Higgs bundles and higher Teichmüller components

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- S oriented smooth compact surface of genus $g \geq 2$
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- $\mathrm{Hom}(\pi_1(S), G)$ is an analytic variety, which is algebraic if G is algebraic
- G acts on $\mathrm{Hom}(\pi_1(S), G)$ by conjugation:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1} \text{ for } g \in G, \rho \in \mathrm{Hom}(\pi_1(S), G)$$

Moduli space of representations

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- Complex algebraic geometry approach: **Higgs bundles**

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- θ Cartan involution of $\mathfrak{g}^{\mathbb{R}}$, Lie algebra of $G^{\mathbb{R}}$, defining the **Cartan decomposition**:

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{h}^{\mathbb{R}} + \mathfrak{m}^{\mathbb{R}}$$

  where $\mathfrak{h}^{\mathbb{R}}$ is the Lie algebra of $H^{\mathbb{R}}$
  We have $[\mathfrak{m}^{\mathbb{R}}, \mathfrak{m}^{\mathbb{R}}] \subset \mathfrak{h}^{\mathbb{R}}$, $[\mathfrak{h}^{\mathbb{R}}, \mathfrak{m}^{\mathbb{R}}] \subset \mathfrak{m}^{\mathbb{R}}$
- The Cartan decomposition is orthogonal with respect to the Killing form of $\mathfrak{g}^{\mathbb{R}}$
- Complexification of **isotropy representation**
  Let H and $\mathfrak{m}$ be the complexifications of $H^{\mathbb{R}}$ and $\mathfrak{m}^{\mathbb{R}}$ respectively

$$\iota : H \to \mathrm{GL}(\mathfrak{m})$$

A $G^{\mathbb{R}}$-**Higgs bundle** on X is a pair (E, φ) consisting of

- E a holomorphic principal H-bundle over X
- φ a holomorphic section of $E(\mathfrak{m}) \otimes K$,
  where $E(\mathfrak{m})$ is the associated vector bundle with fibre $\mathfrak{m}$
  via the complexified isotropy representation
  and K is the canonical line bundle of X

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- There are notions of **stability**: consider for $s \in i\mathfrak{h}^{\mathbb{R}}$:
- Parabolic subgroup
  $P_s = \{g \in H \; : \; e^{ts}ge^{-ts}$ is bounded as $t \to \infty\}$
- Character $\chi_s : \mathfrak{p}_s \to \mathbb{C}$ defined by s ($\mathfrak{p}_s$ Lie algebra of P_s)
- Subspace $\mathfrak{m}_s = \{Y \in \mathfrak{m} \; : \; \iota(e^{ts})Y$ is bounded as $t \to \infty\}$
- For σ a reduction of E to P_s

$$\deg(E)(\sigma, s) := \frac{i}{2\pi} \int_X \chi_s(F).$$

F: curvature of a connection on the P_s-bundle defined by σ

Stability of $G^{\mathbb{R}}$-Higgs bundles

(E, φ) is:

- **stable** if

$$\deg(E)(\sigma, s) > 0$$

for any $s \in i\mathfrak{h}^{\mathbb{R}}$ and any holomorphic reduction $\sigma \in \Gamma(E(H/P_s))$ such that $\varphi \in H^0(X, E_\sigma(\mathfrak{m}_s) \otimes K)$

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- **polystable** if (E, φ) can be reduced to a $G'^{\mathbb{R}}$-Higgs bundle, with $G'^{\mathbb{R}} \subset G^{\mathbb{R}}$ reductive and (E, φ) stable as a $G'^{\mathbb{R}}$-Higgs bundle

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  The **moduli space of polystable $G^{\mathbb{R}}$-Higgs bundles** $\mathcal{M}(X, G^{\mathbb{R}})$ is the set of isomorphism classes of polystable $G^{\mathbb{R}}$-Higgs bundles

- $\mathcal{M}(X, G^{\mathbb{R}})$ is as complex algebraic variety

Higgs bundles

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- When $G^{\mathbb{R}}$ is a clasical group we can formulate the theory in terms of vector bundles
- In this case $H^{\mathbb{R}} = \mathrm{SU}(n)$, $H = \mathrm{SL}(n, \mathbb{C})$ and $\mathfrak{m} = \mathfrak{sl}(n, \mathbb{C})$
  Hence, an $\mathrm{SL}(n, \mathbb{C})$-**Higgs bundle** is equivalent to a pair (V, φ)
  V rank n holomorphic vector bundle with $\det V = \mathcal{O}$
  $\varphi : V \to V \otimes K$ with $\mathrm{Tr}\, \varphi = 0$
- (V, φ) is **stable**:
  $\deg(V') < 0$ for every $V' \subset V$ such that $\varphi(V') \subset V' \otimes K$
  (V, φ) is **polystable**:
  $(V, \varphi) = \oplus(V_i, \varphi_i)$ with $\deg V_i = 0$ and (V_i, φ_i) stable
- We recover the original notions introduced by **Hitchin** (1987)

