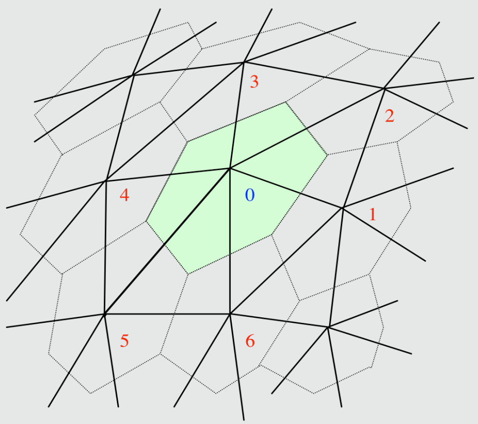
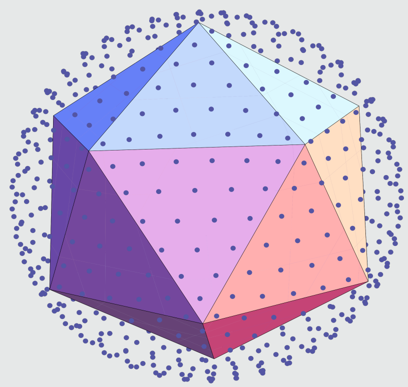
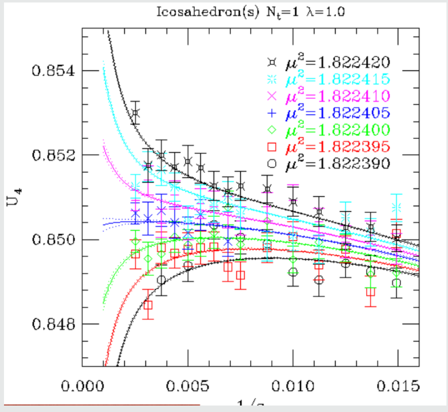
LATTICE FIELD THEORY FOR CFT on CURVED RIEMANN MANIFOLD







