Random matrix

Maximal eigenvalue of an $(N\times N)$ Random matrix.

(1994) Tracy-Widom dist^n & Large deviations.

$\longleftrightarrow$

Baik-Deift-Johansson, 1999

Random Permutation

Longest Increasing Subsequence (LIS)

$\downarrow$

Ulam problem (1960)

$\longleftrightarrow$

Lattice gauge th.

U(N) lattice gauge in 2-d with Wilson action

Gross, Witten, Wadia } 1980.

$\downarrow$

Periwal & Sevitz, 90

Gross & Matytsin, 94

growth models -- Sequence matching .. particle systems -- directed polymer

Statistical Mechanics.

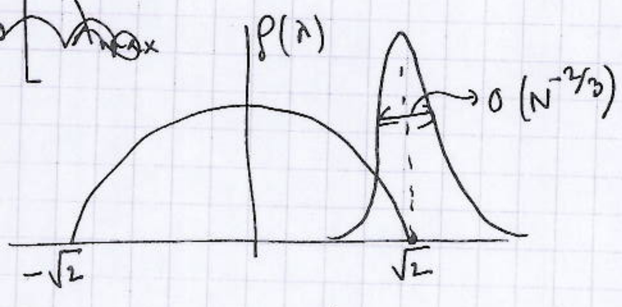
S.M. Les Houches lecture notes (2006)

arXiv: cond-mat/0701193

Reminder: $(N\times N)$ $X = \begin{bmatrix} X_{11} & X_{12} & \\ & \ddots & \\ & & \ddots \end{bmatrix}$, $\quad P_r(X) \sim e^{-N\frac{\beta}{2}\mathrm{Tr}(X^\dagger X)}$

$$P(\lambda_1,\dots\lambda_N) = \frac{1}{Z_N}\, e^{-\frac{\beta N}{2}\sum \lambda_i^2} \prod_{i<j} |\lambda_i-\lambda_j|^\beta$$

Prob$[\lambda_{max}$



$\lambda \to$

$$\rho_N(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\langle \delta(\lambda-\lambda_i)\rangle \xrightarrow[N\to\infty]{} \frac{1}{\pi}\sqrt{2-\lambda^2}$$

$\to\infty$

$\langle \lambda_{max}\rangle = \sqrt{2}$

$$\mathrm{Prob}[\lambda_{max}\le w,\ N\to\text{large}] \to \begin{cases} e^{-\beta N^2 \Phi_-(w)} & \sqrt{2}-w \sim O(1). \\ F_\beta\left[\sqrt{2}\, N^{2/3}(w-\sqrt{2})\right] & |w-\sqrt{2}| \sim O(N^{-2/3}) \\ e^{-\beta N \Phi_+(w)} & w-\sqrt{2}\sim O(1) \end{cases}$$

$F_\beta \to$ Tracy-Widom f^n: $\quad F_2(x) = \exp\left[-\int_x^\infty (z-x)\, q^2(z)\, dz\right]$

with $\quad q''(z) = 2q^3(z) + z\,q(z)$, $\quad z\to\infty$. $\quad q(z)\sim Ai(z)$.

Painlevé-II

$w<\sqrt{2}$

$$\phi_-(w) = \frac{w^2}{3} - \frac{w^4}{108} - \sqrt{w^2+6}\;\frac{(w^3+15z)}{108} - \frac{1}{2}\ln\left[\frac{\sqrt{w^2+6}+w}{\sqrt{2}}\right] + \frac{\ln 3}{2}$$

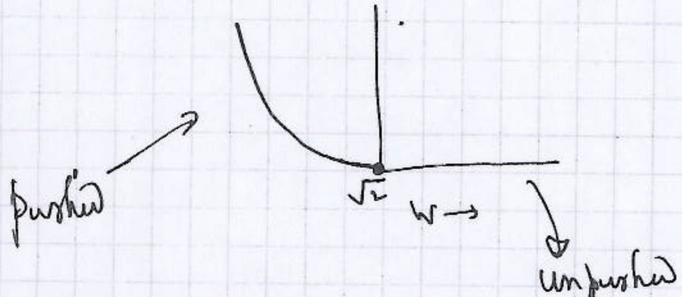
$$\xrightarrow[w\to\sqrt{2}]{} \frac{1}{6\sqrt{2}}\left(\sqrt{2}-w\right)^3.$$