Higgs bundles

$G^{\mathbb{R}} = \mathrm{SU}(p,q)$

- In this case $H^{\mathbb{R}} = \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$, $H = \mathrm{S}(\mathrm{GL}(p) \times \mathrm{GL}(q))$, and $\mathfrak{m} = \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p) \oplus \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q)$
  Hence, an $\mathrm{SU}(p,q)$-**Higgs bundle** is equivalent to a tuple (V, W, β, γ)
  V and W are rank p and q holomorphic vector bundles, respectively, with $\det V \otimes \det W = \mathcal{O}$
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- (V, W, β, γ) is **stable**:
  $\deg(V') + \deg(W') < 0$ for every $V' \subset V$ and $W' \subset W$ such that $\beta(W') \subset V' \otimes K$ and $\gamma(V') \subset W' \otimes K$
  (V, W, β, γ) is **polystable** if the associated $\mathrm{SL}(p+q, \mathbb{C})$-Higgs bundle

$$V \oplus W \quad \text{and} \quad \varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$$

  is polystable

Higgs bundles

Theorem

A $G^{\mathbb{R}}$-Higgs (E, φ) is polystable if and only if there exists a reduction h of the structure group of E from H to $H^{\mathbb{R}}$, such that

$$F_h - [\varphi, \tau_h(\varphi)] = 0 \quad \textbf{(Hitchin equation)}$$

- $\tau_h : \Omega^{1,0}(E(\mathfrak{m})) \to \Omega^{0,1}(E(\mathfrak{m}))$ is the combination of the anti-holomorphic involution in $E(\mathfrak{m})$ defined by the compact real form at each point determined by h and the conjugation of 1-forms
- F_h is the curvature of the unique $H^{\mathbb{R}}$-connection compatible with the holomorphic structure of E

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Proved by: **Hitchin** (1987) for $G^{\mathbb{R}} = \mathrm{SL}(2,\mathbb{C})$, **Simpson** (1988) for general $G^{\mathbb{R}}$, and **Bradlow–G–Mundet** (2003) & **G–Gothen–Mundet** (2009) for general $G^{\mathbb{R}}$ (direct proof)

Non-abelian Hodge correspondence

Let S be a smooth compact surface and J be a complex structure on S. Let $X = (S, J)$. There is a homeomorphism

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- Let (E, φ) be a polystable G-Higgs bundle and h a solution to Hitchin equations

$$\nabla = \bar{\partial}_E - \tau_h(\bar{\partial}_E) + \varphi - \tau_h(\varphi)$$

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- **Converse**: Existence of a **harmonic metric** on a reductive flat $G^{\mathbb{R}}$-bundle. Proved by **Donaldson** (1987) for $G^{\mathbb{R}} = \mathrm{SL}(2, \mathbb{C})$ and **Corlette** (1988) for real reductive $G^{\mathbb{R}}$

Topological invariants

- Given $\rho\colon \pi_1(S) \to G^{\mathbb{R}}$, there is an associated flat $G^{\mathbb{R}}$-bundle on S, defined as $E_\rho = \widetilde{S} \times_\rho G^{\mathbb{R}}$ ($\widetilde{S}$: universal cover of S):

  $\mathrm{Hom}(\pi_1(S), G^{\mathbb{R}})/G^{\mathbb{R}} \cong H^1(S, G^{\mathbb{R}}) =$ iso. classes of flat $G^{\mathbb{R}}$-bundle

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- Let $\widetilde{G}^{\mathbb{R}}$ be the universal covering group of $G^{\mathbb{R}}$. We have an exact sequence
$$1 \to \pi_1(G^{\mathbb{R}}) \to \widetilde{G}^{\mathbb{R}} \to G^{\mathbb{R}} \to 1$$
which gives rise to the (pointed sets) cohomology sequence
$$H^1(S, \widetilde{G}^{\mathbb{R}}) \to H^1(S, G^{\mathbb{R}}) \xrightarrow{c} H^2(S, \pi_1(G^{\mathbb{R}}))$$

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- **topological invariant** of ρ:
  $c(\rho) := c(E_\rho) \in H^2(X, \pi_1(G^{\mathbb{R}})) \cong \pi_1(G^{\mathbb{R}})$
- We can define the subvariety
  $$\mathcal{R}_c(S, G^{\mathbb{R}}) := \{\rho \in \mathcal{R}(S, G^{\mathbb{R}}) \ : \ c(\rho) = c\}$$