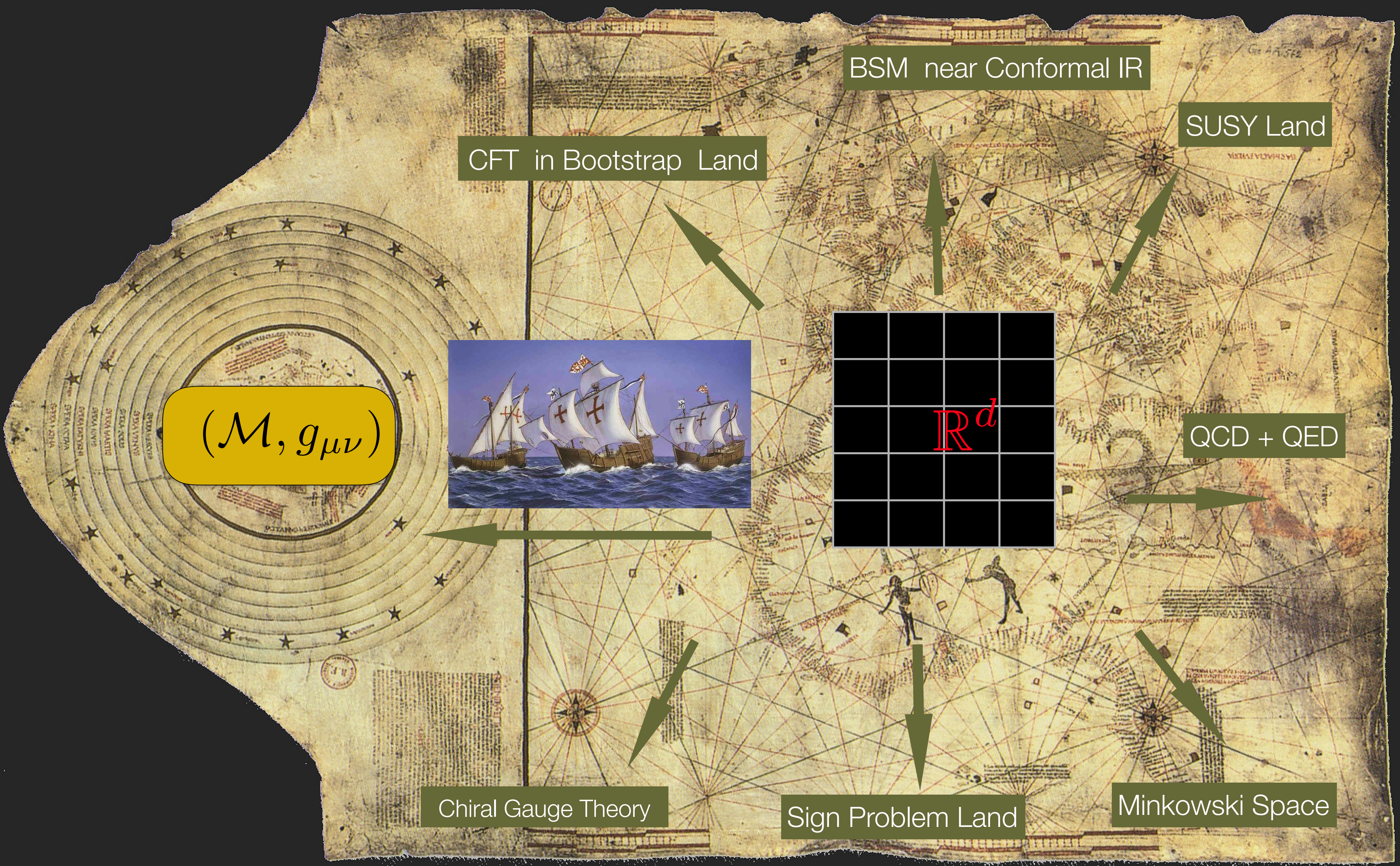
REGGE + FINITE ELEMENT (FEM) + QUANTUM CONTER TERM = QFE

Rich Brower, Boston University ICTS Bangalore Feb. 3, 2018

with G. Fleming, A. Gasbarro, T. Raben, C-I Tan, E. Weinberg

Nonperturbative and Numerical Approaches to Quantum Gravity, String Theory and Holography

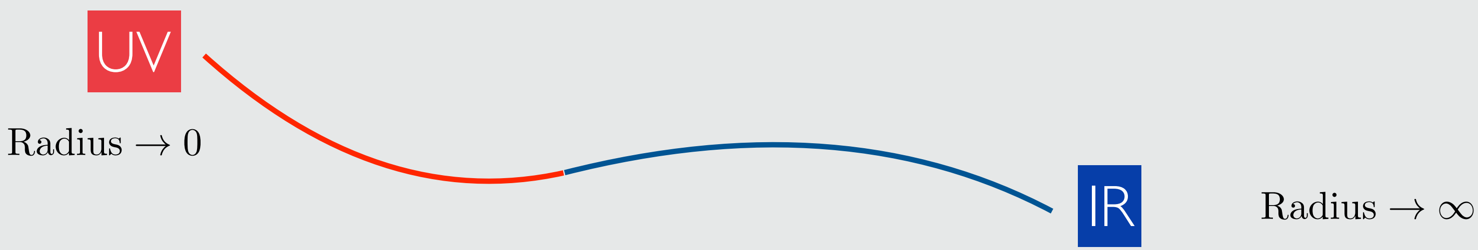
BREAKING OUT OF FLAT LAND LATTICE FIELD THEORY



Christopher Colombus map c.1490.

