Granular Materials III

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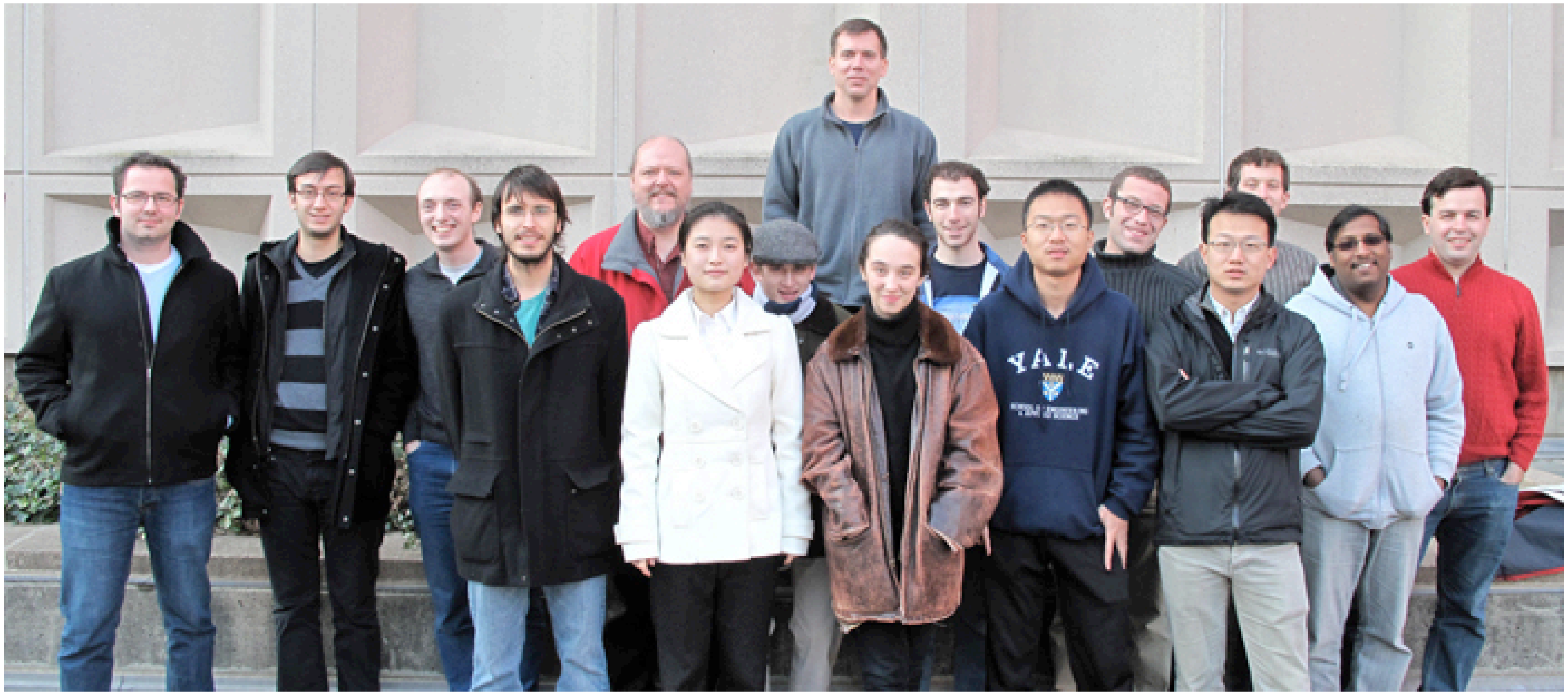
The O'Hern Group



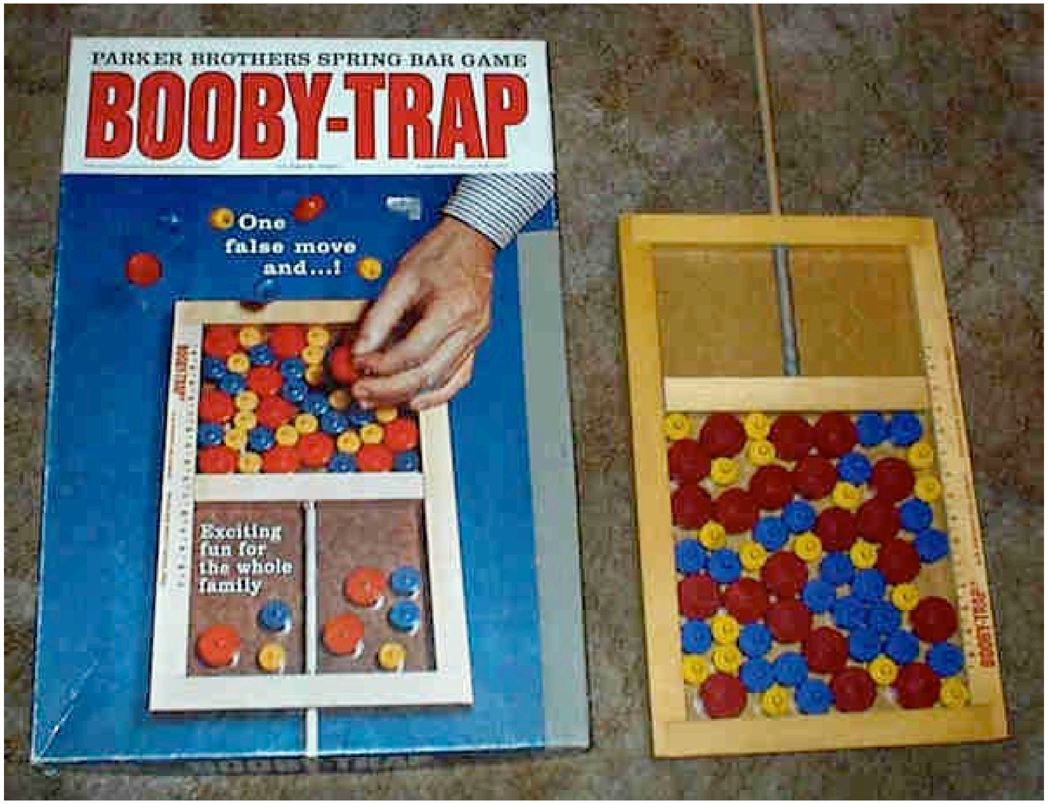
http://jamming.research.yale.edu/

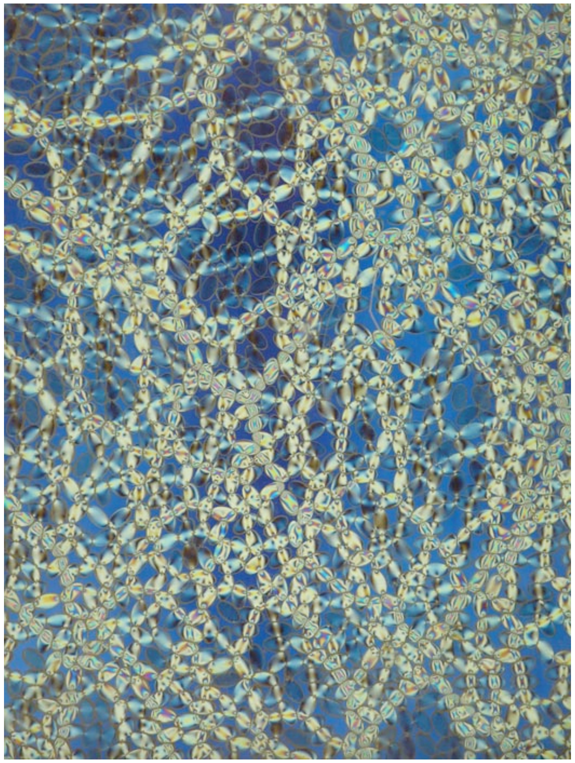
The O'Hern group in the Fall 2011: (from left to right) Thibault Bertrand, Diego Caballero, Wendell Smith, Mate Nagy, Mark Shattuck, Alice Zhou, Jared Harwayne-Gidansky, Corey O'Hern, Georgia Lill, Maxwell Micali, Minglei Wang, Robert Hoy, Tianqi Shen, Carl Schreck, S. S. Ashwin, and Stefanos Papanikolaou

The O'Hern Group

1. Dr. S. S. Ashwin, Ph.D. in Physics, Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore, India (protocol dependence in granular media)
2. Dr. Robert Hoy, Ph.D. in Physics, The Johns Hopkins University (polymer collapse, colloidal clusters)
3. Dr. Stefanos Papanikolau, Ph.D. in Physics, University of Illinois, Urbana-Champaign (frictional granular packings)
4. Minglei Wang, 1st year Ph.D. student in Mechanical Engineering & Materials Science (frictional granular packings)
5. Thibault Bertrand, 1st year Ph.D. student in Mechanical Engineering & Materials Science (vibrations in granular packings)
6. Wendell Smith, 2nd year Ph.D. student in Physics (intrinsically disorder proteins)
7. Diego Caballero, 2nd year Ph.D. student in Physics (modeling of chromatin)
8. Jared Harwayne-Gidansky, 3rd year Ph.D. student in Electrical Engineering (polymer collapse and packing)
9. Alice Zhou, 3rd year Ph.D. student in Molecular Biophysics & Biochemistry (protein-protein interactions)
10. Tianqi Shen, 4th year Ph.D. student in Physics (contact percolation)
11. Carl Schreck, 6th year Ph.D. student in Physics (granular packings)

Static Granular Packings





Stress-induced
birefringence

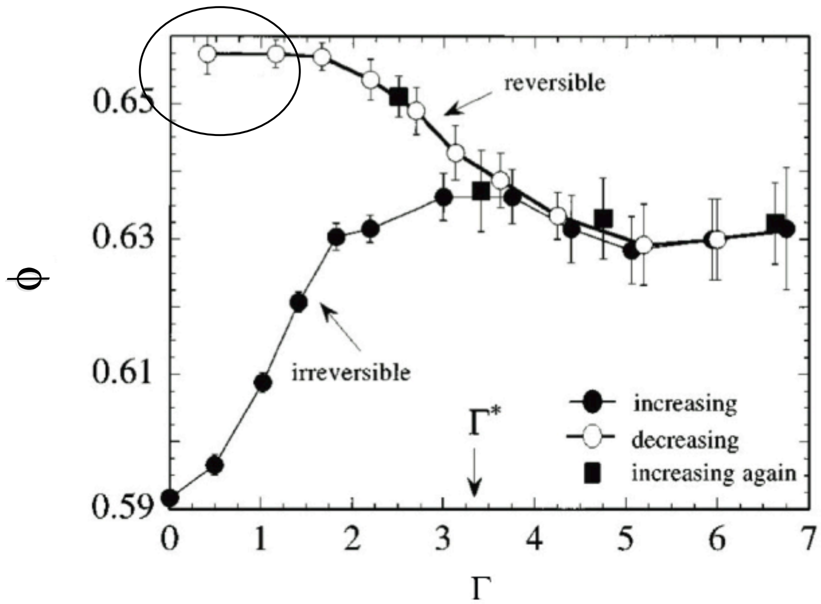
T. S. Majmudar and R. P. Behringer, "Contact force measurements and stress induced anisotropy in granular materials", Nature 43 (2005) 1079.

Birds' Nests?

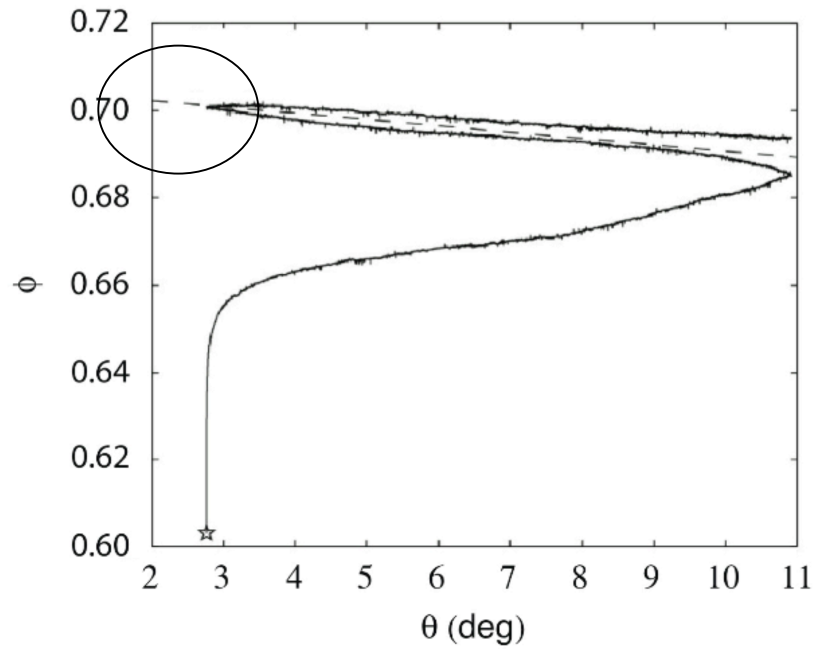
Despite the fact that granular materials are the second-most human-manipulated material (behind water), we waste billions of dollars in processing, conveying, and separating granular materials in the pharmaceutical, oil, agriculture, and other industries. Further, we are not able to effectively and efficiently design and make granular composite materials with precisely targeted structural and mechanical properties. For example, granular media with novel acoustic properties, such as band pass and filter capabilities would have many commercial and military applications.

We propose a multi-pronged effort to identify and understand the strategies that many bird species employ to make robust, mechanically stable nests. We will then mimic these design strategies so that we can reliably produce mechanically stable structures from component materials that possess a wide variation of structural and material properties, yet possess robust mechanical properties on the large scale of the structure. These studies will dramatically improve our ability to build composite granular metamaterials with precisely tuned structural and mechanical properties, yet using inexpensive and interchangeable components.

Statistical Mechanics of Granular Media



PRE 57 (1998) 1971



EPJE 3 (2000) 309

- Apply driving to attain reversible set of states
- Different driving mechanisms lead to different sets of states!

"One must understand static packings of frictionless particles to understand frictional granular packings."

C. S. O'Hern

Outline

1. What are the microstates for mechanically stable (MS) packings? Are MS packings points or continuous geometrical families in configuration space?

2. Are MS packings equally probable? If not, what determines their probabilities? How do the probabilities depend on the packing-generation protocol?

3. How do particle shape and friction affect the microstate statistics?

4. Can the vibrational response (or heat flow) be determined from *static* MS packings?

Attributes of Simple Granular Materials

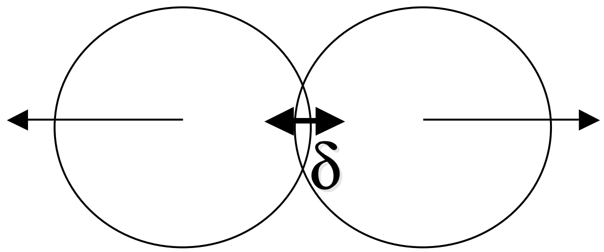
1. Finite number of macroscopic grains with dissipative interactions; exist at `zero temperature' unless driven by external forces

2. Contact interactions; no harmonic approximation?

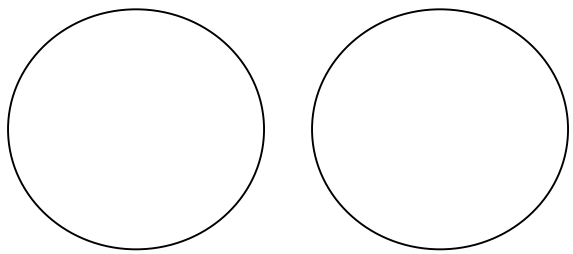
3. Non-spherical particle shapes

4. Frictional or `history-dependent' interactions

Granular Model I: Frictionless Disks



repulsive central forces,
$F_{ij} \sim \delta^{\alpha} \sim (1-r_{ij}/\sigma_{ij})^{\alpha}$, $\alpha=1$



zero force, $F_{ij} = 0$

$$m\vec{a}_i = \sum_j \vec{F}_{ij} - b\vec{v}_i$$

Minimize energy V(r)
to reach T=0 at each ϕ

$$V(\vec{r}) = \sum_{i>j} V(r_{ij})$$

$$V(r_{ij}) = \frac{\varepsilon}{\alpha}\left(1 - \frac{r_{ij}}{\sigma_{ij}}\right)^{\alpha}$$

for overlapping particles

Conservation of Total Energy; NVE

$$E = K + V \qquad K = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} \qquad V = \sum V\left(r_{ij}\right)$$

$$\frac{dE}{dt} = \sum_i \left(\frac{\partial K}{\partial \vec{p}_i} \frac{\partial \vec{p}_i}{\partial t} + \frac{\partial V}{\partial \vec{r}_i} \frac{\partial \vec{r}_i}{\partial t} \right)$$

$$\frac{dE}{dt} = \sum_i \frac{\vec{p}_i}{m} \left(\frac{d\vec{p}_i}{dt} - \vec{F}_i \right) = 0$$

$$\dot{\vec{r}}_i = \frac{\vec{p}_i}{m}$$

$$\dot{\vec{p}}_i = \vec{F}_i$$

Newton's equations of motion

- Newton's equations of motion conserve energy, not T

Integration of Equations of Motion

$$x_i(t+\Delta t) = x_i(t) + v_i(t)\Delta t + \frac{1}{2}a_i(t)(\Delta t)^2$$

$$v_i(t+\Delta t) = v_i(t) + \frac{a_i(t) + a_i(t+\Delta t)}{2}\Delta t$$

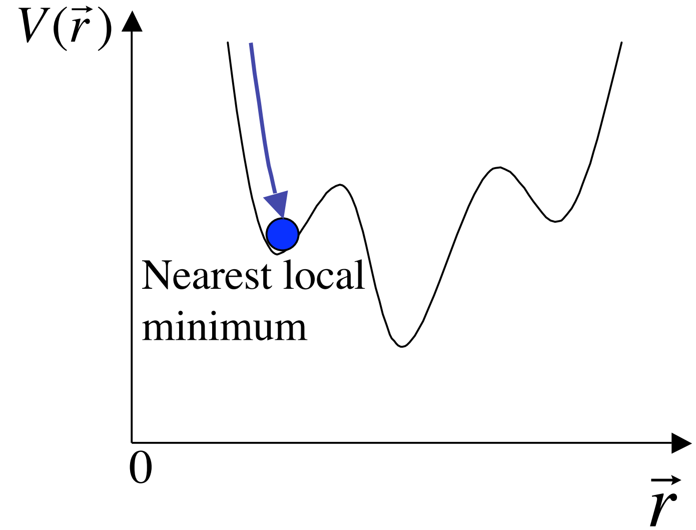
velocity verlet algorithm

Leapfrog Verlet Algorithm; calculate unconstrained v(t+Δt/2), Correct v(t+Δt); or Gear Predictor-corrector steps

`Two' Types of Energy Minimization

1. Conjugate Gradient (CG)

- Numerical method to find local minima of total potential energy
- Overdamped, infinite quench rate b⟶∞

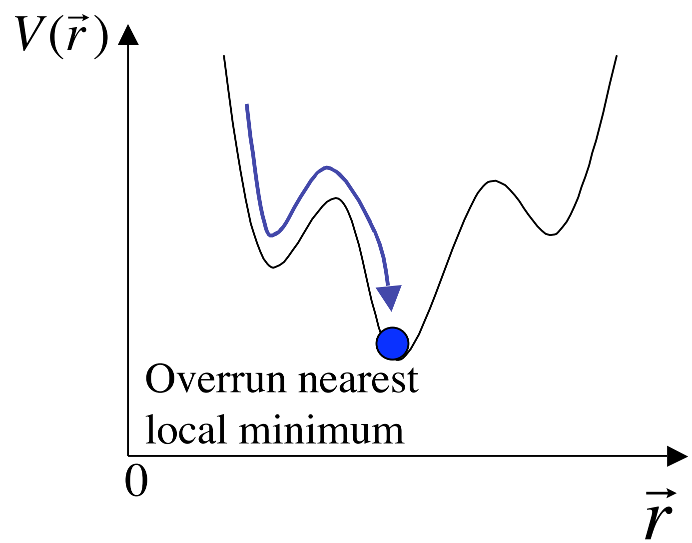


2. Molecular Dynamics (MD)

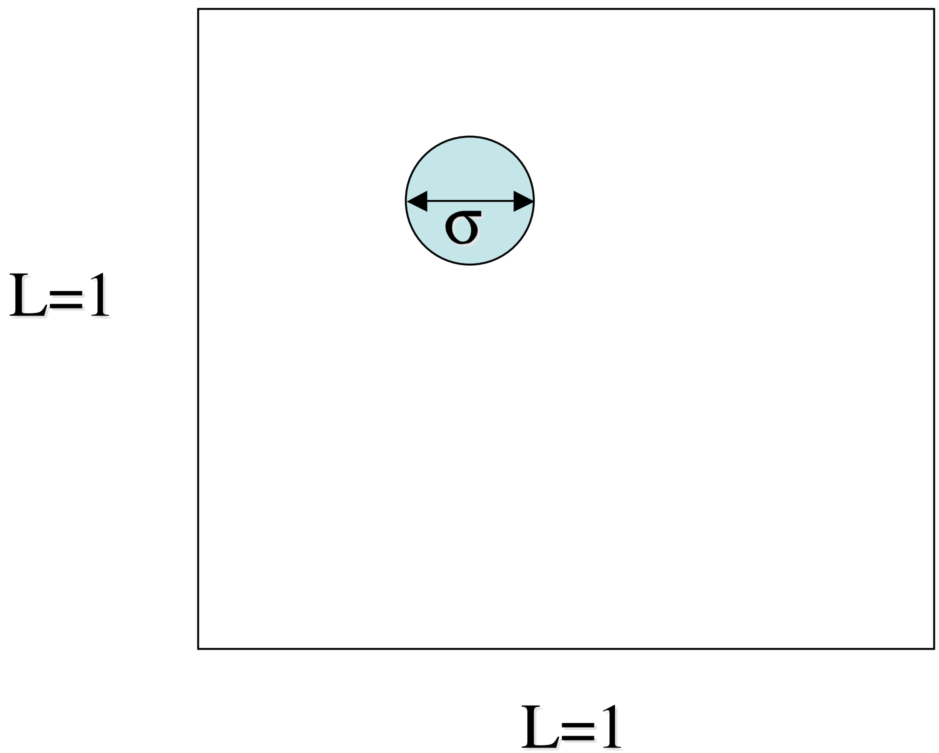
- Solve Newton's eqns. of motion

$$m\vec{a}_i = \vec{F}_i = \vec{F}_{spring} - b\vec{v}_i$$

- Finite damping, quench rate b=0.5



What is the packing fraction ϕ?

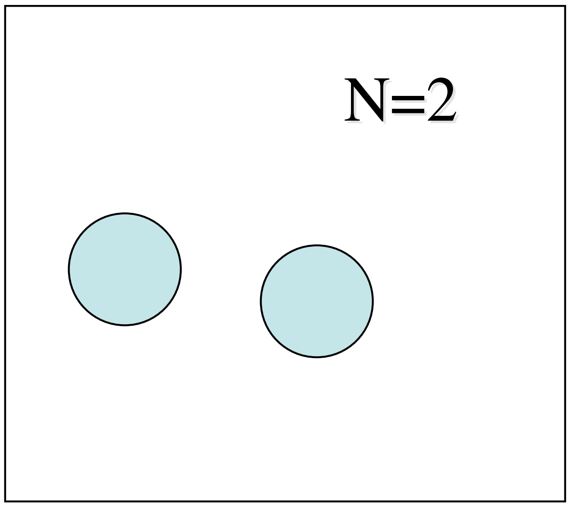


$$\phi = \frac{A_{disks}}{A_{box}} = \frac{\sum_{i=1}^{N} \pi \frac{\sigma_i^2}{4}}{L^2} = \frac{N\pi\sigma^2}{4}$$

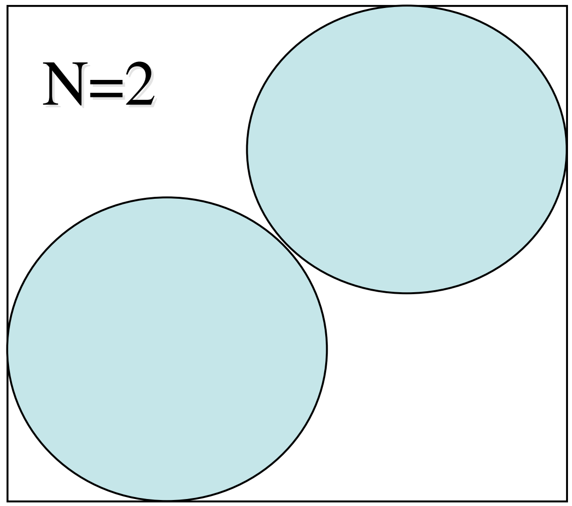
$$\sigma = \sqrt{\frac{4\phi}{N\pi}}$$

What are the allowed static packings for a given N with vanishing particle overlaps?

$\phi < \phi_c$

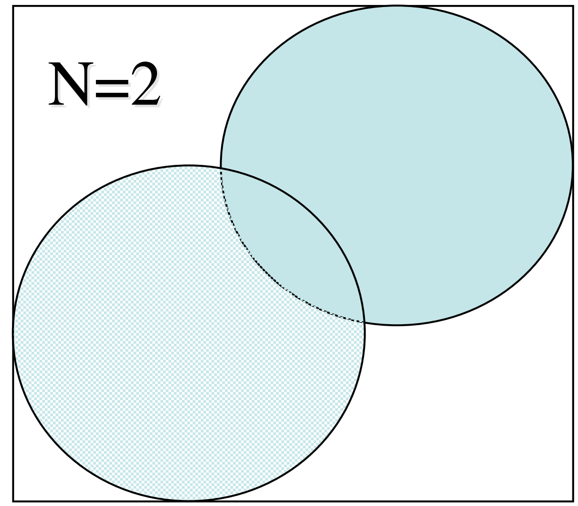


$\phi = \phi_c = 0.539$



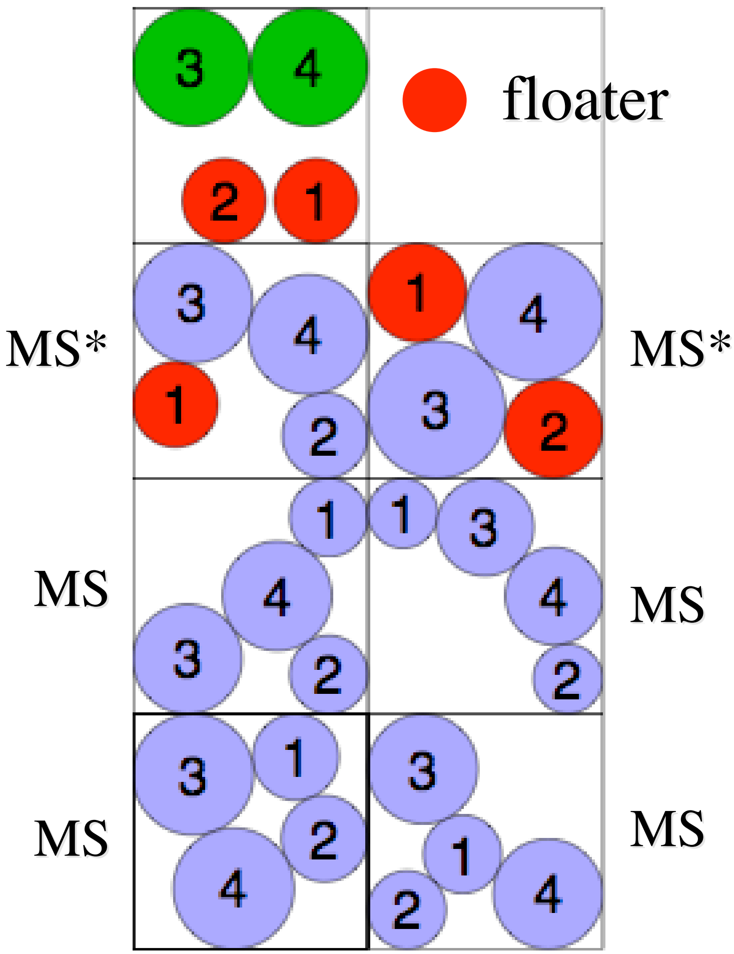
Mechanically stable packing with fixed boundaries

$\phi > \phi_c$



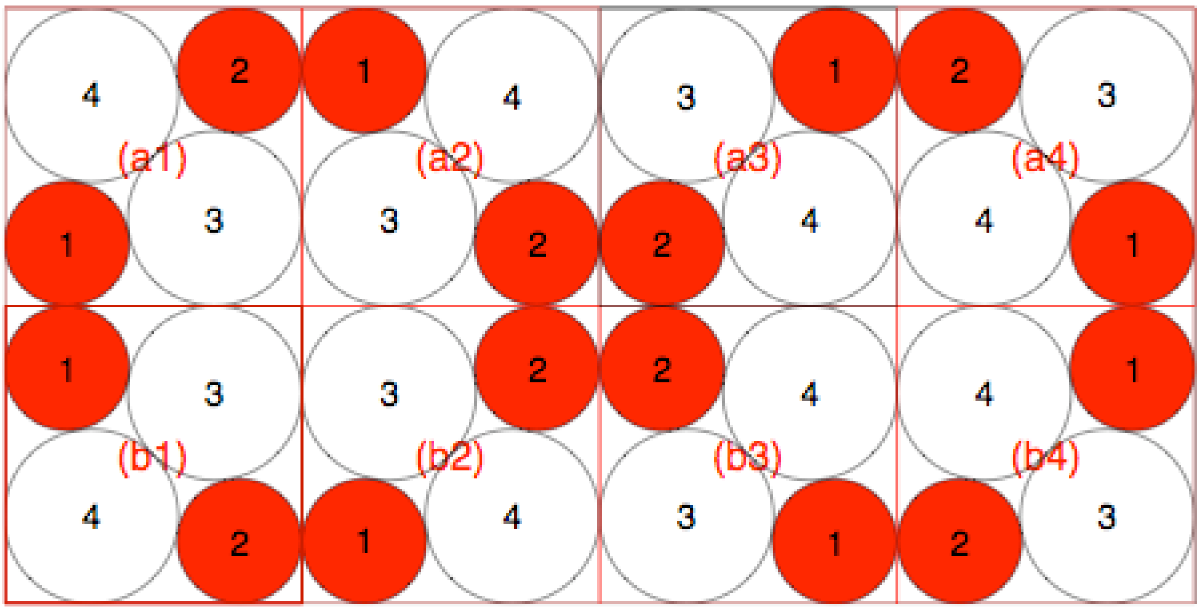
Overlapped soft disks; disallowed hard-disk configs

Distinct Mechanically Stable Macrostates and Microstates



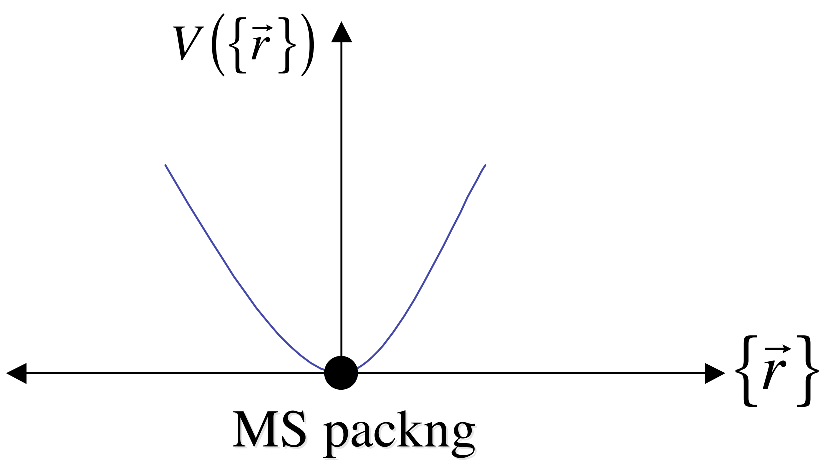
polarizations/permutations

N	N_s	N_m
2	1	4
4	6	136
6	80	19440



8 microstates

Jammed = mechanically stable (MS) configuration with extremely small particle overlaps; net forces (and torques) are zero on each particle; quadratically stable to small perturbations



Isostaticity

$$N_c \geq N_c^{iso} = Nd_f - d + 1$$

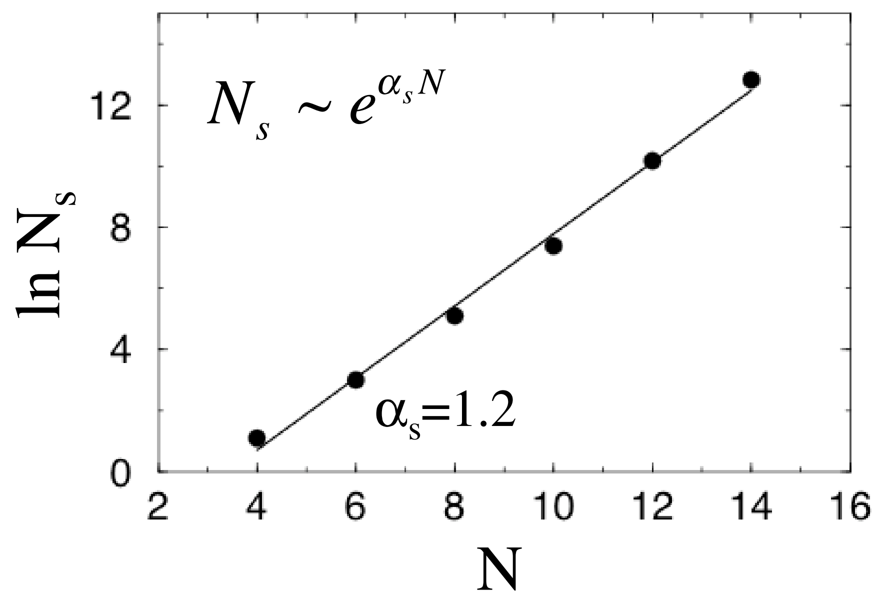
$$z \geq z_{iso} = 2d_f; \qquad z = \frac{2N_c}{N}$$

Gold Standard:

Configuration is mechanically stable if dynamical matrix contains dN-d eigenvalues $\omega^2 > 0$ (periodic b.c.s)

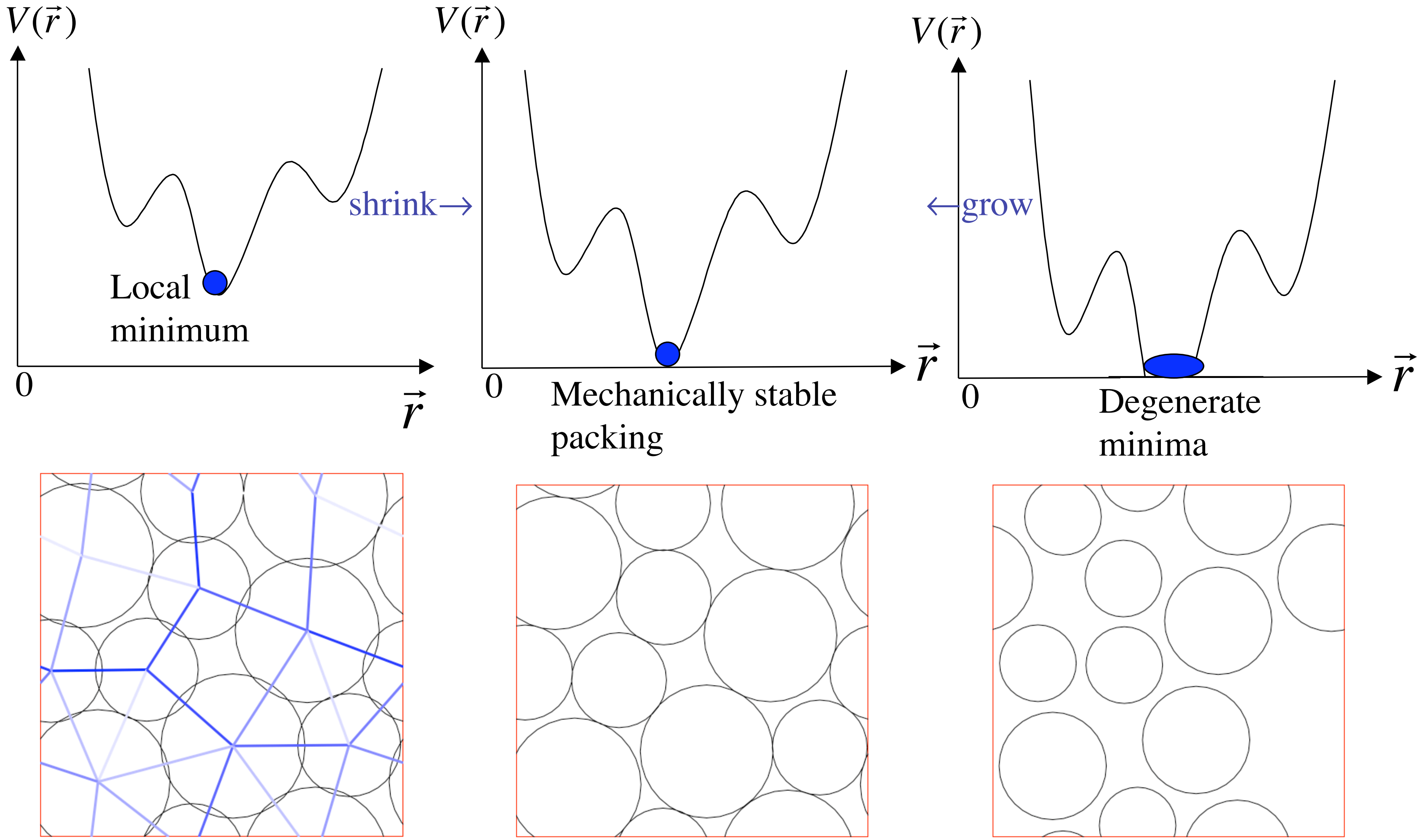
http://gibbs.engr.ccny.cuny.edu/technical/DisorderedSolids/DOSvPcorr.php

Exponential Growth of MS macrostates



- What about placement of rattler particles; does this increase scaling of N_s, *i.e.* $\begin{pmatrix} N_{cav} \\ N_r \end{pmatrix}$?
- What about scaling of N_c^{min}-1,N_c^{min}-2,… and N_c^{min}+1,N_c^{iso}+2,… with N?