Topological invariants

- Similarly, we can define a topological invariant of a $G^{\mathbb{R}}$-Higgs bundle (E, φ) over X as the topological class of the H-bundle E (recall $H^{\mathbb{R}} \subset G^{\mathbb{R}}$ is a maximal compact subgroup)

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- $H^1(X, \underline{H})$ = isomorphisms classes of H-bundles
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Topological invariants

- Recall $\pi_1(G^{\mathbb{R}}) \cong \pi_1(H^{\mathbb{R}}) \cong \pi_1(H)$
- For $c \in \pi_1(G^{\mathbb{R}}) \cong \pi_1(H)$ we have de homeomorphism

$$\mathcal{R}_c(S, G^{\mathbb{R}}) \cong \mathcal{M}_c(X, G^{\mathbb{R}})$$

Theorem

If $G^{\mathbb{R}}$ is compact (Ramanathan, 1975) or complex (J. Li, 1993; G-Oliveira, 2017)

$$\pi_0(\mathcal{R}(S, G^{\mathbb{R}})) = \pi_0(\mathcal{M}(X, G^{\mathbb{R}})) \cong \pi_1(G^{\mathbb{R}})$$

- The story is very different for **non-compact** real Lie groups (non-complex): The map

$$\pi_0(\mathcal{R}(S, G^{\mathbb{R}})) = \pi_0(\mathcal{M}(X, G^{\mathbb{R}})) \to \pi_1(G^{\mathbb{R}})$$

is neither injective, nor surjective in general

$G^{\mathbb{R}} = \mathrm{SL}(2,\mathbb{R})$

- The topological invariant of $\rho \in \mathcal{R}(S, \mathrm{SL}(2,\mathbb{R}))$ in this case is an integer (basically the **Euler class**) $d \in \mathbb{Z} \cong \pi_1(G^{\mathbb{R}})$
- $$\mathcal{R}_d := \{\rho \in \mathcal{R}(S, \mathrm{SL}(2,\mathbb{R})) \quad : \quad \text{with Euler class } \ d\}$$

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- An $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle is a tuple (L, β, γ)
  L line bundle over X $\quad \beta \in H^0(X, L^2K)$ and
  $\gamma \in H^0(X, L^{-2}K)$
  Equivalently it can be seen as an $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle
  (V, φ) with $V = L \oplus L^{-1}$ and $\varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$
- Milnor's inequality follows from the semistability of (V, φ) (**Hitchin**, 1987)

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Theorem (Goldman, 1988; Hitchin 1987)

- $\mathcal{R}_d$ is connected if $|d| < g - 1$
- $\mathcal{R}_d$ has 2^{2g} connected components if $|d| = g - 1$

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- Let $\mathcal{R}_{\max} := \mathcal{R}_d$ for $|d| = g - 1$
- Each connected component of $\mathcal{R}_{\max}$ consists entirely of **Fuchsian representations** (discrete and faithful) and can be identified with the **Teichmüller space** $\mathcal{T} = \mathcal{T}(S)$ of the surface S (**Goldman**, 1980)

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- **Question**: Are there other simple groups with similar features to those of $\mathrm{SL}(2, \mathbb{R})$. More precisely, whose moduli space has connected components not distinguished by the topological invariant and consisting entirely of discrete and faithful representations?

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- **Split** real groups
- Non-compact groups of **Hermitian type**

Split real forms

- **Split** real form: in the Cartan decomposition $\mathfrak{g}^{\mathbb{R}} = \mathfrak{h}^{\mathbb{R}} \oplus \mathfrak{m}^{\mathbb{R}}$, the space $\mathfrak{m}^{\mathbb{R}}$ contains a maximal abelian subalgebra of $\mathfrak{g}^{\mathbb{R}}$
- Every complex semisimple Lie group has a split real form
  Examples: $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}(n,n)$, $\mathrm{SO}(n,n+1)$

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- Consider $G^{\mathbb{R}} = \mathrm{SL}(n, \mathbb{R})$
  A basis for the **invariant polynomials** on $\mathfrak{sl}(n, \mathbb{C})$ is provided by the coefficients of the characteristic polynomial of a trace-free matrix,

$$\det(x - A) = x^n + p_1(A)x^{n-2} + \ldots + p_{n-1}(A),$$

  where $\deg(p_i) = i + 1$.

