) \arrow[d, "\alpha"] \\
\mathbb{Z}_s \arrow[r, equal] & \mathbb{Z}_s
\end{tikzcd}$$



$$\phi_c(w x^{-2}) = \phi_c^{(s)}(w x^{-2})$$

$$\shortparallel$$

$$\phi_c(w)\, \zeta_a^{-2}$$

$$\therefore \quad \phi_c(w) = \phi_c^{(s)}(w x^{-2})\, \zeta_a^{2}$$

We have to check that ϕ_c is a homomorphism

We need $\phi^{(s)}(x^{-1} w x) = \xi_a^{-1} \phi^{(s)}(w) \xi_a$

$$\begin{aligned} &\phi^{(s)}(x^{-1} w x) = \xi_a^{-1} \phi^{(s)}(w) \xi_a \\ &\qquad\qquad\qquad\quad \| \\ &\qquad\qquad\qquad (\phi^{(s)}(w))^{\mu} \\ &\qquad\qquad\qquad\quad \| \\ &\qquad\qquad\qquad \phi^{(s)}(w^{\mu}) \qquad \cdots \text{(ii)} \end{aligned}$$

$$\begin{aligned} \phi_c(w_1 w_2) &= \phi^{(s)}(w_1 w_2 x^{-\varepsilon_1 - \varepsilon_2}) \xi_a^{\varepsilon_1 + \varepsilon_2} \\ &= \phi^{(s)}(w_1 x^{-\varepsilon_1} x^{\varepsilon_1} w_2 x^{-\varepsilon_2} x^{-\varepsilon_1}) \xi_a^{\varepsilon_1 + \varepsilon_2} \\ &= \phi^{(s)}(w_1 x^{-\varepsilon_1}) \underbrace{\phi^{(s)}(x^{\varepsilon_1} w_2 x^{-\varepsilon_2} x^{-\varepsilon_1})}_{\xi_a^{\varepsilon_1} \phi^{(s)}(w_2 x^{-\varepsilon_2}) \xi_a^{-\varepsilon_1}} \xi_a^{\varepsilon_1 + \varepsilon_2} \\ &= \phi^{(s)}(w_1 x^{-\varepsilon_1}) \xi_a^{\varepsilon_1} \cdot \phi^{(s)}(w_2 x^{-\varepsilon_2}) \xi_a^{\varepsilon_2} \\ &= \phi_c(w_1) \cdot \phi_c(w_2) \end{aligned}$$

$$\begin{tikzcd}
\pi_K^{(s)} \arrow[r, "\phi^{(s)}"] \arrow[d] & \ker\alpha \arrow[d] \\
\pi_K^{(1)} \arrow[r, "\exists!\, \phi_c"] \arrow[d] & \mathrm{Inn}(X) \arrow[d, "\alpha"] \\
\mathbb{Z}_s \arrow[r, equal] & \mathbb{Z}_s
\end{tikzcd}$$

lifts

$\pi_K^{(1)} \ni x \longmapsto \xi_a \in \mathrm{Inn}(X)$

$\phi^{(s)}$ is a lift of exactly p X-colorings

∵ $\# X = p$

//