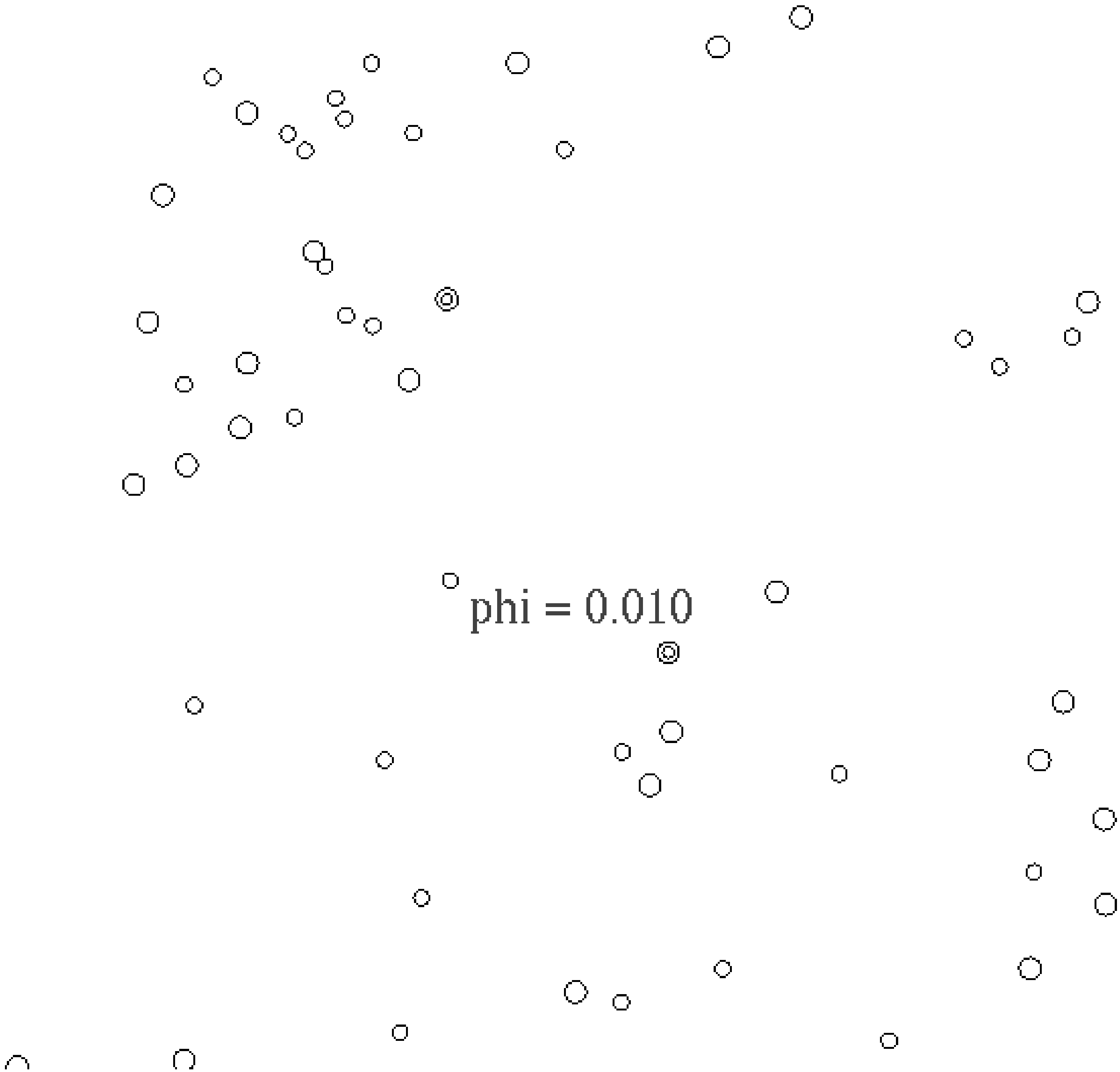
MS Packing-Generation Algorithm



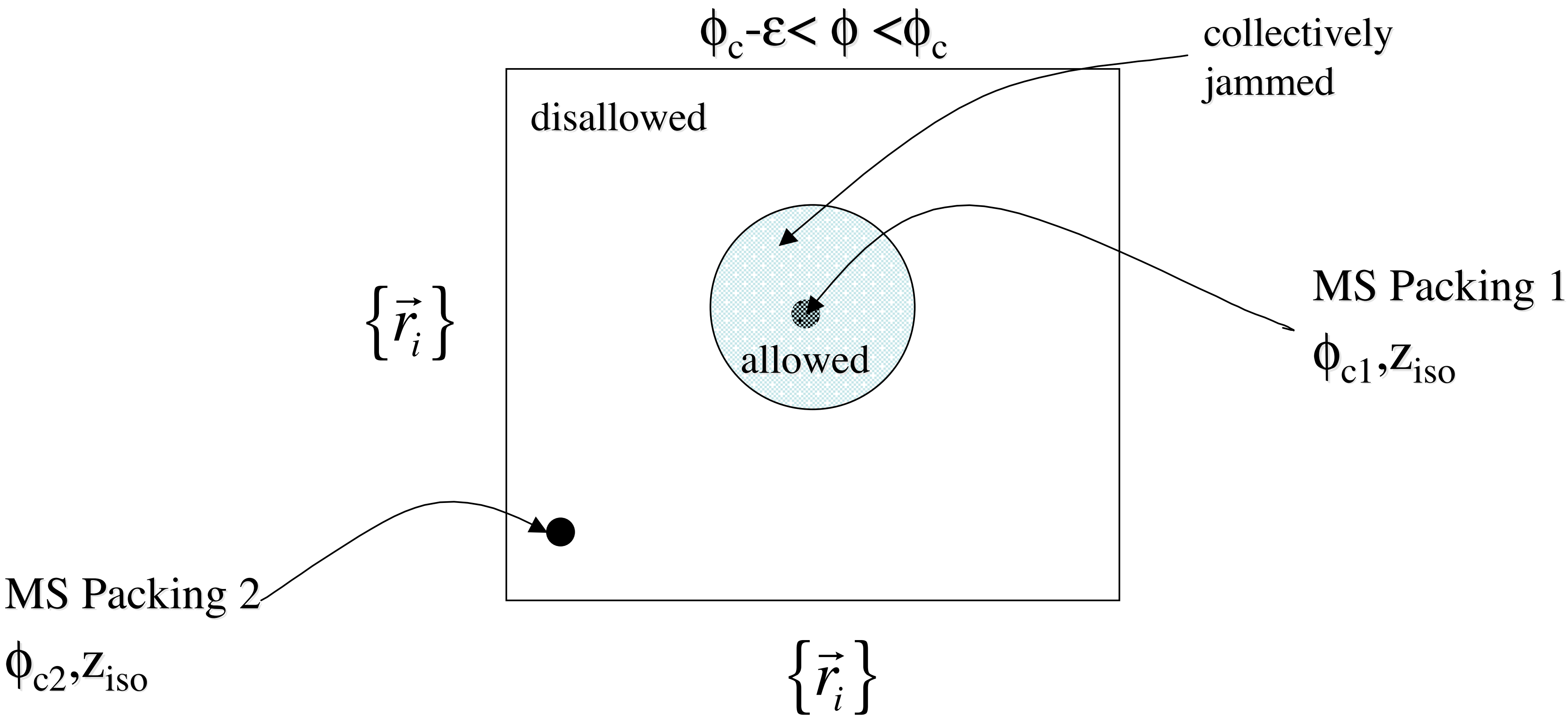
overlapped

Mechanically stable packing

non-overlapped


phi = 0.010

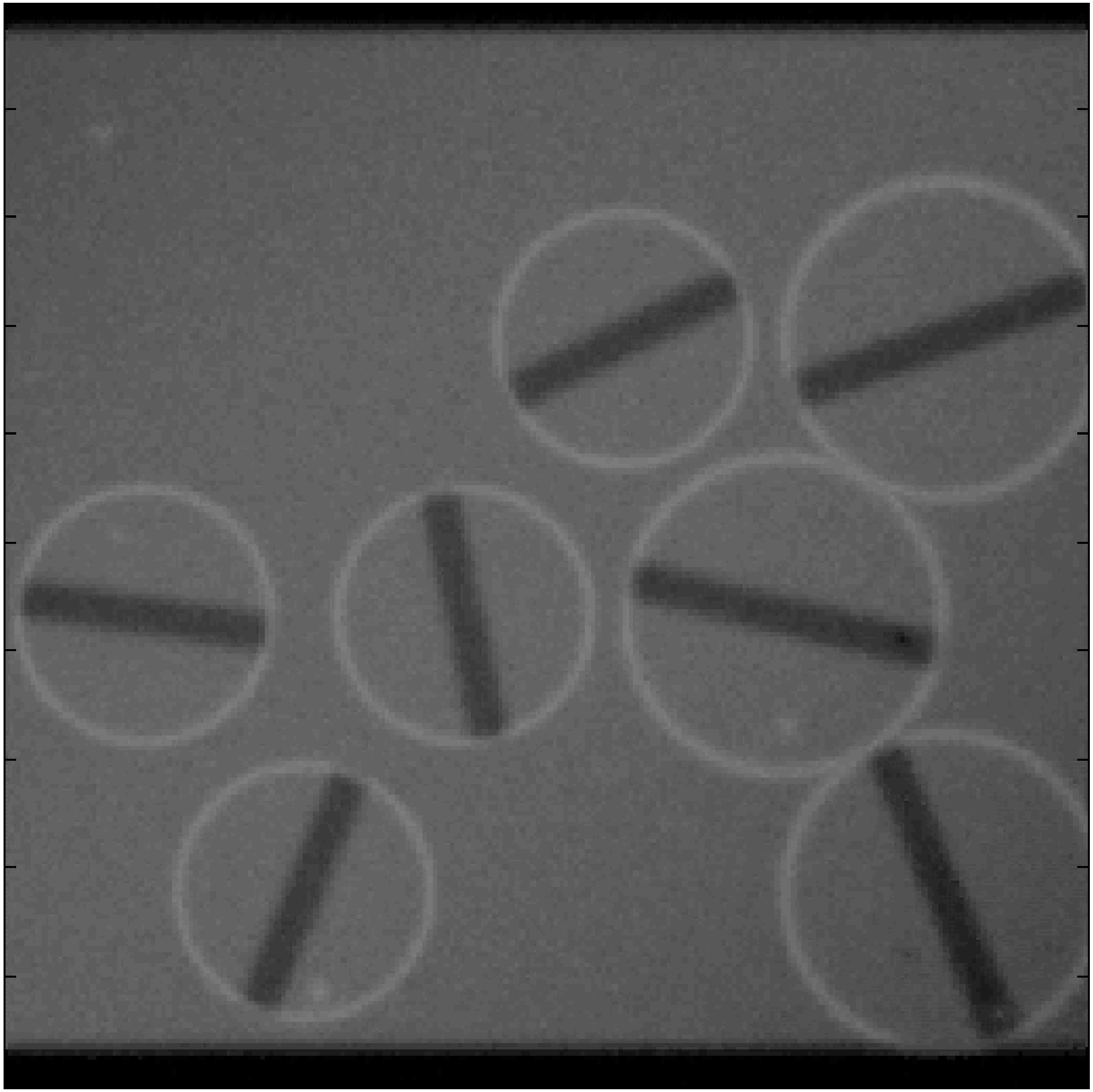
Collectively Jammed Hard Spheres = Mechanically Stable Soft Sphere Packings



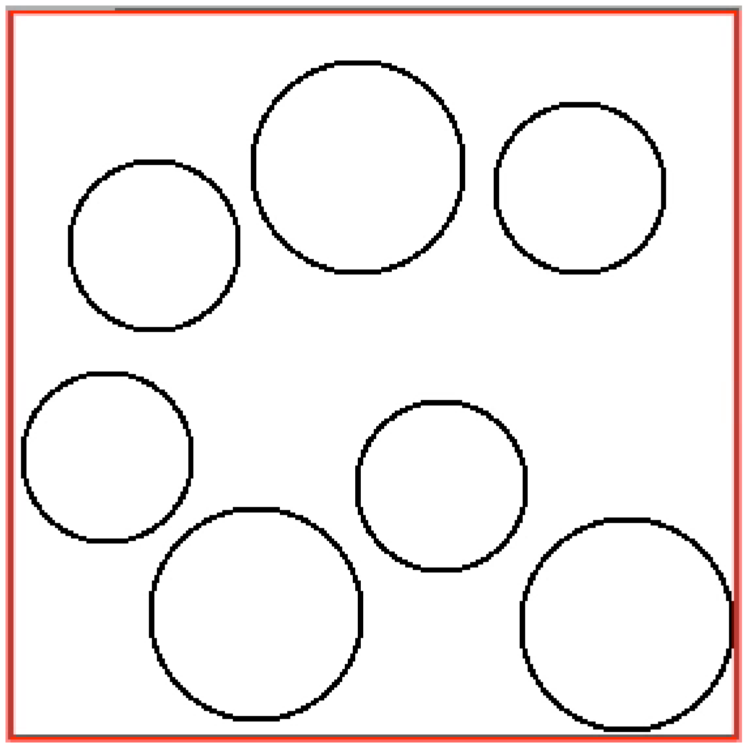
A. Donev, S. Torquato, F. H. Stillinger, and R. Connelly, Jamming in Hard Sphere and Disk Packings, Journal of Applied Physics, 95 (2004) 989.

Are MS packings points in configuration space?
Can we show this experimentally?

G.-J. Gao, J. Blawzdziewicz, C. S. O'Hern, and M. Shattuck, ``Experimental demonstration of nonuniform frequency distributions of mechanically stable granular packings'', Phys. Rev. E 80 (2009) 061304.



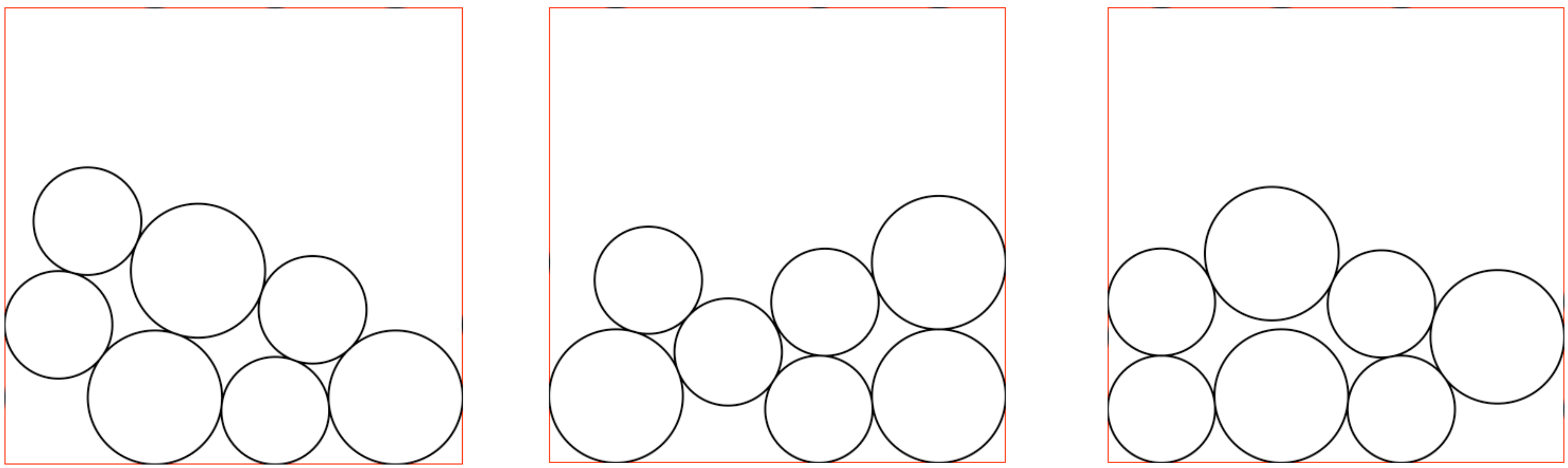
Deposition Algorithm in Simulations



$$\bar{g} = \frac{m_s g}{k\sigma_s}$$

- All geometric parameters identical to those for experiments
- Terminate algorithm when $F_{tot} < F_{max} = 10^{-14}$
- Vary random initial positions and conduct $N_{trials} = 10^8$ to find 'all' mechanically stable packings for small systems N=3 to 10.

Mechanically Stable Frictionless Packings

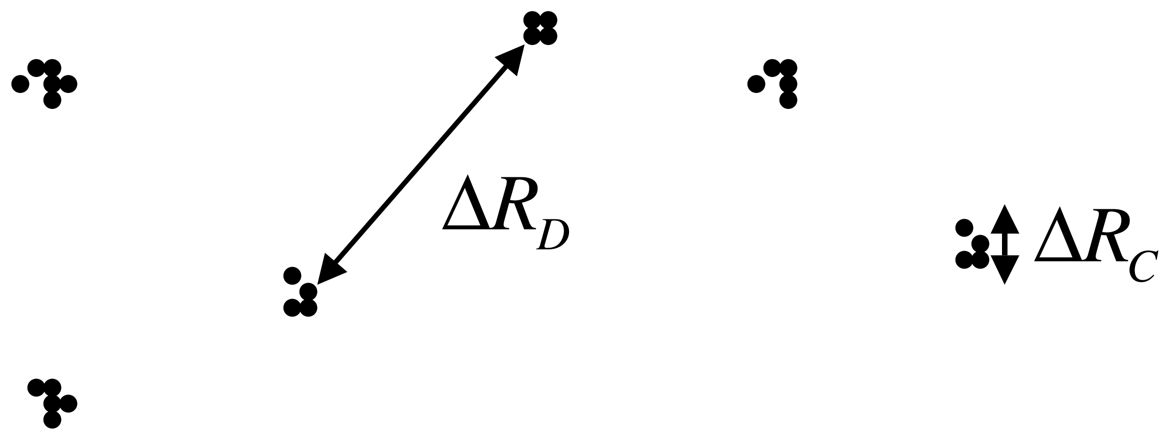


1 2 3

- Distinct MS packings distinguished by particle positions $\{\vec{r}_i\}$
- # of constraints ≥ # of degrees of freedom

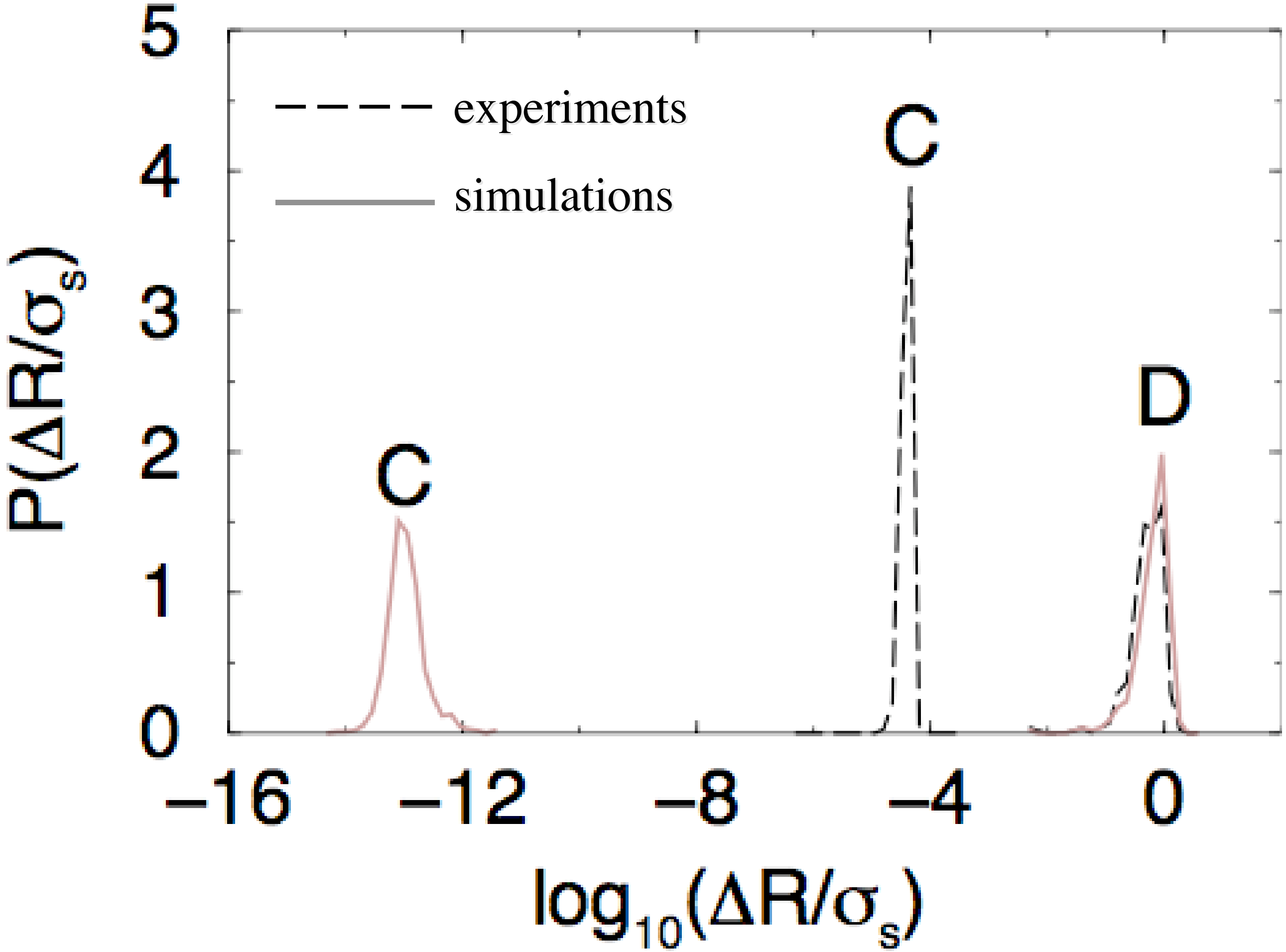
Configuration Space of Mechanically Stable Packings

$$R = \{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N\}$$



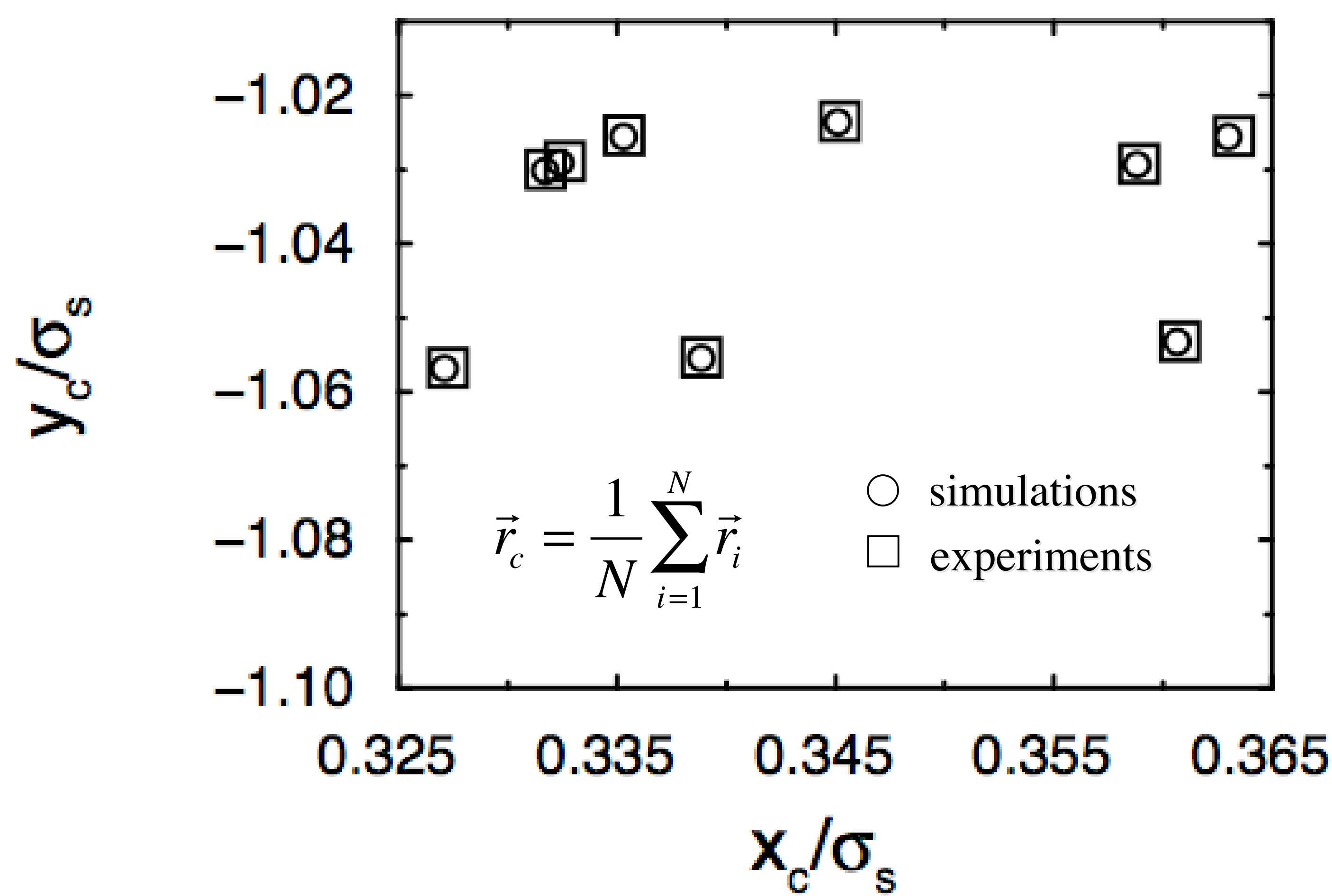
- ΔR_D= distance in configuration space between distinct MS packings
- ΔR_C= error in measuring distinct MS packings

Separation in Configuration Space

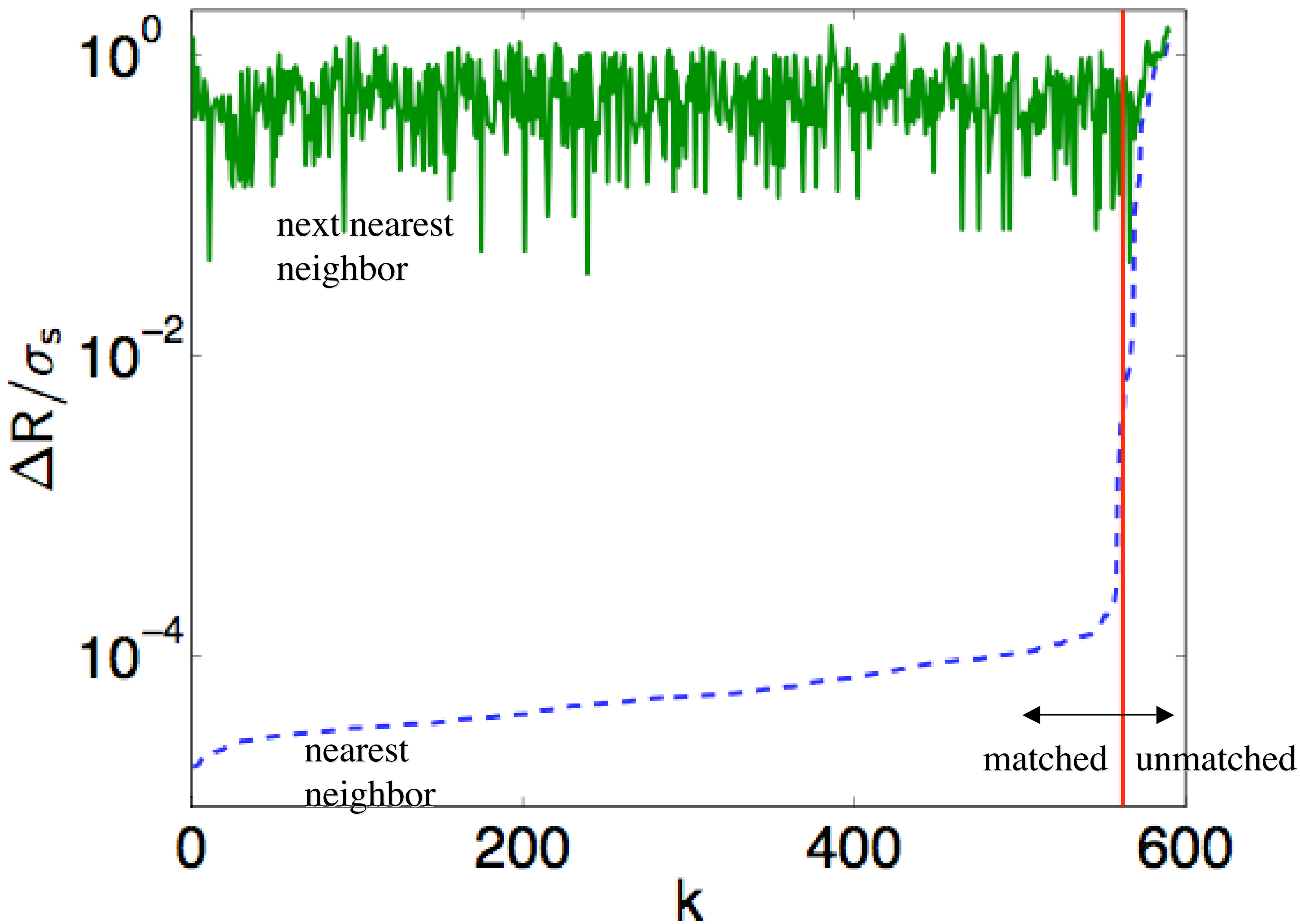


- MS frictionless packings are discrete points in configuration space

Discrete MS Packings



How is the quantitative agreement between sims and exps?



- 95% of distinct MS packing match; others are unstable in sims

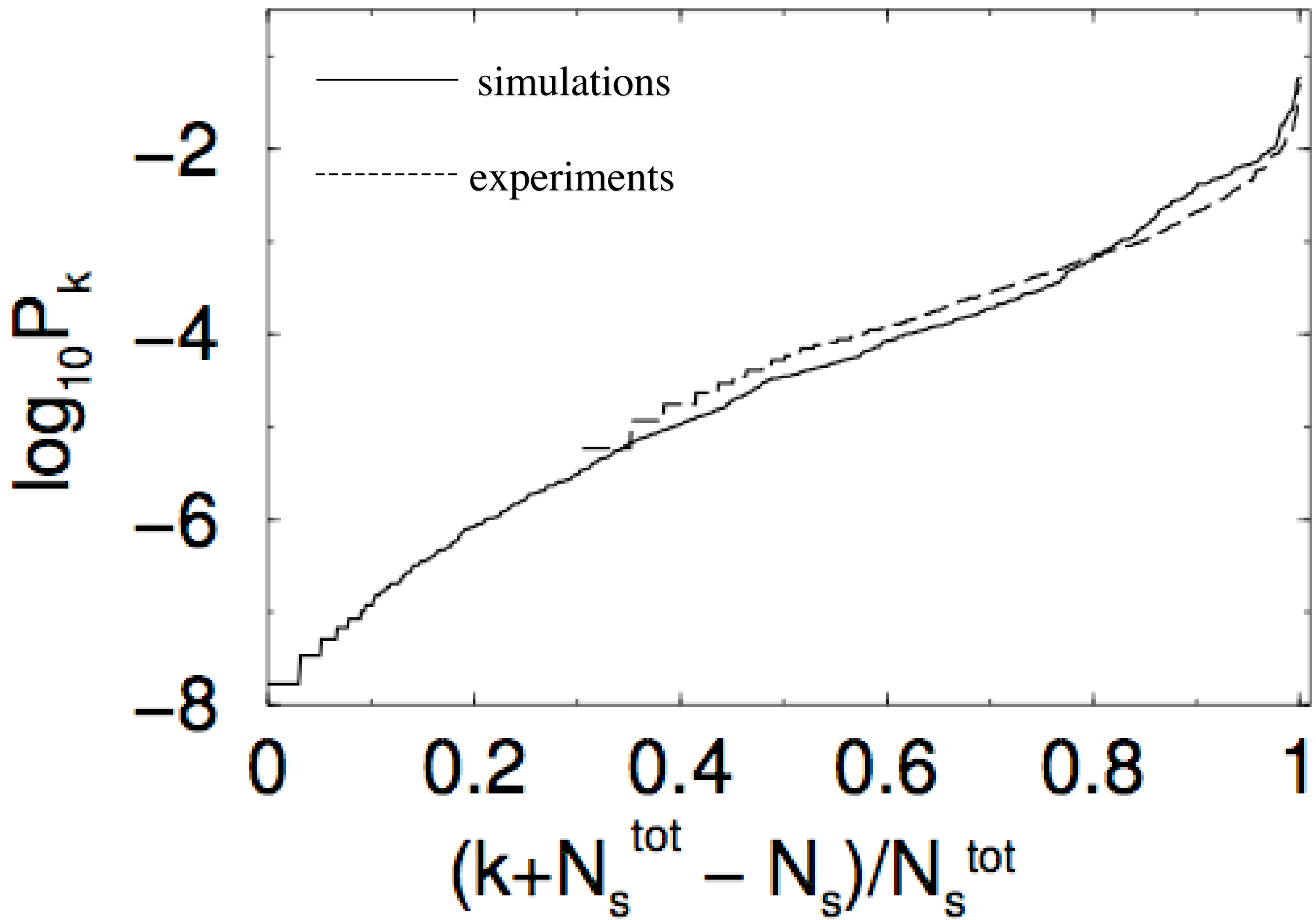
Are MS packings equally probable?