Split real forms

- Consider the **Hitchin map**

$$p : \mathcal{M}(X, \mathrm{SL}(n,\mathbb{C})) \to \bigoplus_{i=1}^{n-1} H^0(K^{i+1})$$

  defined by $p(E,\varphi) = (p_1(\varphi), \ldots, p_{n-1}(\varphi))$,
- **Hitchin** (1992) constructed a **section** of this map giving an isomorphism between the vector space $\bigoplus_{i=1}^{n-1} H^0(K^{i+1})$ and a **connected component** of the moduli space $\mathcal{M}(X, \mathrm{SL}(n,\mathbb{R})) \subset \mathcal{M}(X, \mathrm{SL}(n,\mathbb{C}))$
- This is called a **Hitchin component** (coincides with a Teichmüller component $\cong H^0(X, K^2)$ when $G^{\mathbb{R}} = \mathrm{SL}(2,\mathbb{R})$)

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- **Hitchin** (1992) constructed a **section** of this map giving an isomorphism between the vector space $\bigoplus_{i=1}^{n-1} H^0(K^{i+1})$ and a **connected component** of the moduli space $\mathcal{M}(X, \mathrm{SL}(n, \mathbb{R})) \subset \mathcal{M}(X, \mathrm{SL}(n, \mathbb{C}))$
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- Hitchin gives a general construction for any split real form

Split real forms

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- Hitchin gives a general construction for any split real form
- General construction of a section of the Hitchin map for arbitrary $G^{\mathbb{R}}$ (**G–Peón-Nieto–Ramanan**, 2018)

Split real forms

- Every representation in the Hitchin component can be deformed to a representation factoring as $\pi_1(S) \to \mathrm{SL}(2,\mathbb{R}) \to G^{\mathbb{R}}$, where $\pi_1(S) \to \mathrm{SL}(2,\mathbb{R})$ is in a Teichmüller component and $\mathrm{SL}(2,\mathbb{R}) \to G^{\mathbb{R}}$ is the **principal representation**

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- The Hitchin component for $\mathrm{PSL}(3,\mathbb{R})$ parameterizes convex projective structures on the surface (**Choi–Goldman**, 1993)
- In general, the Hitchin component parameterizes certain type of **geometric structures** modeled on certain homogeneous spaces (**Guichard–Wienhard**, 2012)

Non-compact real forms of Hermitian type

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- Two **exceptional** real forms of E_6^{-14} and E_7^{-25}

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- Let (E,φ) be a $G^{\mathbb{R}}$-Higgs bundle over X.
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- The torsion-free part of $\pi_1(H)$ is isomorphic to $\mathbb{Z}$ (most of the time $\pi_1(H^{\mathbb{R}}) \cong \mathbb{Z}$) and hence the topological invariant of either a representation of $\pi_1(X)$ in $G^{\mathbb{R}}$, or of a $G^{\mathbb{R}}$-Higgs bundle, is essentialy given by an **integer** $d \in \mathbb{Z}$, known as the **Toledo invariant**

Non-compact real forms of Hermitian type

Theorem (Milnor–Wood inequality)

The Toledo invariant d satisfies

$$|d| \leq \operatorname{rank}(G^{\mathbb{R}}/H^{\mathbb{R}})(g-1)$$

- Proved for the classical groups for representations by **Domic–Toledo** (1987) and for Higgs bundles by **Bradlow–G–Gothen** (2001)
- General proof for representations by **Burger–Iozzi–Wienhard** (2003), and for Higgs bundles by **Biquard–G–Rubio** (2017)

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- $\mathcal{D}$ is of **tube type** if and only if $\check{S}$ is a **compact symmetric space** of the form $H^{\mathbb{R}}/H'^{\mathbb{R}}$. In this case $\Omega = H^{\mathbb{R}}_*/H'^{\mathbb{R}}$ is its **non-compact dual symmetric space**

Non-compact real forms of Hermitian type

- The symmetric spaces defined by $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}_0(2,n)$ are of tube type.
- The symmetric space defined by $\mathrm{SU}(p,q)$ is of tube type if and only if $p=q$.
- The symmetric space defined by $\mathrm{SO}^*(2n)$ is of tube type if and only if n is even.
- The E_7^{-25} Hermitian real form is of tube type
- The E_6^{-14} Hermitian real form is **not** of tube type
- Every bounded symmetric domain has a maximal tube subdomain

Non-compact real forms of Hermitian type

- Want to study the **Maximal Toledo invariant** moduli space in the tube case (non-tube reduces to the tube case)

$$\mathcal{M}_{\max}(X, G^{\mathbb{R}}) := \mathcal{M}_d(X, G^{\mathbb{R}}) \text{ for } |d| = \operatorname{rank}(G^{\mathbb{R}}/H^{\mathbb{R}})(g-1)$$

