Calculus of Variations to Optimal Control: A short Introduction *

A. K. Nandakumaran†

1 Introduction

Optimal control problems in some sense can be viewed as *dynamic optimization* and hence it is considered as a generalization of the classical *calculus of variations (CV)*. The subject CV is a pretty old subject (250 years or so) and it is motivated by well known problems from science and engineering. Some of the fundamental example are: *Brachistochrone problem* due to Johann Bernoulli in the late 17th century, *Fermat's Principle in optics* which predicts Snell's law, *Dirichlet principle* (minimizes an energy functional over surfaces leading to Poisson equation), the *Action Principle* (as a particular case, we get Newton's second law). We will present some details about a few of the examples at a later time.

Whereas in CV, one minimizes certain associated functional over a class of trajectories, the optimal control (OC) problems deal with a more wider class of minimization problems where the trajectories are defined via certain dynamic constraints. The dynamic constraints may be ordinary differential equation (ODE) or partial differential equation (PDE) giving rise to trajectories, but the crucial point is that these trajectories can be varied by suitable action on the constraint system by applying what is known as *controls*. Thus the application of optimal control problems is much wider especially in engineering sciences as problems are modeled via differential equations with controls.

A systematic study of CV was initiated by Euler as early as in 1740's, but such a systematic study for OC problems emerged much later in 1950's. This is mainly due to its applications in engineering problems especially aerospace applications after the world war II which gained popularity and it became a distinct field of research. Two school of thoughts developed more or less at the same time. One was the *Maximum*

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†Department of Mathematics, Indian Institute of Science, Bangalore-560012, India. Email: nands@math.iisc.ernet.in

Principle (a set of necessary conditions for control to be optimal) due to *Pontryagin* and his group in USSR. The second one was *Dynamic Programming Principle (DPP)* due to *Bellman* in USA. DPP leads the search of optimal controls to that of solving certain PDEs knows as *Hamilton-Jacobi / Hamilton -Jacobi - Bellman (HJ/HJB)* equations which gives sufficient conditions as well.

Though the OC problems as a minimization problem looks similar to CV problems, it was not easy to adopt the techniques from CV to give rigorous proofs in OC theory due to many reasons like non-smoothness etc. It is here that the Pontryagin Maximum Principle (PMP) gained its significance and became very useful and powerful in applications in engineering even though it is a set of necessary conditions. As usual engineers were not very keen on getting just sufficient condition (of course it is acceptable) as theoretical results, the necessary conditions provided them a place to look for controls with suitable feedback laws. Indeed, theoretically the controls obtained by necessary conditions need not be optimal unless, at least one proves the existence of a minimizer. Nevertheless, PMP worked quite well in many applications.

But treating DPP in a rigorous footing was much more a challenging task as the idea is to relate the value function (minimum value of the cost as a variable function of the initial points) as a solution to non-linear PDEs. Indeed a decade prior to these developments, the theory of *Distributions and Sobolev spaces* made PDE as a topic of analysis. But the theory of distribution is basically a linear theory (one can treat non-linear problem as well, like conservation laws) and it is not suited to the highly non-linear HJB equations. Hence need of the hour was to invent new techniques. It is in this context that one has to view the developments of non-smooth calculus and later interpreting non-smooth functions as solutions (viscosity solutions) in a weak sense to non-linear PDEs. Again the viscosity solutions became an independent area of research where one could study non-linear PDEs in general, not necessarily HJB equations.

Of course, one need to go through few courses to understand the topics discussed above. Hence we are not making an attempt to define all the notions described above, leave alone the proofs of theorems. Our idea is to introduce some of the fundamental concepts like Lagrange multipliers starting with finite dimensional optimization and its generalization to infinite dimensional optimization, namely Calculus of Variations. Later, we will introduce the concept of adjoint state and present PMP in certain situations without proofs. Here, note that one can study various types of minimization problems and one need to prove PMP accordingly. We also introduce DPP and HJB though, we are not planning to define the concept of viscosity solutions which you may see in other lectures. In a smooth situation, we will also see the connection between PMP and DPP. Lastly, if time permits, we specialize to the well known *Linear-Quadratic Regulator (LQR)* problem and in this special case the above results take much simpler form.

There are other developments related and parallel to optimal control, namely *differential game theory, controllability and observability* etc. Differential games were motivated from military applications and in the modern day it plays a vital role in financial market. On the other hand, the *stochastic version* of these theories are much more useful in the present day including economics. The controllability is very important and in fact in OC problem with ODE/PDE constraints one minimizes over trajectories with initial and final points (end points) lie in suitable sets. First of all it is necessary to know that, do such trajectories exist? In other words, is it possible to reach a desired point or a desired set by applying control on the dynamics, a relevant question in space dynamics of satellites and also in missile technology. These are the questions one addresses in controllability. Kalman's necessary and sufficient condition for controllability of finite dimensional systems was the first step forwards the study of controllability of infinite dimensional systems whose dynamics are determined by PDEs. Today the controllability of various systems, in particular, fluid flow control problems both deterministic and stochastic is a very active area of research. A majority of talks in this school are devoted to the control of deterministic and stochastic Navier-Stokers equations.

In my last two talks, we discuss the exact controllability and may introduce the *Hilbert Uniqueness Method (HUM)* for linear wave equation.

2 Extremal Problems (Finite Dimensional)

The problems that involve finding maxima and minima are called *extremal problems*. The optimization has the same meaning. We now describe two of the ancient problems. The observation is that nature is guided by extremal principles.

2.1 Isoperimetric Inequalities (Dido's Problem)

Probably the oldest maximizing problem goes back to the story of the phoenician princess Dido around 1000 BC. Her husband was killed by her own brother and she left the kingdom, set off westward along the Mediterranean shore in search of a place. A certain place, now the Bay of Tunis, caught her attention and Dido negotiated the sale of land with a leader. She was offered a deal where she could take as much place *encircled with a bull's hide*. The clever lady cut the bull's hide into narrow strips tied them together and enclosed a large area of land. This is nothing but the famous isoperimetric inequality: Among all closed plane curves with a fixed length, find the one that encloses the largest area. In modern language: Find a domain Ω that satisfies $\max\{|\Omega| : |\partial\Omega| = l\}$, where $l > 0$, a given fixed number. The isoperimetric inequality is given by $l^2 \geq 4\pi|\Omega|$ and the equality is achieved if and only if Ω is a circle. This is a pretty difficult problem, but Dido had a very good approximate solution. In 3-dimensions, the sphere has the largest volume among all domains of fixed surface area. A much easier problem is related to n-gon (polygon with n-sides). It is known that a regular polygon (i.e. with equal sides and equal angles) has the largest area among all n-gons of given length.

2.2 Heron's Problem

Suppose A and B are two points on the same side of a line. Find a point C on the line so that the sum $(AC + CB)$ is minimum. The solution is simple and beautiful. To see this, let B_1 be the reflection point with respect to the line and join AB, to cut the line at the point at C. This point will give the solution (Why?). The beauty is that the problem is associated to optics.

This probably shows that nature is guided by extremal principles and Heron's idea was developed by Fermat in the 17th century. Fermat's principle in optics is: The light ray follows a path which takes the shortest time, not the shortest path when it suffers reflection and refraction. The principle predicts Snell's Laws of reflection and refraction when the light rays move in through an inhomogeneous media.