"for a given volume all [jammed] configurations are equally probable"

S. F. Edwards and R. B. S. Oakeshott, "Theory of Powders", Physica A 157 (1989) 1080

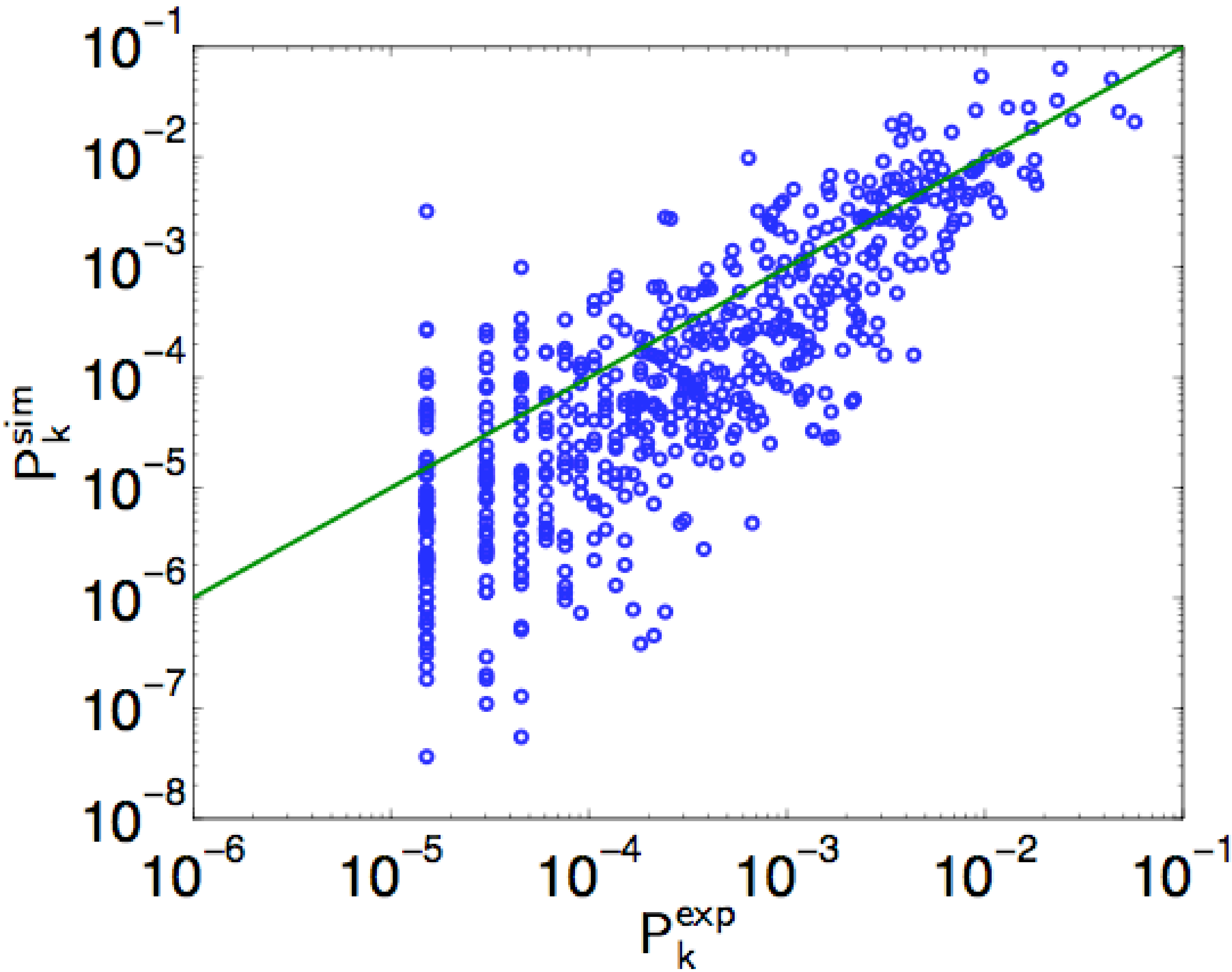
G.-J. Gao, J. Blawzdziewicz, C. S. O'Hern, and M. Shattuck, ``Experimental demonstration of nonuniform frequency distributions of mechanically stable granular packings'', Phys. Rev. E 80 (2009) 061304.

Sorted Probabilities



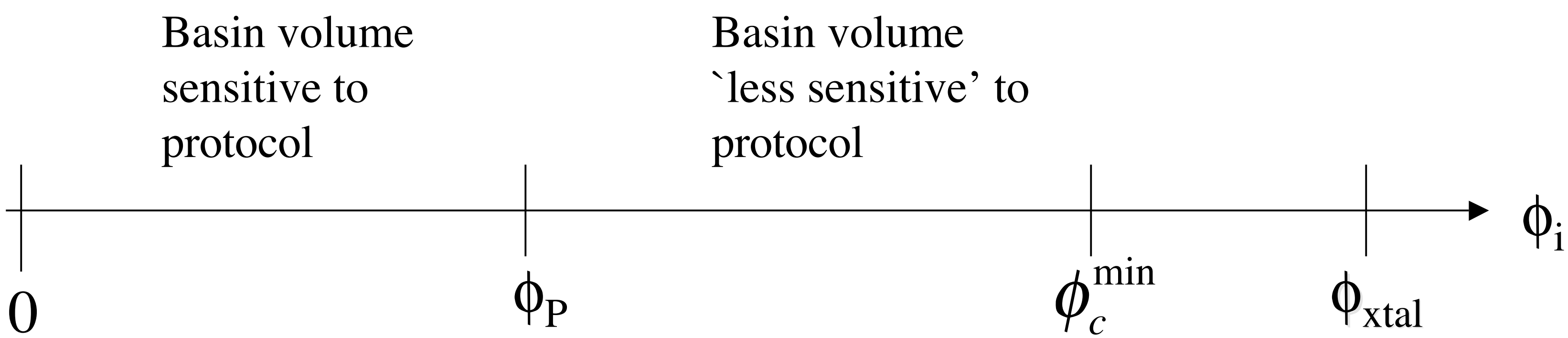
- 7 (4) orders of magnitude variation in probabilities in simulations (experiments)

MS Packing Probabilities Are Robust



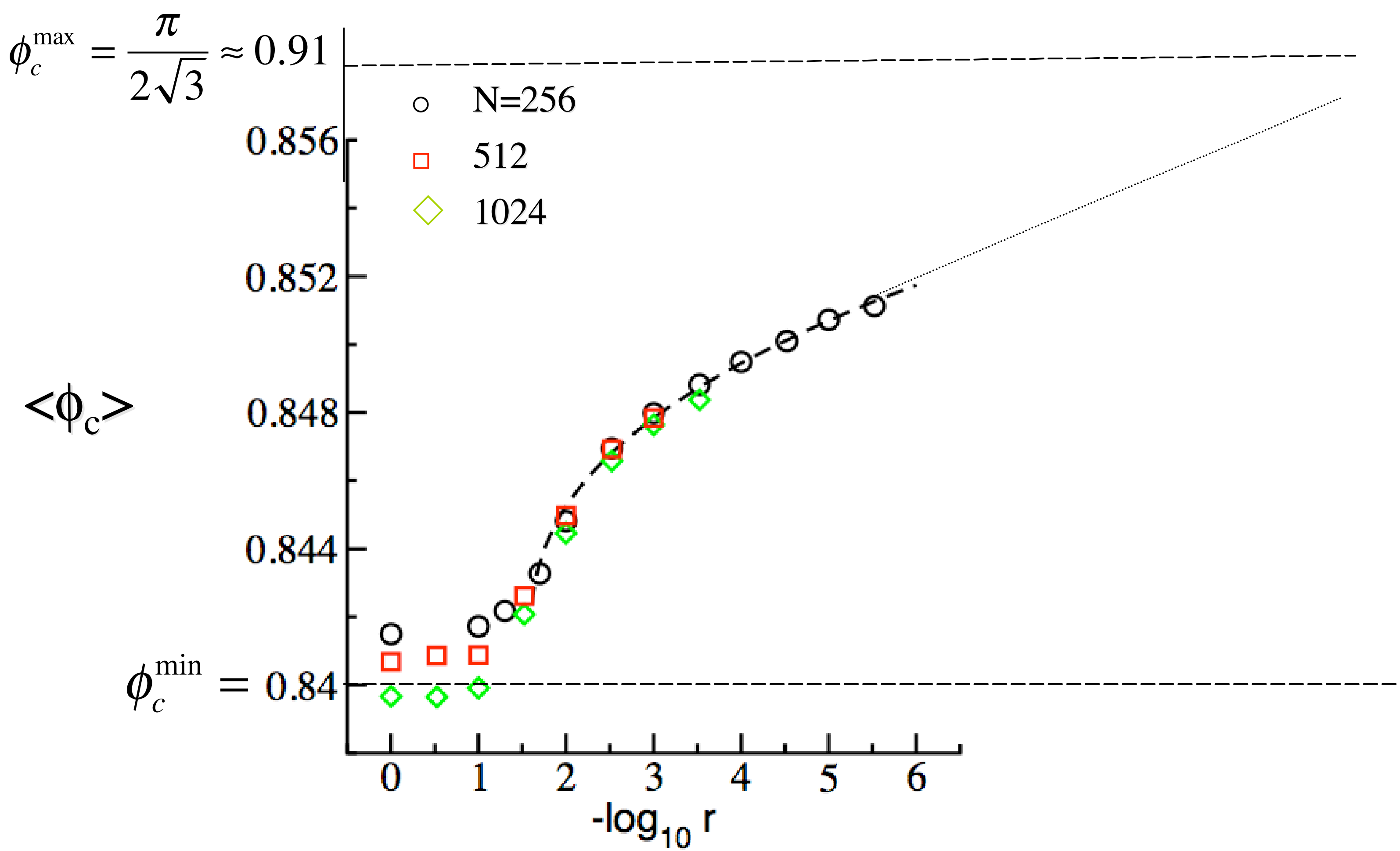
- Rare MS packings in exps are rare in sims; frequent MS packings in exps are frequent in sims

What determines MS packing probabilities?



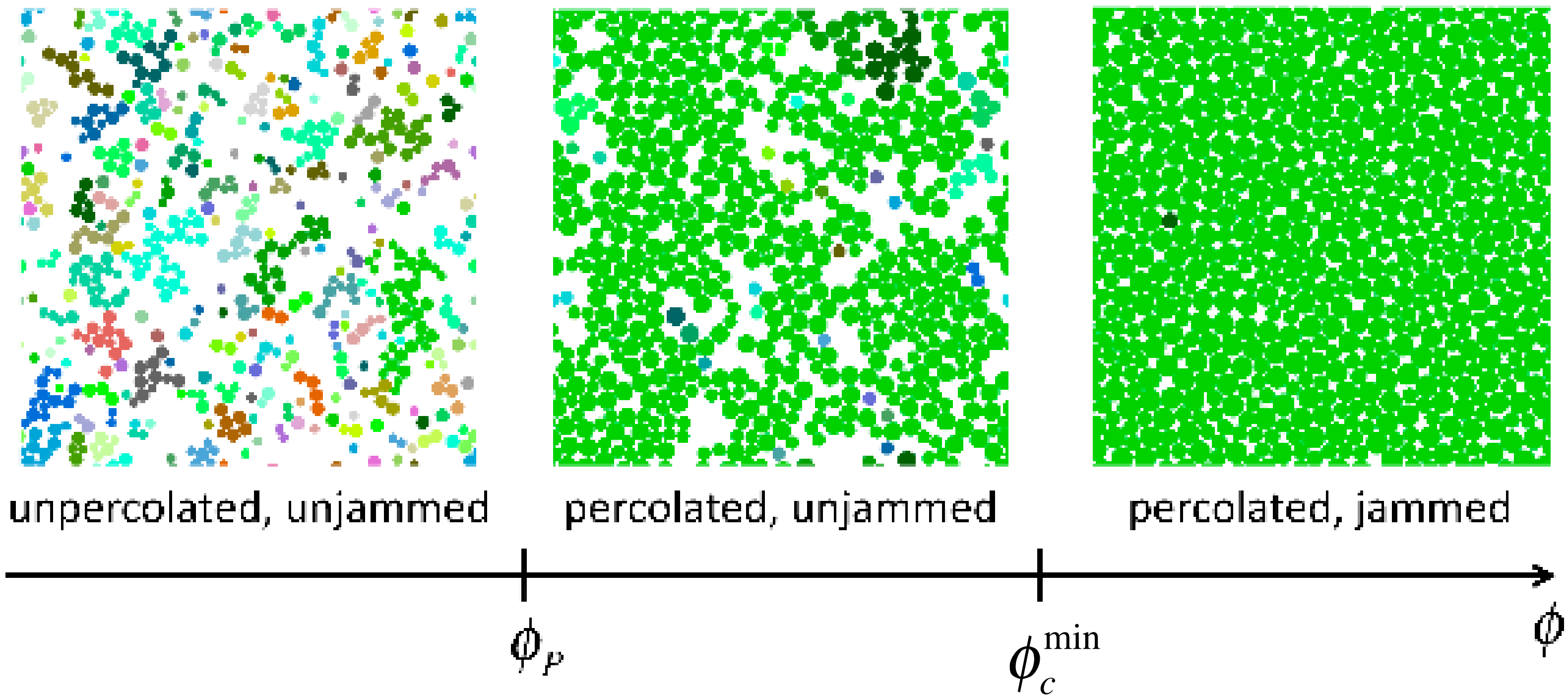
- What are the important packing fractions?

MS packings occur over a range of packing fractions



P. Chaudhuri, L.Berthier, & S. Sastry, Phys. Rev. Lett., 104, 165701 (2010)

Contact Percolation Transition



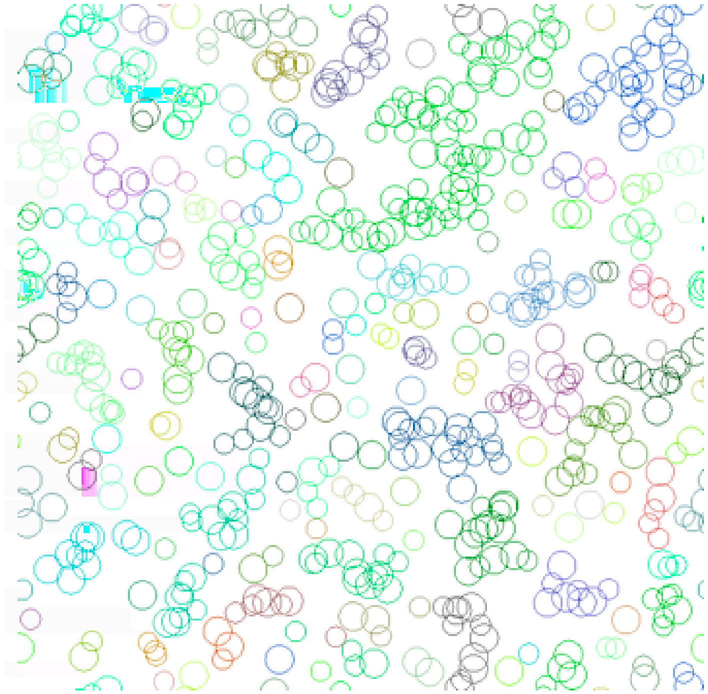
- Onset of cooperative and non-affine motion, irreversibility and weak protocol-dependence for $\phi > \phi_P << \phi_c^{min}$

T. Shen, C. S. O'Hern, & M. D. Shattuck, "The contact percolation transition in athermal particulate systems," Phys. Rev. E 85 (2012) 011308.

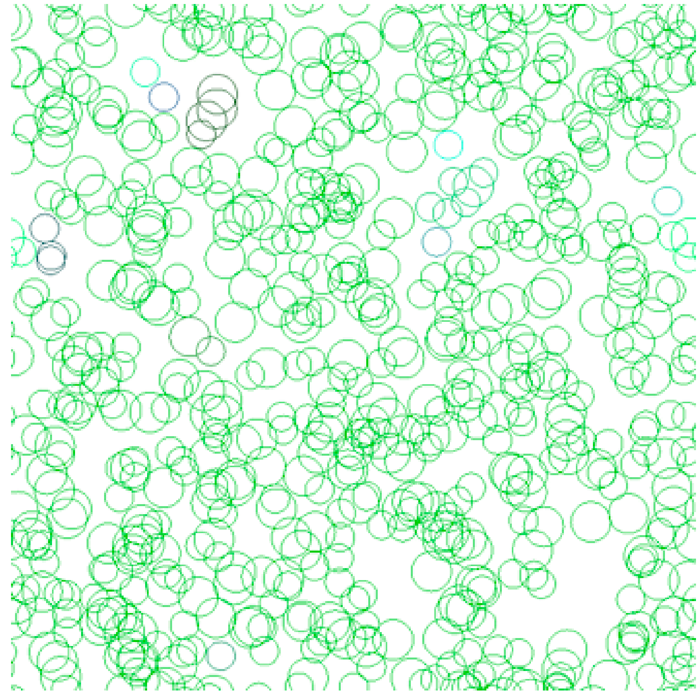
"Unjammed frictionless packings with $\phi_P < \phi < \phi_J$ can be made mechanically stable by adding static friction"

‘Random’ Continuum Percolation

unpercolated

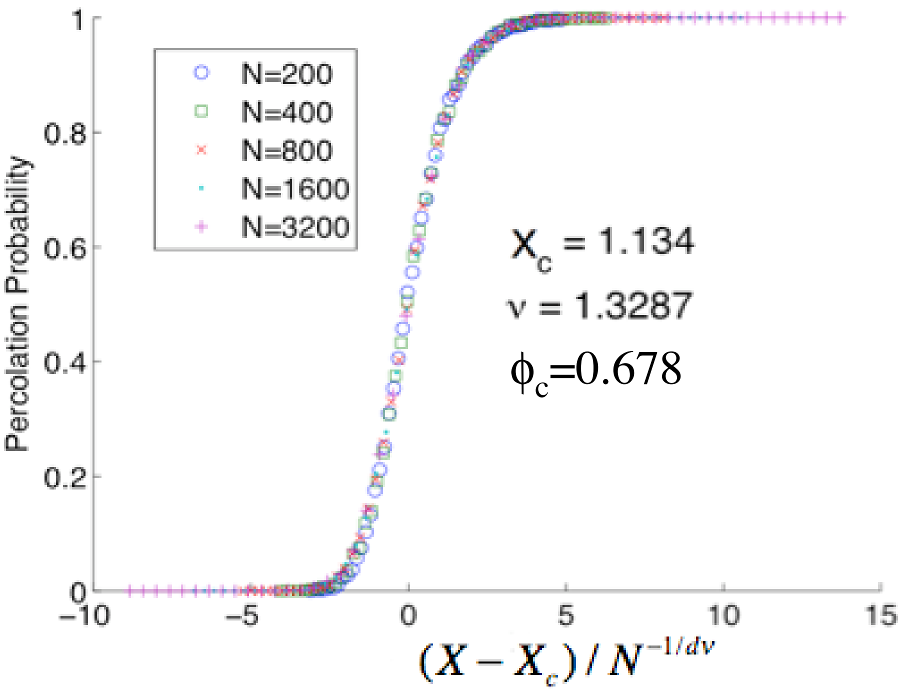


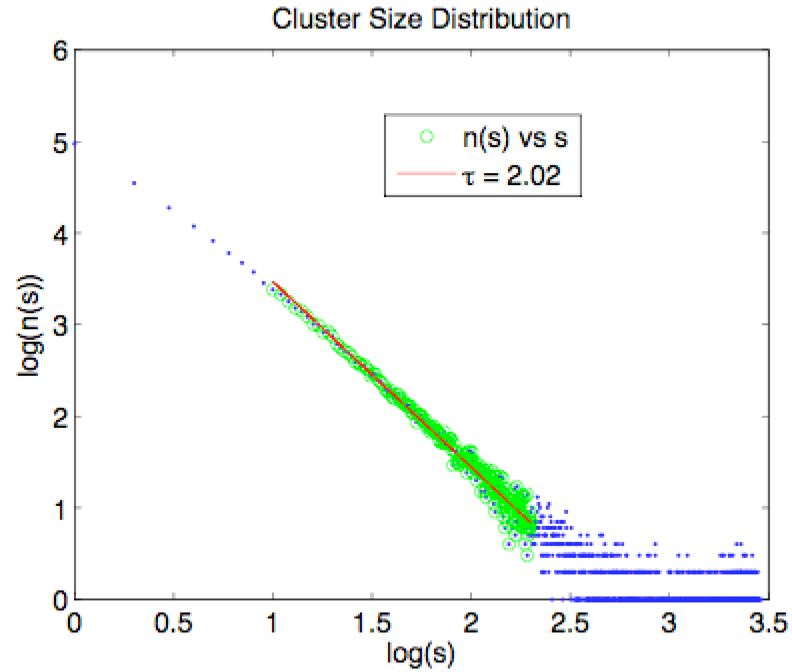
percolated

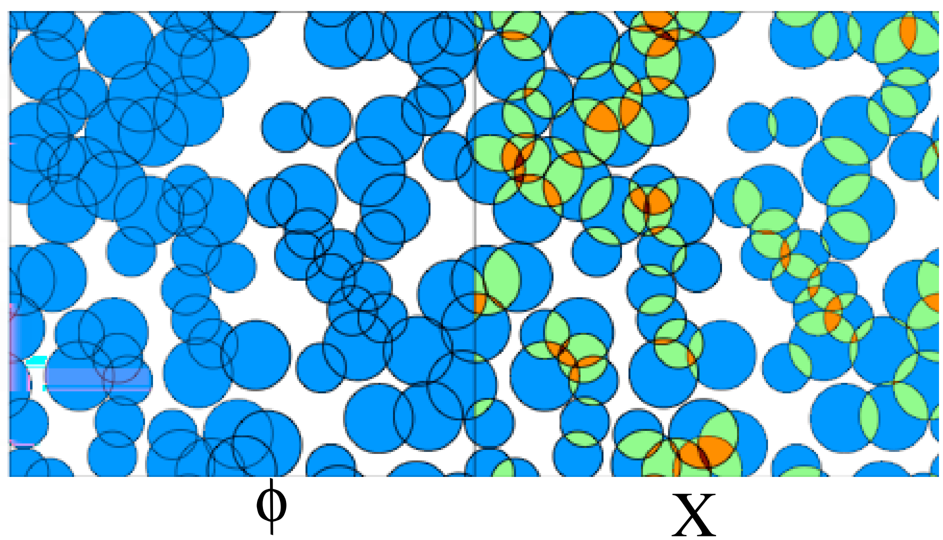


$$X_p = N\pi R_p^2 / L^2$$

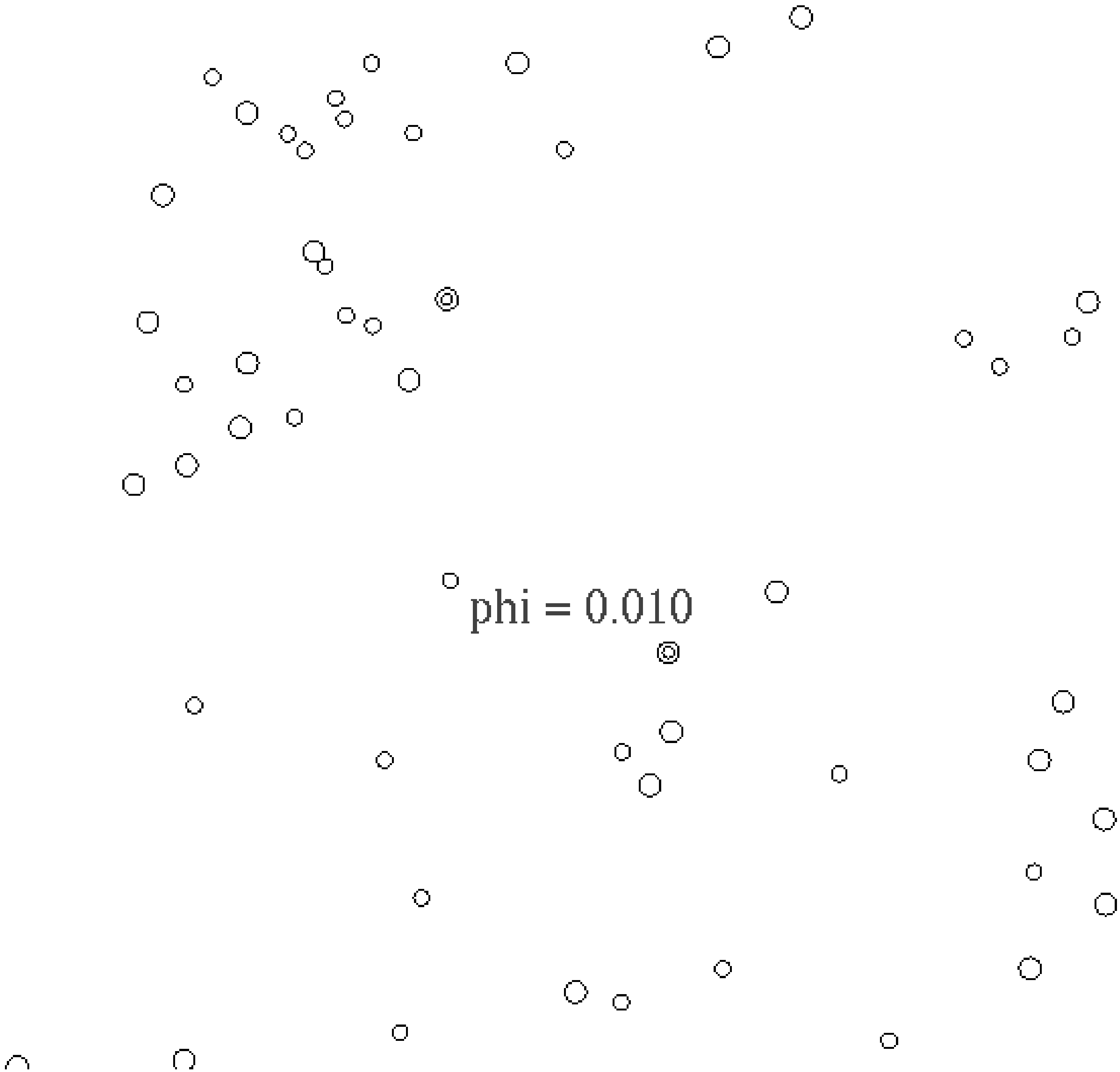
Critical Scaling Exponents



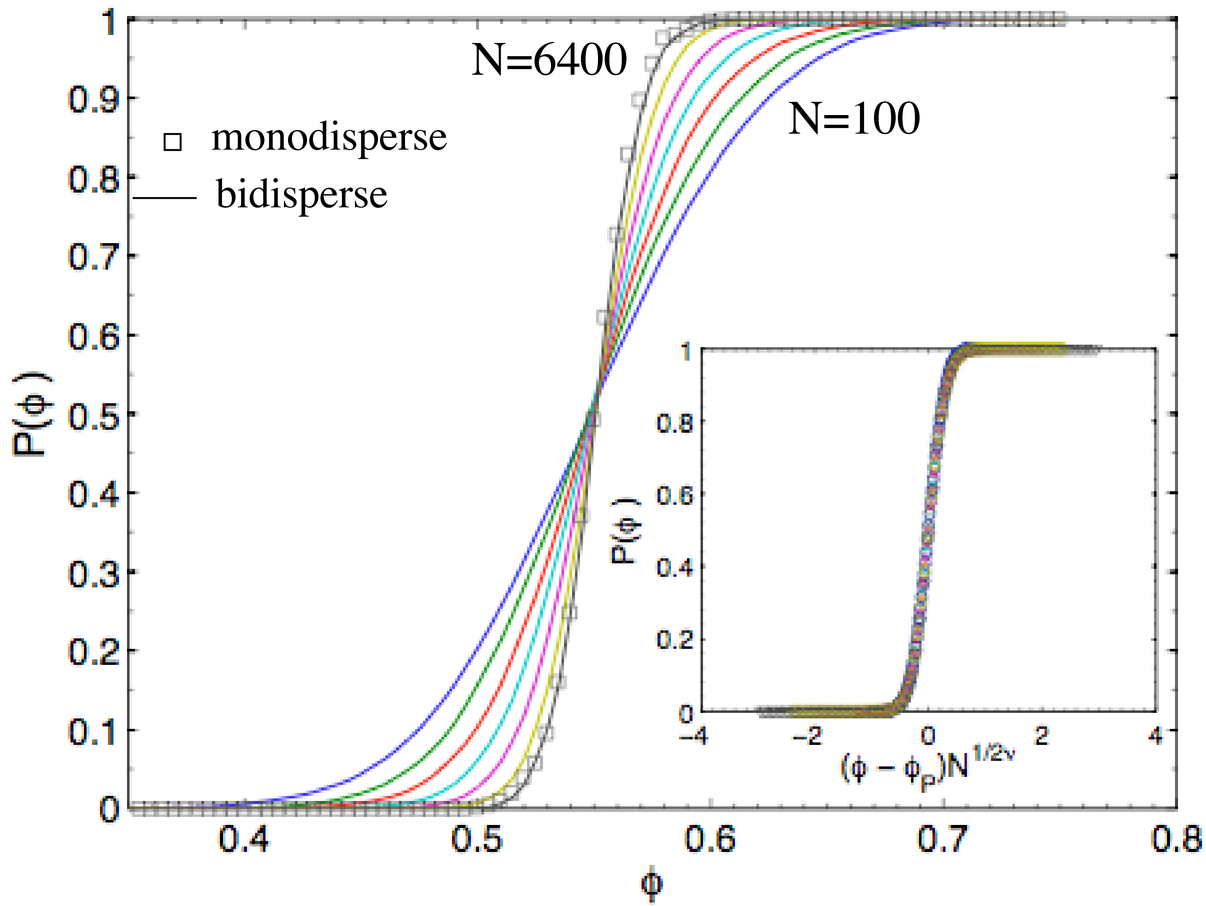




X_c	1.13
ϕ_c	0.678
D	1.91
τ	2.02
ν	1.33


phi = 0.010

Finite-size scaling analysis for contact percolation



Percolation Exponents

	continuum percolation	athermal repulsive	athermal attractive
D	1.91	1.89	1.88
τ	2.02	2.01	2.04
ν	1.33	1.68	1.92
ϕ_P		0.549	0.558

…but percolation is purely geometrical, so what?

Non-floppy Modes of Dynamical Matrix

$$M_{\alpha,\beta} = \left.\frac{\partial^2 V(\vec{r})}{\partial r_\alpha \partial r_\beta}\right|_{\vec{r}=\vec{r}_0}$$

α,β=x, y particle index

$\vec{r}_0$ = positions of MS packing

Calculate 2 N- 2 eigenvalues, ω

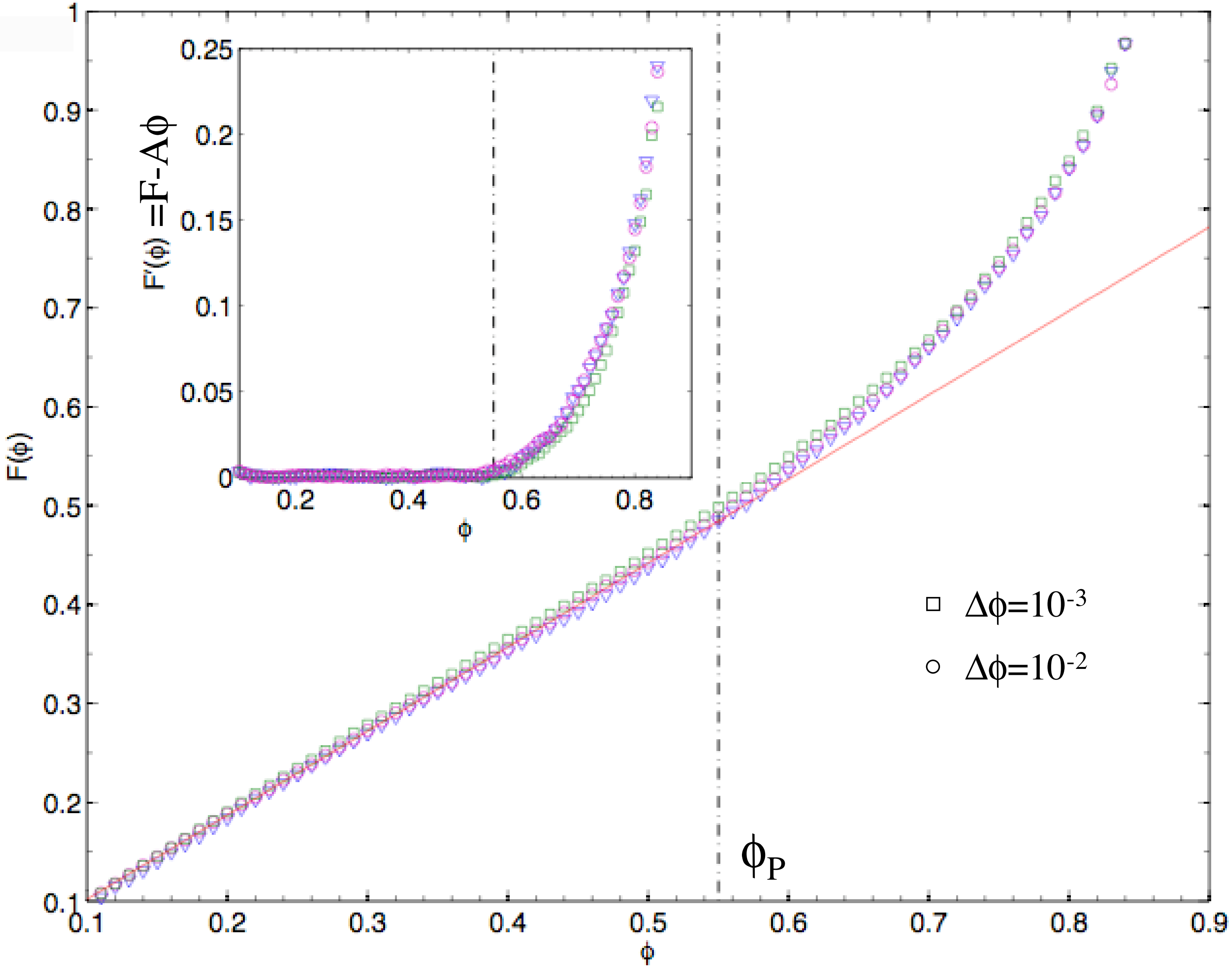
Number of non-floppy modes with $\omega>0$

$$F(\phi) = \frac{N(\omega > 0)}{2N - 2}$$

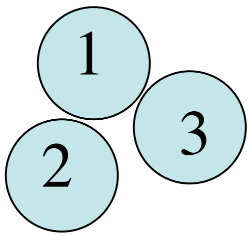
$F(\phi_J)=1$

$F(\phi < \phi_J) < 1$

Fraction of Non-Floppy Modes of Dynamical Matrix



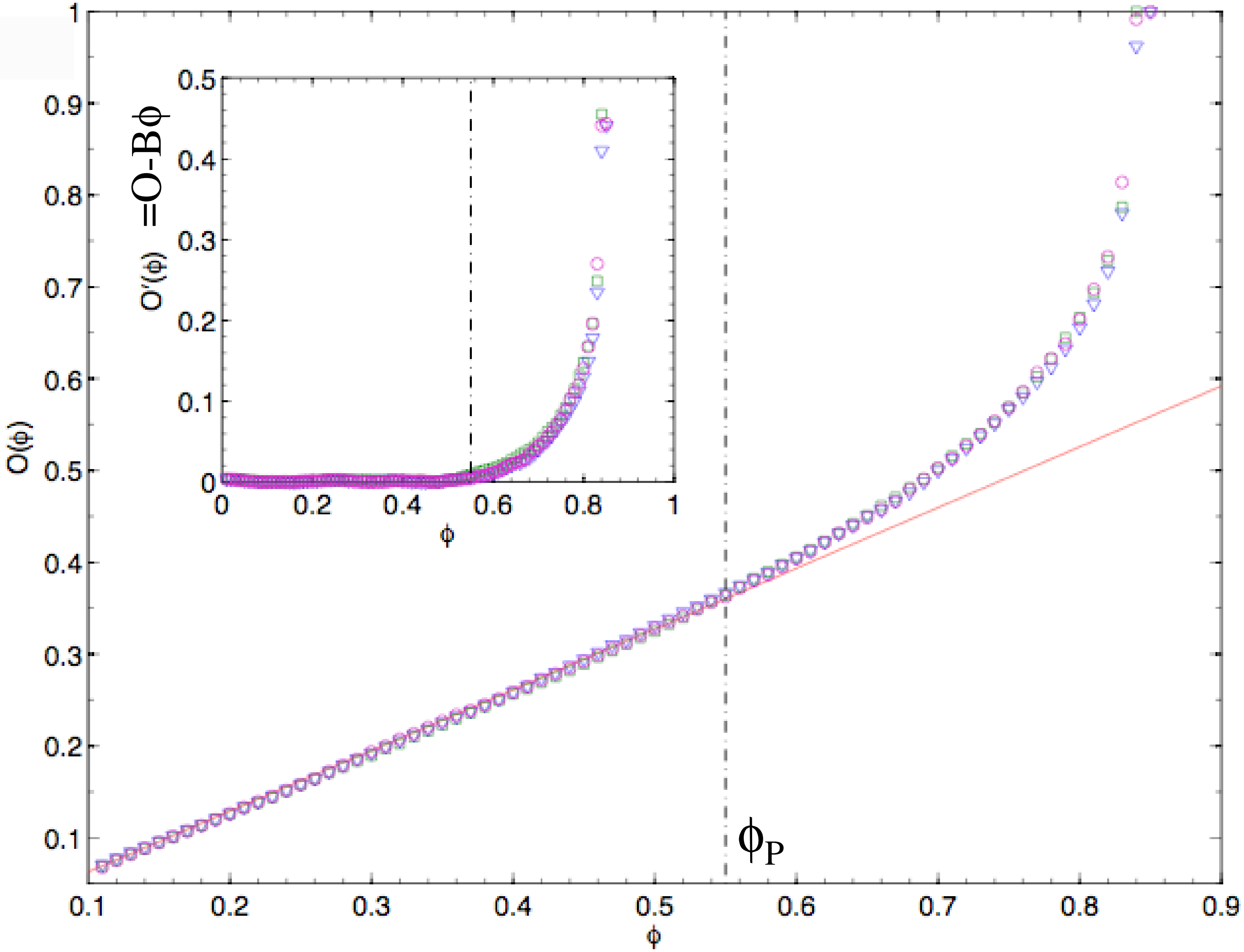
How similar are the contact networks at ϕ and ϕ_J?