$w>\sqrt{2}$

$$\phi_+(w) = \frac{w\sqrt{w^2-2}}{2} + \ln\left(\frac{w-\sqrt{w^2-2}}{2}\right) \underset{w\to\sqrt{2}}{\sim} \frac{2^{7/4}}{3}\left(z-\sqrt{2}\right)^{3/2}$$

$$-\lim_{N\to\infty}\frac{\ln \mathrm{Prob}[\lambda_{max}\le w]}{N^2} = \phi_-(w) \qquad w<\sqrt{2}$$

$$= 0 \qquad w>\sqrt{2}.$$



$\phi_-(w)\sim\left(\sqrt{2}-w\right)^3$

↳ 3rd-order phase transition.

<u>Random Permutation</u>: Longest Increasing Subsequence (LIS)

$\longrightarrow$ Ulam's problem.

Consider M integers $(1, 2, 3, \cdots M)$

" all $M!$ permutations.

- For each permutation, find all the increasing subsequences.

ex: $\{8, 2, 7, 1, 3, 4, 10, 6, 9, 5\}$ $M = 10$.

increasing subsequences: $\{8, 10\}$, $\{2, 3, 4, 5\}$, $\{1, 4, 9\}, \cdots$

- Find the longest ~~...~~ $\rightarrow$ may be degenerate.

$\{1, 3, 4, 6, 9\}$, $\{2, 3, 4, 6, 9\}$

$l_M \rightarrow$ length of the longest increasing subsequence $= 5$.

<u>Simplest algorithm:</u> Patience sorting algorithm (Aldous & Diaconis, '99; Mallows, '73)

$$\begin{array}{ccccc} 1 & & & 5 & \\ 2 & 3 & & 6 & \\ 8 & 7 & 4 & 10 & 9 \end{array}$$

$\longrightarrow$ 5 piles

$l_M =$ no. of piles.

<u>R-S-K ~~algorithm~~ correspondence to Young tableaux</u>

$$\begin{array}{c} 2 \\ 8 \end{array} \longrightarrow \begin{array}{cc} 2 & 7 \\ 8 & \end{array} \longrightarrow \begin{array}{cc} 1 & 7 \\ 2 & \\ 8 & \end{array} \longrightarrow \begin{array}{cc} 1 & 3 \\ 2 & 7 \\ 8 & \end{array} \longrightarrow \begin{array}{cccc} 1 & 3 & 4 & 10 \\ 2 & 7 & & \\ 8 & & & \end{array}$$

$$\downarrow$$

$$\begin{array}{ccccc} 1 & 3 & 4 & 6 & 9 \\ 2 & 7 & 10 & & \\ 8 & & & & \end{array}$$

$$\begin{array}{ccccc} 1 & 3 & 4 & 5 & 9 \\ 2 & 6 & 10 & & \\ 7 & & & & \\ 8 & & & & \end{array}$$

no. of elements in the first row $= l_M$

• Consider all permutations equally likely → uniform measure on permutations.

$l_M \to$ random variable, → fluctuates from one permutation to another.

What is the statistics of l_M?