We will see few more examples like Dirichlet Principle, Least action principle in mechanics, Brachistochrone problem, Catenary etc. when we come to the study calculus of variations.

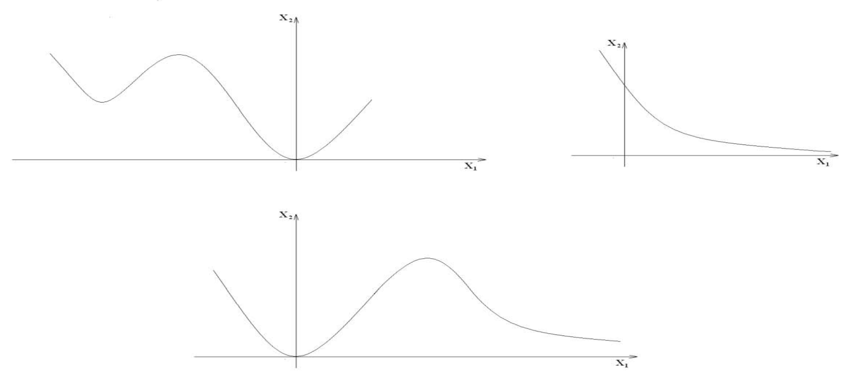
This further justifies that the variety of natural problems leads to optimization problems. In the 18th and 19th centuries, the significance became evident that the laws of physics that identified states of nature by minimizing curves, surfaces and energy functionals.

2.3 Finite Dimensional Optimization

The more familiar problem to the modern students is finding a minimum point of a differentiable function f in a closed interval $[a, b]$. This is done by analyzing the first order condition $f'(x) = 0$. A second order condition $f''(x_0) > 0$ satisfying $f'(x_0) = 0$, $x_0 \in (a, b)$ will then determine that x_0 is a minimum.

Suppose now $f : \mathbb{R} \to \mathbb{R}$. Are there maxima, minima? Essentially, we have 3 situations which are crucial in the modern non-linear analysis. This is depicted in the following figure.

Indeed $\inf f(x) = 0$ in all the cases. The first figure shows that $x_0 = 0$ is a minimum point and all minimizing sequences x_n converges to 0, whereas $x_n \to \infty$ is a minimizing sequence (see the second figure), but it also has minimizing sequences that goes to the minimum point. The third figure shows that there are no minimizing sequences that goes to 0. The above simple examples shows that the study of optimization problems is not an easy subject. The behaviors in figures (2) and (3) is due



to the non-compactness properties which has to be overcome by other assumptions.

Theorem 2.1. *(Weierstrass): If* $f : [a, b] \to \mathbb{R}$ *is continuous, then* f *has a solution to minimization (maximization) problem. If* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and coercive, that is* $f(x) \to \infty$ *as* $x \to \pm\infty$*, then* f *has a point of global minimum.*

Theorem 2.2. *(Fermat): If* x_0 *is a local extremum of a differentiable function* f*, then* $f'(x_0) = 0$*.*

2.4 Unconstrained Optimization

The above results are true in higher dimensions as well. Let Ω be an open set in $\mathbb{R}^n$, f be a C^1 function and assume x^* is a local minimum. Note that x^* is called a local minimum of f if $\exists$ a neighborhood of U of x_0 in Ω such that $f(x^*) \leq f(x)$ for all $x \in U$. One can derive a *first order necessary condition* as $\nabla f(x^*) = 0$ and assuming that $f \in C^2$, a *second order necessary condition* can also be derived as $\nabla^2 f(x^*) \geq 0$ (positive semi-definite) that is $\langle \nabla^2 f(x^*)h,\ h\rangle \geq 0$, for all $h \in \mathbb{R}^n$.

A sufficient condition for optimality can be obtained by strengthening the second order condition; that is if $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) > 0$, then x^* is a strict local minimum of f.

We do not prove it here, but we have the following remarks

Remark 2.1. The idea of the proof is to consider any fixed vector $d \in \mathbb{R}^n$. Then for any α sufficiently close to 0, we have $x^* + \alpha d \in \Omega$ (unconstrained case) and consider the one dimensional function $g(\alpha) = f(x^* + \alpha d)$. Then g has a minimum at $\alpha = 0$ and $g'(\alpha) = \nabla f(x^* + \alpha d) \cdot d,\ g'(0) = \nabla f(x^*) \cdot d$. To derive second order conditions, one has to expand g upto order 2.

Remark 2.2. If x^* is a point on the boundary of Ω, the above analysis is not true. In this case one cannot take all directions $d \in \mathbb{R}^n$ because $x^* + \alpha d$ need not be in Ω. A direction d is called a *feasible direction* at $x^* \in \partial\Omega$ if $x^* + \alpha d \in \Omega$ for small enough $\alpha > 0$. One can derive $\nabla f(x^*) \cdot d \geq 0$ for every feasible direction d as a necessary condition. Further $\langle \nabla^2 f(x^*)d,\ d\rangle \geq 0$ for all feasible directions d satisfying $\nabla f(x^*) \cdot d = 0$

Remark 2.3. Convex functions and convex sets plays an important role in optimization problems and quite often, it is easily tractable. The first order necessary condition is also sufficient, a local minimum is automatically a global one and there is uniqueness of the minimum. The main fact is that the function f lies above the linear function $L(x) = f(x^*) + \nabla f(x^*) \cdot (x - x^*)$ which satisfies $L(x^*) = f(x^*)$. Further the numerical algorithms like steepest descent (gradient) method converges to x^* satisfying $\nabla f(x^*) = 0$ and leads to global minimum in convex problems.

A point x^* satisfying $\nabla f(x^*) = 0$ is called a *stationary point*.

2.5 Constrained Optimization and Lagrange Multipliers

Quite often one may not minimize over all points in a full neighborhood, but may be minimizing with a constraint say on a surface (manifold) of dimension less than n.

Example 2.4. (Linear Constraint). Consider a simple problem of minimizing $f(x, y) = x^2 + y^2$ with the constraint $y = x + 1$. In other words, minimizing along the straight line $y = x + 1$, not on the whole space $\mathbb{R}^2$.

Method 1 (Reduction Method) The idea is to reduce the number of variables and minimize over $\mathbb{R}$. Define $g(x) = f(x, x+1)$. It is easy to see that a minimization of g over $\mathbb{R}$ yields the minimum is achieved at $(-\frac{1}{2}, \frac{1}{2})$ and the minimum value is $\frac{1}{2}$.

Method 2: One can add the constraint with the minimizing function with the help of a new variable λ. Define

$$F(x, y, \lambda) = x^2 + y^2 - \lambda(y - x - 1)$$

Now treat F as an unconstrained minimization problem over $\mathbb{R}^3$ instead of $\mathbb{R}^2$. By applying $\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial \lambda} = 0$ one obtains $\lambda = -1$ and $x = -1/2,\ y = 1/2$ as obtained earlier with $y = x + 1$.

This is the idea behind the powerful Lagrange multipliers.