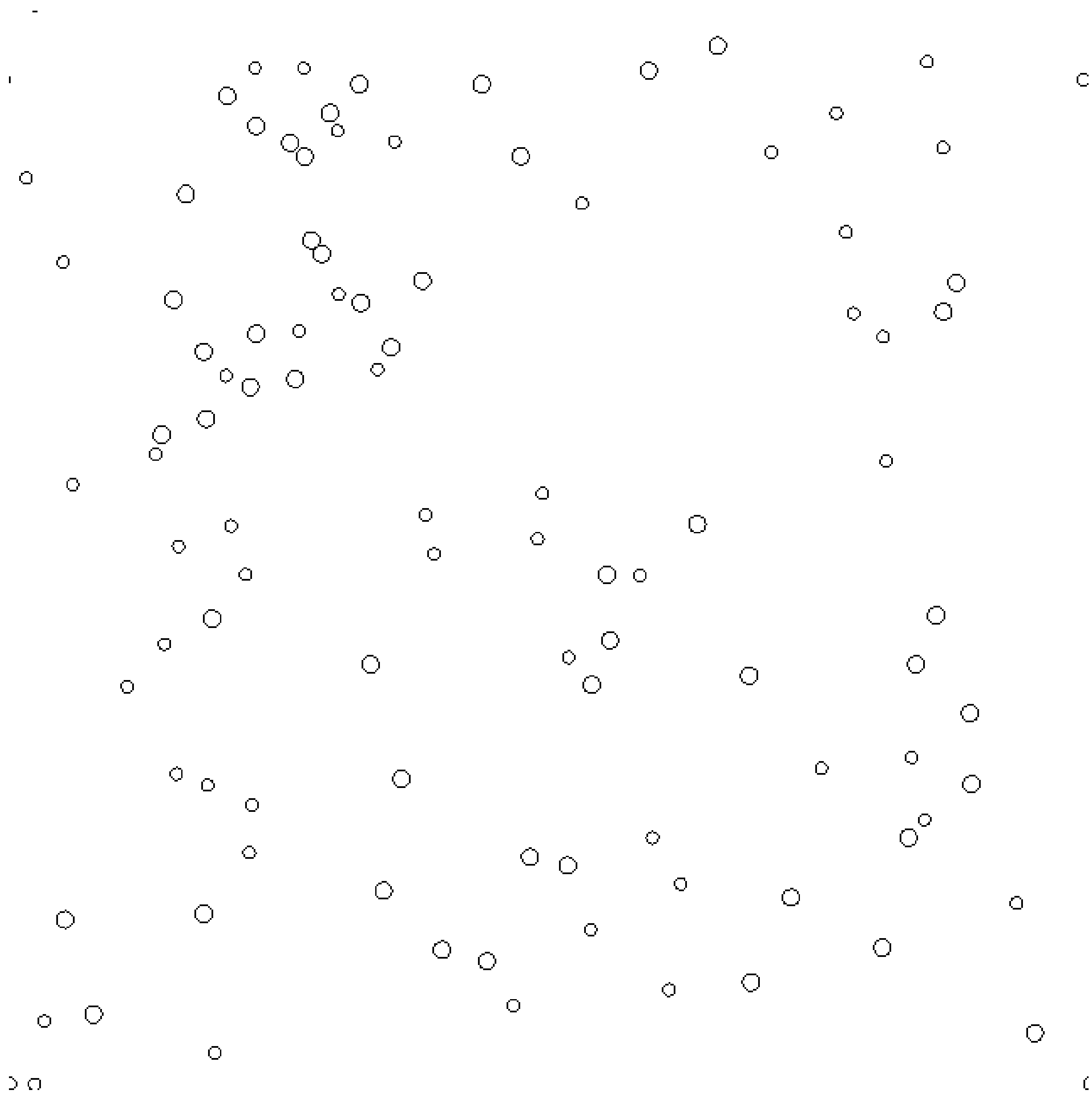
$$A_{ij} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$O(\phi) = \frac{1}{N_c} \sum_{i>j} A_{ij}(\phi) A_{ij}(\phi_J) \qquad O(\phi_J) = 1$$

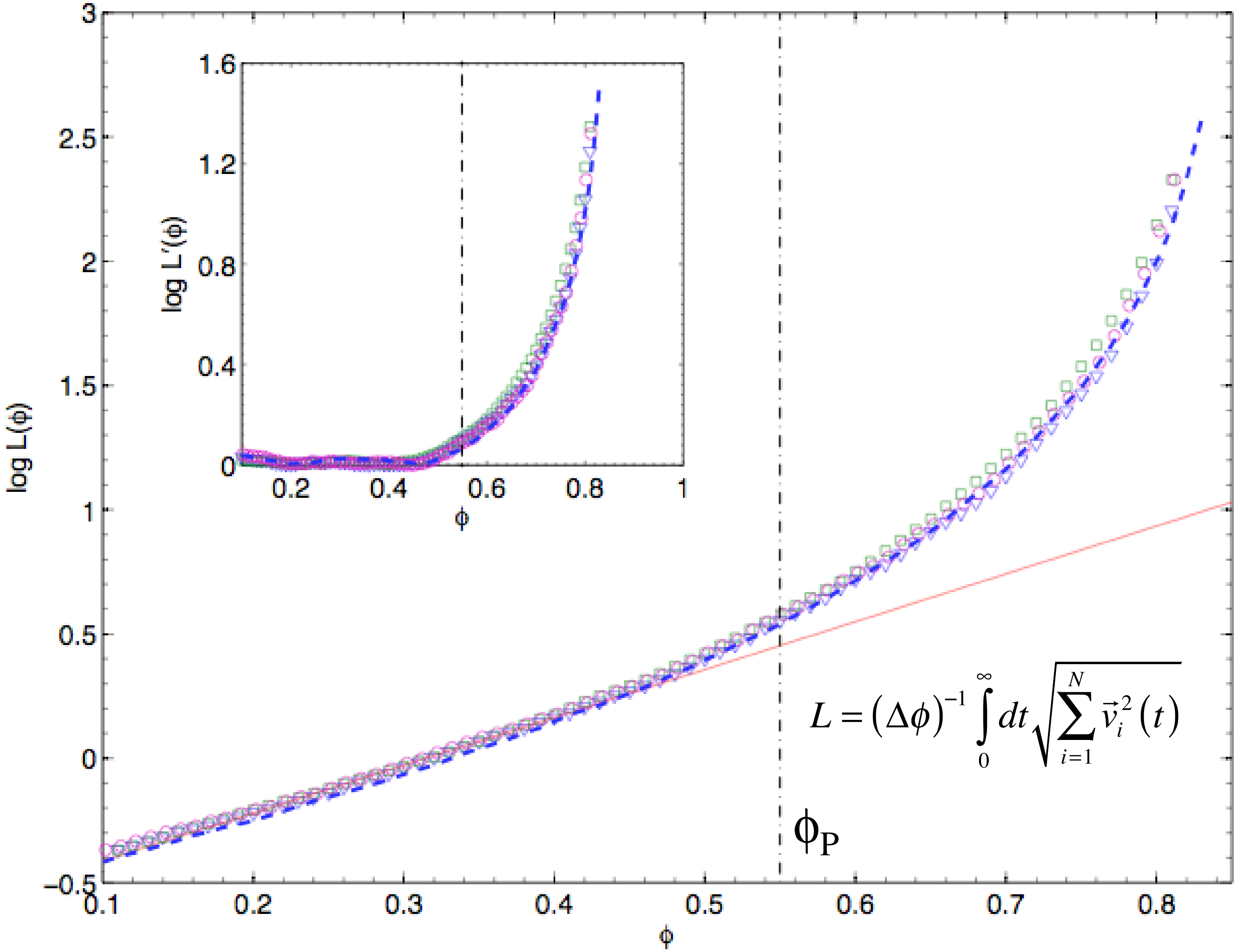
Adjacency Matrix Overlap



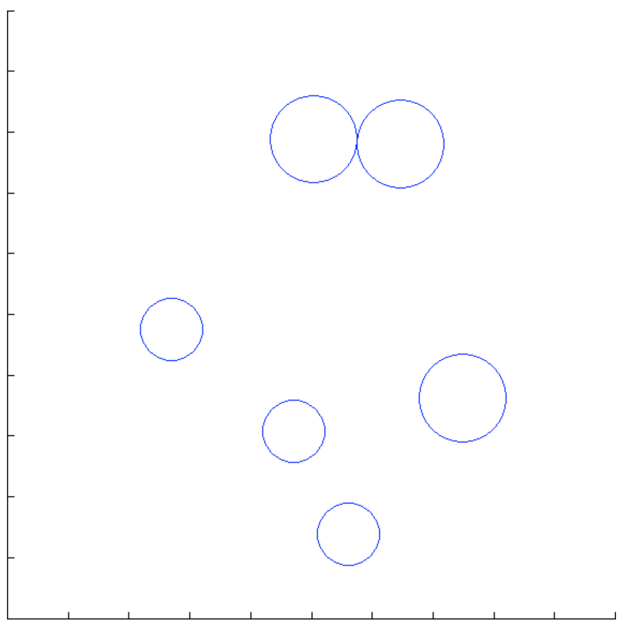
Cooperative Motion



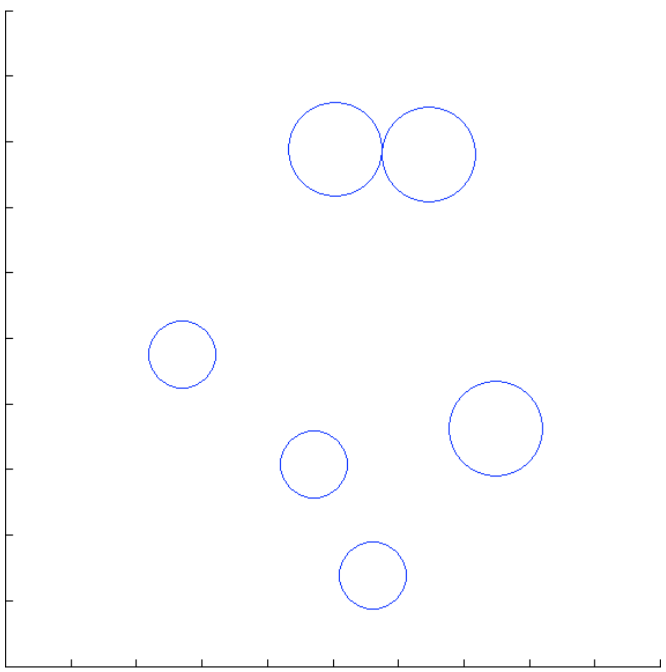
Incremental Contour Length of Trajectory



What determines MS packing probabilities?



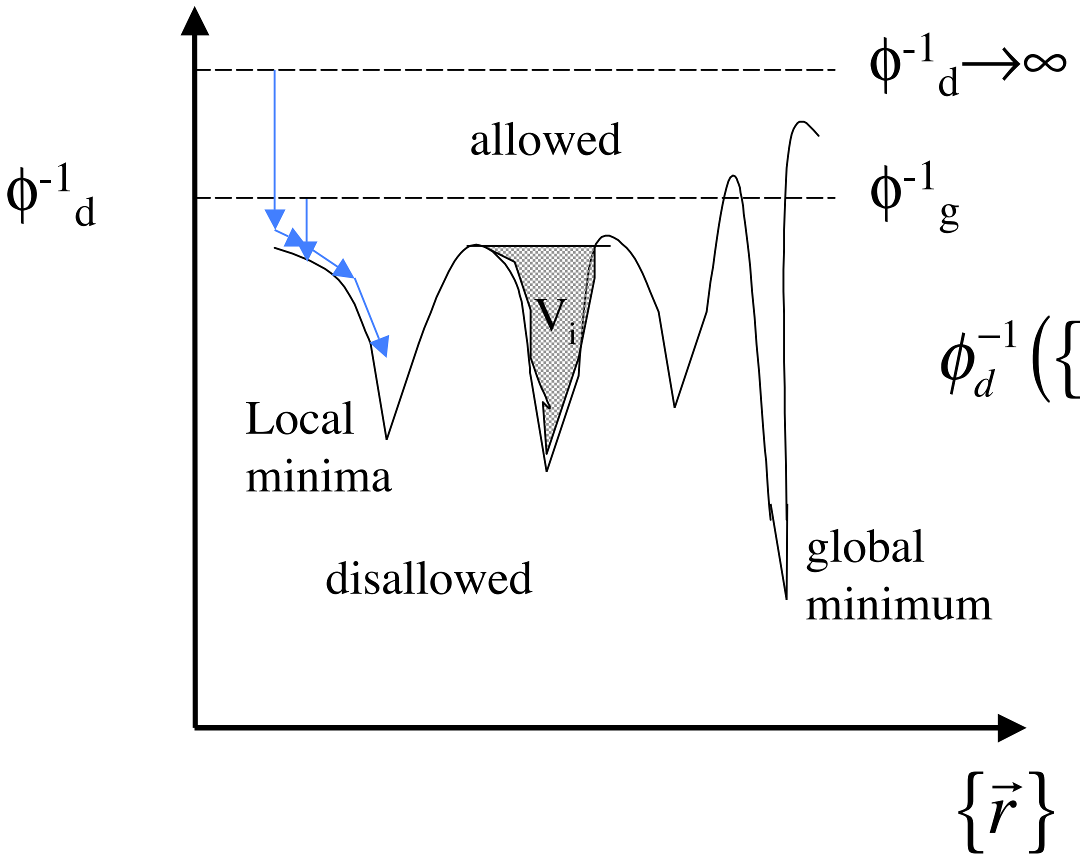
fast rate; ϕ_f=0.622



slow rate; ϕ_f=0.730

- Protocol-dependent basin volume…

Density landscape for hard spheres

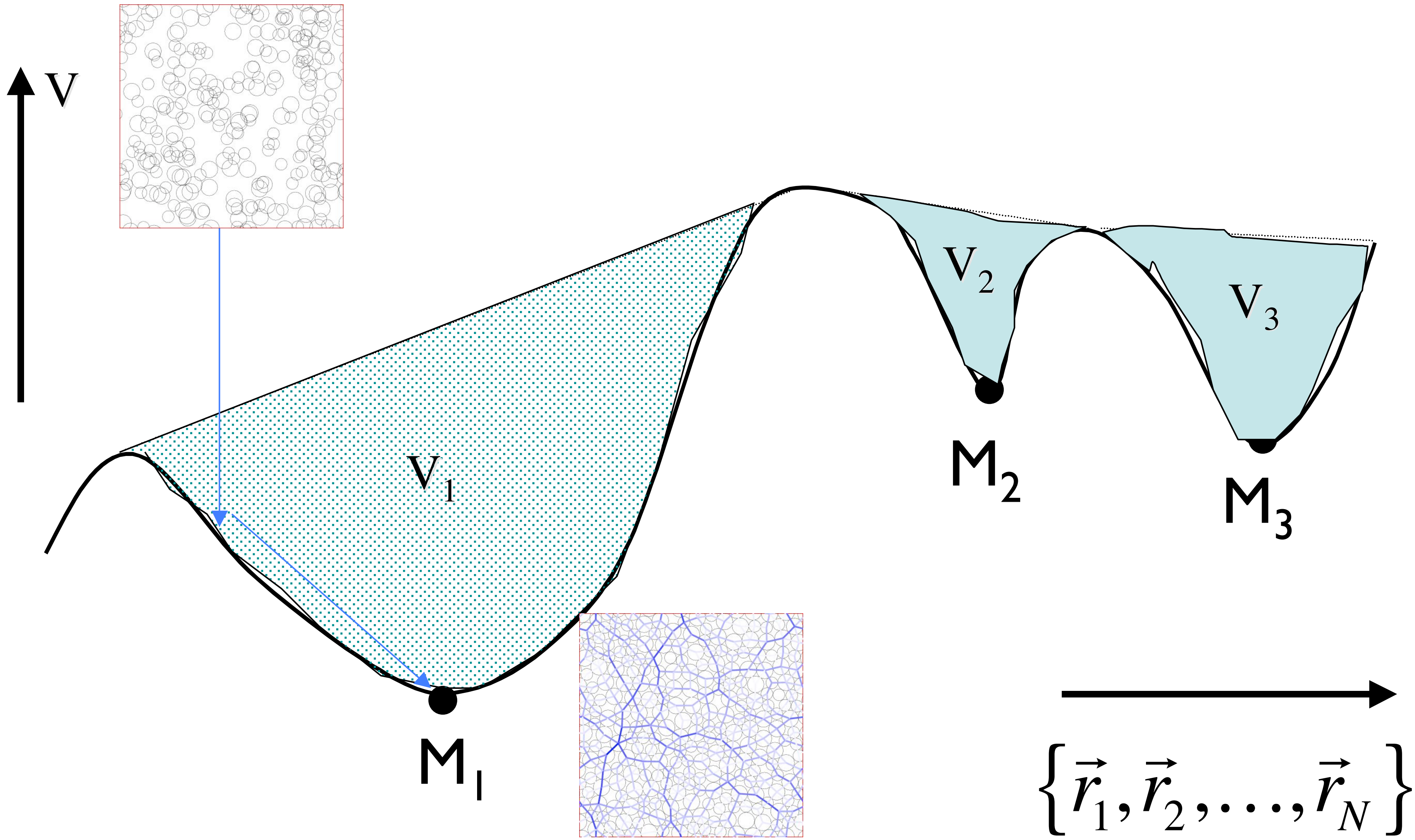


$$\phi_d^{-1}\left(\{\vec{r}\}\right) = \frac{6V}{\pi N \min_{mn} \left|\vec{r}_m - \vec{r}_n\right|^3}$$

- Granular media are out-of-equilibrium at all ϕ during packing-generation procedure

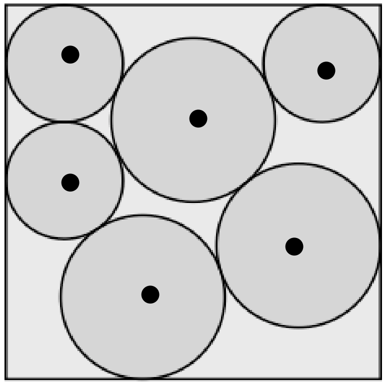
S. S. Ashwin, C. S. O'Hern, & M. D. Shattuck, submitted to Phys. Rev. E (2011).

Energy Landscape for Soft Compressed Spheres at Fixed ϕ

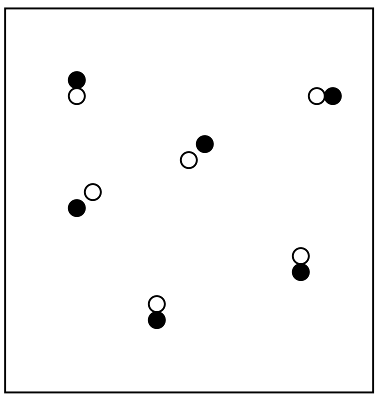


N. Xu, D. Frenkel, and A. J. Liu, Phys. Rev. Lett. 106 (2011) 245502.

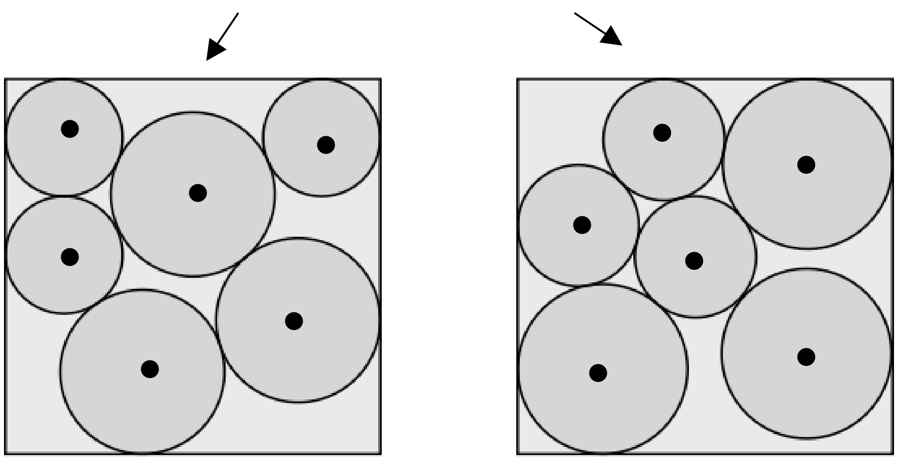
Method 1 (small l): Probability to return to a given MS packing



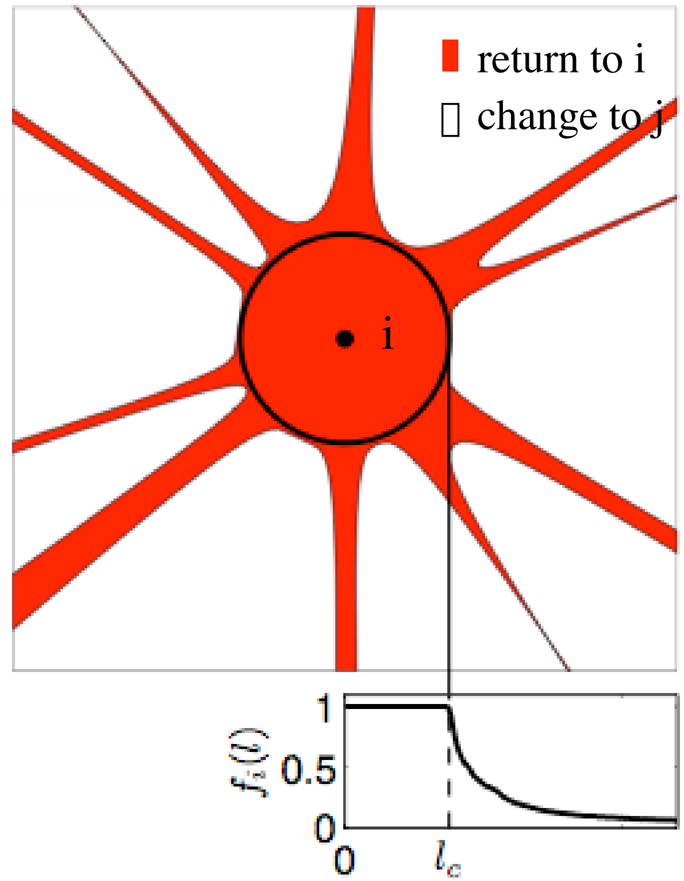
$$\phi_i^{MS}, \{\vec{r}\}_i^{MS}$$



$$\{\vec{r}\} = \{\vec{r}\}_i^{MS} + l\hat{e}_r$$



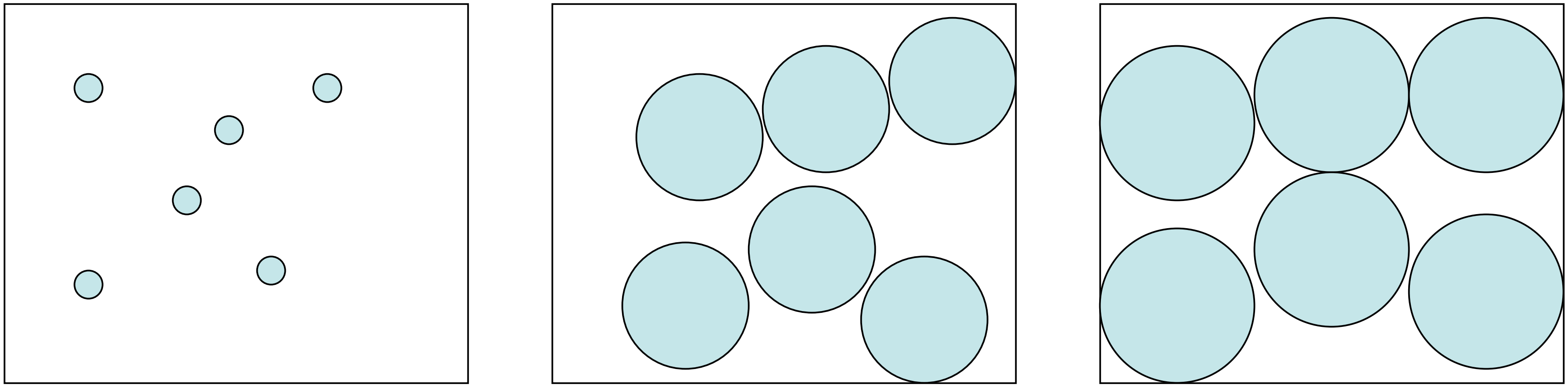
$$f_i(l) = \frac{M_i}{M}$$



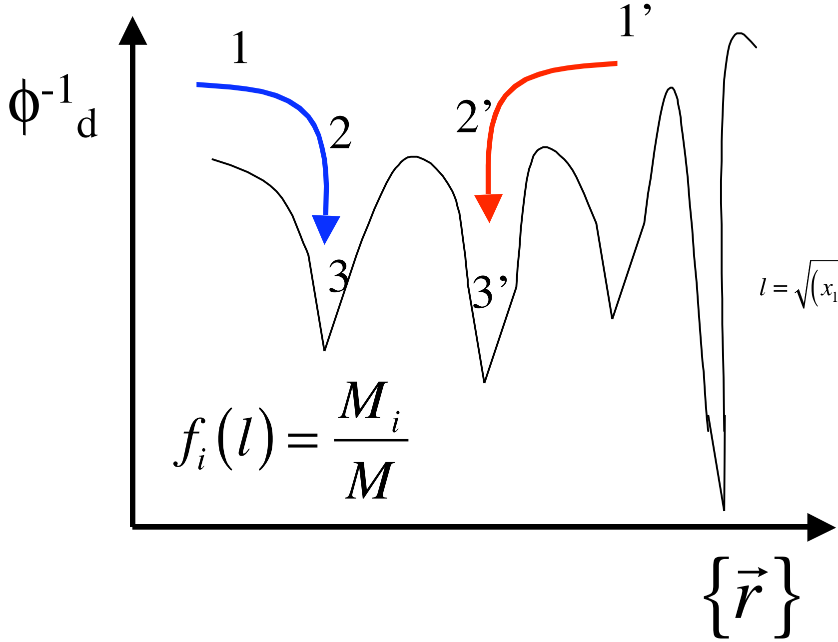
$$l = \sqrt{(x_{1f}-x_{10})^2 + (x_{2f}-x_{20})^2 + \cdots + (x_{Nf}-x_{N0})^2 + (y_{1f}-y_{10})^2 + (y_{2f}-y_{20})^2 + \cdots + (y_{Nf}-y_{N0})^2}$$

Distance in config. space

Method 2 (large l): Random initial conditions



$\phi_1 , \{\vec{r}\}_1$ $\phi_2 , \{\vec{r}\}_2$ $\phi_3 , \{\vec{r}\}_3$



Distance in config. space

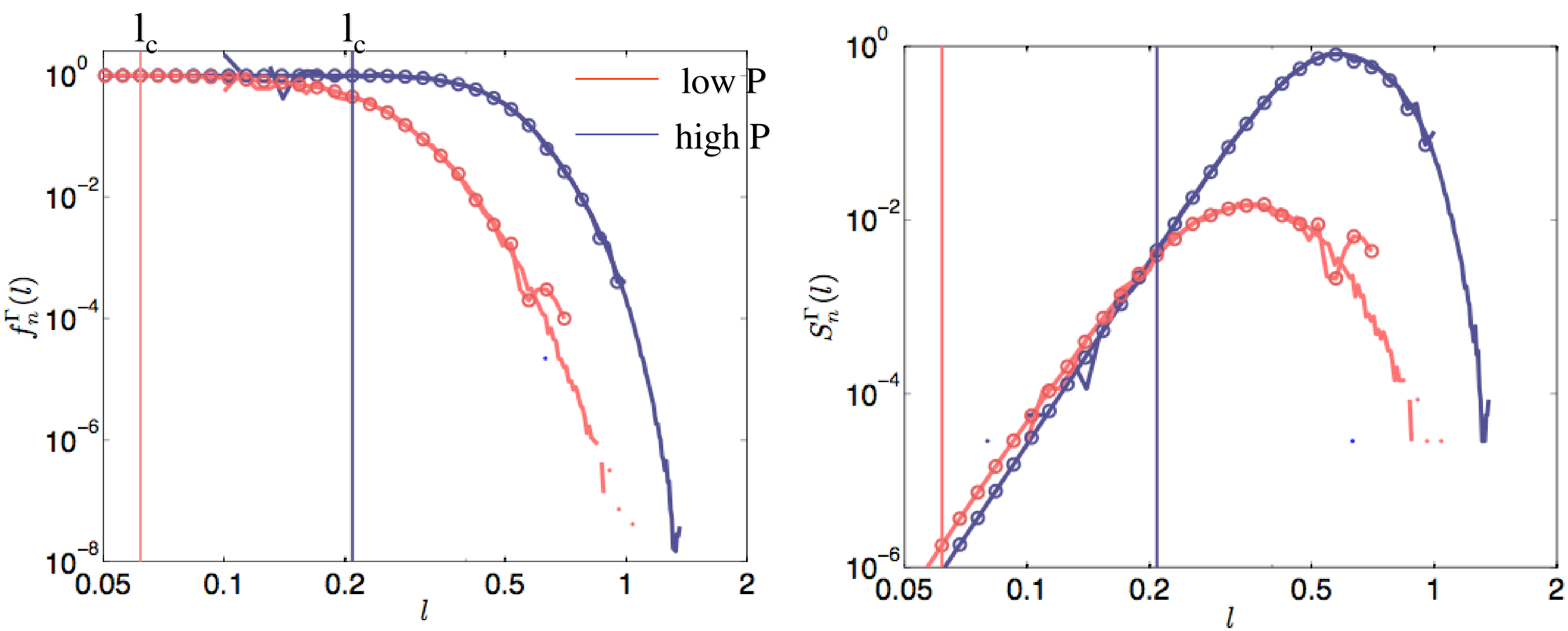
$$l = \sqrt{(x_{1f}-x_{10})^2 + (x_{2f}-x_{20})^2 + \cdots + (x_{Nf}-x_{N0})^2 + (y_{1f}-y_{10})^2 + (y_{2f}-y_{20})^2 + \cdots + (y_{Nf}-y_{N0})^2}$$