Theorem (Cayley Correspondence)

Let $G^{\mathbb{R}}$ be a such $G^{\mathbb{R}}/H^{\mathbb{R}}$ is a Hermitian symmetric space of tube type, and let $\Omega = H_*^{\mathbb{R}}/H'^{\mathbb{R}}$ be the non-compact dual of the Shilov boundary $\check{S} = H^{\mathbb{R}}/H'^{\mathbb{R}}$ of $G^{\mathbb{R}}/H^{\mathbb{R}}$. Then

$$\mathcal{M}_{\max}(X, G^{\mathbb{R}}) \cong \mathcal{M}_{K^2}(X, H_*^{\mathbb{R}}),$$

where $\mathcal{M}_{K^2}(H_*^{\mathbb{R}})$ is the moduli space of K^2-twisted $H_*^{\mathbb{R}}$-Higgs bundles

- Proved for the classical groups by **Bradlow–G–Gothen** (2006) **G–Gothen–Mundet** (2013) ($G^{\mathbb{R}} = \mathrm{Sp}(2n, \mathbb{R})$)
- General case proved by **Biquard–G–Rubio** (2017)

Non-compact real forms of Hermitian type

- The connected components of $\mathcal{M}(X, G^{\mathbb{R}})$ are not fully distinguished by the usual topological invariants. The dual group $H_*^{\mathbb{R}}$ detects **new hidden invariants** (for example for $G^{\mathbb{R}} = \mathrm{Sp}(2n, \mathbb{R})$, $H_*^{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$ — **Stiefel–Whitney classes**
- $\mathcal{R}_{\max}(S, G^{\mathbb{R}})$ consists entirely of discrete and faithful representations (**Burger–Iozzi–Labourie–Wienhard**, 2006)
- The mapping class group of S acts properly on $\mathcal{R}_{\max}(S, G^{\mathbb{R}})$ (**Wienhard**, 2006)
- All common features with **Hitchin components**

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- **Question:** Are there other groups besides **split** and **hermitian** real forms for which **higher Teichmüller components** exist?

- Joint work with **M. Aparicio, S. Bradlow, B. Collier, P. Gothen and A. Oliveira**, Comptes Rendus Mathematiques (2018), and Inventiones Math. (2019)
- $\mathrm{SO}(p,q)$-**Higgs bundle**: triple (V, W, η) where V and W are respectively rank p and rank q vector bundles with orthogonal structures such that $\det(W) \simeq \det(V)$, and $\eta : W \to V \otimes K$

Higher Teichmüller components for $G^{\mathbb{R}} = \mathrm{SO}(p,q)$

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- For $p > 2$, rank p orthogonal bundles on X are classified topologically by their **first and second Stiefel–Whitney classes**, $sw_1 \in H^1(X, \mathbb{Z}_2)$ and $sw_2 \in H^2(X, \mathbb{Z}_2)$

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- Since $\det(W) \simeq \det(V)$ $sw_1(V) = sw_1(W)$, the components of the moduli space $\mathcal{M}(\mathrm{SO}(p, q))$ are thus **partially** labeled by triples $(a, b, c) \in \mathbb{Z}_2^{2g} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $a = sw_1(V) \in H^1(X, \mathbb{Z}_2)$, $b = sw_2(V) \in H^2(X, \mathbb{Z}_2)$, and $c = sw_2(W) \in H^2(X, \mathbb{Z}_2)$

Higher Teichmüller components for $G^{\mathbb{R}} = \mathrm{SO}(p,q)$

$$\mathcal{M}(\mathrm{SO}(p,q)) = \coprod_{(a,b,c)\in\mathbb{Z}_2^{2g}\times\mathbb{Z}_2\times\mathbb{Z}_2} \mathcal{M}^{a,b,c}(\mathrm{SO}(p,q))$$

Theorem

Assume that $2 < p \leq q$. For every $(a,b,c) \in \mathbb{Z}_2^{2g} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ the space $\mathcal{M}^{a,b,c}(\mathrm{SO}(p,q))$ has a non-empty connected component denoted by $\mathcal{M}^{a,b,c}_{\mathrm{top}}(\mathrm{SO}(p,q))$

- Define

$$\mathcal{M}_{\mathrm{top}}(\mathrm{SO}(p,q)) = \coprod_{a,b,c} \mathcal{M}^{a,b,c}_{\mathrm{top}}(\mathrm{SO}(p,q))$$

- Our main result shows that the moduli space $\mathcal{M}(\mathrm{SO}(p,q))$ has additional **exotic** components disjoint from the components of $\mathcal{M}_{\mathrm{top}}(\mathrm{SO}(p,q))$