Linear Constraints: Let $f : \mathbb{R}^n \to \mathbb{R}$ to be minimized over a plane

$$L = \left\{ x \in \mathbb{R}^n : x = x_0 + \sum_{i=1}^{k} \alpha_i u_i,\ \alpha_i \in \mathbb{R} \right\}$$

of dimension k, where $u_1, \cdots u_k$ are k-linearly independent vectors in $\mathbb{R}^n$. The space N normal to L will be spanned by $n-k$ vectors, say $b_1, \cdots b_{n-k}$. Then we can write

$$L = \{x \in \mathbb{R}^n : \langle b_i,\ x - x_0\rangle = 0,\ 1 \le i \le n-k\}$$

Theorem 2.3. *If $f \in C^2(\Omega)$, $\Omega_L = \Omega \cap L$ and f has a minimum at $x^* \in \Omega_L$, then $\nabla f(x^*)$ is normal to L.*

The reduction method will give an unconstrained problem in $\mathbb{R}^{n-k}$ which will yield $\langle \nabla f(x^*),\ u_i\rangle = 0,\ 1 \le i \le k$ which means that $\nabla f(x^*)$ is normal to L.

Using the Lagrange multiplies, we have the following theorem.

Theorem 2.4. *Let $f \in C^2(\mathbb{R})$ and x^* minimizes f subject to k linear constraints*

$$g_i(x) = c_i + \sum_{j=1}^{k} b_{ij}x_j = c_i + \langle b_i, x\rangle = 0,\ 1 \le i \le k \tag{2.1}$$

Then there exists k multipliers $\lambda_1, \cdots, \lambda_k$ such that

$$\nabla F(x^*) = 0,\ \langle \nabla^2 F(x^*)h,\ h\rangle \ge 0 \tag{2.2}$$

for $h \neq 0$ and orthogonal to $b_1, \cdots b_k$. Here F is the augmented function $F(x) = f(x) + \sum_{i=1}^{k} \lambda_i \cdot g_i(x)$. Conversely, if $\lambda_1, \cdots \lambda_k$ and x^* satisfying $\nabla F(x^*) = 0$ and $\langle \nabla^2 F(x^*)h,\ h\rangle > 0$ for all $h \neq 0$ orthogonal to $b_1, \cdots b_k$, then x^* is a local minimum of f satisfying the constraints.

Non-linear Constraints: Let Ω be a surface in $\mathbb{R}^n$ with the equality constraints

$$h_1(x) = \cdots = h_k(x) = 0$$

where $h_i \in C^1(\mathbb{R}^n, \mathbb{R})$. Let $x^* \in \Omega$ be a local minimum of f over Ω. Assume x^* is a regular point, that is the set $\{\nabla h_i(x^*), 1 \le i \le k\}$ is an independent set in $\mathbb{R}^n$. First order necessary condition (Lagrange multipliers): Since it is a surface one need to consider curves lying in Ω passing through x^* than the line segments. Let $x(\alpha)$ be a curve in Ω such that $x(0) = x^*$ and let $g(\alpha) := f(x(\alpha))$, where α is a small parameter varying in the neighborhood of 0. A simple calculation will give us

$$g'(\alpha) = \nabla f(x(\alpha)) \cdot x'(\alpha)$$

and at $\alpha = 0$, we have

$$g'(0) = \nabla f(x^*) \cdot x'(0) = 0 \tag{2.3}$$

since x^* is a minimum over Ω. Note that, the vector $x'(0)$ is a tangent vector in the tangent space $T_{x^*}\Omega$. Since $h_i(x(\alpha)) = 0$ for all α and i, a further calculation will show that

$$\nabla h_i(x^*) \cdot x'(0) = 0 \tag{2.4}$$

In fact, the converse is also true. That is if $d \in \mathbb{R}^n$ satisfying $\nabla h_i(x^*) \cdot d = 0,\ i = 1,\ \cdots\ k$, then d is a tangent vector to Ω at x^* corresponding to some curve. In other words, tangent vectors to Ω are exactly the vector d satisfying (2.4), that is $T_{x^*}\Omega = \{d \in \mathbb{R}^n : \nabla h_i(x^*) \cdot d = 0\}$. In view of this characterization, we can rewrite the condition (2.3) as

$$\nabla f(x^*) \cdot d = 0, \text{ for all } d \text{ such that } \nabla h_i(x^*) \cdot d = 0,\ i = 1, \cdots k \tag{2.5}$$

But (2.5) then implies (leave it as an exercise) that

$$\nabla f(x^*) \in \text{ Span } \{\nabla h_i(x^*) : 1 \le i \le k\} \tag{2.6}$$

Geometrically, the the equation (2.6) says that $\nabla f(x^*)$ is normal to Ω at x^* similar to the linear constraint case. The condition (2.6), implies the existence of Lagrange multiplier $\lambda_1, \cdots \lambda_k$ such that

$$\nabla f(x^*) + \sum_{i=1}^{k} \lambda_i \nabla h_i(x^*) = 0 \tag{2.7}$$

Remark 2.5. The above proof is geometrical, but we can indeed prove it analytically using implicit function theorem which we do not do it here. One can also derive in an analogous fashion second order necessary condition as well as sufficient conditions.

An application (Eigenvalue representation): Let A be a symmetric matrix and λ be an eigenvalue with eigenvector x_0. Then $\lambda = \frac{\langle Ax_0,\ x_0\rangle}{\langle x_0,\ x_0\rangle}$. This motivates the definition of Rayleigh quotient $R(x) = \frac{\langle Ax, x\rangle}{\langle x, x\rangle}, x \neq 0$. Can we characterize λ as an optimization problem of $R(x)$?

Theorem 2.5. *The minimum point $x^* \neq 0,\ x^* \in \mathbb{R}^n$ of $R(x)$ is an eigenvector of A and the minimum value $\lambda_1 = R(x^*) = \min_{x \in \mathbb{R}^n} R(x)$ is the least eigenvalue of A.*

To see this, one can easily compute $\nabla R(x) = \frac{2}{|x|^2}(Ax - R(x)x)$. Thus for $x \neq 0$, $\nabla R(x) = 0$ if and only if x is an eigenvector with $\lambda = R(x)$ as the corresponding eigenvalue. In fact, it will be the least eigenvalue. A maximization of $R(x)$ will give the largest eigenvalue.

One can also obtain the other eigenvalues via the constraint optimization. Let E_k be the span of the orthonormal set of vectors $x_1, \cdots x_k$ of first k eigenvectors.

Theorem 2.6. *There exists a unit vector $x_{k+1} \in E_k^{\perp}$ which minimizes $R(x)$ on $E_k^{\perp}$ and it is the eigenvector corresponding to λ_{k+1}.*

3 Calculus of Variations

We begin with few examples from classical variational problems where a curve or a path to be chosen from a given family of admissible curves so that it minimizes or maximizes a given functional. We already stated the Dido's and Heron's problems. To see more clearly, the isoperimetric problem in a simpler form, consider a function $y : [a,b] \to \mathbb{R},\ y \geq 0$ such that $y(a) = y(b) = 0$. We would like to maximize the area functional $J(y) = \int_a^b y(x)dx$. Assuming y is C^1, the length constraint can be written as $\displaystyle\int_a^b \sqrt{1 + y'(x)^2}dx = l$. The solution to this problem, namely the semi-circle to known to Zenodorus (around 200 BC) though rigorous proof requires tools from CV.

Example 3.1 (**Catenary).** Consider a chain with uniform mass density of given length hanging freely (under gravity) between two fixed points. What is the shape of the chain? The question is due to Galileo in 1630's and the anticipated answer parabola was wrong.