SPHERES AND CYLINDERS ARE NICE*

- Spheres and Cylinders are Weyl trans* & CFT are "preserved".
- Sphere: For CFT, no finite volume approx & define: "c-theorems"
- Cylinders: Radial Quantized: Bndry of global AdS (H = Dilatations)



$\mathbb{R}^d \to \mathbb{S}^d$

$$ds^2_{flat} = \sum_{\mu=1}^{d} dx^\mu dx^\mu = e^{\sigma(x)} d\Omega^2_d \xrightarrow{Weyl} d\Omega^2_d .$$

$\mathbb{R}^{d+1} \to \mathbb{R} \times \mathbb{S}^d$

$$ds^2_{flat} = \sum_{\mu=1}^{d+1} dx^\mu dx^\mu = e^{2t}(dt^2 + d\Omega^2_d) \xrightarrow{Weyl} (dt^2 + d\Omega^2_d) \; .$$

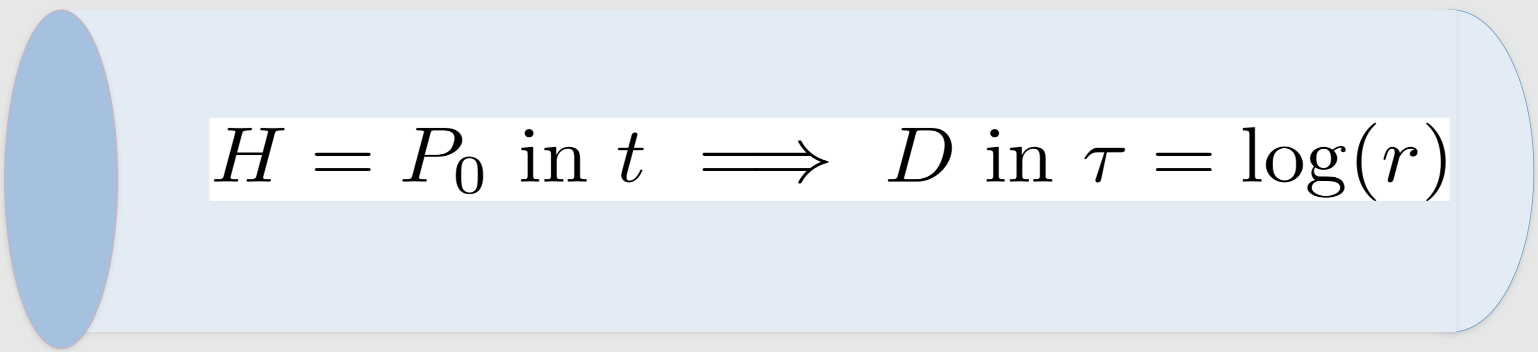
	$\mathbb{R}^d$	$\mathbb{S}^d$	$\mathbb{H}^d$
*MAX SYM SPACE ARE NICE:	Flat,	de Sitter	AdS

ORIGNINAL MOTIVATON: Radial Quantization

Conformal (near conformal) for

- BSM composite Higgs
- AdS/CFT weak-strong duality
- Model building & Critical Phenomena in general

$$\mathbb{R} \times \mathbb{T}^3 \qquad \text{vs} \qquad \mathbb{R} \times \mathbb{S}^3$$



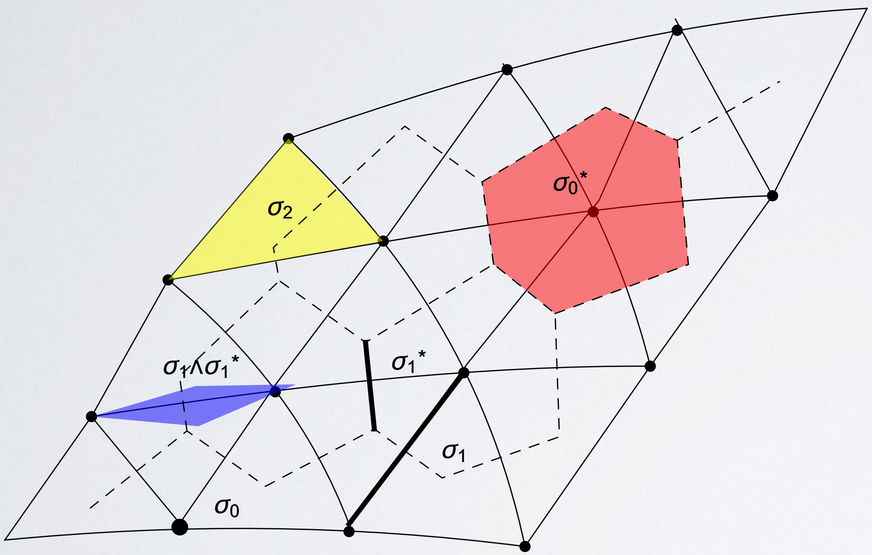
$$H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$$