Ulam conjectured: $\langle l_M \rangle \sim \sqrt{M}$.

$$\lim_{M\to\infty} \frac{\langle l_M \rangle}{\sqrt{M}} = c, \to \text{Hammersley ('72)}.$$

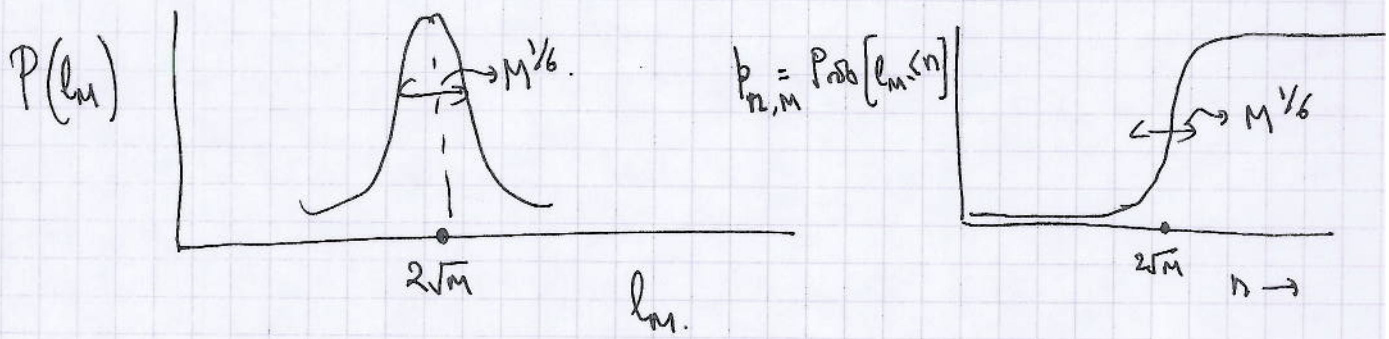
Logan & Shepp, $c \geq 2$

Vershik & Kerov, $c = 2$

$$\Rightarrow \boxed{\langle l_M \rangle = 2\sqrt{M}} \quad \text{as } M \to \infty.$$

$$\langle l_M \rangle = 2\sqrt{M} - c_0 M^{1/6} + o(M^{1/6}).$$

$$\sqrt{\mathrm{var}(l_M)} \to c_1 M^{1/6}.$$



$$p_{n,M} = \mathrm{Prob}\left[l_M \leq n, M\right] \xrightarrow[M\to\infty]{} F_2\left[\frac{n - 2\sqrt{M}}{M^{1/6}}\right]$$

$F_2(x) \to$ Tracy-Widom $\beta = 2$.

Baik, Deift & Johansson, '99.

* Anisotropic BD model

To make connection to the lattice gauge theory:

- Consider a poissonized version,

$M \to$ size of the original permutation seq.

assume that $M \to$ itself is a random variable drawn from a poisson dist^n.

$$\text{Prob}[M] = \frac{e^{-\mu}\mu^M}{M!}$$

$\mu = \langle M \rangle \to$ mean length of the perm.

"grand canonical" ensemble.

$p_{n,M} = \text{Prob}[l_M \le n]$

$$\Phi_n(\mu) = \sum_{M=0}^{\infty} p_{n,M} \frac{e^{-\mu}\mu^M}{M!}$$

{ Odlyzko et. al. '98
Rains → '98
Johansson → '98 }

$$\downarrow$$

$$\frac{e^{-\mu}}{(2\pi)^n n!}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} e^{2\sqrt{\mu}\sum_{i=1}^{n}\cos\theta_i} \prod_{1\le j<k\le n} \left|e^{i\theta_j}-e^{i\theta_k}\right|^2 d\theta_1\ldots d\theta_n$$

$$= e^{-\mu}\int dU\, e^{\sqrt{\mu}\,\text{Tr}(U+U^\dagger)}$$

$U \to (n\times n)$ unitary matrix.
$\to$ Haar measure

$$= e^{-\mu}\left\langle e^{\sqrt{\mu}\,\text{Tr}(U+U^\dagger)}\right\rangle$$

setting $\mu = \frac{n^2}{g^2}$

$$\boxed{e^{\frac{n^2}{g^2}}\,\Phi_n\!\left(\mu=\frac{n^2}{g^2}\right) = \left\langle e^{\frac{n}{g}\text{Tr}(U+U^\dagger)}\right\rangle} = \boxed{G(g,n)}$$