How do we describe it as an optimization problem and what is to be minimized? Let $y : [a,b] \to (0, \infty)$ with $y(a) = A, y(b) = B$ as the end points of the possible shape of the chain. According to physics, the chain will take the shape of minimal potential energy. The potential energy functional is given by $J(y) = \int_a^b y(x)\sqrt{1 + y'(x)^2}dx$.

The catenary curve obtained by Johann Bernoulli in 1670's is given by $y(x) = c\cos h(x/c),\ c > 0$ unless the chain touches the ground. The name catenary is derived from the Latin word Catena (chain).

Example 3.2. Brachistochrone Problem This well known and famous problem again probably due to Johann Bernoulli. A frictionless bead located in a vertical plane at a point $A(x_0, y_0)$ slides along a wire under force of gravity whose other end is fixed in the vertical plane at $B(x_f, y_f)$. What is the shape of the curve (wire) so that bead arrives at its destination B in minimum time?

Formulate the problem using the conservation of energy. Let positive $y - axis$ points downwards and let $A(a, 0)$ and $B(x_f, y_f),\ a < x_f,\ y_f > 0$. Since the total energy initially is zero, we have at any point of time

$$\frac{mv^2}{2} - mgy = 0$$

Normalize with suitable units, assume $m = 1,\ g = 1/2$, we get $v = \sqrt{y}$. Thus our problem reduces to that of minimizing the functional $J(y) = \displaystyle\int_a^b \frac{1 + y'(s)^2}{\sqrt{y(s)}}ds$. Johann Bernoulli in 1696 posed this problem to his contemporaries and correct solutions were obtained by Leibnitz, Newton, Jacob Bernoulli and others including J. Bernoulli. The optimal curves are *cycloids* given by $x(t) = a + c(t - \sin t),\ y(t) = c(1 - \cos t)$. This is the locus of a fixed point on a circle when it rolls without slipping on the horizontal axis.

It is interesting to remark that the first solution by Bernoulli was based on Snell's law for light refraction.

Example 3.3 (Dirichlet Principle). This optimization problem involves finding a surface that minimizes a given integral functional $J(y) = \displaystyle\int_{\Omega} \nabla y(x) \cdot \nabla y(x) dx$ over surfaces y satisfying $y(x) = \overline{y}(x)$ on $\partial\Omega$. Here Ω is a given open bounded set in $\mathbb{R}^2$, $\partial\Omega$ is the boundary of Ω and $\overline{y}$ are given. If y is an optimal solution, then y satisfies the Laplace equation $\Delta y(x) = 0$ in Ω, $y = \overline{y}$ on $\partial\Omega$. This is also given as the equation satisfied by the electric potential in which a static two dimensional electric field is distributed. The functionals of the form $\displaystyle\frac{1}{2}\int \nabla y \cdot \nabla y - \int fv$ also represents the strain energy functional.

Remark 3.4. In the second half of the 19th century, this forced mathematicians like Green, Riemann, Poincare to view minimizing sequences in the gradient norm not in classical smooth spaces and looking for limiting points (completion) under this convergence (topology). This eventually lead to the introduction of generalized functions and Sobolev spaces in 1900-1930s. In this context, It is worth mentioning the problems 19 and 20 of Hilberts' 23 problems posed in 1900 regarding regularity and existence of solutions to partial differential equations.

Example 3.5 (The least action principle). Let $q(t)$ denotes generalized coordinates of a conservative mechanical system. According to the action principle $q(t)$ moves in such a way that the action integral $\displaystyle\int [T(q(t),\ \dot{q}(t)) - V(q(t))]dt$ is minimized. According to the physics terminology T is called the kinetic energy and V is the potential energy. For example, in Newton's law of motion $T = \frac{m}{2}|\dot{q}|^2$ from which one can obtain $\frac{d}{dt}(m\dot{q}) = -V(q)$. We may elaborate on this at a later time.

Action principle is useful for deriving dynamical equations of complex interacting systems and its qualitative properties.

3.1 Calculus of Variations (General Framework)

Among all C^1 curves, $y : [a, b] \to \mathbb{R}^n$ satisfying the boundary conditions $y(a) = y_0,\ y(b) = y_1$, find (local) minima of the functional

$$J(y) := \int_a^b L(x, y(x),\ y'(x))dx$$

Here $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is known as *Lagrangian* or the *running cost*, the terminologies derived from applications. This setting is originally due to Lagrange (1780's).

Let $\overline{y}$ be a minimizer of $J(y)$. The question is that can we derive optimality conditions similar to Fermats conditions as earlier.

3.2 Optimality Conditions (Euler-Lagrange equations)

One need to have appropriate smoothness assumptions on L. We do not concentrate on this aspect and the reader can refer to books for rigorous analysis. In any case, we are not planning to prove theorems in this article. The fundamental idea is to look for variations from the minimizer $\overline{y}$ which will increase the cost. Given any C^1 curve η satisfying $\eta(a) = \eta(b) = 0$ and $\varepsilon > 0$, the curve $\overline{y} + \varepsilon\eta$ is an admissible curve and hence $J(\overline{y} + \varepsilon\eta) - J(\overline{y}) \geq 0$. A formal Taylor expansion (which of course can be justified) with little computation, one can derive

$$\int_a^b \left[L_y(x, \overline{y}(x),\ \overline{y}'(x)) - \frac{d}{dx} L_{y'}(x, \overline{y}(x),\ \overline{y}'(x)) \right] \eta(x) dx = 0 \tag{3.1}$$

for all C^1 curves η with $\eta(a) = \eta(b) = 0$. From the above identity, one can indeed derive the *Euler-Lagrange equations (E-L)* providing the first order necessary condition as

$$L_y(x, \overline{y}(x),\ \overline{y}'(x)) = \frac{d}{dx} L_{y'}(x, \overline{y}(x), \overline{y}'(x)) \tag{3.2}$$

This is actually a second order equation for $\bar{y}$.

Remark 3.6. Trajectories y satisfying the Euler-Lagrange equation (3.2) are called *extremals*. Since it is only a necessary condition, not every extremal is an extremum. Indeed, if an optimal curve is known to exist and the Euler-Lagrange equation has a unique extremal, then this extremal gives the optimal solution. Thus proving the existence of solutions of minimization problem in a rigorous setting is important, but it not be trivial.

Example 3.7. Consider $J(y) = \displaystyle\int_0^1 y(y')^2 dx$ subject to $y(0) = y(1) = 0$. The E-L equation is $(y')^2 = \frac{d}{dx}(2yy')$ and $y = 0$ is, in fact, a solution. But the reader can prove that $y = 0$ is neither a minimum nor maximum.

The E-L equation was first derived by Euler in 1740's based on approximation method and later by Lagrange in 1750's at a very young age and subsequently gave the name Calculus of Variations. We now consider two special cases which are quite important in the future discussion.

Case1 ($L = L(x, y')$ is independent of y) In this case, from the E-L equations, one can derive that $\frac{d}{dx} L_{y'} = 0$ which gives $L_{y'}(x, y'(x)) = C$, a constant.

Case2 ($L = L(y, y')$ is independent of x) The equation obtained in the case is $\frac{d}{dx}(L_{y'} y' - L) = 0$ and hence $L_{y'}(y, y')y' - L(y, y') = C$, a constant.

Now consider the least action principle, in particular the Newton's law of motion. Here the Lagrangian L is given by

$$L(x, y(x), y'(x)) = L(y, y') = \frac{m}{2}|y'|^2 - V(y)$$

Then $L_{y'} = my'$ which is the momentum introduced in physics. Analogous with this terminology, we call in general $L_{y'}$ evaluated along a given curve the *momentum* associated with the Lagrangian. So the first case tells us that if L is independent of y, then the momentum is constant along optimal trajectories.