Basin Volumes

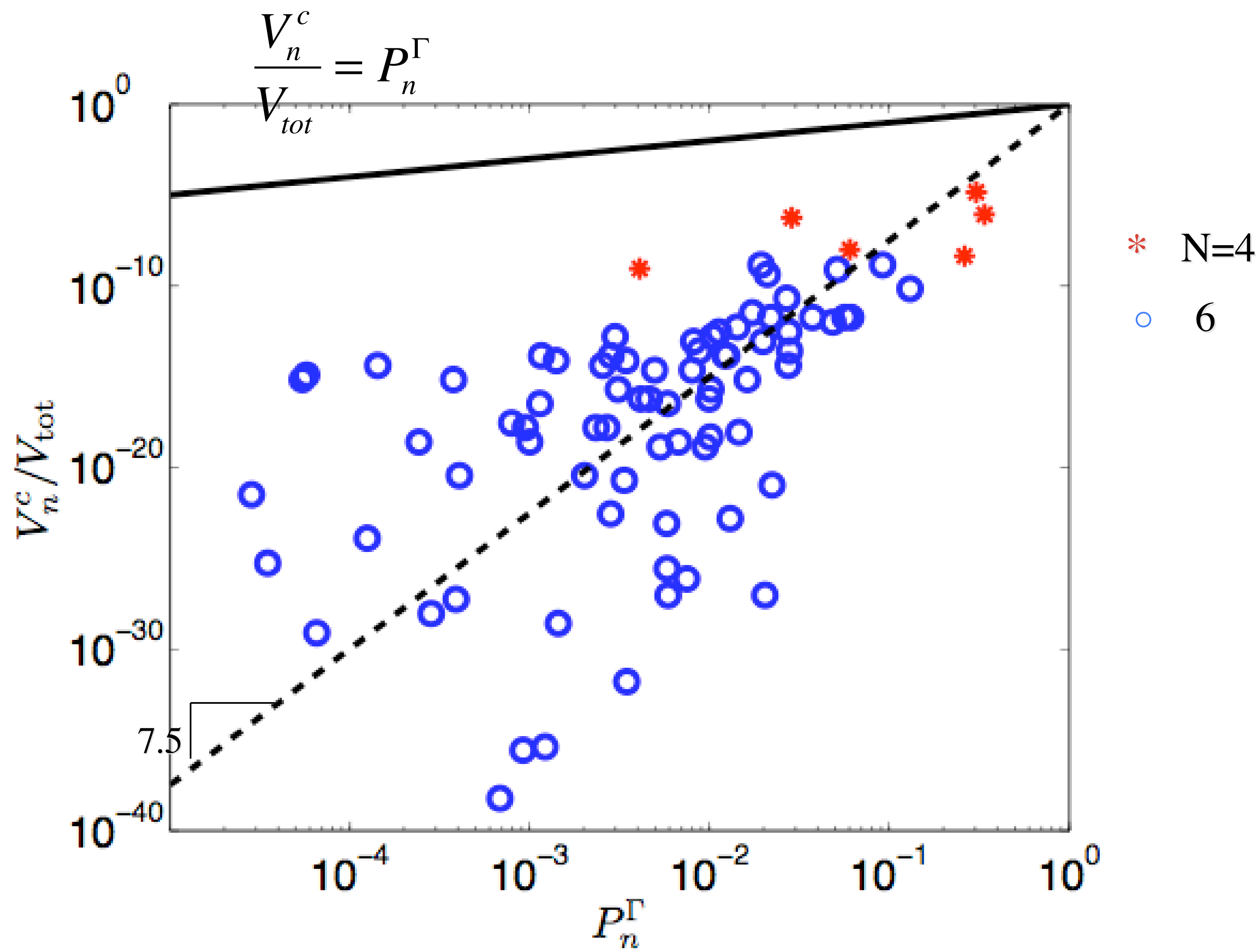
$$P_i = \frac{V_i}{L^{dN}} \qquad V_i = \int_0^{\sqrt{dN}} S_i(l)\,dl$$

$$S_i(l) = A_{dN} f_i(l)\, l^{dN-1} \mathrm{P}_i N_s!N_l! \qquad f_i(l) = \frac{M_i}{M}$$

Unweighted and Weighted Basin Profile Functions

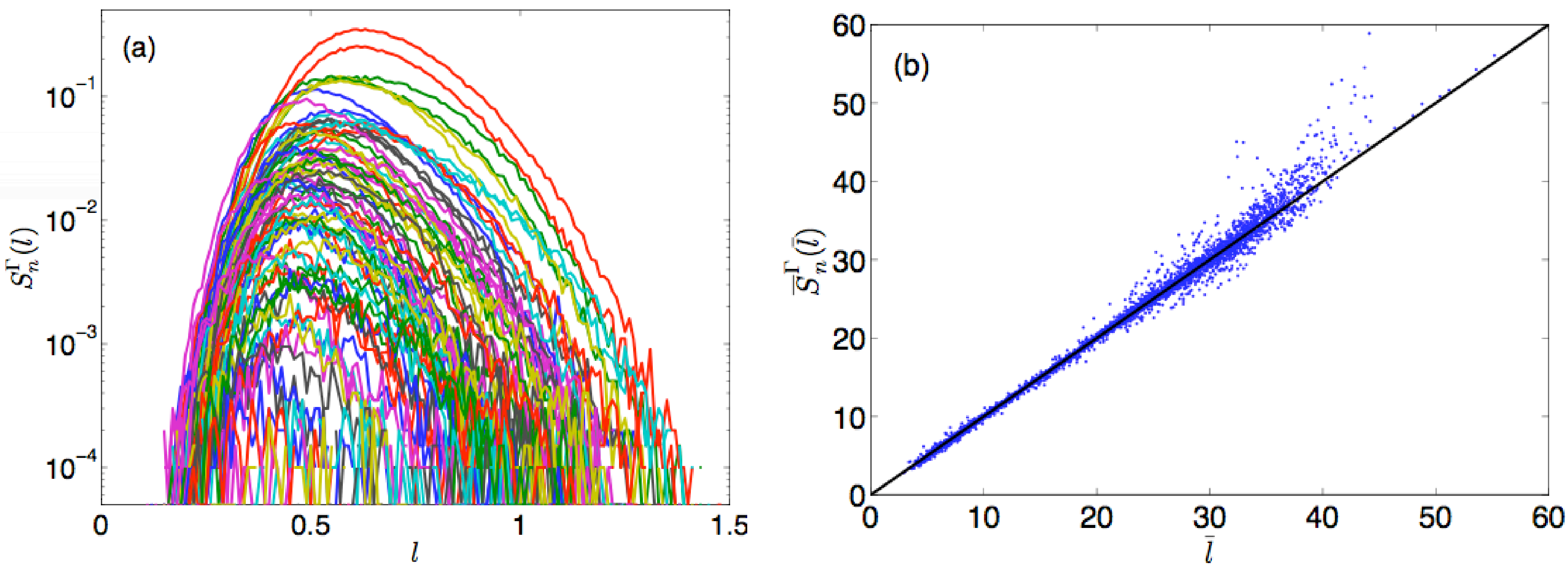


- Probability is determined by large $l > l_c$



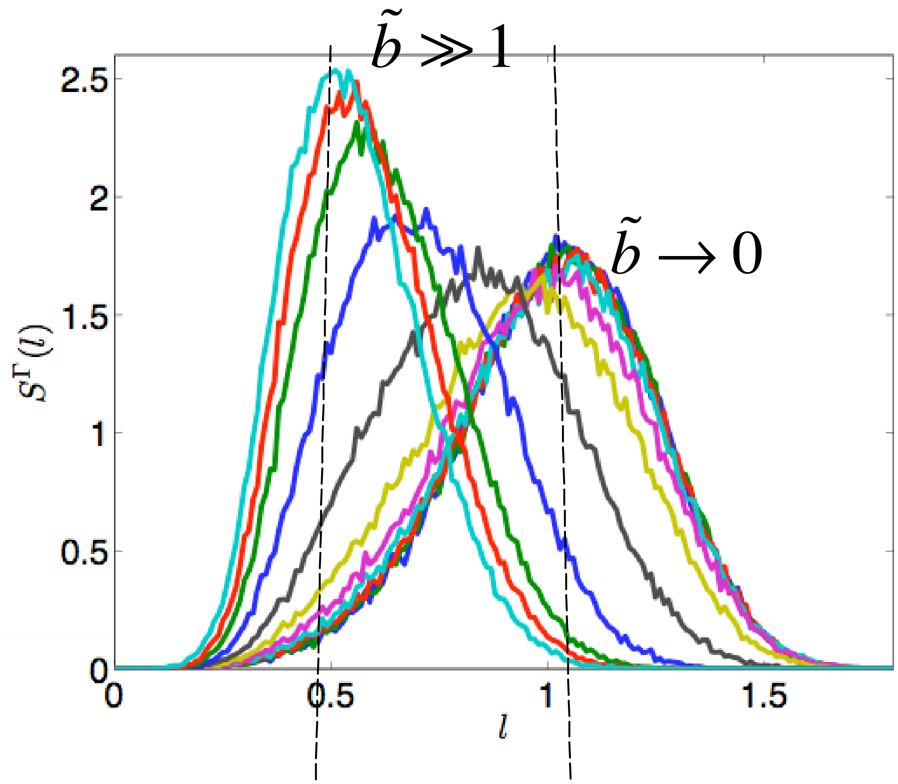
- Volumes of hyperspherical cores are much smaller than basin volumes!

Form of weighted basin profile functions



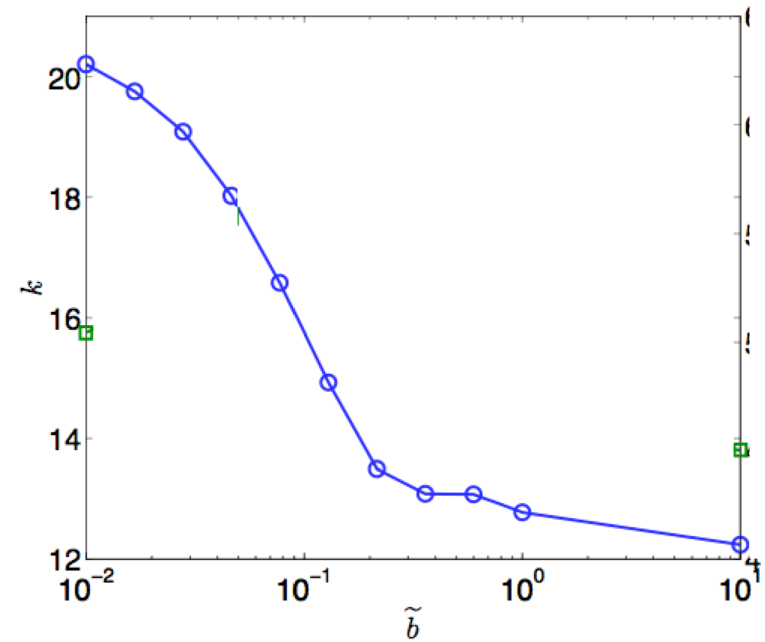
$$S_n^\Gamma(l) = \frac{\left(\frac{l}{\theta}\right)^{k-1} e^{-\frac{l}{\theta}}}{\theta\Gamma(k)} \qquad \theta = \frac{\langle l^2\rangle - \langle l\rangle^2}{\langle l\rangle} \qquad k = \langle l\rangle/\theta$$

Protocol dependence of average weighted basin profile function

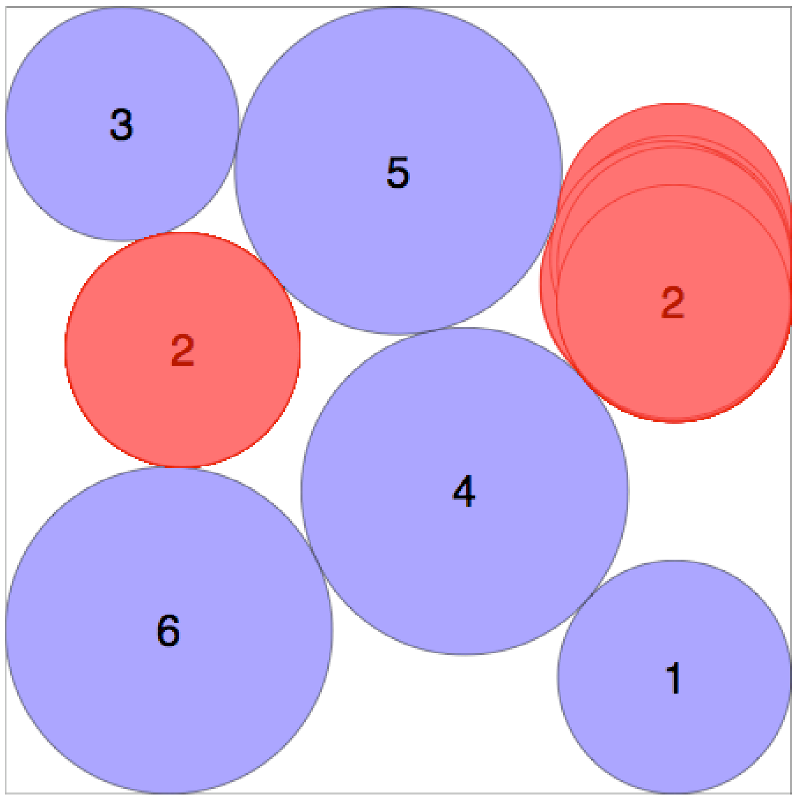


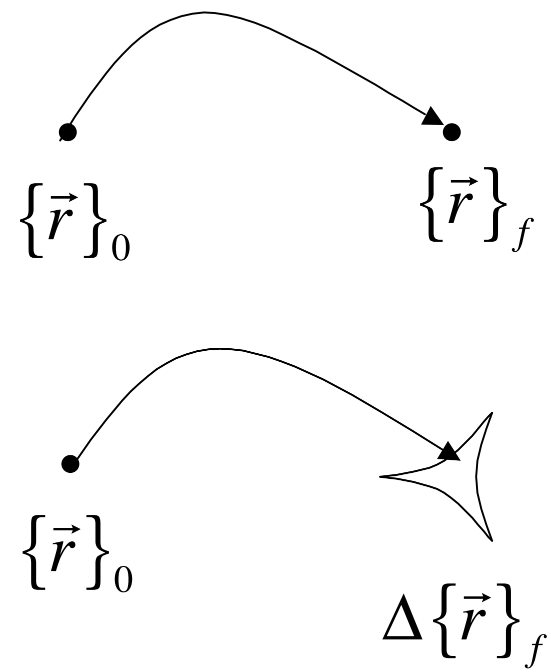
Average distance between two random points in a single polarization

Average distance between two random points in random polarizations

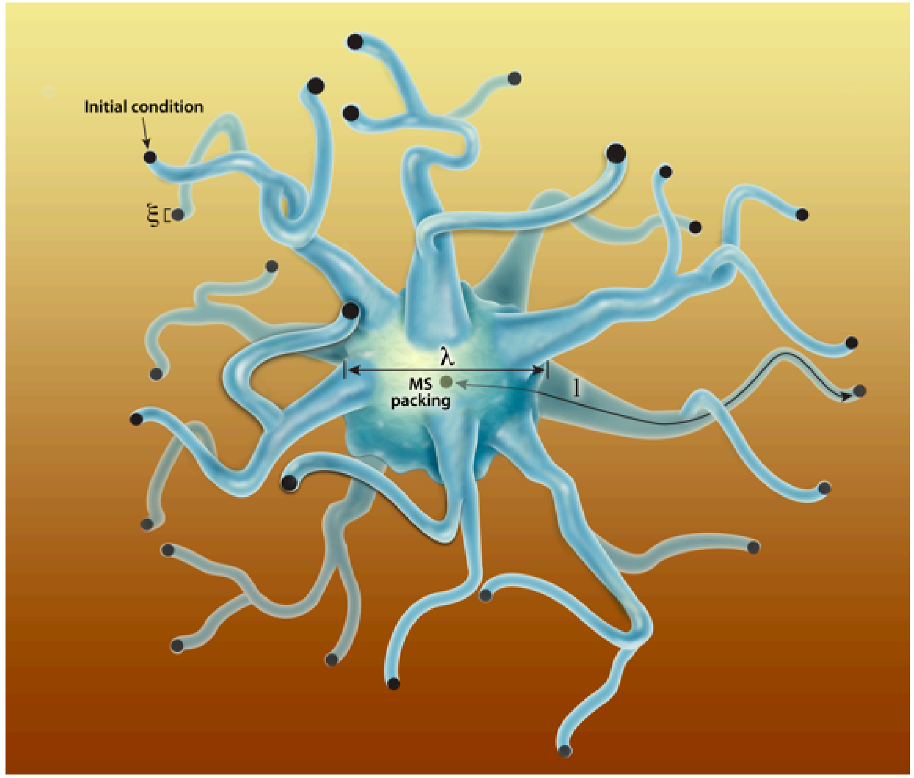


Floaters





Particle 2 can exist in two locations with fewer than 3 contacts

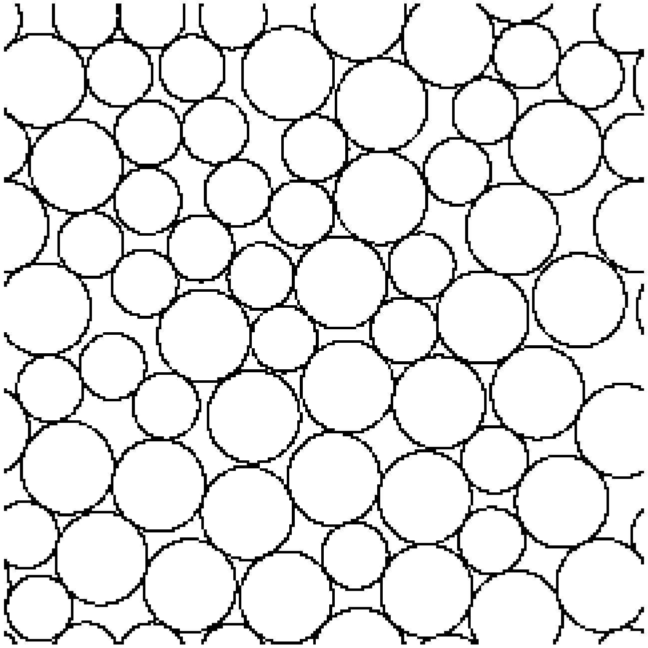


•Probability for MS packings determined by large l, not nearby regions of configuration space

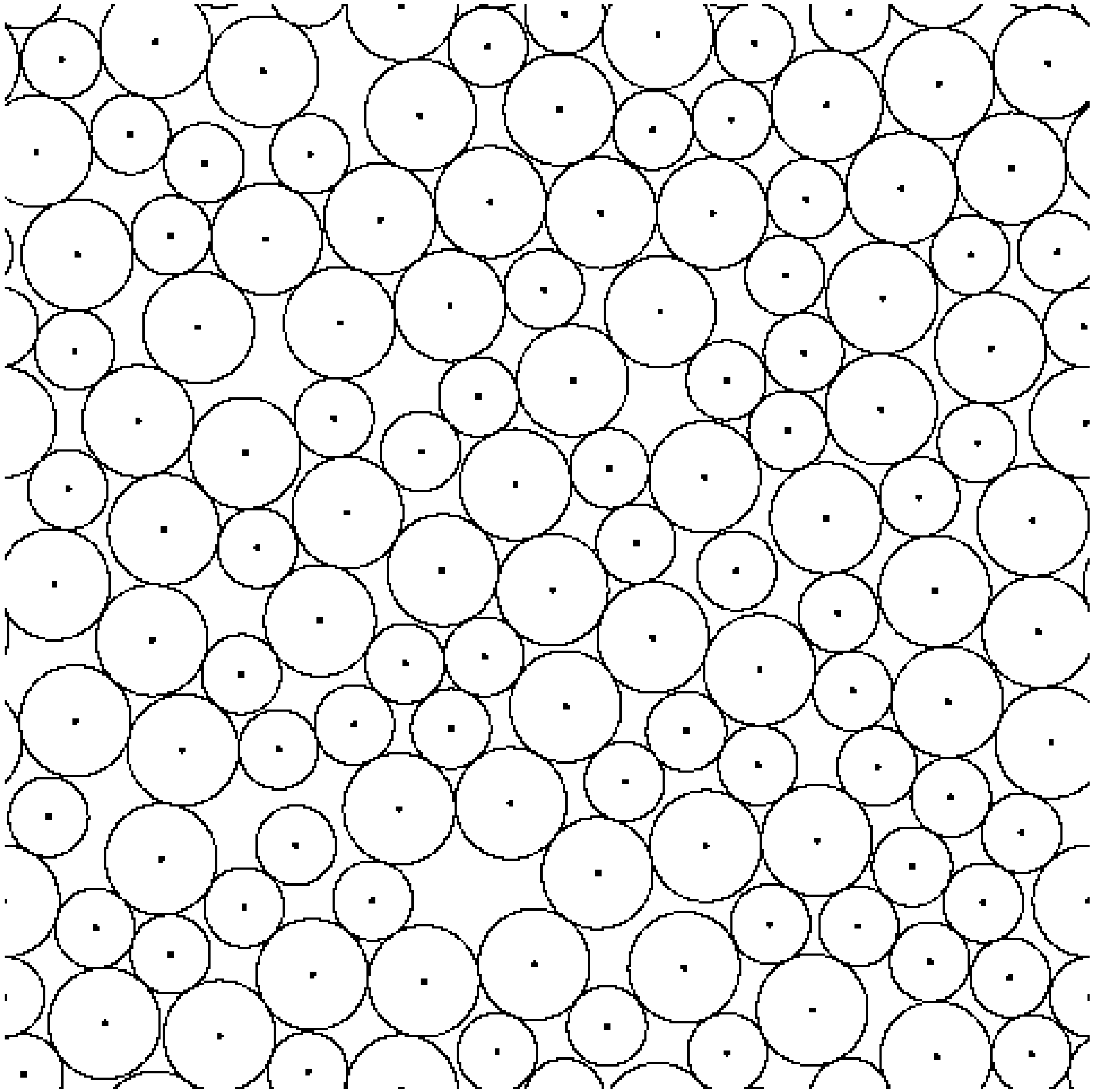
•Universal form for $S_n(l)$

•At what ϕ_i do basin volumes become hyperspherical?

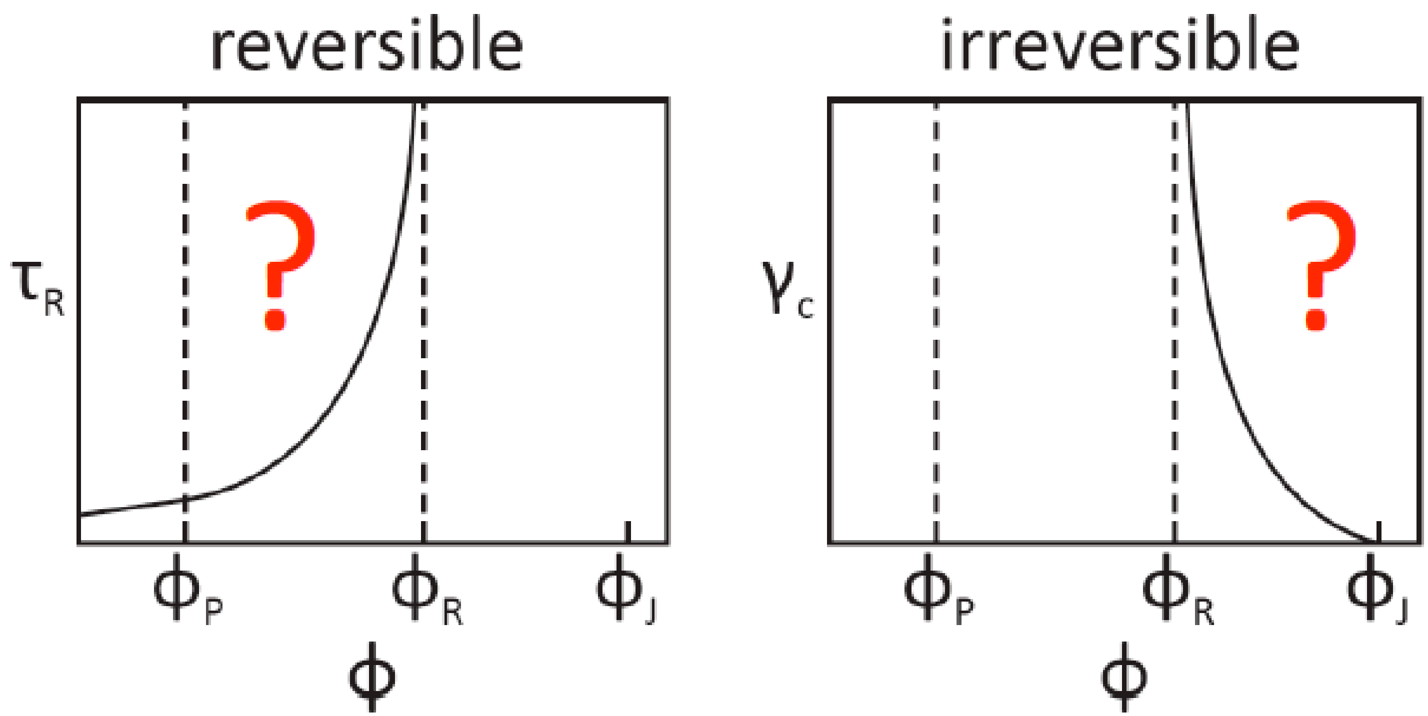
New Directions: Reversibility/Irreversibility



Displacements from strain=0 from previous cycle

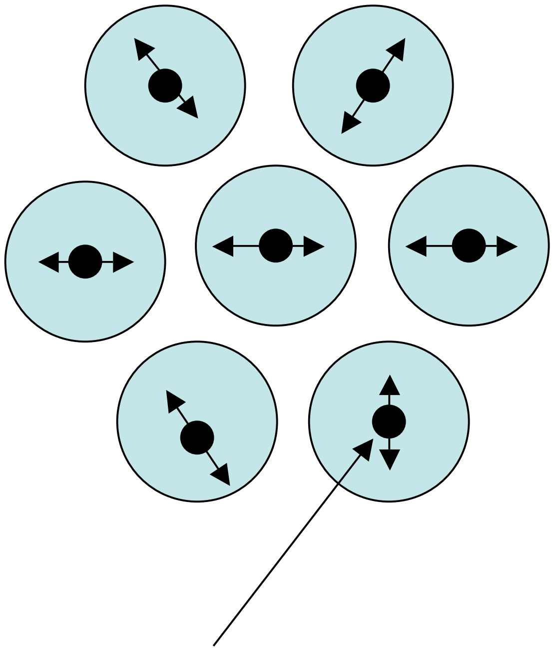


‘Phase Digram’ for Irreversible Flow

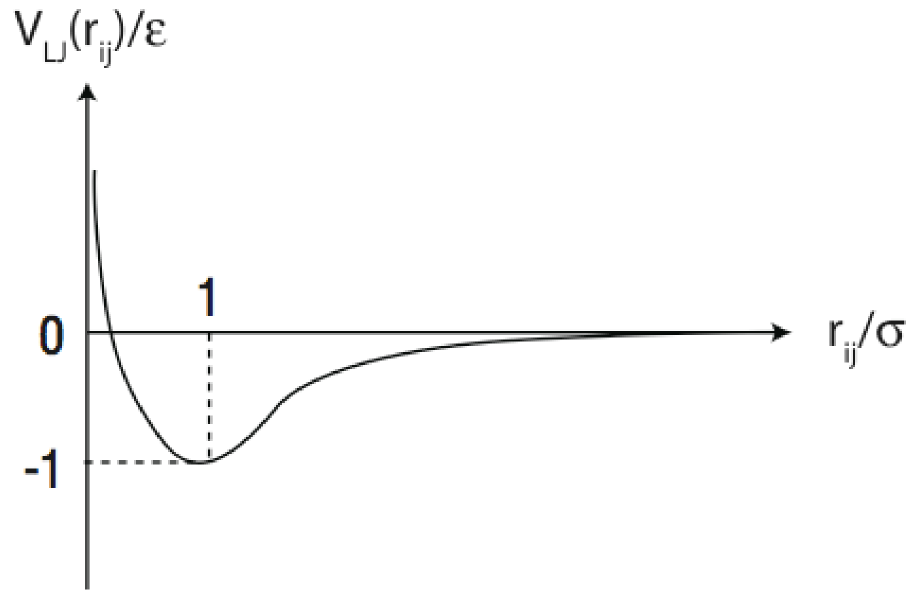


Do static granular packings possess a harmonic vibrational response?

Harmonic Solids



$$\vec{r}_0^{\,i} = \left\langle \vec{r}^{\,i}(t) \right\rangle_t$$



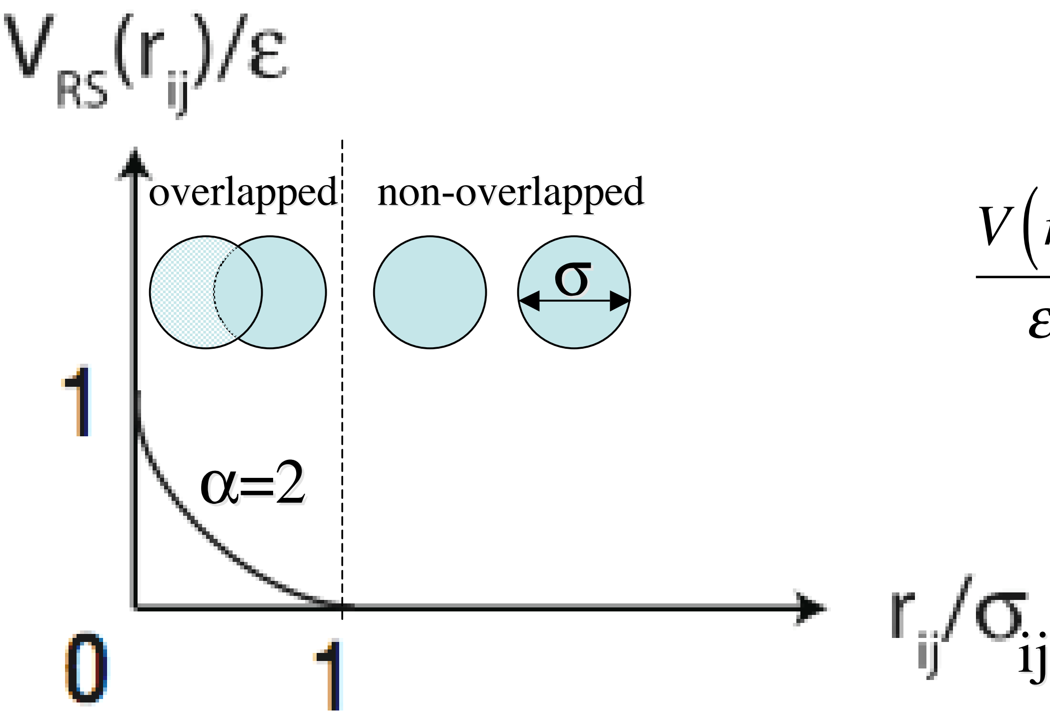
- Atomic and molecular systems
- Pair potentials have `double-sided' minimum and are long-ranged
- Equilibrium positions are well-defined
- Vibrations at low T captured using harmonic approximation

Causes of nonharmonicity in granular solids

- Nonlinear Hertzian interaction potential ✗
- Dissipation from normal contacts ✗
- Sliding and rolling friction ✗

- *Breaking existing contacts and forming new contacts*

Model Particulate Media



$$\frac{V(r_{ij})}{\varepsilon} = \begin{cases} \alpha^{-1}\left(1 - \dfrac{r_{ij}}{\sigma_{ij}}\right)^{\alpha} & r_{ij} \le \sigma_{ij} \\ 0 & r_{ij} > \sigma_{ij} \end{cases}$$

α=2 linear

α=5/2 Hertzian

Total potential energy $V = \sum_{\langle i,j \rangle} V(r_{ij})$

Harmonic approximation: Normal Modes from Dynamical Matrix

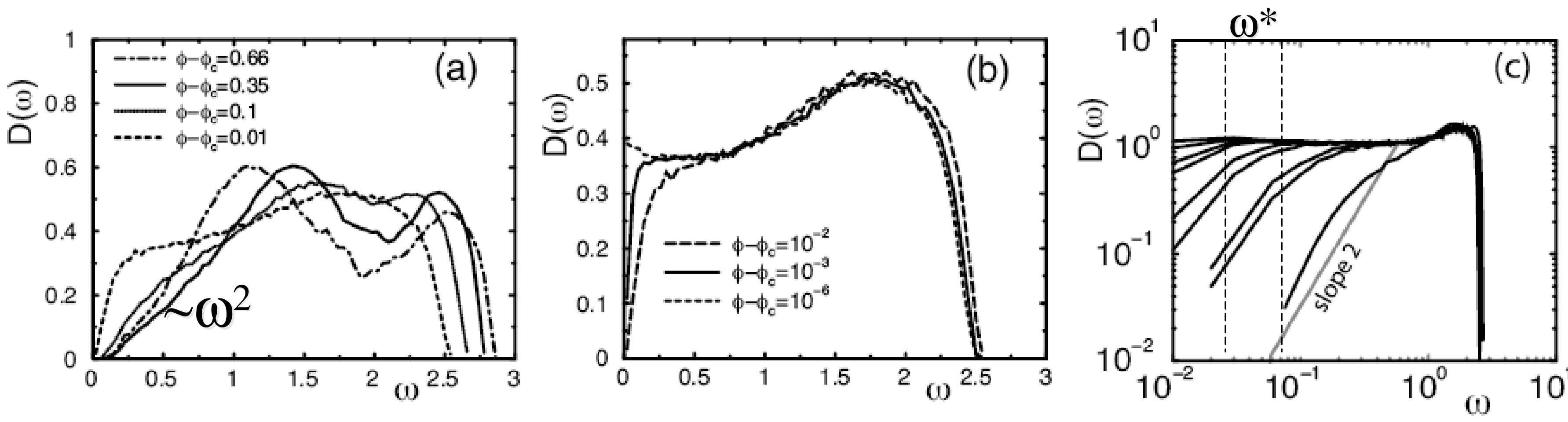
$$M_{\alpha,\beta} = \left. \frac{\partial^2 V(\vec{r})}{\partial r_\alpha \partial r_\beta} \right|_{\vec{r}=\vec{r}_0}$$

α,β=x, y, z, particle index

$\vec{r}_0$ = positions of MS packing

Calculate d N- d eigenvalues; $m_i = \omega^2_i > 0$.

Density of Vibrational Modes via Dynamical Matrix



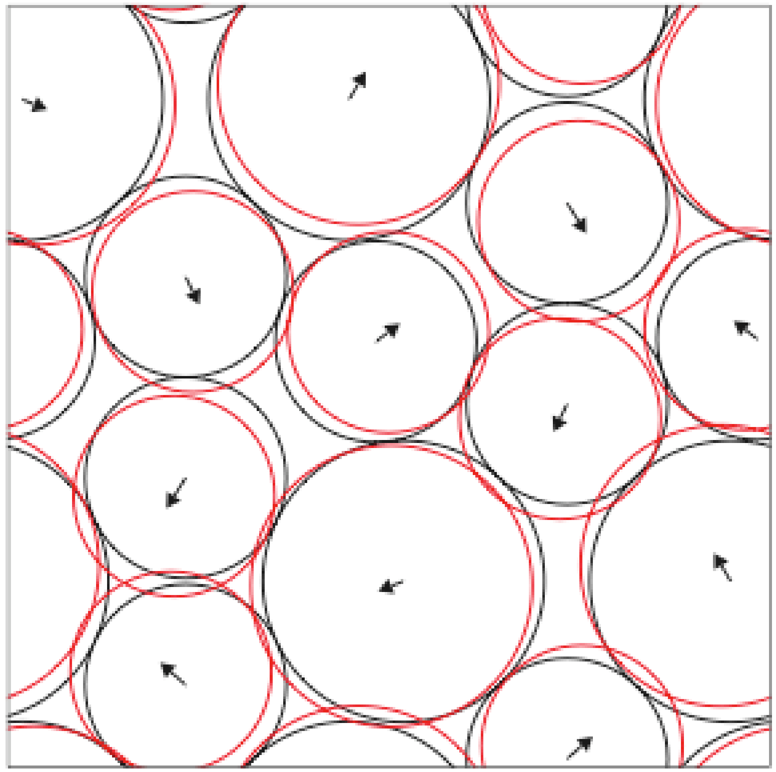
$$D(\omega)d\omega = N(\omega + d\omega) - N(\omega)$$

- Why $D(\omega)$?
- Formation of plateau in $D(\omega)$ (excess of low-frequency modes) as $\Delta\phi=\phi-\phi_J\rightarrow 0$

A. J. Liu, S. R. Nagel, W. van Saarloos, and M. Wyart, "The jamming scenario--an introdcution and outlook," Soft Matter (2010).

Are jammed particulate systems harmonic?