$$\Psi_n(\mu) = \sum_{M=0}^{\infty} \overset{\text{Prob}[\ell_M \leq n]}{p_{n,M}} \frac{e^{-\mu}\mu^M}{M!}$$

$$\longrightarrow F_2\left[\frac{n - 2\sqrt{\mu}}{\mu^{1/6}}\right]$$

setting $\mu = \frac{n^2}{g^2}$

$$\longrightarrow F_2\left[\frac{n - 2\cdot\frac{n}{g}}{\left(\frac{n}{g}\right)^{1/3}}\right]$$

$$= F_2\left[g^{-2/3}(g-2)\, n^{2/3}\right]$$

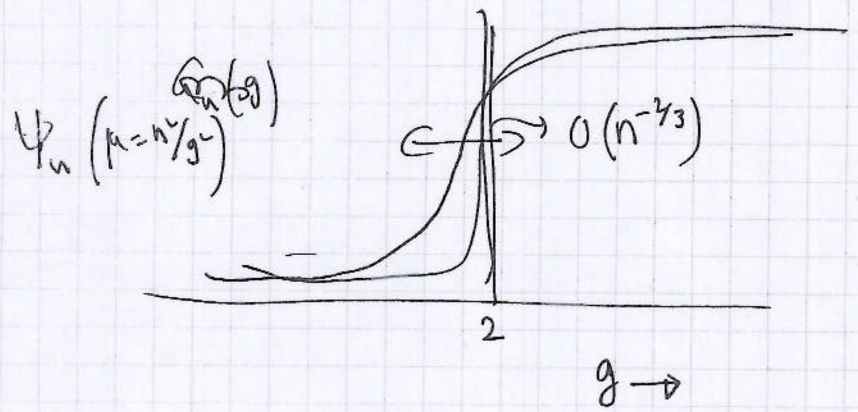
$$\xrightarrow[g\to 2]{} F_2\left[2^{-2/3}(g-2)\, n^{2/3}\right].$$

$M = \frac{n^2}{g^2}$

$n^2 = g^2 M$

$n = g\sqrt{M}.$

$g > 2.$



$$\frac{n}{g} = \sqrt{\mu} \to \sqrt{M}.$$

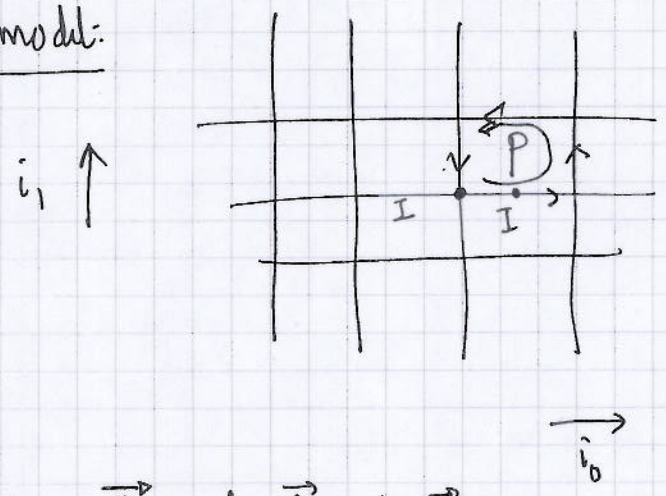
$\boxed{g = \frac{n}{\sqrt{M}}}$ $\qquad n = g\sqrt{M}.$

(7)

$$G(g,n) = \left\langle \exp\left[\frac{n}{g}\,\mathrm{Tr}(U+U^\dagger)\right]\right\rangle \longrightarrow$$ appears in lattice gauge th. with Wilson action,

Gross, Witten, Wadia '80.