Potential advantage: Scales increases exponentially in lattice size L!

$$1 < t < aL \implies 1 < \tau = log(r) < L$$

REGGE: Piecewise linear metric

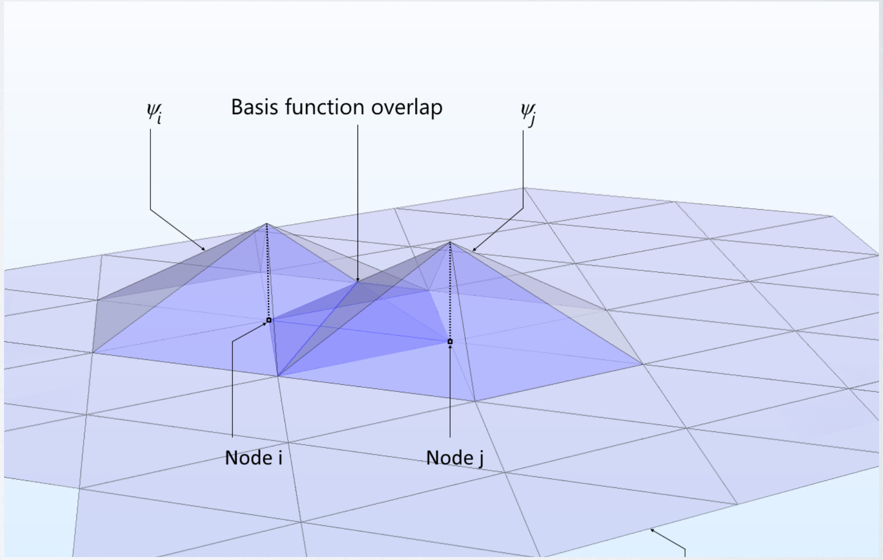
$$(\mathcal{M}, g_{\mu\nu}(x)) \leftrightarrow (\mathcal{M}_\sigma, g_\sigma = \{l_{ij}\})$$



Simplicial Complex/Delaunay Dual Complex + Regge flat metric on each Simplex

FEM: Piecewise linear fields

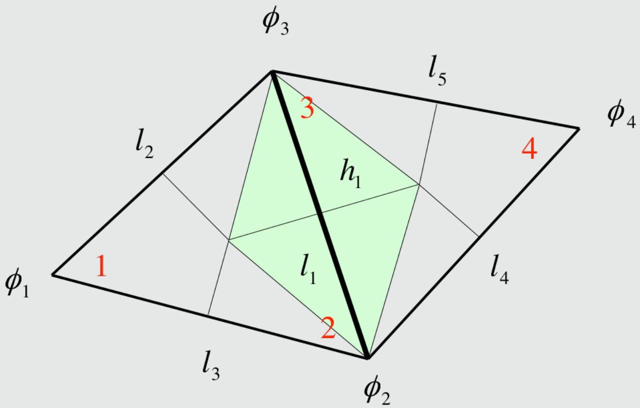
$$\phi(x) \leftrightarrow \phi = \sum_i \phi_i W_i(\xi)$$