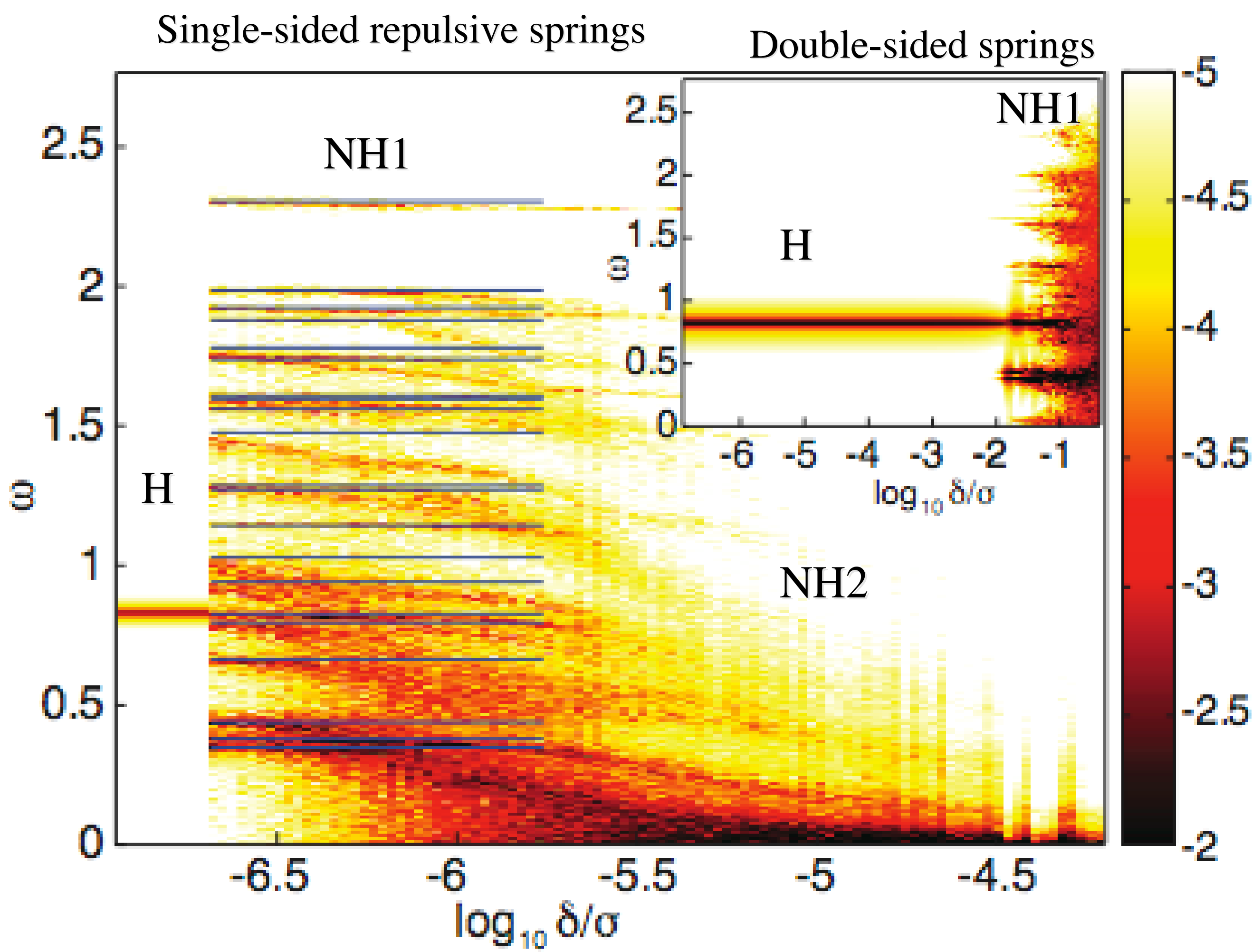
$$\vec{r}_i^{'} = \vec{r}_i + \delta \hat{e}_6$$



- Deform system along each 'eigenmode' ω_i
- Run at constant NVE, measure power spectrum of grain displacements
- Does system oscillate at frequency ω_i from dynamical matrix?

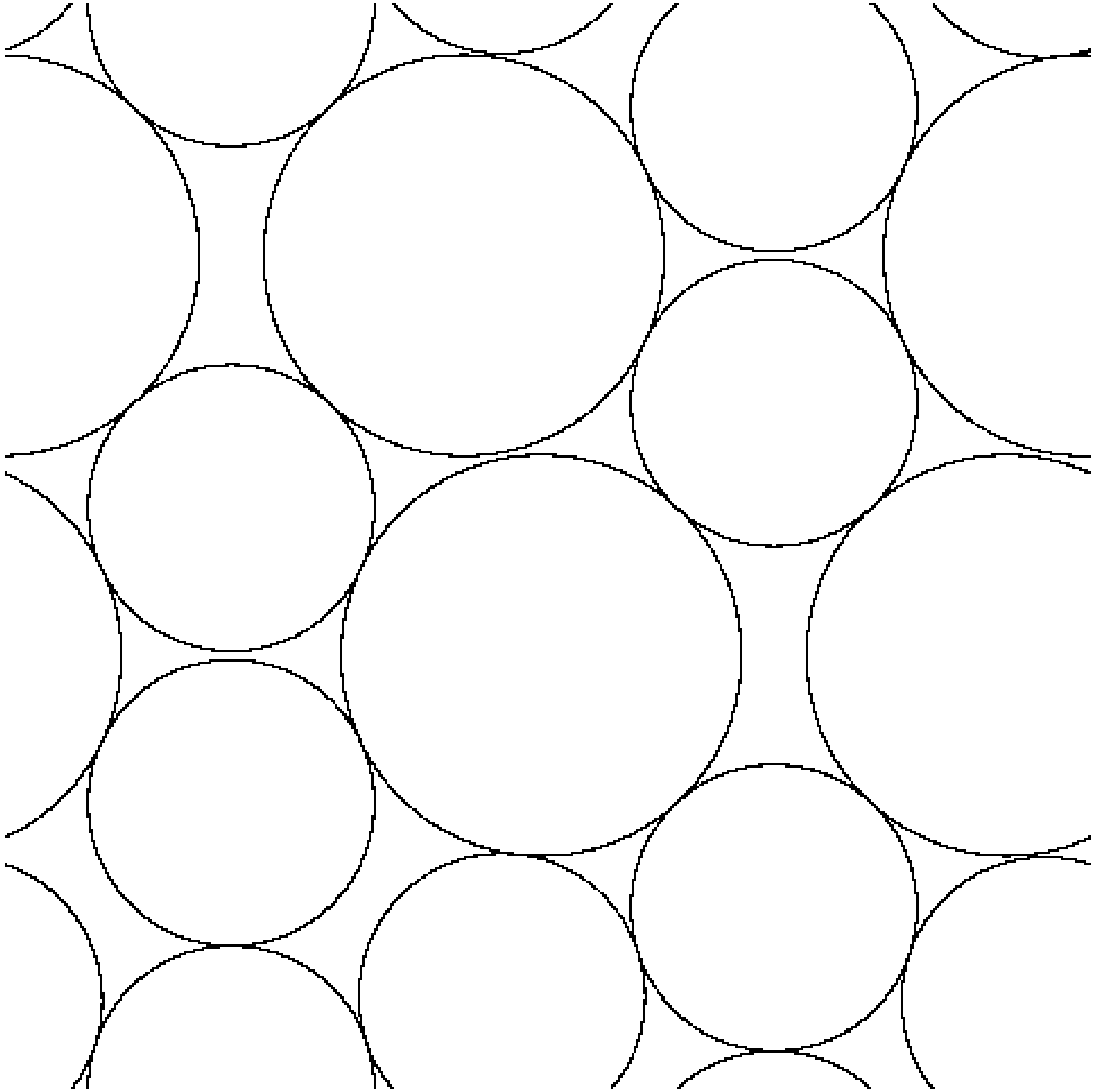
Power-spectrum of particle displacements

N=12
$\Delta\phi=10^{-5}$
Mode=6
Steady-state

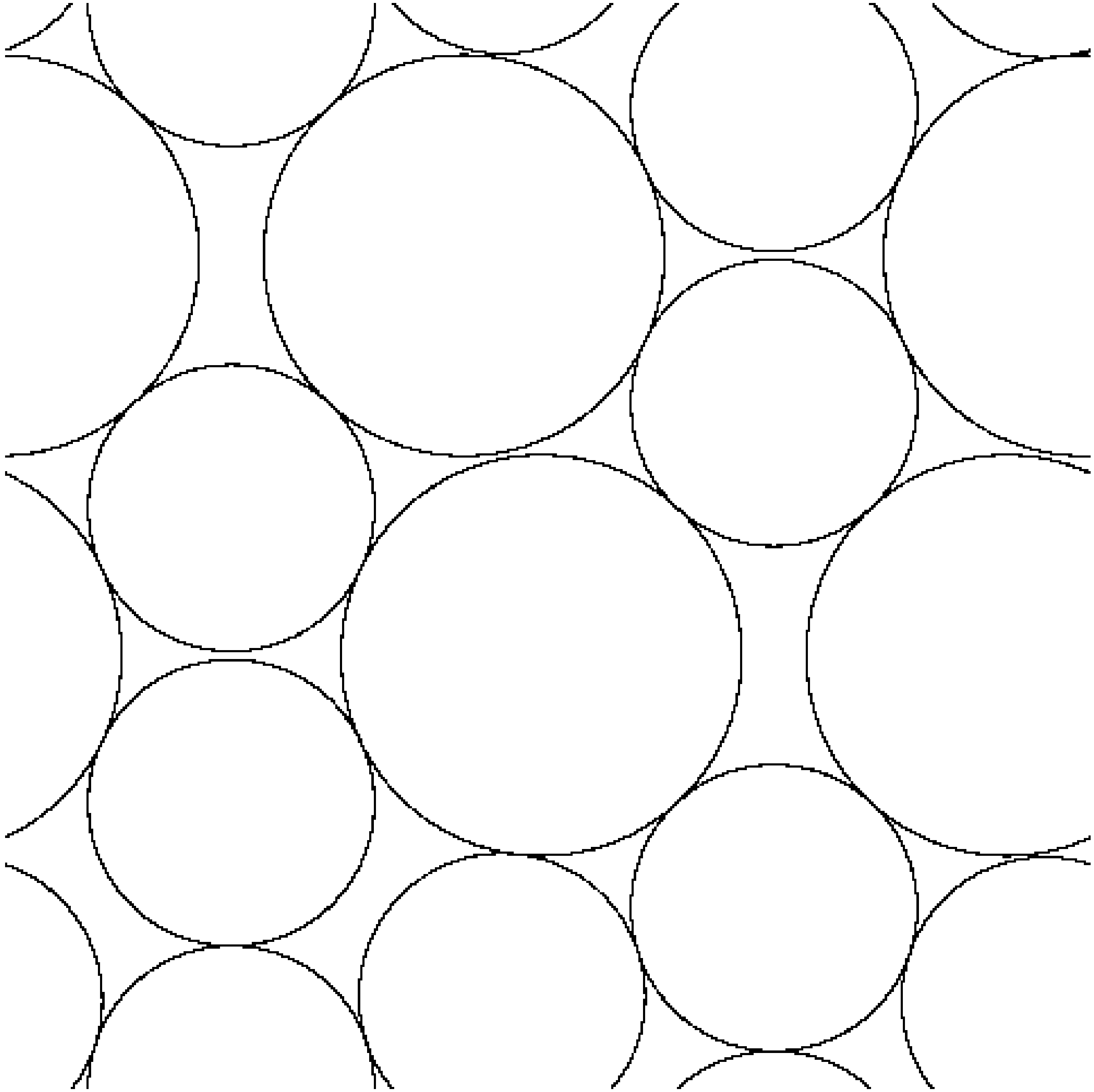


- System becomes strongly nonharmonic at extremely small δ
- First spreads to `harmonic' set of ω (NH1); then continuum of ω (NH2)

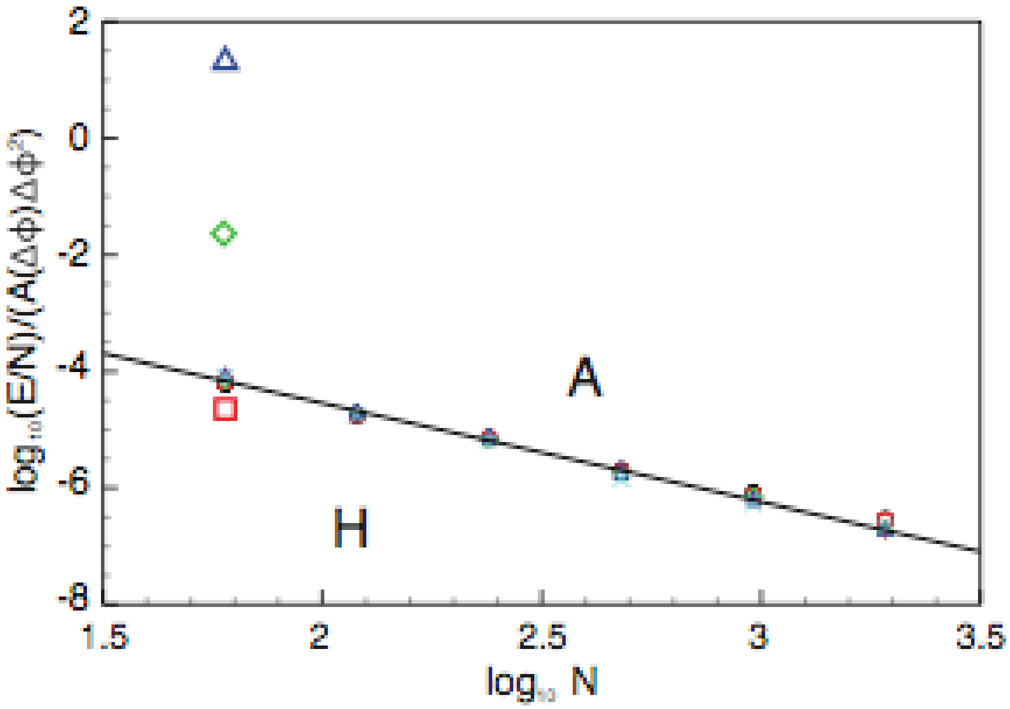
N=12
$\Delta\phi=10^{-5}$
Mode=6
$\delta/\sigma=10^{-5}$

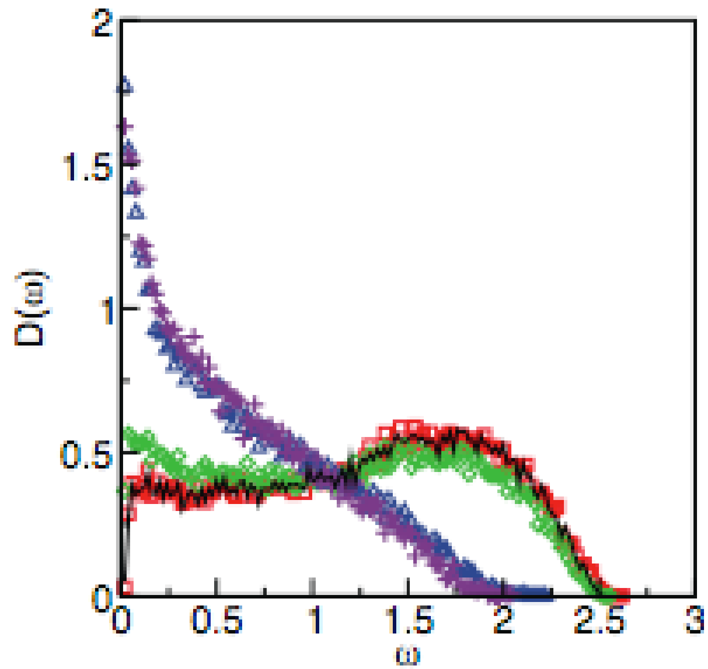


N=12
$\Delta\phi=10^{-5}$
Mode=6
$\delta/\sigma=10^{-3}$

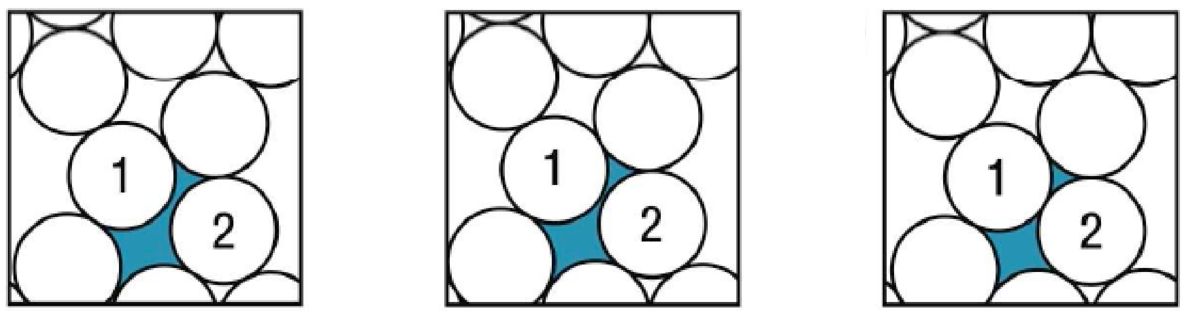


Strongly Anharmonic Behavior





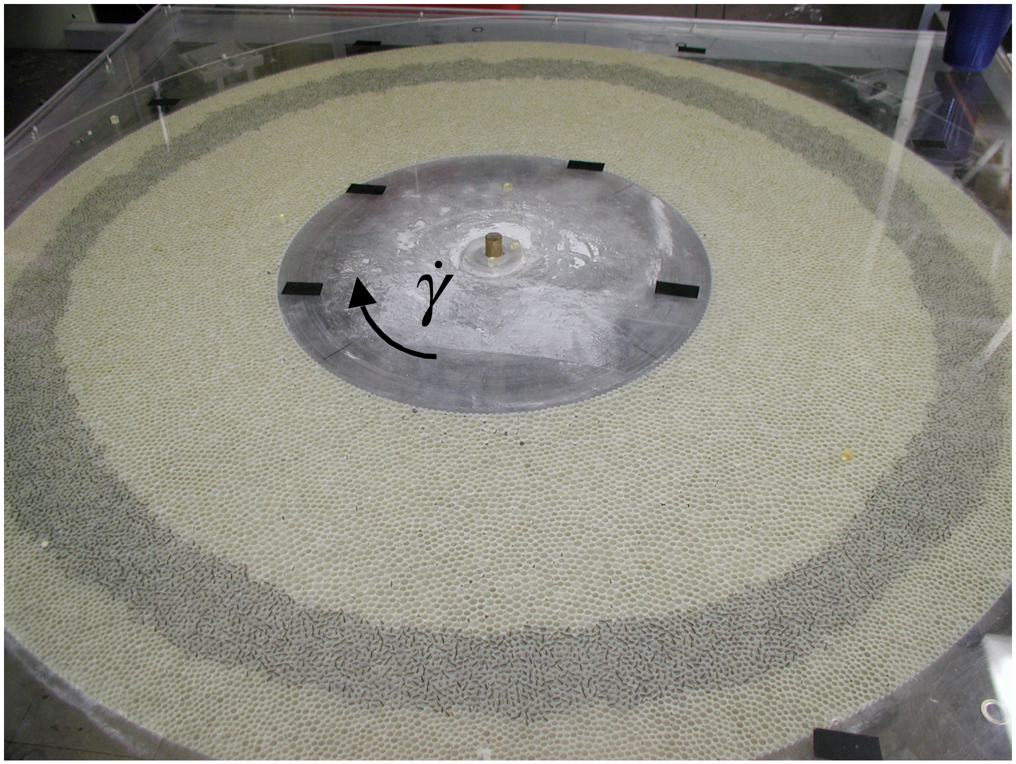
Geometrical Families



$\phi_7 = 0.7290$ $\phi_7 = 0.7320$ $\phi_7 = 0.7365$

- How can this happen? Boundary conditions, tangential forces

How do slow, dense shear flows sample MS packings…with equal probability?

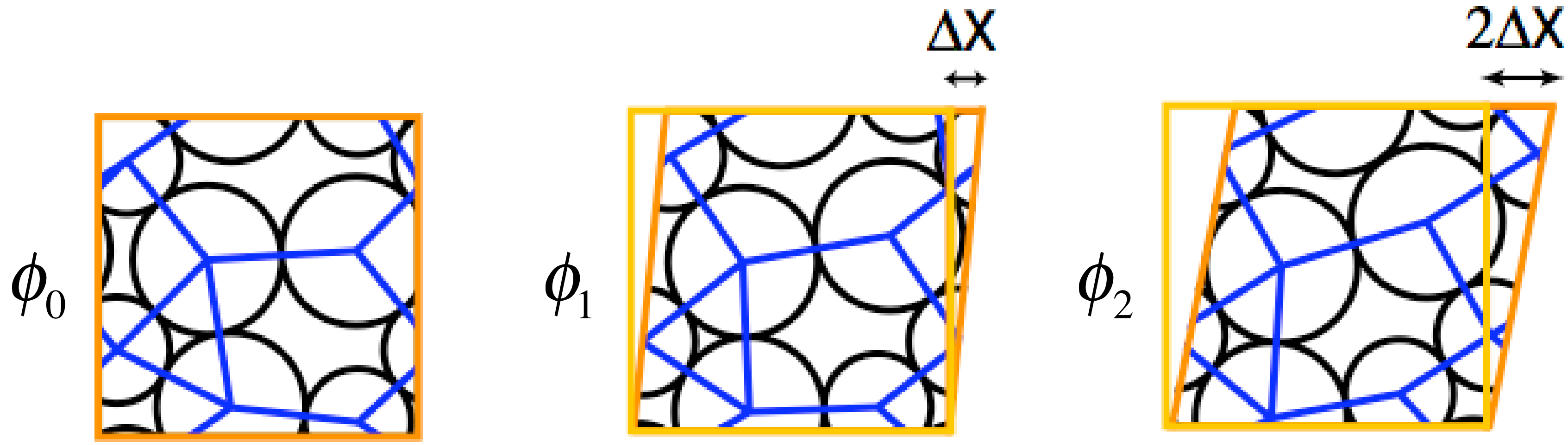


Quasi-static Couette Shear Flow $\dot{\gamma} \rightarrow 0$

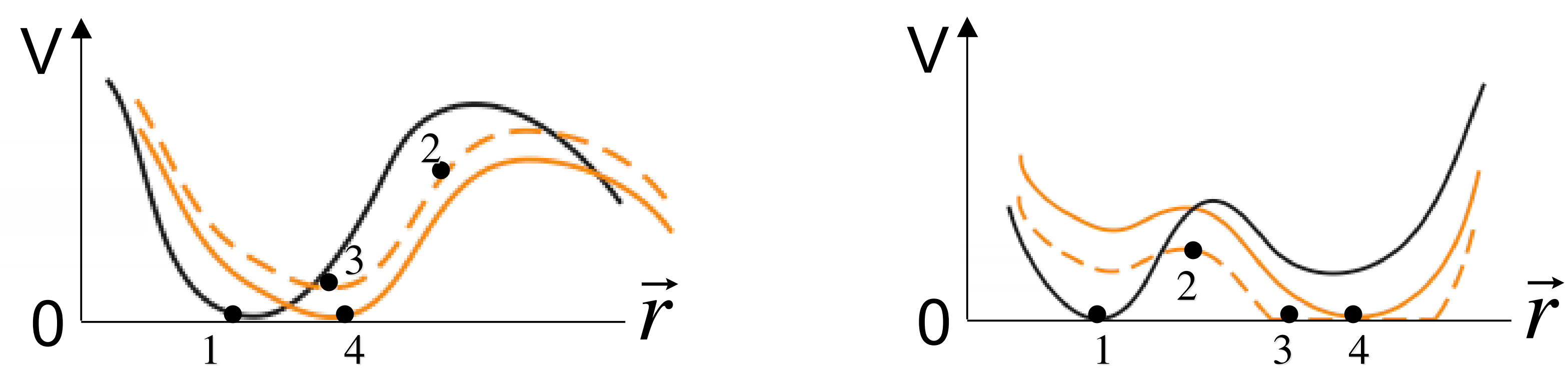
B. Utter and R. P. Behringer Phys. Rev. Lett. 100 (2008) 203302

H. A. Makse and J. Kurchan Nature 415 (2001) 614

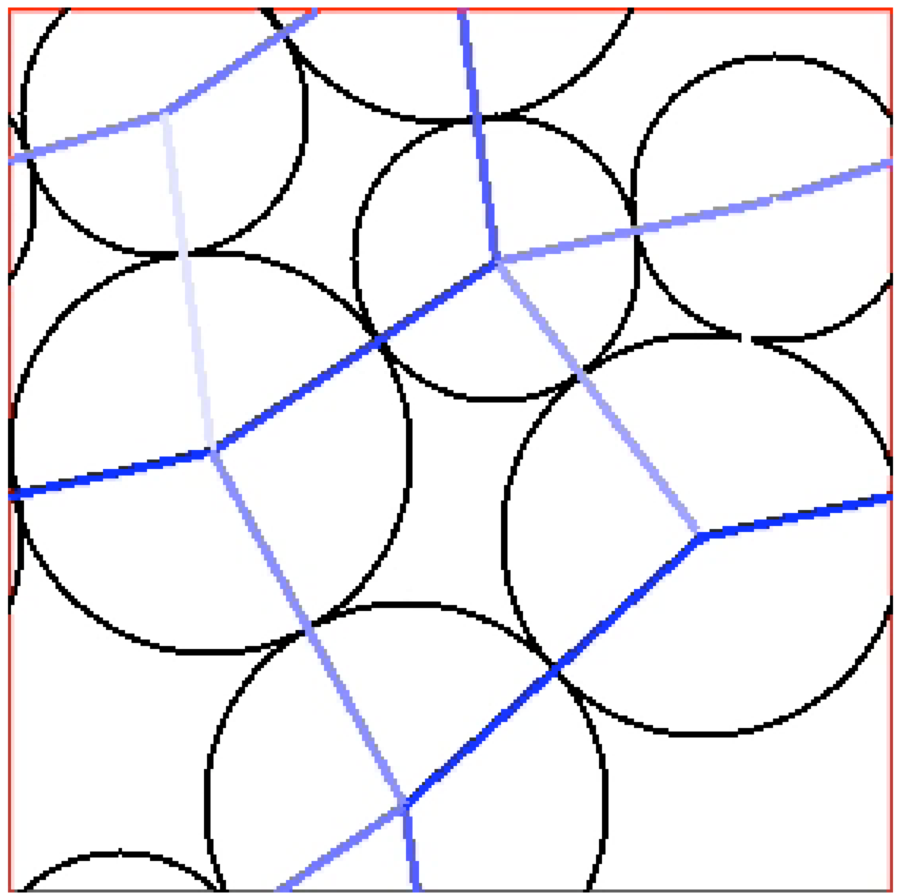
Quasi-static shear flow at zero pressure



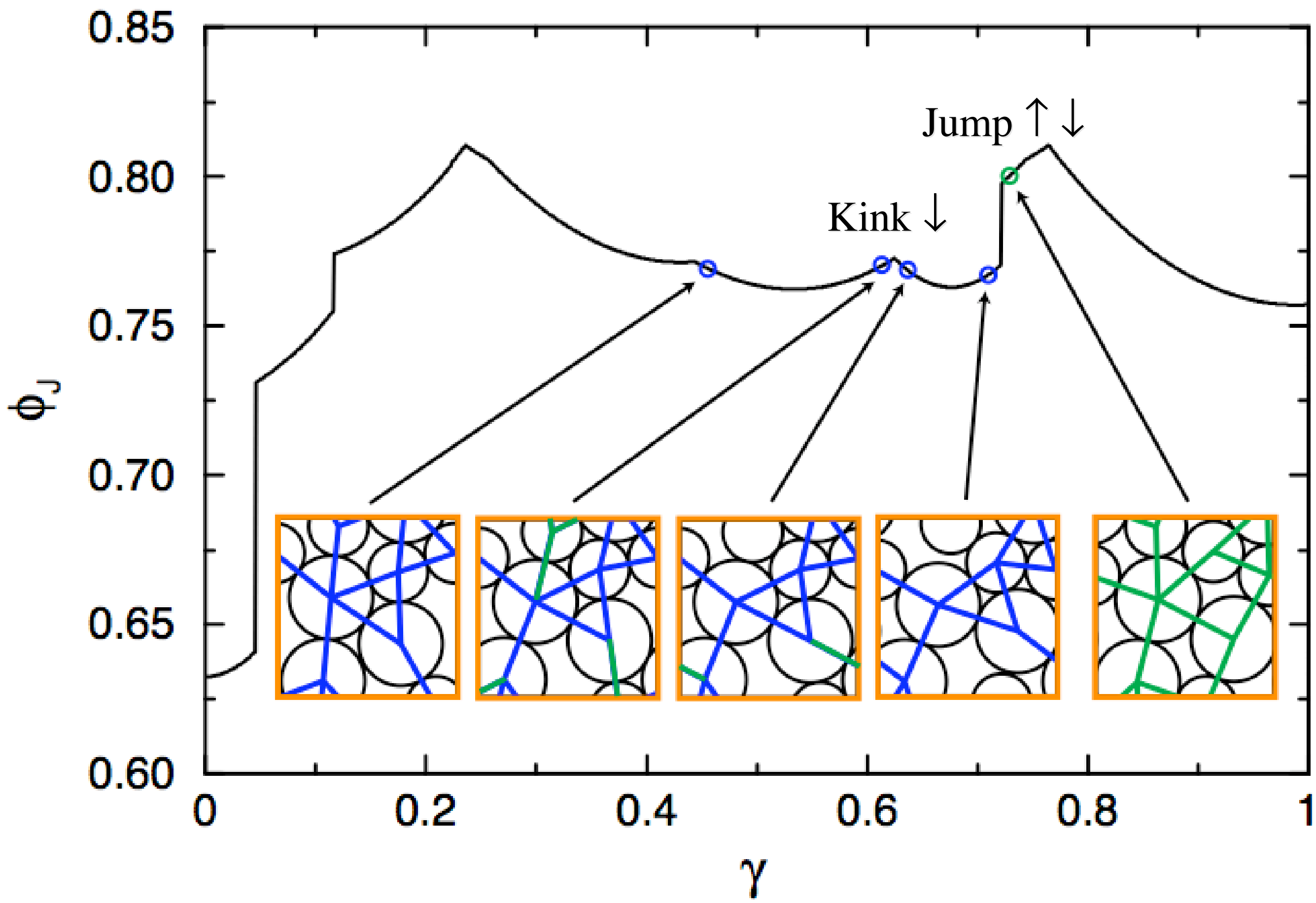
1. Initialize MS packing at zero shear strain
2. Take small step shear strain $x_i' = x_i + \Delta\gamma\, y_i$
3. Minimize energy
4. Find nearest MS packing at P=0 using growth/shrink procedure
5. Repeat steps 2, 3, 4



Quasistatic Shear Flow at Zero Pressure

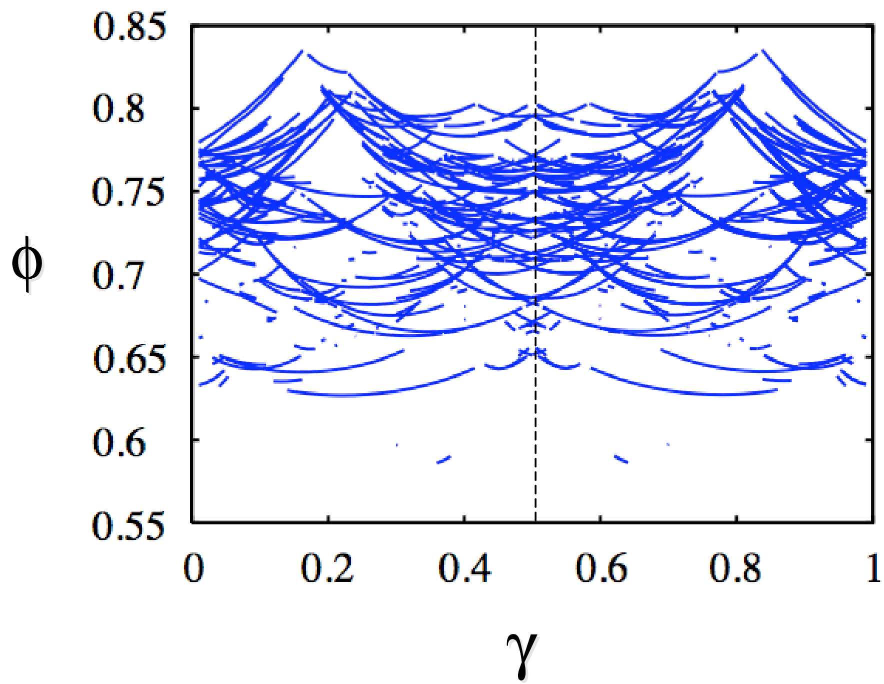


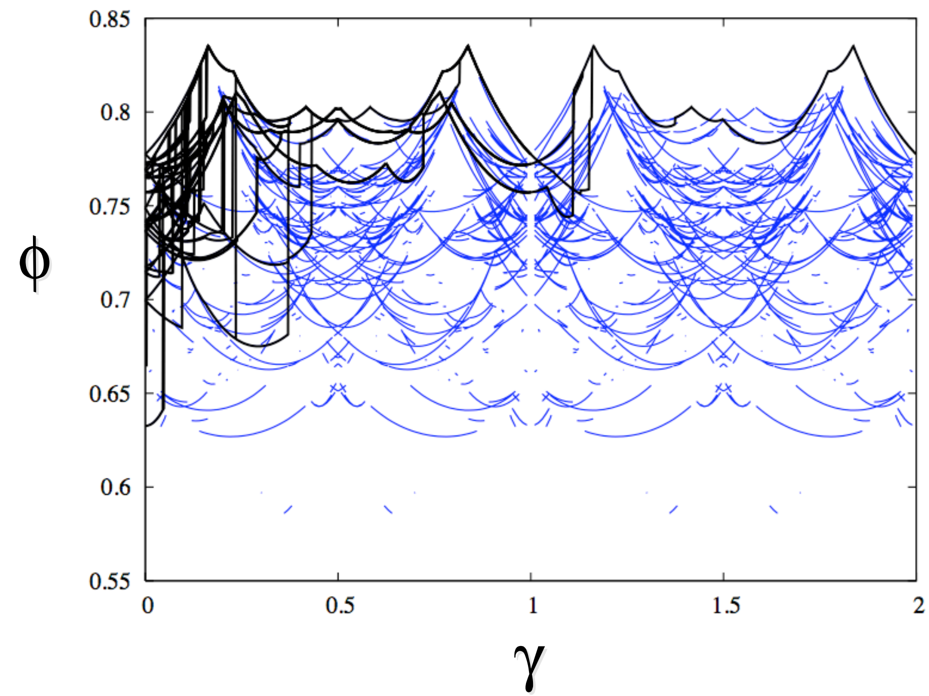
Geometric Families Exist over Continuous Range of γ