Wilson model:



$U_{\vec{\ell},\vec{i}_0}$

any point: $\vec{\ell} = \ell_0\,\vec{i}_0 + \ell_1\,\vec{i}_1$

On every link puts a unitary $(n\times n)$ matrix: $\longrightarrow U_{\vec{\ell},\vec{i}}$

$$U_p = U_{\vec{\ell},\vec{i}_0}\; U_{\vec{\ell}+\vec{i}_0,\vec{i}_1}\; U_{\vec{\ell}+\vec{i}_0+\vec{i}_1,-\vec{i}_0}\; U_{\vec{\ell}+\vec{i}_1,-\vec{i}_1}$$

$$U^\dagger_{\vec{\ell},\vec{i}} = U_{\vec{\ell}+\vec{i},-\vec{i}}$$

Wilson action
$$\boxed{S[U] = \frac{n}{g}\sum_p \mathrm{Tr}\left(U_p + U_p^\dagger\right)}$$

$$Z = \int \mathcal{D}U\; e^{S[U]}$$

$\hookrightarrow$ can be reduced to a single matrix integral by exploiting the gauge invariance of the theory.

i.e. invariance under
$$U_{\vec{\ell},\vec{i}} \longrightarrow V_{\vec{\ell}}\; U_{\vec{\ell},\vec{i}}\; V^\dagger_{\vec{\ell}+\vec{i}}$$

for arbitrary unitary matrices "V"

A convenient gauge is one where

$$U_{\vec{l},\vec{i}_0} = \mathbb{1} \quad \text{for all } \vec{n}_l$$

- In this gauge:

$$S[U] = \frac{n}{g} \sum_{\vec{l}} \mathrm{Tr}\left[U_{\vec{l},\vec{i}_1} U^{\dagger}_{\vec{l}+\vec{i}_0,\vec{i}_1} + h.c. \right]$$

- Change of variable. $U_{\vec{l}+\vec{i}_0,\vec{i}_1} = W_{\vec{l}}\, U_{\vec{l},\vec{i}_1}$

so that:

$$Z = \int \prod dW_{\vec{l}} \; \exp\left[\frac{n}{g} \mathrm{Tr}\left(W_l + W_l^{\dagger} \right) \right]$$

$$= \int \prod dU_{\vec{l}} \; \exp\left[\frac{n}{g} \mathrm{Tr}\left(U_l + U_l^{\dagger} \right) \right]$$

$$= \left[G(g,n) \right]^{A/a^2}$$

$A \to$ total area.
$a^2 \to$ area of a plaq.
$\frac{A}{a^2} \to$ no. of plaq...

$$G(g,n) = \int dU \; \exp\left[\frac{n}{g} \mathrm{Tr}\left(U + U^{\dagger} \right) \right]$$

$$= \frac{1}{(2\pi)^n n!} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} d\theta_1 \, d\theta_2 \cdots d\theta_n \; \cancel{\prod_i} \; e^{\frac{2n}{g}\sum_i \cos\theta_i} \prod_{i<j} \sin^2\left(\frac{\theta_i - \theta_j}{2} \right)$$

saddle point for large n

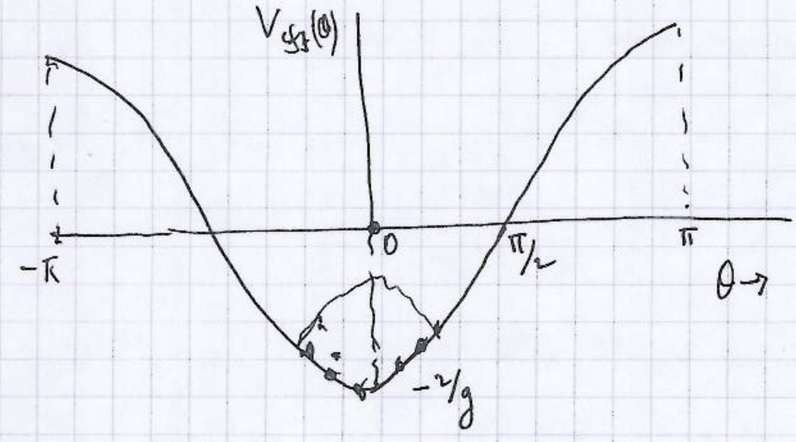
$$= \int \mathcal{D}\rho(\theta) \; e^{n^2 \left[\underbrace{ \frac{2}{g} \int \cos\theta \, \rho(\theta) \, d\theta + \int \rho(\theta)\rho(\theta') \log\left| \sin\left(\frac{\theta_i - \theta_j}{2} \right) \right| d\theta \, d\theta' + c_0 \left[\int \rho(\theta) d\theta - 1 \right] }_{S[\rho(\theta)]} \right]}$$