Now, for the above example $L_{y'}y' - L = \frac{m}{2}|y'|^2 + V(y)$ is the *total energy* and it is known as *Hamiltonian*. Thus if L is independent of x, then the total energy is conserved.

3.3 Hamiltons' Canonical Equations

Define the momentum p and Hamiltonian H as

$$p = p(x, y, y') := L_{y'}(x, y, y')$$

and

$$H = H(x, y, y', p) := py' - L(x, y, y')$$

The variable y and p are called the *canonical variables*. Let y be an extremal, that is y satisfies E-L equations (3.2). Then, it follows that y and p satisfies,

$$\left.\begin{aligned} \frac{dy}{dx} &= H_p \\ \frac{dp}{dx} &= -H_y \end{aligned}\right\} \tag{3.3}$$

The above system is known as *Hamiltons' canonical system of equations.*

Remark 3.8. In the special cases, we have observed that if the Lagrangian is independent of x, then the momentum is constant along extremals whereas the Hamiltonian is constant if L is independent of y. Such a result is not true in general.

Remark 3.9. The Lagrangian L and the Hamiltonian H are related mathematically via *Legendre transformation.* Given $f : \mathbb{R} \to \mathbb{R}$, the Legendre transform of f, denoted by $f^* : \mathbb{R} \to \mathbb{R}$ is defined as

$$f^*(p) = \max_{\xi} \{p \cdot \xi - f(\xi)\} \tag{3.4}$$

There are nice properties of f^*. For example, f^* is convex even if f is not, but if f is convex, then the $*$ operation is *involutive*, that is $f^{**} = f$.

The following computations are formal to convey some ideas. For fixed p, suppose $\xi = \xi(p)$ solves uniquely the maximization problem (3.4), then

$$f^*(p) = p \cdot \xi(p) - f(\xi(p)).$$

Since $\xi(p)$ is a maximal point, we have $\frac{d}{d\xi}(\xi \cdot p - f(\xi))|_{\xi(p)} = 0$, that is

$$p - f'(\xi(p)) = 0 \tag{3.5}$$

Let x, y be fixed and take $\xi = y'$. Now consider the Lagrangian $L(x, y, y\prime) = L(x, y, \xi) = L(\xi)$ as f above. Then, applying (3.5) to $L(\xi)$, we get

$$p - L_{y'}(x, y, y'(p)) = 0 \tag{3.6}$$

which is the definition of the momentum. Further

$$L^*(x, y, p) = p \cdot y'(p) - L(x, y, y'(p)) \tag{3.7}$$

$$= H(x, y, y'(p), p) \tag{3.8}$$

according to the earlier definition. Thus we obtain the Hamiltonian H, in some sense, as the Legendre transform of the Lagrangian.

But, there is a fundamental difference from the earlier case, where y' was an independent argument of H. In the present case, y' is a dependent variable expressed in terms of x, y, p by the relation (3.6). In other words, the Legendre transformation of $L(y') = L(x, y, y')$ as a function of y' is $H(p) = H(x, y, p)$ which is a function of p and no longer has y' as argument. The solvability of y' in (3.6) is crucial. This together with the canonical transformation will lead to the maximum principle. Probably this was the difficulty that the maximum principle had to wait until the late 1950's.

We do not get into the details of constraint variational problem, but we make some remarks.

3.4 Constrained Calculus of Variations

One can consider various type of constraints. Suppose we want to minimize $J(y)$, where in addition to the ODE constraints, we also have a integral constraint of the form $C(y) := \displaystyle\int M(x, y(x), y'(x)dx = C$. Then, there exists a Lagrange multiplier λ^* such that the Euler-Lagrange equation holds for the augmented Lagrangian $L + \lambda^* M$. The analysis is bit more involved than the unconstrained problem.

On the other hand, if we take a point-wise constraint of the form $M(x, y(x), y'(x)) = C$, the intuitive idea is to consider a multiplier for each point x. Thus, the L-M will be function $\lambda* : [a, b] \to \mathbb{R}$ and the Euler-Lagrange equation holds for the augmented Lagrangian $L + \lambda^*(x)M$.

Remark 3.10. Suppose, we want to solve y' in terms of x and y from the point-wise constraint. In general, one may have fewer constraint equations, say k equations, which may be less than the dimension of y'. Thus, it will be an under determined system which gives us some free variables which we denote by u. In other words, we will have a system of the from $y' = F(x, y, u)$ leading to control problem. Unconstrained problem corresponds the case $y_1' = y_2$ and $y_2 = u$ (free variable, control).

Example 3.11. Consider the functional $I(u) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 - \dfrac{1}{3}\displaystyle\int_\Omega u^3$. This functional is neither bounded below nor above on $H_0^1(\Omega)$. Hence no question of maximizing or minimizing in $H_0^1(\Omega)$. We use the constraint set $M = \{u \in H_0^1(\Omega) : \int G(u) = 1\}$, where $G(u) = u^3$ and minimize $J(u) = \frac{1}{2}\displaystyle\int_\Omega |\nabla u|^2$ on M.

Theorem 3.1. *Let $u \in M$ be the minimizer of J. Then there exist a real number λ such that*

$$\int_\Omega \nabla u \cdot \nabla w = \lambda \int_\Omega DG(u) \cdot w$$

for all $w \in H_0^1(\Omega)$. We can absorb λ by scaling as $v = \lambda^{1/3} u$, then v solves the non-linear problem $-\Delta v = v^2$ in Ω, $\quad v = 0$ on $\partial\Omega$.

We leave the details to the reader.

4 Optimal Control

This is too vast a subject with too many generalizations and diversities, hence it is impossible to present a reasonably short review. We present certain important issues may be with special cost functional. The main idea is to introduce the Pontryagin Maximum Principle (PMP), namely the optimality conditions. Indeed, the form of PMP may change according to the form of cost functional, but we try to present the fundamental theme behind it. In the process, we will also see that the tools from CV are not at all enough to handle the OC problems.

4.1 Calculus of Variation to Optimal Control

In CV, we have considered cost functional of the form $J(y) = \displaystyle\int_a^b L(x, y(x), y'(x))dx$ and aim was to minimize $J(y)$ among a given class of trajectories. But quite often the trajectories need not be completely defined earlier or one may need to take instantaneous decision so that the trajectories move according to our need like in the trajectories of satellites and missiles. Such a system is called a *control system.* For example, if we take $y' = u$, where $u(t)$ is the decision we take at time t regarding the slope of the trajectory. One need to take optimal control decision to minimize the cost involved at the same time achieving our goal.

In constraint CV problems which we did not discuss, note that the constraints may be given by a relation of the form $M(x, y(x),\ y'(x)) = 0$. If we solve and parameterize with respect to a variable u, we end up with $y' = f(x, y, u)$. This is the dynamic view given in OC problems.

Thus a control system consists a system of constraints given by

$$\dot{x}(t) = f(t, x(t), u(t)),\ x(t_0) = x_0 \tag{4.1}$$

Here $x = x(t) \in \mathbb{R}^n$ is called the state, $u(t) \in U \subset \mathbb{R}^m$ is the control, $t \in \mathbb{R}$ is the time variable, t_0 is the initial time with initial state $x_0 \in \mathbb{R}^n$.