Theorem (Generalized Cayley Correspondence)

Fix integers (p,q) such that $2 < p < q-1$. For each choice of $a \in \mathbb{Z}_2^{2g}$ and $c \in \mathbb{Z}_2$, the moduli space $\mathcal{M}(\mathrm{SO}(p,q))$ has a connected component disjoint from $\mathcal{M}_{\mathrm{top}}(\mathrm{SO}(p,q))$. This component is isomorphic to

$$\mathcal{M}^{a,c}_{K^p}(\mathrm{SO}(1,q-p+1)) \times H^0(K^2) \times \cdots \times H^0(K^{2p-2})$$

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- **Conjecture**: These exotic components are higher Teichmüller components (i.e. consist entirely of discrete and faithful representations)
- **Evidence**: Notion of **positivity** recently introduced by Guichard and Wienhard
- The **only** classical groups admiting positive structures are: **split groups, hermitian groups of tube type** and groups locally isomorphic to $\mathrm{SO}(p,q)$!!!

Anosov and positive representations

- Let $P \subset G^{\mathbb{R}}$ be a parabolic subgroup. Let $L \subset P$ be the Levi factor of P. The homogeneous space $G^{\mathbb{R}}/L$ is the unique open $G^{\mathbb{R}}$ orbit in $G^{\mathbb{R}}/P \times G^{\mathbb{R}}/P$, and points $(x, y) \in G^{\mathbb{R}}/P \times G^{\mathbb{R}}/P$ in this open orbit are called **transverse**.
- Let $\partial_\infty \pi_1(S)$ be the **Gromov boundary** of $\pi_1(S)$. Topologically $\partial_\infty \pi_1(S) \cong \mathbb{RP}^1$.

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- A representation $\rho : \pi_1(S) \to G^{\mathbb{R}}$ is P**-Anosov** if there exists a unique continuous boundary map $\xi_\rho : \partial_\infty \pi_1(S) \to G^{\mathbb{R}}/P$ satisfying
  - Equivariance: $\xi(\gamma \cdot x) = \rho(\gamma) \cdot \xi(x)$ for all $\gamma \in \pi_1(S)$ and all $x \in \partial_\infty \pi_1(S)$.
  - Transversality: for all distinct $x, y \in \partial_\infty \pi_1(S)$ the generalized flags $\xi(x)$ and $\xi(y)$ are transverse.
  - Dynamics preserving

  The map ξ_ρ is called the P**-Anosov boundary curve**

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- Anosov representations were introduced by **Labourie** (2006) and have many interesting geometric and dynamic properties
- Anosov representations are discrete and faithful and define an open subset of the moduli space of representations $\mathcal{R}(G^{\mathbb{R}})$. The set of Anosov representations is however not closed.

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- The special cases of Hitchin representations and maximal representations define connected components of Anosov representations. Both Hitchin representations and maximal representations satisfy an additional "positivity" property which is a closed condition. For Hitchin representations this was proved by **Labourie** (2006) and **Fock–Goncharov** (2006), and for maximal representations by **Burger–Iozzi–Wienhard** (2010).

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- A pair $(G^{\mathbb{R}}, P)$ admits a positive structure if there is a certain semigroup $U_{>0} \subset U$, which gives rise to a well defined notion of **positively oriented triples** of pairwise transverse points in $G^{\mathbb{R}}/P$

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- If the pair $(G^{\mathbb{R}}, P)$ admits a positive structure, then a P-Anosov representation $\rho : \pi_1(S) \to G$ is called **positive** if the Anosov boundary curve $\xi : \partial_\infty \pi_1(S) \to G^{\mathbb{R}}/P$ sends positively ordered triples in $\partial_\infty \pi_1(S)$ to positive triples in $G^{\mathbb{R}}/P$

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- **Conjecture (Guichard–Labourie–Wienhard)**: If $(G^{\mathbb{R}}, P)$ admits a notion of positivity, then the set of P-positive Anosov representations is open and closed in $\mathcal{R}(S, G^{\mathbb{R}})$, and hence define connected components

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- **Higher Teichmüller components**: connected components of $\mathcal{R}(S, G^{\mathbb{R}})$ consisting of positive Anosov representations. These components are not labeled by primary topological invariants

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- **Conjecture**: Higher Teichmüller components (:= those consisting of positive representations) coincide with **Cayley components**,

Nilpotents and embeddings of Teichmüller space

- $e \in \mathfrak{g}$ nilpotent i.e. $\mathrm{ad}_e : \mathfrak{g} \to \mathfrak{g}$ nilpotent endomorphism.
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- This gives $\iota_e : \mathcal{T} \hookrightarrow \mathcal{M}(G^{\mathbb{R}})$ (where $\mathcal{T}$ be the Teichmüller component in $\mathcal{M}(\mathrm{PSL}_2\,\mathbb{R})$), whose image depends only on the conjugacy class of e.