- Rearrangement events cause system to switch geometric families

Complete Family Tree



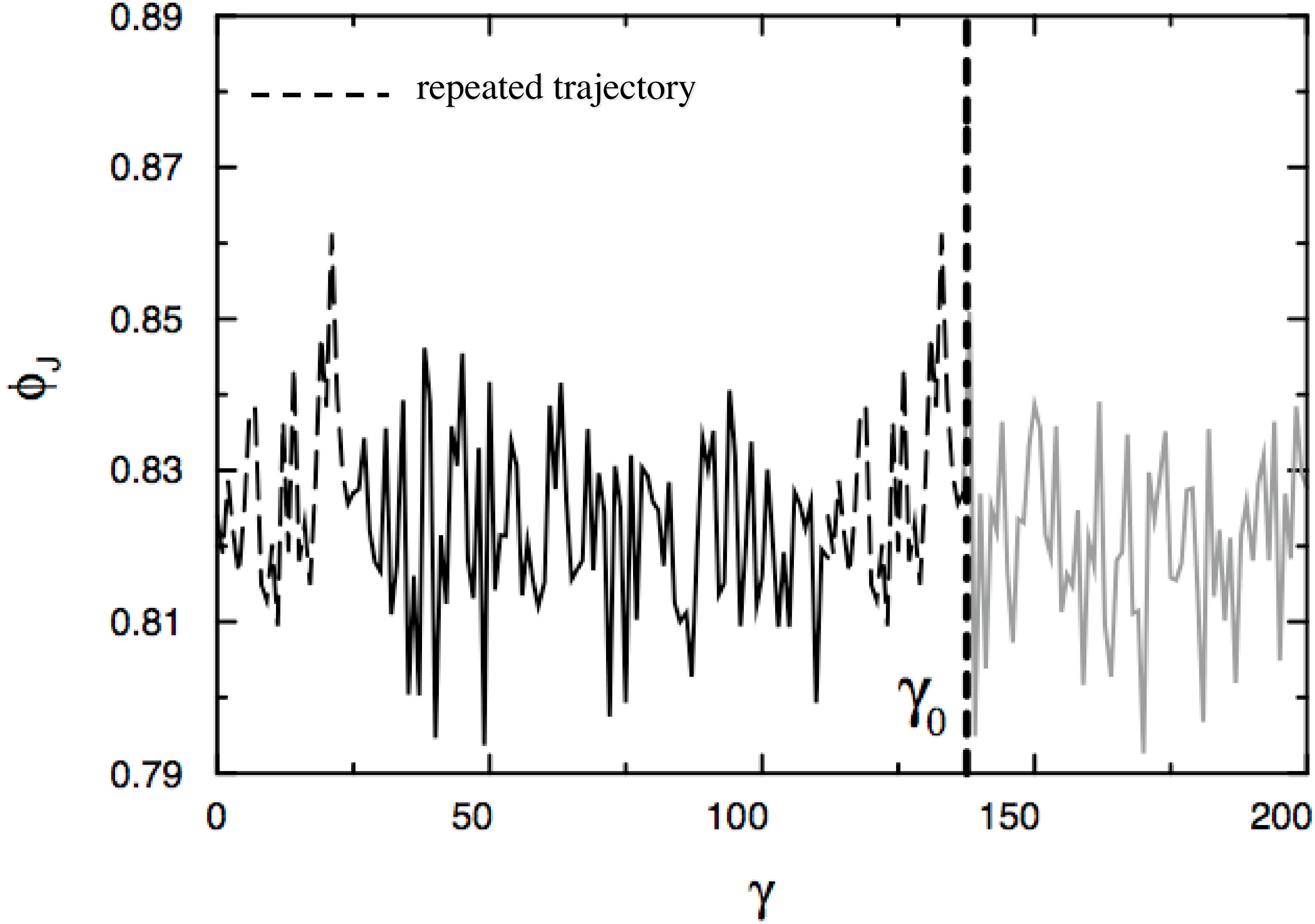


complete family tree

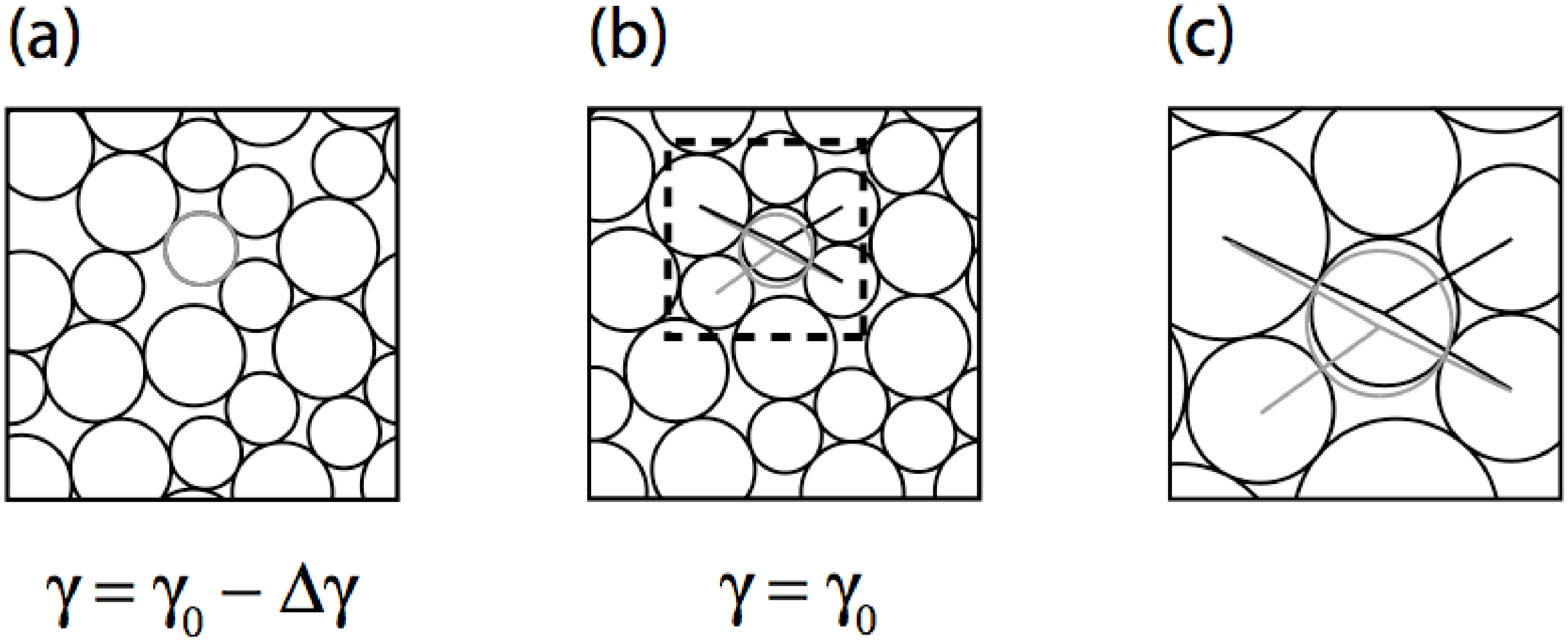
deterministic evolution of all γ=0 packings

Small systems sample only negligible fraction of available geometric families!

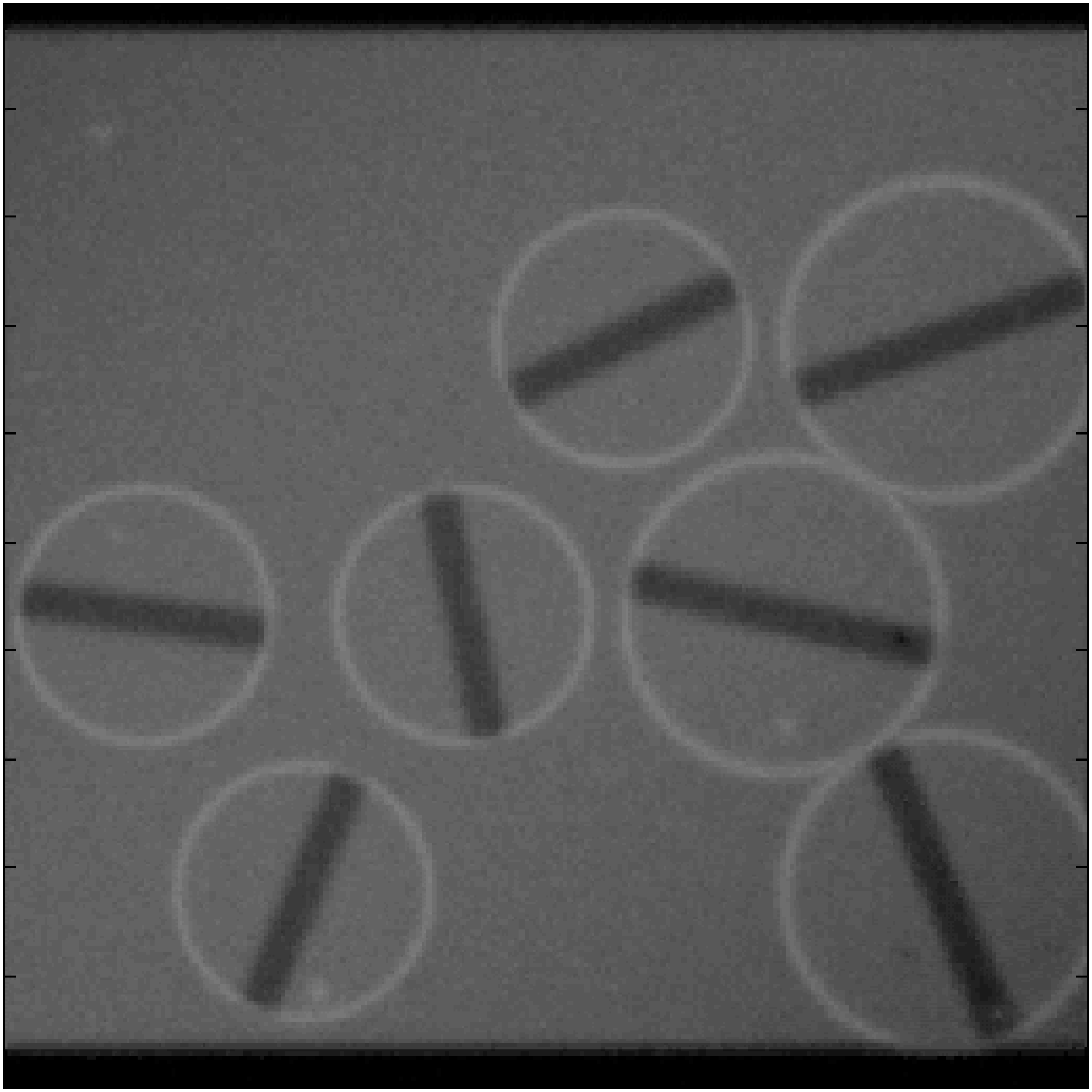
Sensitivity to Initial Conditions: N≥12



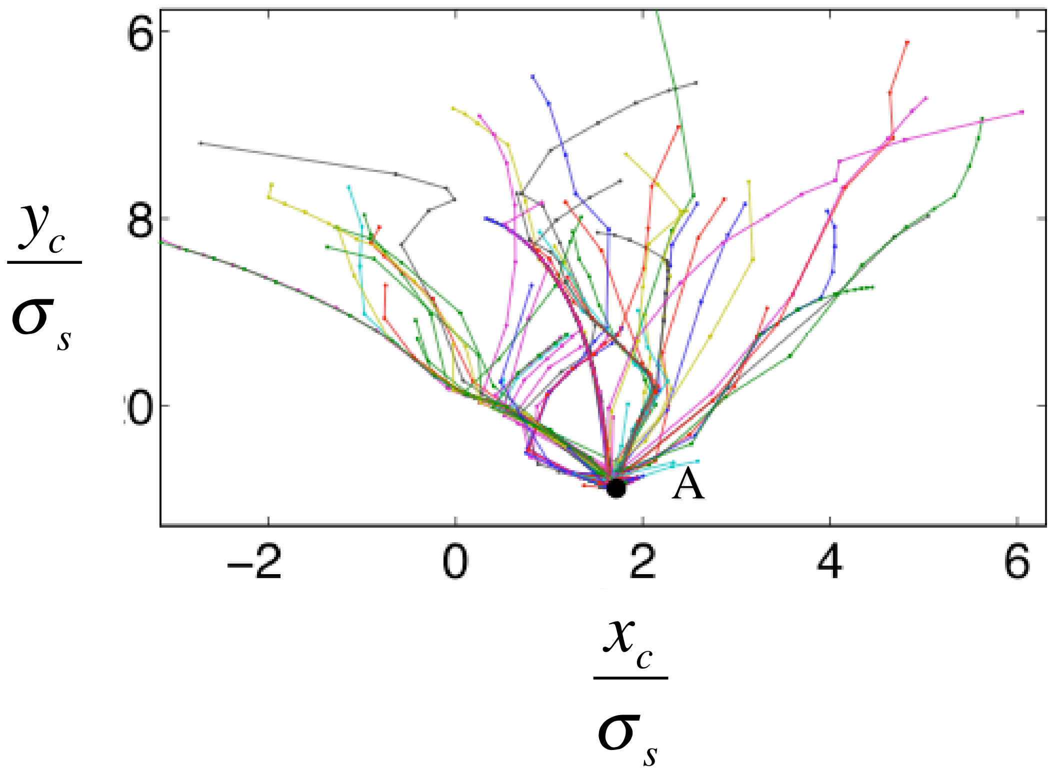
Noise-generation Mechanism: Collinear Particles

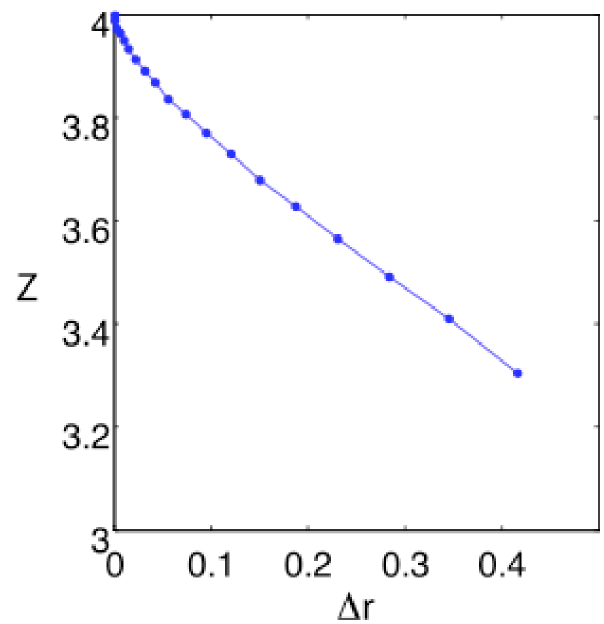


Frictional Geometrical Families



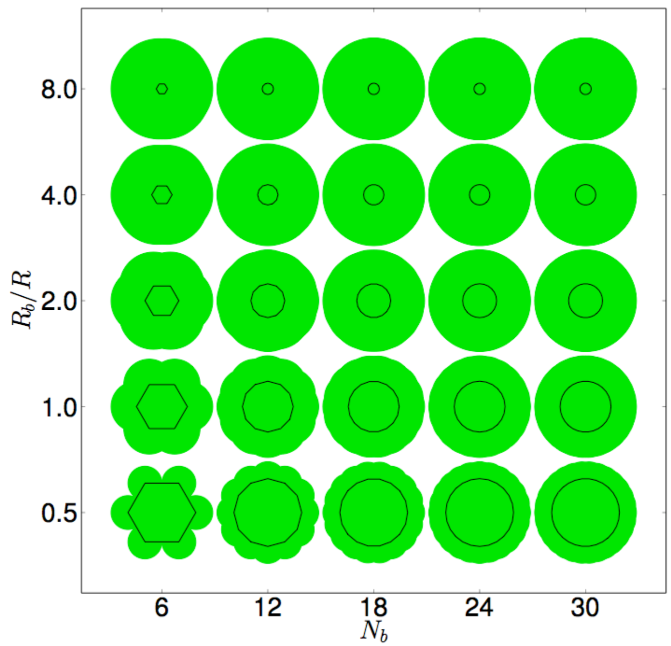
Frictional Geometric Families

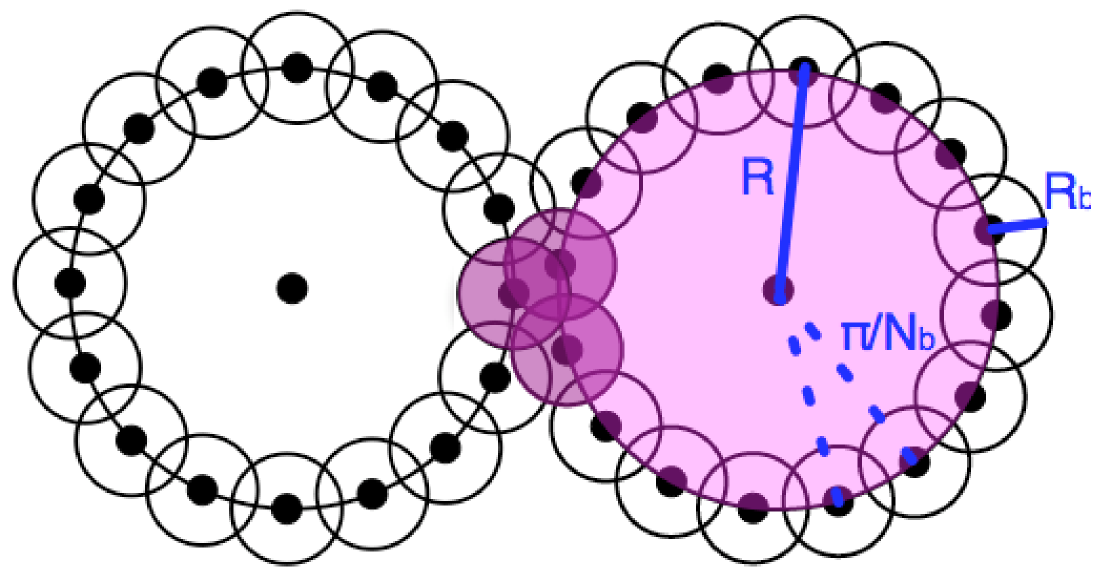




- Plot of all centers of mass that evolve to MS packing A

Bumpy Particle Model for Friction



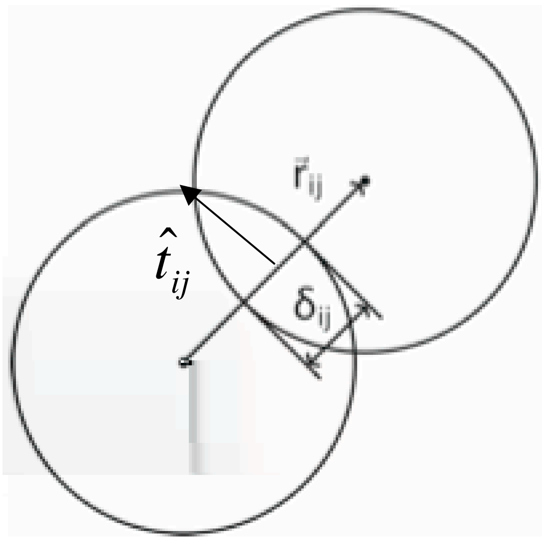


$$\mu_{\max} \cong \frac{\pi R}{2N_b R_b}$$

- Linear repulsive spring bump-bump, bump-particle, and particle-particle interactions

Hertz-Mindlin Friction Model

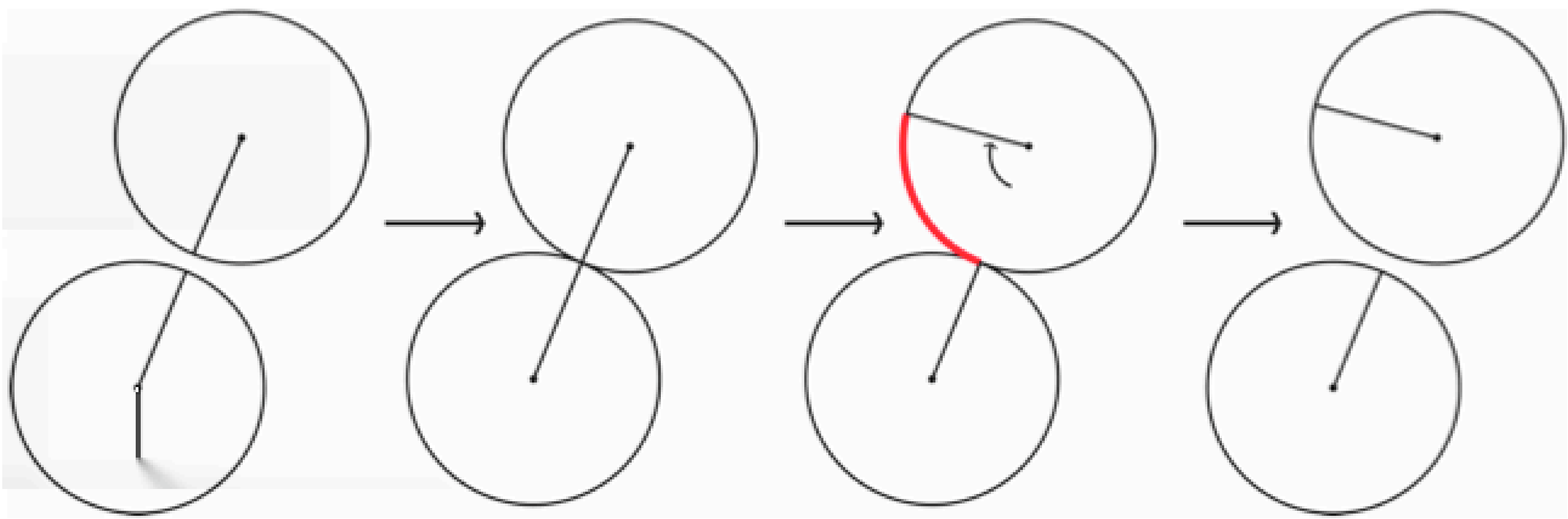
Elastic force law



$$
\begin{aligned}
\vec{F}_{n_{ij}} &= k_n \delta_{ij} \hat{r}_{ij} \\
\vec{F}_{t_{ij}} &= k_t \vec{u}_{t_{ij}} \\
\frac{d\vec{u}_{t_{ij}}}{dt} &= \vec{v}_{t_{ij}} - \frac{(\vec{u}_{t_{ij}} \cdot \vec{v}_{ij})\hat{r}_{ij}}{r_{ij}}
\end{aligned}
$$

$$\left|F_t\right| \leq \mu \left|F_n\right|$$

Tangential displacement resets after contact breaks

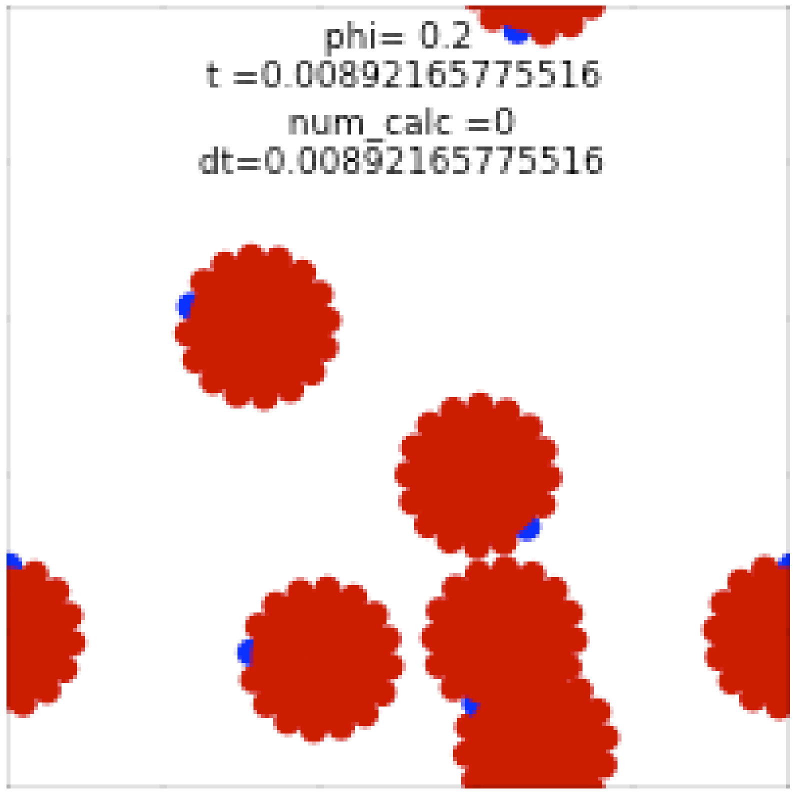


Advantages of Bumpy-Particle Model over Hertz-Mindlin

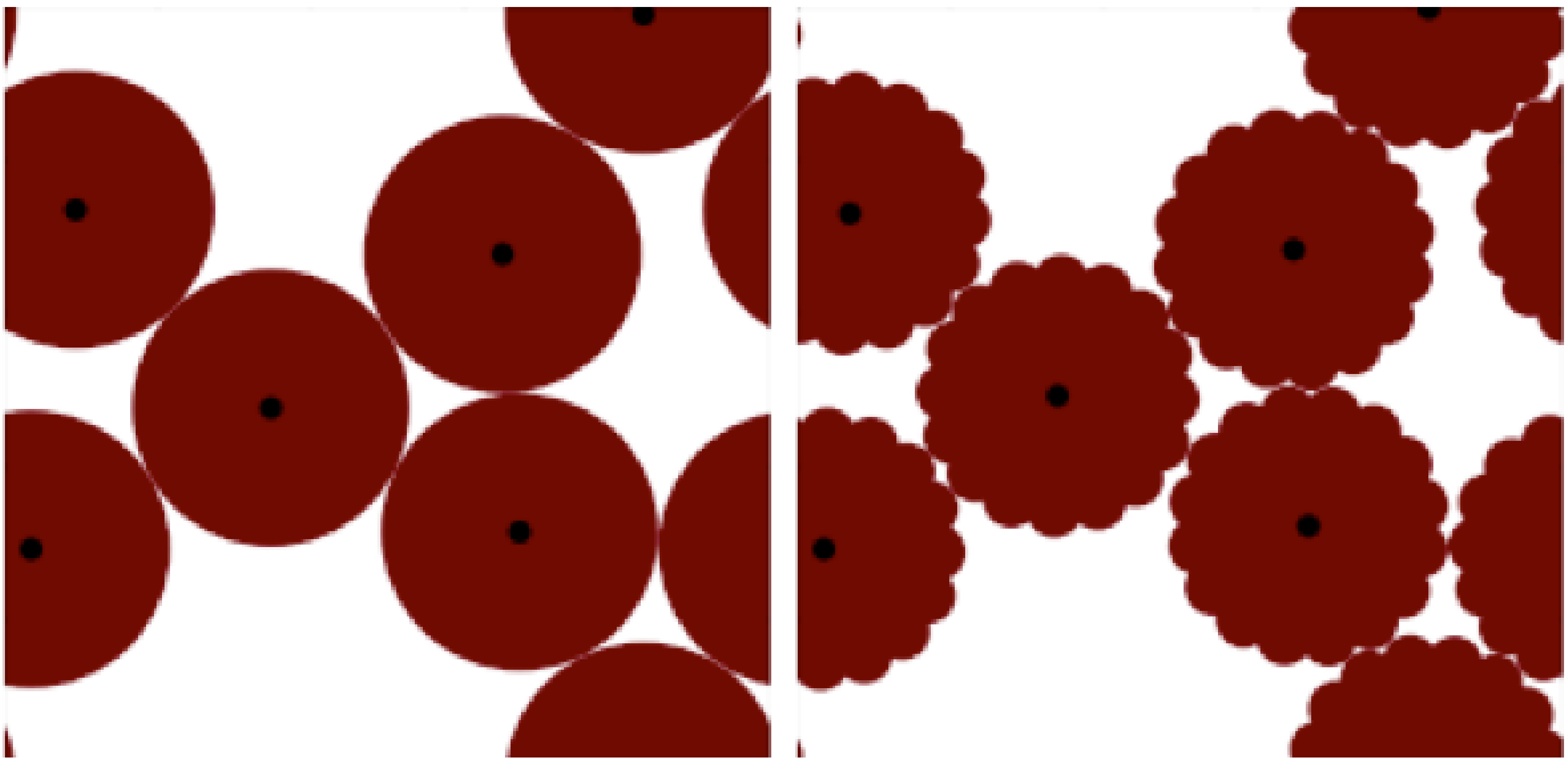
- No *ad hoc* sliding, history dependence

- Forces depend only on particle positions and orientations;
Use dynamical matrix to calculate vibrational response

- Test Hertz-Mindlin mobility distribution, P(m) $m = \frac{F_t}{\mu F_n}$


phi= 0.2
t =0.00892165775516
num_calc =0
dt=0.00892165775516

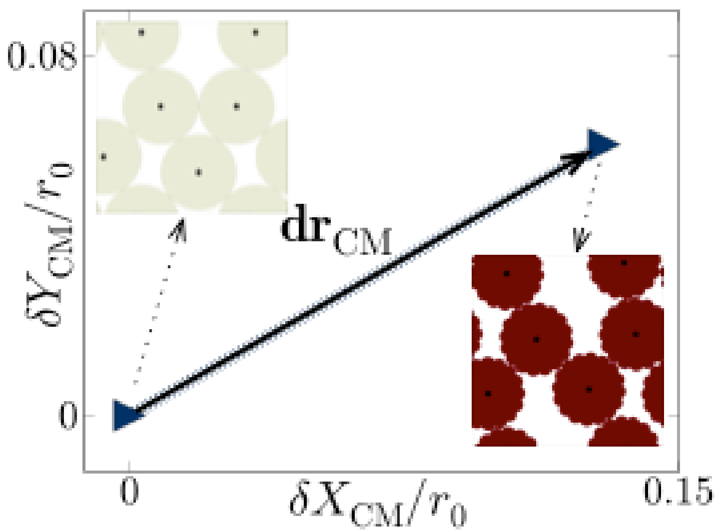
ϕ_c=0.6131, N_c=10, $N_c^{bb} = 17$

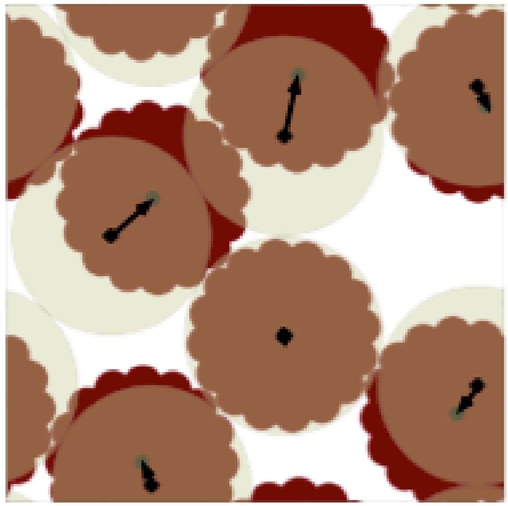


Hertz-Mindlin

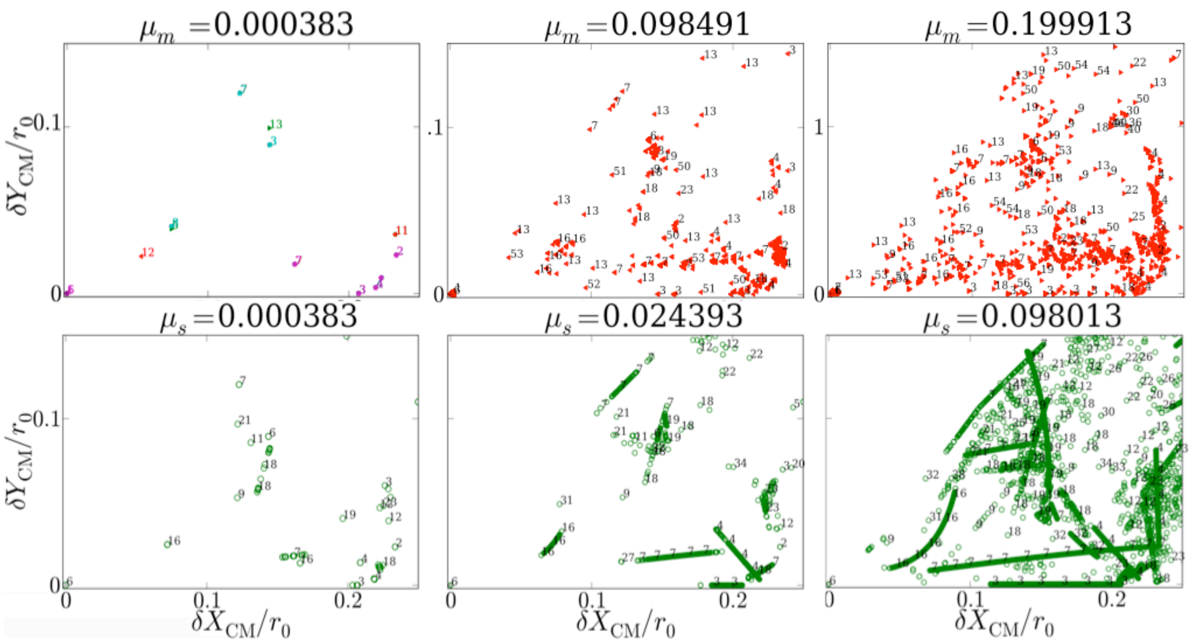
Bumpy-particle model

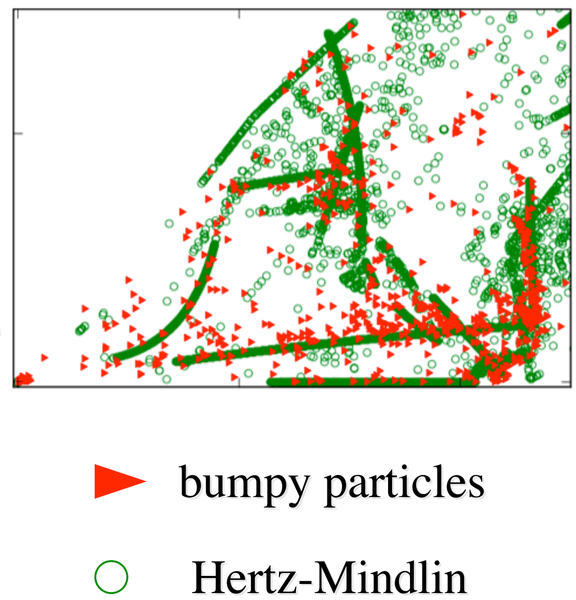
`Minimum Distance' from Reference MS Packing