Together with control system, there will be an associated cost functional. The OC problem is to minimize this cost according to the dynamic constraint (4.1). There are various forms of cost functionals and one of the standard forms is known as *Bolza form* given by (Bolza problem)

$$J(u) = \int_{t_0}^{t_1} L(t, x(t), u(t))dt + K(t_1, x_1) \tag{4.2}$$

Here t_1 is the final time with the final state x_1, L is the running cost and K is the terminal cost.

When there is no terminal cost ($K \equiv 0$), the problem is known as *Lagrange problem* and it is called a *Meyer problem* if there is no running cost ($L \equiv 0$). It is interesting to note that it is easy to go from one formulation to another. To see this, given a Mayor problem with terminal cost K, we write

$$\begin{aligned} K(t_1, x_1) &= K(t_0, x_0) + \int_{t_0}^{t} \frac{d}{dt} K(t, x(t))dt \\ &= K(t_0, x_0) + \int_{t_0}^{t_1} (K_t + K_x \cdot f)dt \end{aligned}$$

Since $K(t_0, x_0)$ independent of the control u, the terminal cost K is replaced by an integral cost. Conversely, for a Lagrangian problem, with running cost L and $K = 0$, introduce a new variable w as $\dot{w} = L(t, x, u),\ w(t_0) = 0$, then the cost $\int_{t_0}^{t_1} L(t, x(t), u(t)) = w(t_1)$ which is like a terminal cost.

There are various issues regarding the final time and final state and each of them will lead to different types of problems. One has to treat it differently. Indeed the problem can be consolidated by introducing the concept of a target set $S \subset [t_0, \infty) \times \mathbb{R}^n$ setting t_1 is the first time such that $(t_1, x_1) \in S$. Of course, it is possible that $(t, x(t))$ does not enter S and such problems are treated in infinite-horizon problem.

Given a fixed $x_1 \in \mathbb{R}^n$, the target set $S = [t_0, \infty) \times \{x_1\}$ gives a free-time, fixed end point problem and a generalization of this is $S = [t_0, \infty) \times M$, where M is a surface (Manifold) in $\mathbb{R}^n$. A fixed-time, free-end point problem is obtained by taking $S = \{t_1\} \times \mathbb{R}^n$. A target set of the form $\{t_1\} \times \{x_1\}$ corresponds to the case of a fixed-time, fixed-end point problem. It is also possible to have moving target problem.

4.2 Maximum Principle

L. S. Portryagins' discovery of Maximum Principle in 1950's was an important milestone in optimal control theory and became a distinct field of research. He derived initially optimality conditions for the Meyer-type problem in reasonable generality by considering $x \in W^{1,1}([t_0, t], \mathbb{R}^n)$. The problem he considered in his formulation was applicable to a wide range of optimization problems. The techniques from traditional CVs were very limited. Indeed one can formally derive many results, but need to have more sophisticated tools to rigorously justify the results.

One of the major issues in classical CV Problems is the requirement of regularity assumptions like differentiability of H, L and f. For example, the differentiability of f with respect to the control is not at all reasonable and it is not necessary for existence and uniqueness results. Regularity also rules out functionals with $L(u) = |u|$.

If we follow the approach from CV, one need to consider small perturbations in both x and u. But it is possible to have closeness in the state trajectories even for large control perturbations and one would wish to capture optimality in this large perturbations of the control. Thus a less restrictive notion of closeness of controls would be welcome. For example, one may consider smallness of the controls in L^1-norm which will produce smallness of the trajectories in L^∞-norm. Final conclusion is that the maximum principle is a highly nontrivial extension of the variational approach. We now present the PMP in one example. We consider the fixed end-point problem with $K = 0$ terminal cost. Consider

$$J(u) = \int_{t_0}^{t_1} L(t, x(t), u(t))dt \tag{4.3}$$

where x, u satisfy the control system

$$\dot{x} = f(t, x(t), u(t)),\ x(t_0) = x_0,\ x(t_1) = x_1. \tag{4.4}$$

4.3 Pontryagin Maximum Principle (PMP)

Let $u^* : [t_0, t_1] \to U$ be an optimal control and $x^* : [t_0, t_1] \to \mathbb{R}^n$ be the corresponding optimal state trajectory. Then there exist a function $p^* : [t_0, t_1] \to \mathbb{R}^n$ and a constant $\lambda^* \leq 0$ satisfying $(\lambda^*, p^*(t)) \neq (0, 0)$ for all t and further

1) x^* and p^* satisfy the Hamiltons' canonical system

$$\begin{cases} \dot{x}^* = \mathcal{H}_p(x^*, u^*, p^*, \lambda^*), \; x^*(0) = x_0, x^*(t_1) = x_1 \\ \dot{p}^* = -\mathcal{H}_x(x^*, u^*, p^*, \lambda^*), \end{cases} \tag{4.5}$$

where $\mathcal{H} : \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is defined by $\mathcal{H}(x, u, p, \lambda) = \langle p, f(x, u)\rangle + \lambda L(x, u)$.

2) For each fixed t, the function $u \to \mathcal{H}(x^*(t), u, p^*(t), \lambda^*)$ has a global maximum at $u = u^*(t)$, i.e.,

$$\mathcal{H}(x^*(t), u^*(t), p^*(t), \lambda^*) = \max_{u \in U} \mathcal{H}(x^*(t), u, p^*(t), \lambda^*) \tag{4.6}$$

3) $H(x^*(t), u^*(t), p^*(t), \lambda^*) = 0$, for all $t \in [t_0, t_1]$.

Remark 4.1. The function p is known as *co-state* or *adjoint vector* which can be viewed as a generalization of Lagrange multipliers. Recall that we have a Lagrange multiplier associated to each constraint. Here, we have an ODE constraint. Intuitively, this can be interpreted as a constraint for each t and thus giving infinitely many (continuum) constraints. Thus, we need to have a Lagrange multiplier for each t which is given by $p(t)$ and we have to augment the cost functional with a term of the form $\displaystyle\int_{t_0}^{t_1} p(t) \cdot (\dot{x}(t) - f(t, x(t), u(t)))$ to get an unconstrained problem. This is done in the analysis.

Remark 4.2. Recall the Hamiltonian $H(t, x, p) = \max_{v \in \mathbb{R}^n} \{p \cdot v - L(t, x, v)\}$ defined for classical form of Hamiltons' equation.

One crucial difference is that in the classical case described earlier in section 3, the Hamiltonian is defined via maximization and then the equations for p and x are given. On the other hand, the process is reversed in PMP, where p and x are defined via an unmaximized Hamiltonian $\mathcal{H}$ and the maximization is a consequence. Postponing the supremum taking has a clear advantage in the analysis that in general the derived H (classical case) need not be differentiable as required in the derivation of Hamilton condition. But if L is smooth, then $\mathcal{H}$ is smooth for PMP.

Another advantage of PMP is that one can consider situations where the trajectories lie in some set U and hence it covers problems with differential constraints.