Nilpotents and embeddings of Teichmüller space

- For most nilpotents, $\iota_e(\mathcal{T})$ lies in a connected component of $\mathcal{M}(G^{\mathbb{R}})$ containing Higgs bundles with $\varphi \equiv 0$, corresponding to compact representations $\pi_1(S) \to H^{\mathbb{R}} \subset G^{\mathbb{R}}$, hence not discrete and faithful.

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- To every magical nilpotent of $\mathfrak{g}$ there is an associated **canonical real form** $\mathfrak{g}^{\mathbb{R}}$ of $\mathfrak{g}$.
- Before properly defining these objects, let us state our main results — **joint with Steve Bradlow, Brian Collier, Peter Gothen and André Oliveira (BCGGO)**

Theorem 1 (BCGGO)

A real form $G^{\mathbb{R}}$ is such that $\mathfrak{g}^{\mathbb{R}}$ arises as the canonical real form of a magical nilpotent of $\mathfrak{g}$ if and only if it is either:

- split;
- Hermitian of tube type;
- locally isomorphic to $\mathrm{SO}(p,q)$ with $1 < p \leq q$;
- locally isomorphic to E_6^2, E_7^{-5}, E_8^{-24} or F_4^4.

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Corollary

A real form $G^{\mathbb{R}}$ arises as the canonical real form of a magical nilpotent of $\mathfrak{g}$ if and only if it admits a positive structure.

Theorem 2 (BCGGO)

Let $e \in \mathfrak{g}$ be a magical nilpotent, with corresponding canonical real form $\mathfrak{g}^{\mathbb{R}}$. If $G^{\mathbb{R}}$ is a Lie group with Lie algebra $\mathfrak{g}^{\mathbb{R}}$, there exists a union of connected components $\mathcal{H}_e(G^{\mathbb{R}})$ of $\mathcal{M}(G^{\mathbb{R}})$ s.t.:

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- $\mathcal{H}_e(G^{\mathbb{R}})$ can be parameterized as

$$\mathcal{H}_e(G^{\mathbb{R}}) \cong \mathcal{M}_{K^{m_c+1}}(G^{\mathbb{R}}_{\mathcal{C}}) \times \bigoplus_{j=1,\ j\neq c}^{N} H^0(X, K^{m_j+1})$$

  with $G^{\mathbb{R}}_{\mathcal{C}}$ a real Lie group — the **Cayley partner** of $G^{\mathbb{R}}$ — and $m_1, \ldots, m_N \in \mathbb{N}$ depending only on (the conjugacy class of) e.

Previously known cases

- The parameterization of Theorem 2 is given by a morphism

$$\Psi : \mathcal{M}_{K^{m_c+1}}(G^{\mathbb{R}}_{\mathcal{C}}) \times \bigoplus_{j=1,\, j\neq c}^{N} H^0(X, K^{m_j+1}) \hookrightarrow \mathcal{M}(G^{\mathbb{R}})$$

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- $G^{\mathbb{R}}$ split $\rightsquigarrow G^{\mathbb{R}}_{\mathcal{C}} = \mathbb{R}^+$, m_i are the exponents of $\mathfrak{g}$ and Ψ is just the Hitchin section.

Using our parameterization, we also proved that

Theorem 3 (BCGGO)

There are no representations in $\mathcal{H}_e(G^{\mathbb{R}})$ which factor through proper parabolic subgroups of $G^{\mathbb{R}}$.

This should imply that every representations in the components are positive.

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Rest of the talk: (1) define the objects appearing in **Theorem 2**; (2) give an idea of the parametrization Ψ of $\mathcal{H}_e(G^{\mathbb{R}})$.

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$$\mathfrak{g} = \bigoplus_{j=0}^{N} W_j = W_0 \oplus \bigoplus_{j=1}^{N} W_j$$

$W_j \rightsquigarrow$ direct sum of $n_j \geq 0$ copies of the unique $(j+1)$-dimensional irreducible $\mathfrak{sl}_2\mathbb{C}$-representation.
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- $\mathfrak{g}^e = \ker(\mathrm{ad}_e) \rightsquigarrow$ Centralizer of e:

$$\mathfrak{g}^e = \oplus_{j=0}^{N} V_j$$

with $V_j = W_j \cap \mathfrak{g}_j$ the highest weight subspaces ($V_0 = W_0$).