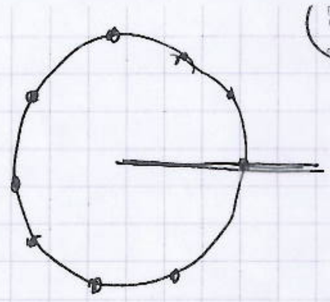
$$\frac{\delta S}{\delta \rho(\theta)} = 0 \Rightarrow \boxed{ \frac{2}{g} \cos\theta + c_0 + 2 \int \rho(\theta') \log\left[\left| \sin \frac{\theta - \theta'}{2} \right| \right] d\theta' = 0 }$$

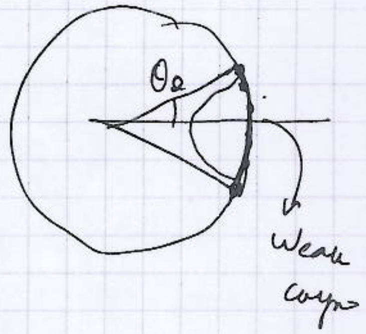
deriving w.r.t. θ:

$$\boxed{ \frac{2}{g} \sin\theta = \mathrm{Pr} \int \rho(\theta') \cot\left(\frac{\theta - \theta'}{2} \right) d\theta' }$$

$$V_{eff}(\theta) = -\frac{2}{g}\cos\theta$$







Sol^n depends on 'g'..

$g < 2 \longrightarrow$ Weak coupling

$g > 2 \longrightarrow$ Strong "

$\boxed{g<2}$.

$$\rho(\theta) = \frac{2}{\pi g}\cos\frac{\theta}{2}\left[\frac{g}{2} - \sin^2\frac{\theta}{2}\right]^{1/2}.$$

$$-\theta_0 < \theta < \theta_0$$

$$\theta_0 = 2\sin^{-1}\left(\sqrt{\frac{g}{2}}\right).$$

when $g \to 2^- \longrightarrow \theta_0 \to \pi$.

↳ gap closes

$\underline{g > 2}$

$$\rho(\theta) = \frac{1}{2\pi}\left[1 + \frac{2}{g}\cos\theta\right] \qquad -\pi \leq \theta \leq \pi$$

spread all over the circle.

$$G(g,n) \simeq e^{n^2/g^2} \qquad g > 2$$

$$\simeq e^{n^2\left[\frac{2}{g} + \frac{1}{2}\log\left(\frac{g}{2}\right) - \frac{3}{4}\right]} \qquad g \lessapprox 2$$

3rd derivative: $\longrightarrow \lim_{n\to\infty} \frac{\log G}{n^2} \to$ discontinuous.

$$\Psi_n\left(\frac{n^2}{g^2}\right) \simeq 1 \qquad g > 2.$$

$$= e^{-n^2\left[\frac{1}{g^2} - \frac{2}{g} - \frac{1}{2}\log\left(\frac{g}{2}\right) + \frac{3}{4}\right]} \qquad g \le .$$

exponential generating f^n:

$$\Rightarrow \sum_{M=0}^{\infty} p_{n,M} \frac{\mu^M}{M!} = \left\langle e^{\sqrt{\mu}\, \mathrm{Tr}(U+U^\dagger)} \right\rangle = \left\langle e^{\frac{n}{g}\mathrm{Tr}(U+U^\dagger)} \right\rangle$$