Remark 4.3. The Hamiltons' ODE defined for x^*, p^* is a $2n$-dimensional system of ODEs and $2n$ boundary conditions are defined through $x^*(t_0)$ and $x^*(t_1)$. But the situation changes if we consider variable-end point control problem. For example, consider the target set $S = [t_0, \infty) \times S_1$, where S_1 is a k-dimensional surface. Then the PMP satisfies (1), (2), (3) as earlier with $x^*(t_1) = x_1$ replaced by $x^*(t_1) \in S_1$ (k-conditions) with additional $n - k$ transversality conditions for p given by

$$\langle p^*(t_1), d\rangle = 0,$$

for $d \in T_{x^*(t_1)}S_1$, the tangent space to S_1 at $x^*(t_1)$.

Note that when $S_1 = \mathbb{R}^n$, we have $p^*(t_1) = 0$ whereas when $S_1 = \{x_1\},\ p^*(t_1)$ is free as in the earlier case.

We now present a simple example to show that even though PMP is a necessary condition, it can be used to get the optimal control.

Example 4.4. Consider a control system

$$\frac{dx}{dt} = x(t)u(t),\ x(0) = x_0$$

with the cost functional $J(u) = \int_0^1 (1 - u(t))x(t)$. Here $x(t)$ represents the output of a factory, where we consume a part of $x(t)$ and reinvest the rest. Let $0 \leq u(t) \leq 1$ be the fraction reinvested and J represent the maximization of the consumption. For this model, $f(x,u) = xu,\ L(x,u) = (1-u)x,\ H(x,p,u) = f(x,u)p + L(x,u) = x + ux(p-1)$. The state and co-state equations are given by

$$\dot{x}(t) = \mathcal{H}_p = u(t)x(t),\ \dot{p}(t) = -\mathcal{H}_x = -1 - u(t)(p(t)-1)$$

with the conditions $x(0) = x_0,\ p(t_1) = 0$.

The maximum principle is given by

$$\mathcal{H}(x(t), p(t), u(t)) = \max_{0 \leq u \leq 1} \left(x(t) + u(p(t) - 1) \right).$$

Thus, we have

$$u(t) = 1 \text{ if } p(t) > 1,\ u(t) = 0 \text{ if } p(t) < 1.$$

To design an optimal control, we need to know $p(t)$. Now we can use the simple nature of the above equations. Since $p(t_1) = 0$, by continuity $p(t) \leq 1$ for t near $t_1,\ t \leq t_1$. Hence $u(t) = 0$ for these values of t at which $\dot{p}(t) = -1$ which implies $p(t) = t_1 - t \leq 1$. Thus for $t_1 - 1 \leq t \leq t_1,\ \ u(t) = 0$. Now t near $t_1 - 1,\ t < t_1 - 1,\ \dot{p}(t) = -p(t)$ with $p(t_1 - 1) = 1$ which can be solved to get p. This is an optimal control of *bang-bang type*.

4.4 Dynamic Programming Principle and HJB

More or less at the same time Pontryagin derived PMP in Soviet union, Bellman in US had different ideas to study optimal control problems. The idea was to consider not only the given fixed initial state x_0, at time t_0, but vary the initial state and time in the neighborhood (or everywhere) and consider all minimal values and view it as a function of t_0, x_0. This lead to the concept of *value function*. This is really important on two counts. First of all this is the way one can understand the *intrinsic* *properties of minimal values*. Secondly, in practical applications the values are always approximations and hence (like disturbances/approximations/errors etc.) it would be better to understand minimal values as a variable of initial values and initial time. This approach is known as *dynamic programming* and leads not only to necessary conditions, but it gives sufficient conditions as well. Further, the value function, at least in the smooth situation can be realized as the solution of the so-called *Hamilton-Jacobi-Bellman (HJB)* equations. Hence the rich theory of PDEs could be used.

The Dynamic Programming Principle which we will state shortly roughly states that *if we have an optimal trajectory starting at an initial state, take any point on the trajectory and treat the same OC problem with the selected point as the initial state, then the remaining part of the trajectory will be optimal for the latter problem.* A motivational example in the discrete case is given below.

Example 4.5. (Motivation): Suppose, we wish to go from an initial state x_0 to x_1, then to x_2 and so on to x_N. Assume the state x_k belongs to a finite set X of cardinality l, the cost to go from x_k to x_{k+1} is represented by u_k which may belong to a finite set U with cardinality of M. There will also be a terminal cost at x_N.

From x_k to x_{k+1}, there are M choices for the cost and thus the number of trajectories is $O(M^N N)$ and the total number of computations will be $O(M^N Nl)$. In other words, we have to look for all possible options and look for the trajectory with the least cost.

We now apply a backward procedure. Starting from the level $N-1$ at each stage k compute the optimal cost to go to $k+1$. But at $k+1$, the optimal cost is already computed as a terminal cost and one need not have to compute the cost in other trajectories. The Figure 2.2 will give a better understanding. The number of operations required is $O(MNl)$. In the forward approach, we are not able to discard any trajectories, whereas in the backward iteration, one can skip many trajectories. The idea is to calculate the optimal cost-to-go.

4.5 The Value Function and DPP

We consider the cost functional

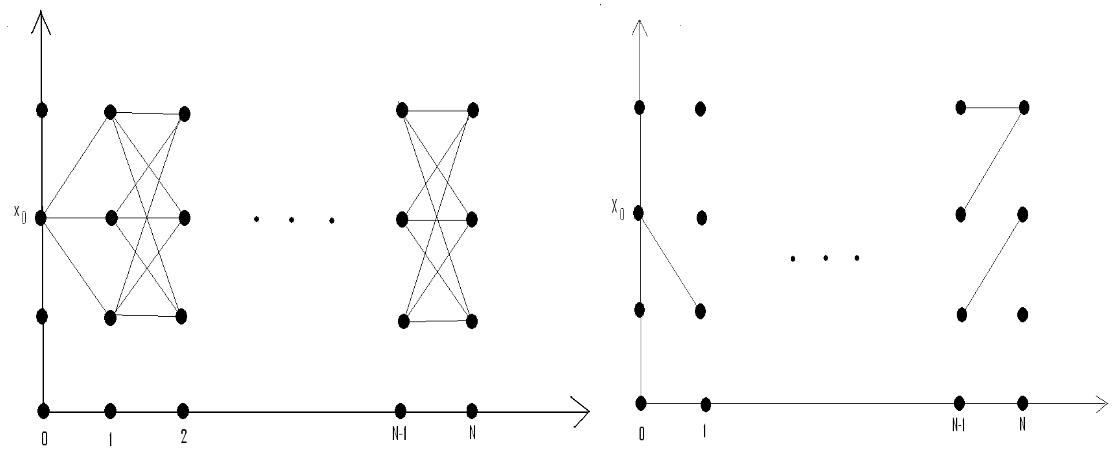
$$J(u) = J(t_0, x_0, u) = \int_{t_0}^{t_1} L(t, x(t), u(t))dt + K(x(t_1)) \tag{4.7}$$

The idea is to vary t_0 and x_0 and introduce the family of minimization problems

$$J(t, x, u) = \int_{t}^{t_1} L(s, x(s), u(s))dt + K(x(t_1)), \tag{4.8}$$

where $t \in [t_0, t_1),\ x \in \mathbb{R}^n$ and $x(s)$ is the solution to the ODE with the initial condition $x(t) = x$ solved for $s \in [t, t_1]$. Introduce, the value function $V : [t_0, t_1) \times \mathbb{R}^n \to \mathbb{R}$ as

$$V(t, x) = \inf_u J(t, x, u) \tag{4.9}$$



which is the optimal cost to go from state x at time t to the state x_1 at time t_1 (cost-to-go). Indeed V satisfies the final condition

$$V(t_1, x) = K(x), \text{ for all } x \in \mathbb{R}^n \tag{4.10}$$

Theorem 4.1 (DPP). *For every $(t, x) \in [t_0, t_1) \times \mathbb{R}^n$ and every $\tau \in (t, t_1)$, the value function V satisfies*

$$V(t, x) = \inf_u \left\{ \int_t^\tau L(s, x(s), u(s)) ds + V(\tau, x(\tau)) \right\} \tag{4.11}$$

where $x(s)$ is the trajectory corresponding to the control u.