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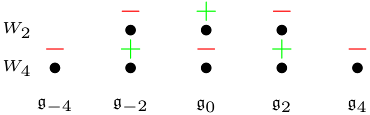
- $\mathfrak{g}^{\mathbb{R}} \rightsquigarrow$ **canonical real form** associated to e.

Examples of magical nilpotents

- $\mathfrak{g} = \mathfrak{sl}_3\mathbb{C}$;

$$e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$



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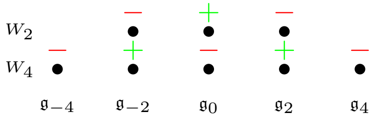

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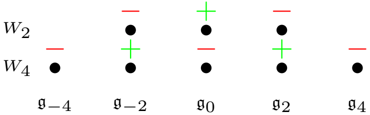

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In general, the canonical real form of the principal nilpotent is the split one.

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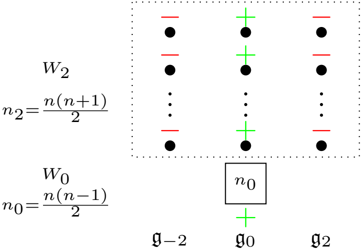

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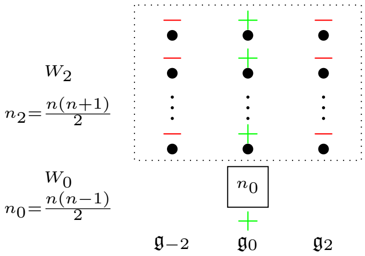

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Examples of magical nilpotents

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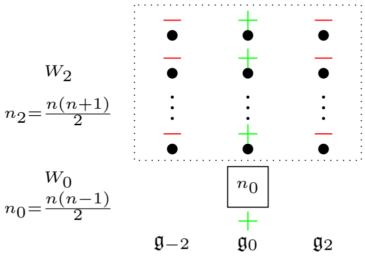

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- The same nilpotent e thought of as an element in $\mathfrak{g} = \mathfrak{sl}_{2n}\mathbb{C}$ is also magical, and $\dim \mathfrak{h}^{\mathbb{R}} = 2n^2 - 1$ so $\mathfrak{g}^{\mathbb{R}} = \mathfrak{su}_{n,n}$.

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- $[Z_{2m_i}, Z_{2m_j}] \subset W_0$ for all i, j ($\Leftrightarrow e$ magical);
- $[Z_{2m_i}, Z_{2m_j}] = 0$ for all $i \neq j$;
- $\exists$ at most one $c \in \{1, \ldots, N\}$ such that $[Z_{2m_c}, Z_{2m_c}] \neq 0$ ($\Leftrightarrow n_{2m_c} > 1$).
- $\Longrightarrow Z_{2m_j} = \mathbb{C}$ for $j \neq c$.

The Cayley partner group

- Define a Lie algebra involution $\theta_e : \mathfrak{g}_0 \to \mathfrak{g}_0$ (recall that $\mathfrak{g}_0 = \mathfrak{g}^h$) by

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- Now we know the objects:

$$\mathcal{H}_e(G^{\mathbb{R}}) \cong \mathcal{M}_{K^{m_c+1}}(G_{\mathcal{C}}^{\mathbb{R}}) \times \bigoplus_{j=1, j\neq c}^{N} H^0(X, K^{m_j+1})$$

$$= \mathcal{M}_{K^{m_c+1}}(G_{\mathcal{C}}^{\mathbb{R}}) \times \prod_{j=1, j\neq c}^{N} \mathcal{M}_{K^{m_j+1}}(\mathbb{R}^+)$$

Back to Higgs bundles

- $(E_T, f) \rightsquigarrow$ the $\mathrm{PSL}_2\,\mathbb{R}$-Higgs bundle induced by the $\mathrm{SL}_2\,\mathbb{R}$-Higgs bundle

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- $C \rightsquigarrow$ the subgroup of G whose Lie algebra is $W_0 \subset \mathfrak{g}$.
- C is the complexification of the maximal compact subgroup of the Cayley partner $G_{\mathcal{C}}^{\mathbb{R}}$.

The parametrization

$$\Psi : \mathcal{M}_{K^{m_c+1}}(G^{\mathbb{R}}_{\mathcal{C}}) \times \bigoplus_{j=1, j\neq c}^{N} H^0(X, K^{m_j+1}) \longrightarrow \mathcal{M}(G^{\mathbb{R}})$$

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Theorem 4 (BCGGO)

Ψ is an isomorphism onto its image, which is open and closed in $\mathcal{M}(G^{\mathbb{R}})$.