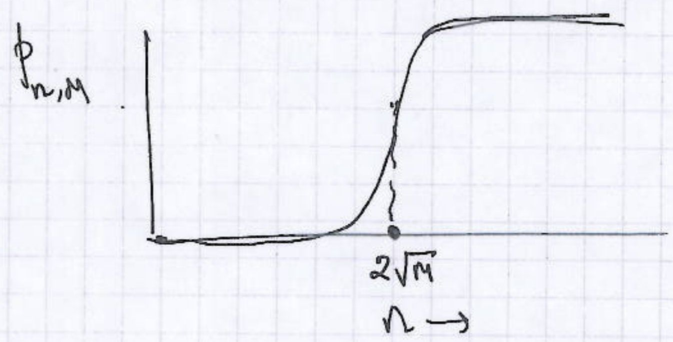
set $\mu = \frac{n^2}{g^2}$ $\qquad = G(g,n).$

$$\sum_M p_{n,M} \frac{(n/g)^M}{M!} = G(g,n) \simeq e^{n^2/g^2} \qquad g > 2$$

$$e^{n^2\left[\frac{2}{g} + \frac{1}{2}\log\left(\frac{g}{2}\right) - \frac{3}{4}\right]} \qquad g \le 2$$

phase transition $\longrightarrow$ transition in M.
in 'g'.

$n \Leftrightarrow g\sqrt{M}$.



$n_c = 2\sqrt{M}$.

$p_{n,M} \to 1 \qquad n > n_c = 2\sqrt{}$

$\longrightarrow e^{-n^2\left[\frac{1}{g^2} - \frac{2}{g} - \frac{1}{2}\log\left(\frac{g}{2}\right)\right.}$