When an optimal control and trajectory exist, then the inf is achieved at the optimal solution and it will become a minimum. The infinitesimal version of DPP is the *Hamilton-Jacobi-Bellman* equation which is a first order PDE satisfied by V and is given in $[t_0, t_1) \times \mathbb{R}^n$by

$$V_t(t, x) = \sup_{u \in U} \{-L(t, x, u) - V_x(t, x) \cdot f(t, x, u)\}, \; V(t_1, x) = K(x) \tag{4.12}$$

Remark 4.6. Note that, the terminal cost appears only at the final condition, not in the PDE. The equation can be derived using Taylor's expansion if the involving quantities V, L, f are smooth.

The following result is a *verification theorem (sufficient condition)* to verify a given $(\bar{x}, \bar{u})$ satisfying the ODE constraint is optimal or not.

Theorem 4.2. *Let $(\bar{x}, \bar{u})$ be a solution to the ODE of the associated OC problem. Suppose $\phi \in C^1$ is a solution to* (4.12) *that satisfies: $\phi(t_1, x) = K(x)$ and*

$$\phi_x(t, \bar{x}(t)) \cdot f(t, \bar{x}(t), \bar{u}(t)) + L(t, \bar{x}(t), \bar{u}(t)) = \min_{u \in U} \{\phi_x(t, \bar{x}) \cdot f(t, \bar{x}(t), u) + L(t, \bar{x}(t), u)\} \tag{4.13}$$

which is equivalent to the Hamiltonian maximization. Then

1. $(\bar{x}, \bar{u})$ *is an optimal solution*

2. $\phi(t_0, x_0)$ *is the minimum cost.*

Remark 4.7. The natural candidate for verification function is the value function V. Here is a theorem.

Theorem 4.3. *Let V be the value function defined earlier and assume that V is continuously differentiable. Further the minimization problem has a minimizer with continuous control function. Then V is a solution to* (4.12).

Comments:

1. Indeed HJB has its roots in Hamilton-Jacobi equations in the context of calculus of variations due to Hamilton and Jacobi in 1830s where they have viewed it as necessary conditions. It is used as a sufficient conditions is due to Caratheodory. In the case of HJB equation, Kalman derived the sufficient conditions for optimal control combining the ideas of Caratheodory and Bellman.

2. Compared to the necessary condition PMP, the progress of DPP was slow as it lacked rigorous foundation in 1960s since it heavily depends on the regularity of V. But unfortunately, the value function quite often is not smooth and the concept of weak solution using distributional theory is not suited to the nonlinear HJB equations. A rigorous treatment had begun much later in 1970's via *non-smooth analysis* and latter the concept of *viscosity solution* was introduced.

   We will not touch upon these concepts, but you may learn elsewhere in the school. We, now present the connection between PMP and HJB.

4.6 HJB and Maximum Principle

Recall that PMP is formulated using the canonical equations

$$\dot{x}^* = \mathcal{H}_p, \ \dot{p}^* = -\mathcal{H}_x \tag{4.14}$$

and says that at each time t, the value $u^*(t)$ of the optimal control must maximize $\mathcal{H}(x^*(t), u, p^*(t))$ with respect to u; that is

$$u^*(t) = \arg\max_{u \in U} \mathcal{H}(x^*(t), u, p^*(t)) \tag{4.15}$$

This is an *open-loop control* because to compute u^*, we need not only the state x^*, but need co-state p^* as well which is given from the adjoint equation.

Let us see the Hamiltonian associated with HJB. The inf in the equation (4.12) becomes a minimum for the optimal control. Thus we get

$$H(t, x^*(t), u^*(t), -V_x(t, x^*(t))) = \max_{u \in U} H(t, x^*(t), u, -V_x(t, x^*(t)), \tag{4.16}$$

where H is given by $H(t,x,u,p) := p\cdot f(t,x,u) - L(t,x,u)$. Thus, the optimal control must satisfy (assume H is independent of t)

$$u^*(t) = \arg\max_{u\in U} H(x^*(t), u, -V_x(t, x^*(t))). \tag{4.17}$$

This is a *closed-loop (feed-back specification)* quite useful in applications. If we know the value everywhere, u^* is determined by the current state $x^*(t)$. *This feature of generating an optimal control policy from the state is the trade mark of Dynamic programming approach.* However, the major draw back is that V is obtained by solving HJB which is a very difficult task. Though this approach is more novel than PMP, the later one involves solving only ODEs and hence computationally more fruitful.

Another question is the derivation of PMP from HJB. The equations (4.15) and (4.17) suggests that we should look for the co-state of the form $p^*(t) := -V_x(t, x^*(t))$. The PMP follows if one proves that p^* satisfies the second equation in (4.14) which can be established as usual in the smooth case.

4.7 The Linear Quadratic Regulator (LQR)

The well-known LQR problem is the OC problem with linear constraint $\dot{x}(t) = A(t)x + B(t)u,\ \ x(t_0) = x_0,\ x(t) \in \mathbb{R}^n,\ \ u \in \mathbb{R}^m$ with the quadratic cost

$$J(u) = \int_{t_0}^{t_1} \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right] dt + x^T(t_1)Mx(t_1)$$

where $Q(\cdot), R(\cdot), M$ are matrices with

$$M = M^T \geq 0,\ \ Q(t) = Q^T(t) \geq 0 \text{ and } R(t) = R^T > 0.$$

Here $f(t,x,u) = A(t)x + B(t)u$ and $L(t,x,u) = x^TQ(t)x + u^TR(T)u$. All the results defined above are applicable in this case, but one can get further information. The Hamiltonian is given by

$$H(t,x,u,p) = p^TA(t)x + p^TB(t)u - x^TQ(t)x - u^TR(t)u.$$

Some computation will yield that an optimal control must satisfy

$$u^*(t) = \frac{1}{2}R^{-1}(t)B^T(t)p^*(t)$$

Further since $H_{uu} = -2R(t) < 0$, the above control maximizes the Hamiltonian globally.

Interestingly, one can do a further analysis on the co-state equation to get

$$p^*(t) = -2P(t)x^*(t)$$

where $P(t)$ is some matrix valued function to be determined. But the crucial point is that p^* and x^* is related by a linear relation and u^* then takes the form

$$u^*(t) = -R^{-1}(t)B^T(t)P(t)x(t)$$

which is a *linear state feed back law.*

One can also deduce that $P(t)$ will satisfy linear matrix differential equation

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) - Q(t) + P(t)B(t)R^{-1}(t)B^T(t)P(t)$$

which is known as *Riccati Differential Equation (RDE).*

This article is based on the books by Richard Vinter and Daniel Liberzon. The reader can also refer the notes by L.C. Evans. There are many other interesting reference books accordingly to the interest of the readers. Viscosity solutions are nicely treated in Bardi and Dolcetta.

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