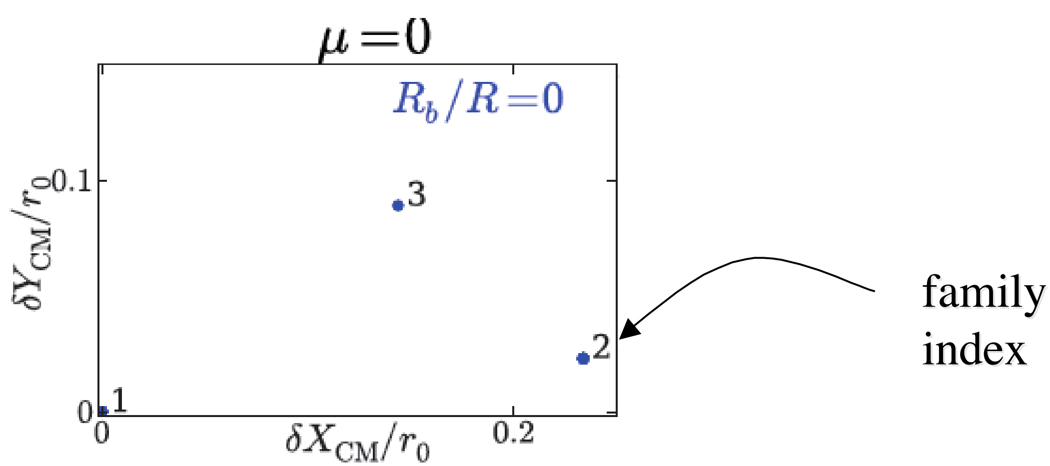


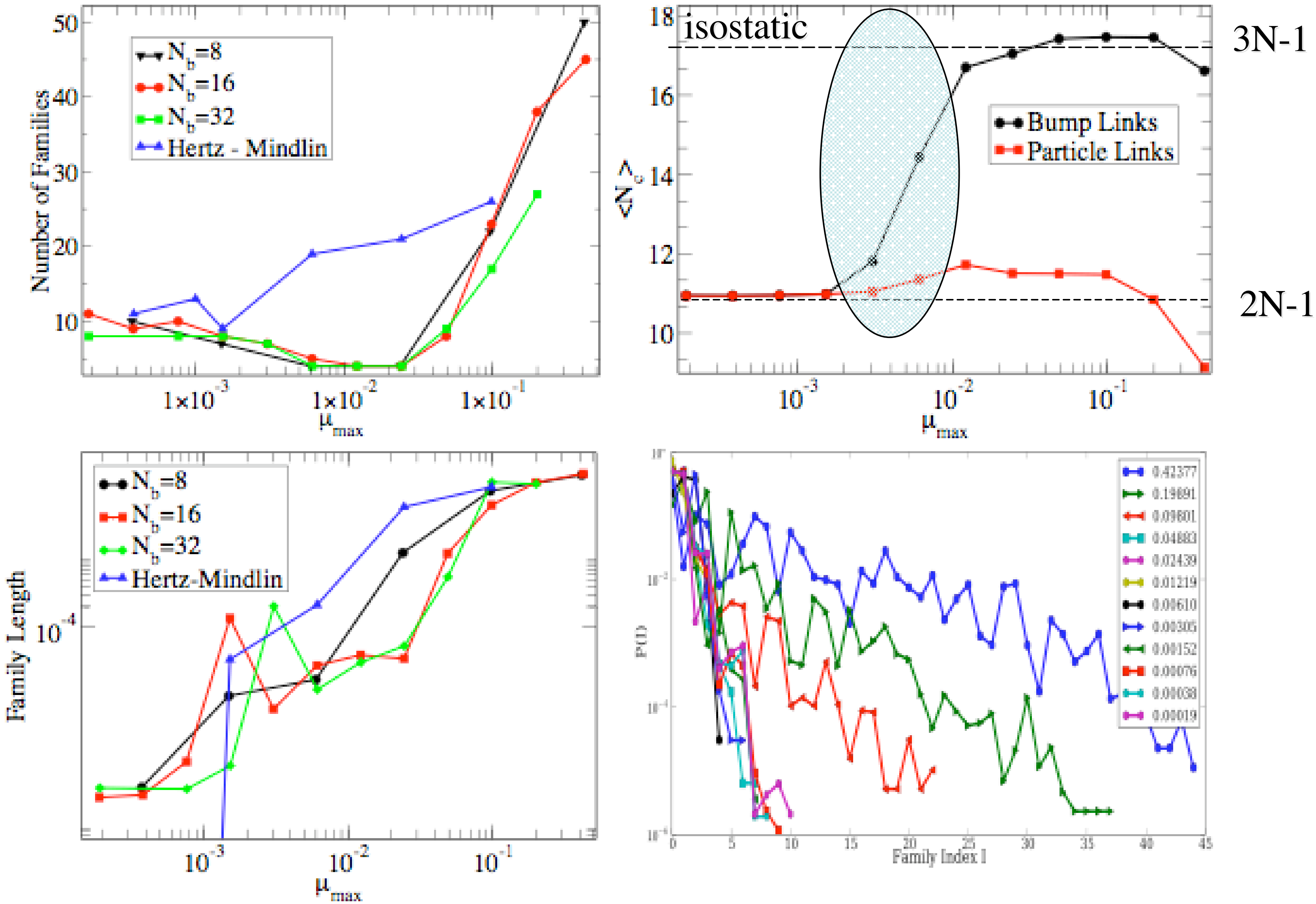


Comparison of Hertz-Mindlin and Bumpy-Particle Minimum-Distance Maps









Number of Families
N_b=8
N_b=16
N_b=32
Hertz - Mindlin
1×10^-3
1×10^-2
1×10^-1
μ_max
isostatic
3N-1
2N-1
<N_c>
Bump Links
Particle Links
10^-3
10^-2
10^-1
μ_max
Family Length
N_b=8
N_b=16
N_b=32
Hertz-Mindlin
10^-4
μ_max
P(I)
Family Index I
0.42377
0.19891
0.09801
0.04883
0.02439
0.01219
0.00610
0.00305
0.00152
0.00076
0.00038
0.00019

Hertz-Mindlin Results

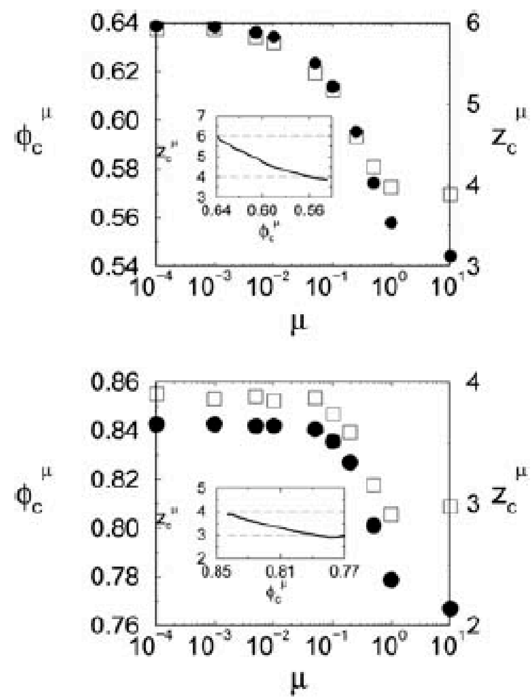
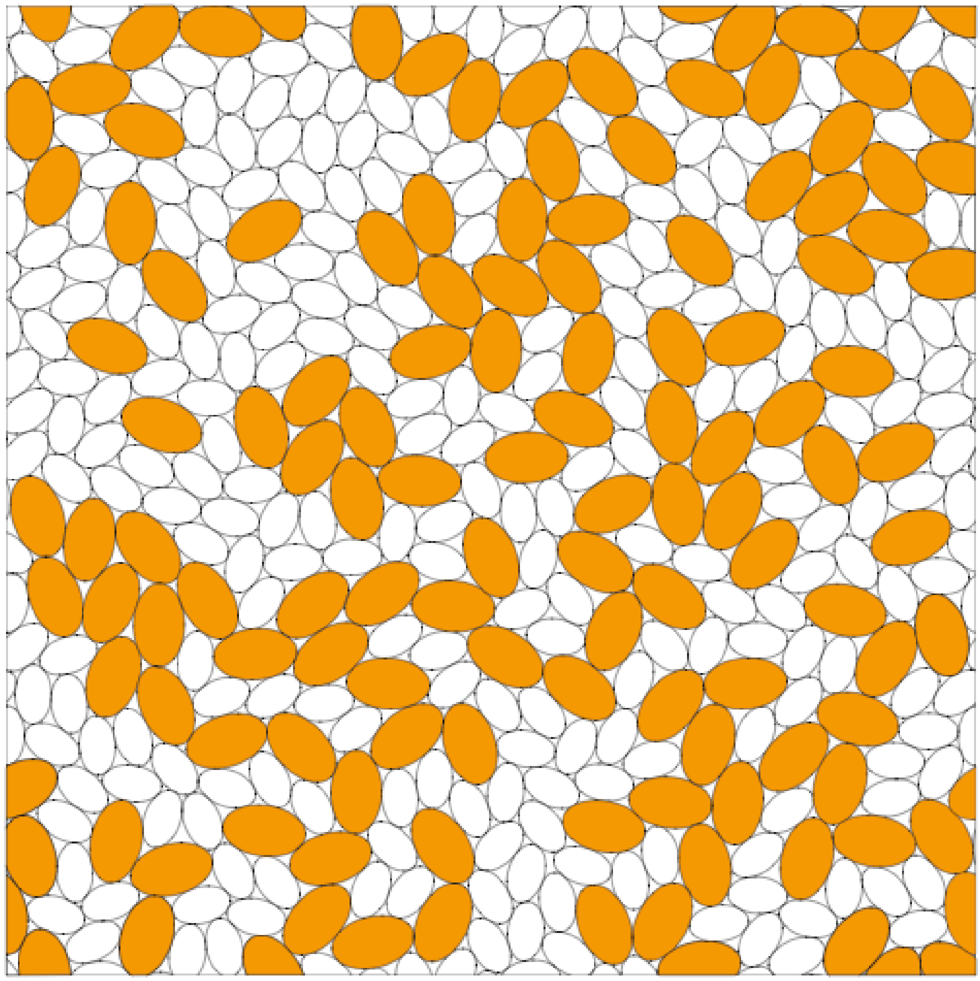


Fig. 1 Dependence of the critical values of the packing fraction ϕ^{μ}_{c} (filled circles), and coordination number z^{μ}_{c} (open squares), on the particle friction coefficient μ, for monodisperse spheres in $3D$ (upper panel) and bidisperse discs in $2D$ (lower panel). The insets are parametric plots of ϕ_c^{μ} against z_c^{μ}. Symbol size is representative of sample-to-sample fluctuations and error bars.

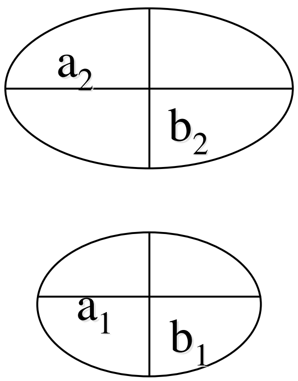
L. Silbert, Soft Matter, 6 (2010) 2918.

Shape Matters: Packings of Frictionless Ellipsoidal Particles Are Stabilized by Quartic Modes



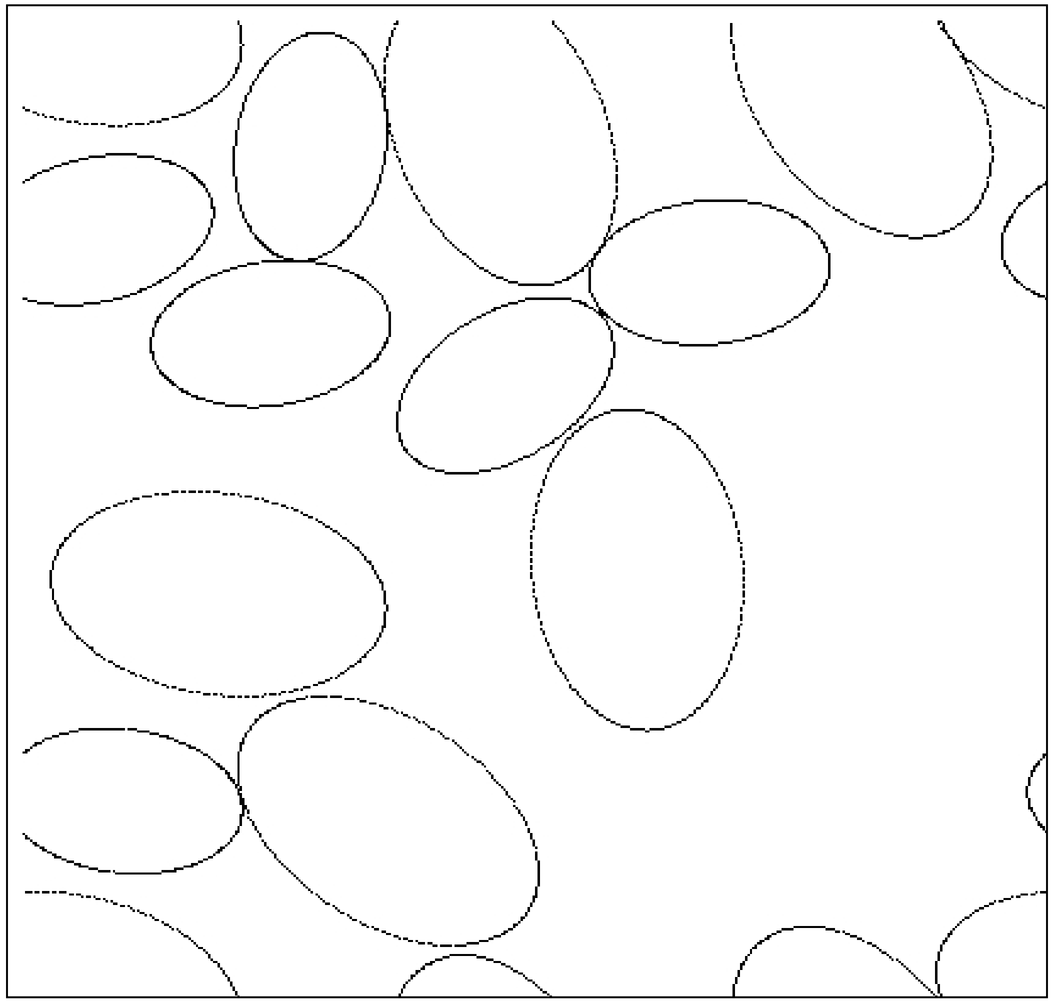
Packings of ellipse-shaped particles

bidisperse



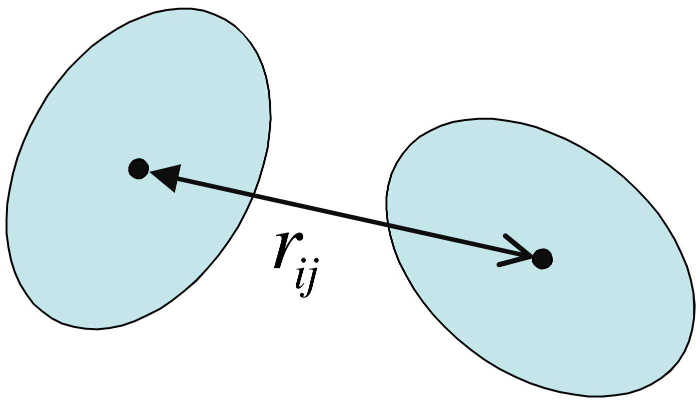
$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \alpha$$

$$\frac{a_1}{a_2} = 1.4$$

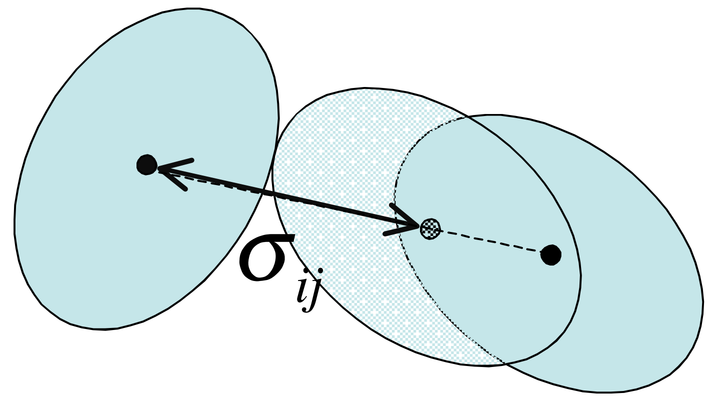


compression method-fixed aspect ratio α

Pairwise Repulsive Interactions: True Contact Distance



$$V(r_{ij}) = 0$$

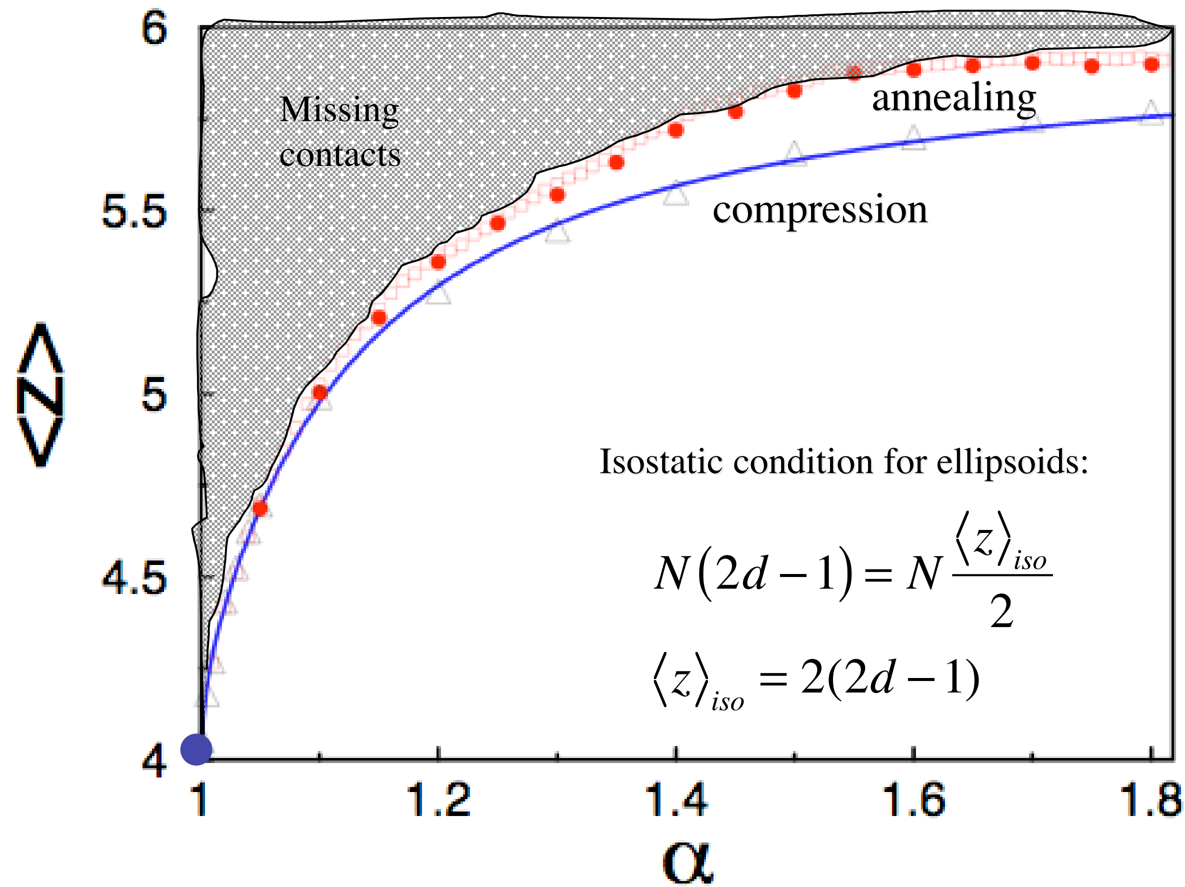


$$V(r_{ij}) > 0$$

$$V(r_{ij}) = \begin{cases} \dfrac{\varepsilon}{\alpha}\left(1 - \dfrac{r_{ij}}{\sigma_{ij}}\right)^{\alpha} & r < \sigma_{ij} \\ 0 & r \geq \sigma_{ij} \end{cases}$$

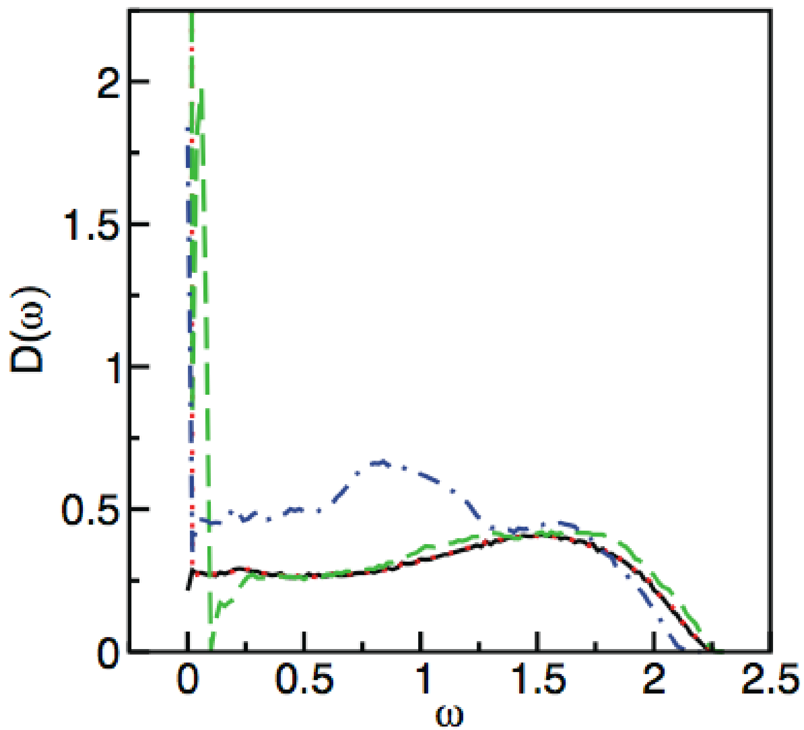
α=2; linear springs

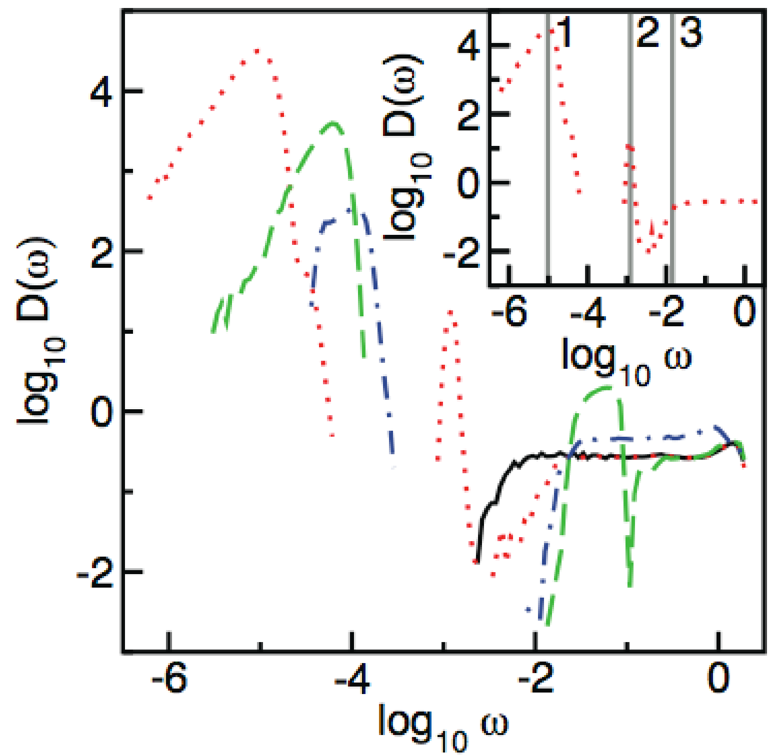
Average Contact Number



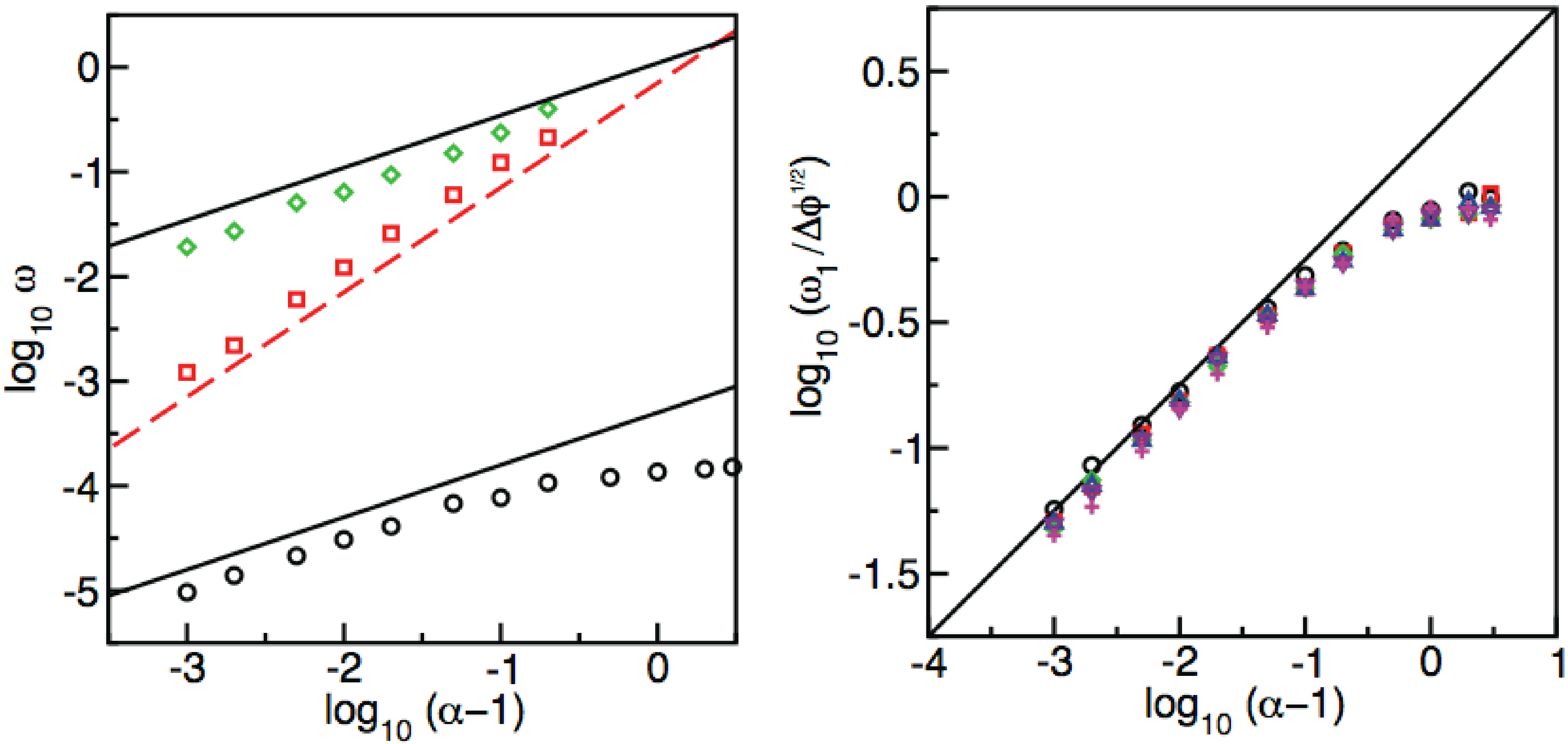
- Not a discontinuous jump from <z> = 4 to 6.
- Quartic modes to the rescue!

Density of Vibrational Modes from Dynamical Matrix

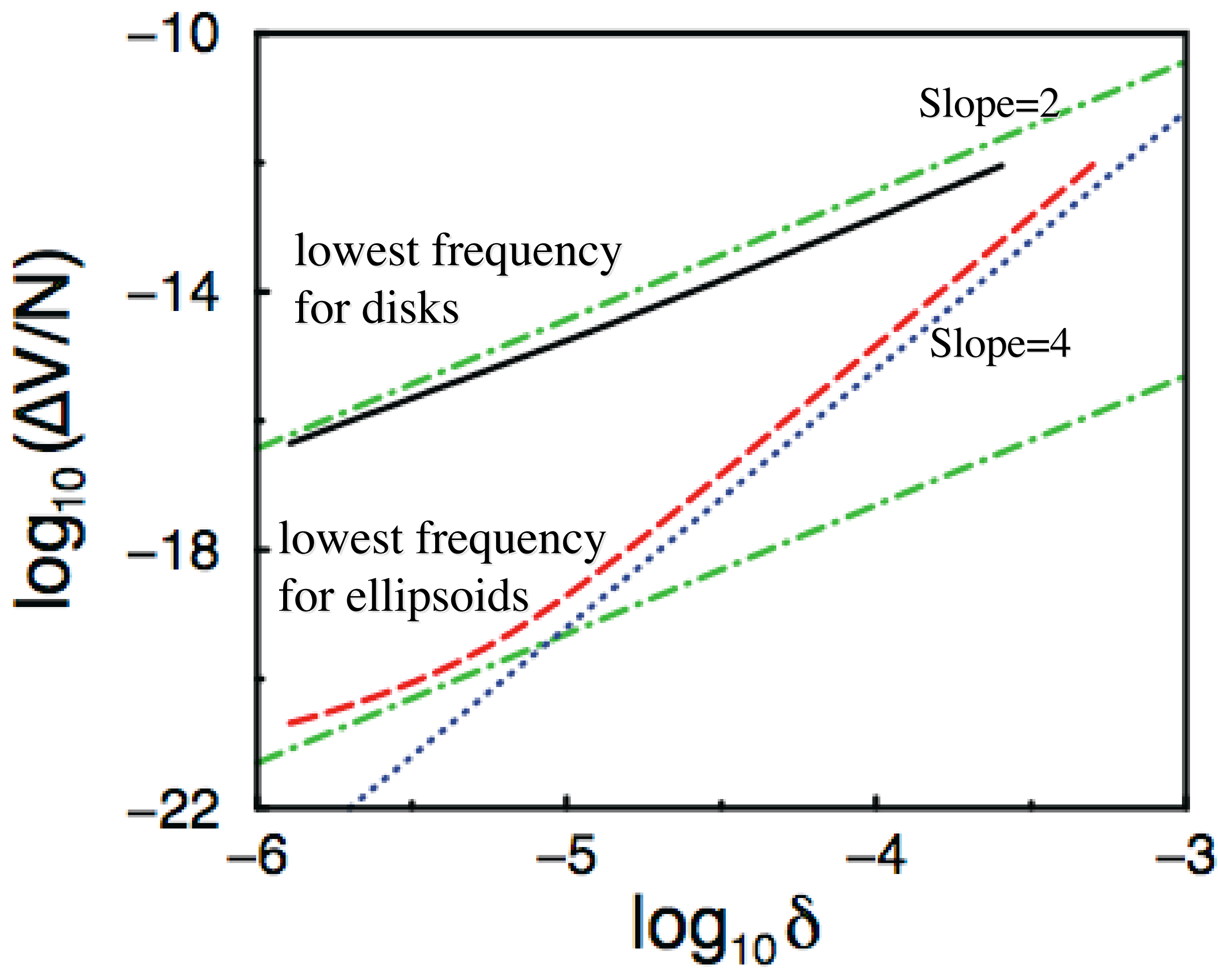




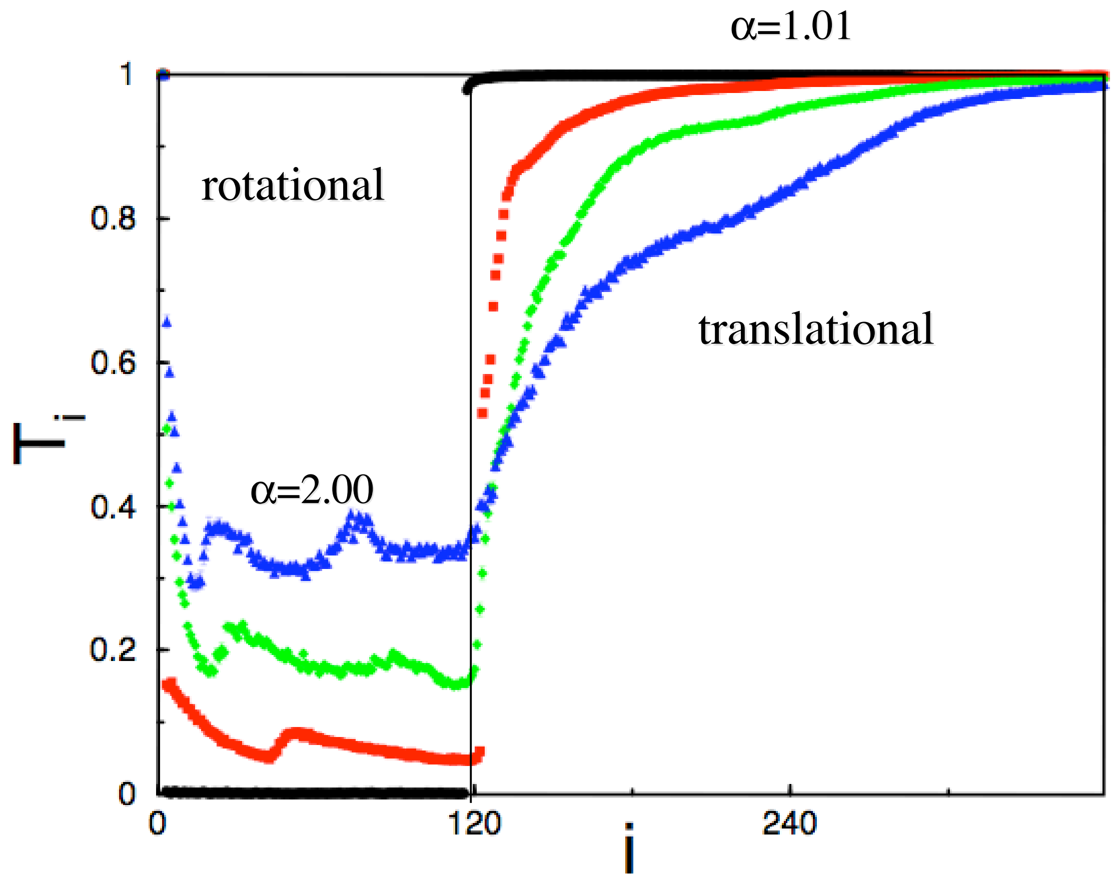
Scaling of Characteristic Frequencies



Perturbations along lowest frequency eigenmodes

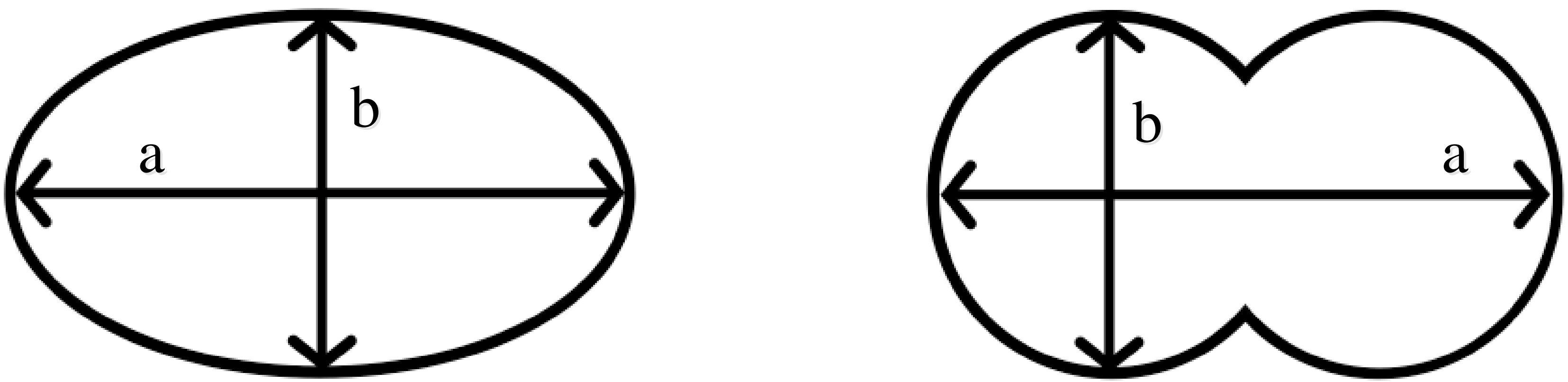


Rotational/Translational Character of Eigenmodes



$$T_i = \sum_{j=1}^{N} \left[\left(e_{xi}^{j} \right)^2 + \left(e_{yi}^{j} \right)^2 \right] \qquad T_i = 1 - R_i$$

What is the difference between between a dimer and an ellipse?



$\alpha = a/b$

Structural Properties