$e^{-n^2[}$

$$e^{-[M - \frac{2}{}}$$

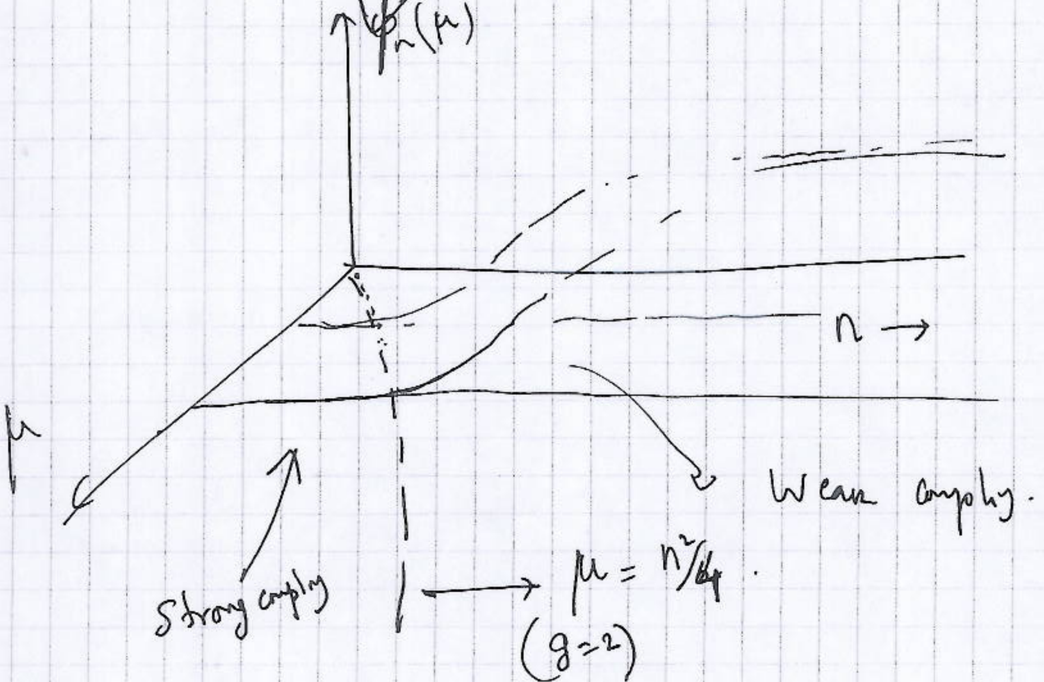
$M \to n^2/g^2$

$\frac{n^2}{g} + n^2 \log g - \frac{3}{4}n^2$ $\qquad n^2 = g^2 M$

$\frac{n^2}{g} = gM =$

$\frac{n^2}{g}$

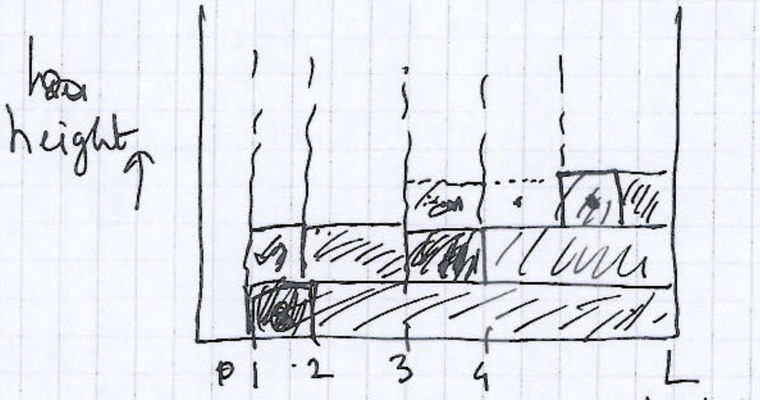
$$\psi_n(\mu) = \sum p_{n,m} \frac{\mu^m}{m!} e^{-\mu} \quad (11)$$



$$\boxed{\mu = n^2/g^2}$$

Anisotropic BD model

S.M. & S. Nechaev, 20..
~~PRE 72, 020901 (..)~~
PRE, 69, 011103 (2004)



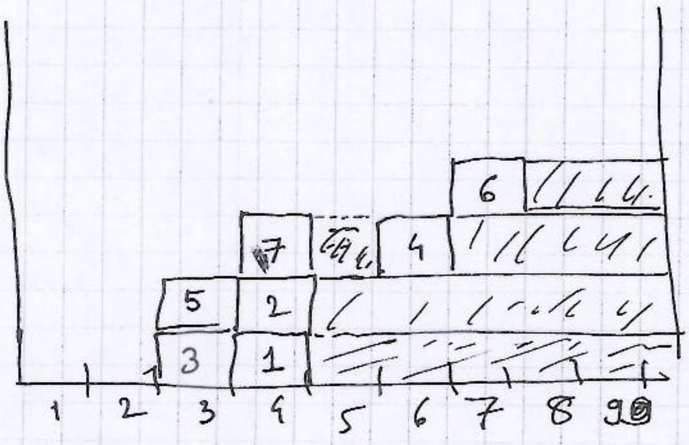
columnar growth

'k' (say) (with prob 1/L)

• Select a site at random, and drop a ~~particle~~ "brick"

$$H_k = H_k + 1$$

$$\cancel{H_k(t+1) = \max\left[H_k(t), H_{k-1}(t), \dots H_1(t)\right] + 1}$$



$$\cancel{H_k(t+1) = \max\left[H_{k-1}(t), H_{k-2}(t), \dots H\right] + 1}$$

k-site is chosen with prob 1/L

heights of all columns ~~to~~ labelled ~~k, k+1, k+2,~~ ....

~~m>k~~, $\cancel{H_m(t+1) = \max\left[H_m(t), H_{m-1}(t), \dots, H_k(t)\right] + 1}$

$$h_k(t) = H_{k+1}(t) - H_1(t)$$

$$h_k(t) \longrightarrow 2\sqrt{\frac{tk}{L}} + \left(\frac{tk}{L}\right)^{1/6} \chi_2 .$$

$$\sigma_k^2(t) = \left\langle \left[h_k(t) - \langle h_k(t)\rangle\right]^2 \right\rangle \sim \langle h_k(t)\rangle^{2\beta} .$$

$$\beta = 1/3 .$$