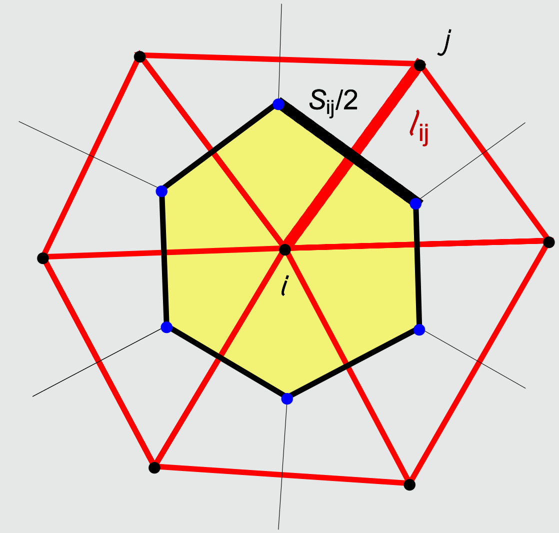


Actually fancier methods: Discrete Exterior Calculus (scalar), Spin connection (Fermion), Wilson links (gauge) , etc.

PART I: FREE THEORIES

REGGE CALCULUS FEM FORMULATION





LINEAR FEM/ REGGE CALCULUS *

$$FEM : A_d \frac{(\phi_2 - \phi_3)^2}{l_1^2}$$

$$*d*d\phi_i$$

$$\text{Delaunay Link Area:} \quad A_d = h_1 l_1$$

DISCRETE EXTERIOR CALCULUS or CHRIST FRIEBERG & LEE

SIMPLICIAL ALGEBRA

Building a Simplicial Complex $\mathcal{S}$ $\quad \sigma_0 \to \sigma_1 \to \sigma_2 \to \cdots \to \sigma_D$

Circumcenter Dual Lattice $\mathcal{S}^*$ $\quad \sigma_0^* \leftarrow \sigma_1^* \leftarrow \sigma_2^* \leftarrow \cdots \leftarrow \sigma_D^*$

boundary operator:

$$\partial\sigma_n(i_0 i_1 \cdots i_n) = \sum_{k=0}^{n} (-1)^k \sigma_{n-1}(i_0 i_1 \cdots \widehat{i}_k \cdots i_n)$$

Orthogonality and proper hybrid tiling :

$$V_{nn^*} = \langle\, \sigma_n | \sigma_n^* \rangle = \int \sigma_n \wedge \sigma_n^* = \frac{n!(D-n)!}{D!} |\sigma_n||\sigma_n^*|$$

Add matter $\quad$ n form ω_n on σ_n

Stokes Theorem for Exterior Derivative

$$\int_{\sigma_n} d\omega(y) = \int_{\partial\sigma_n} \omega(y) \quad \text{or} \quad \langle\, \sigma_n | d\omega \rangle = \langle\, \partial\sigma_n | \omega \rangle$$

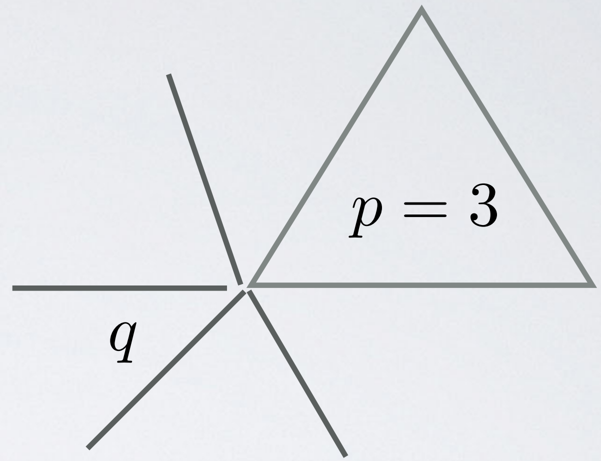
$$\delta = *d* \quad \Longrightarrow \quad \delta^2 = d^2 = 0$$

Beltrami-Laplace $\quad (\delta + d)^2 \phi = *d * d\phi$

Kahler-Dirac $\quad (\delta + d)\psi = 0$

REGULAR TRIANGULATIONS!

Triangle case



Preserves Discrete
Subgroup of Isometries

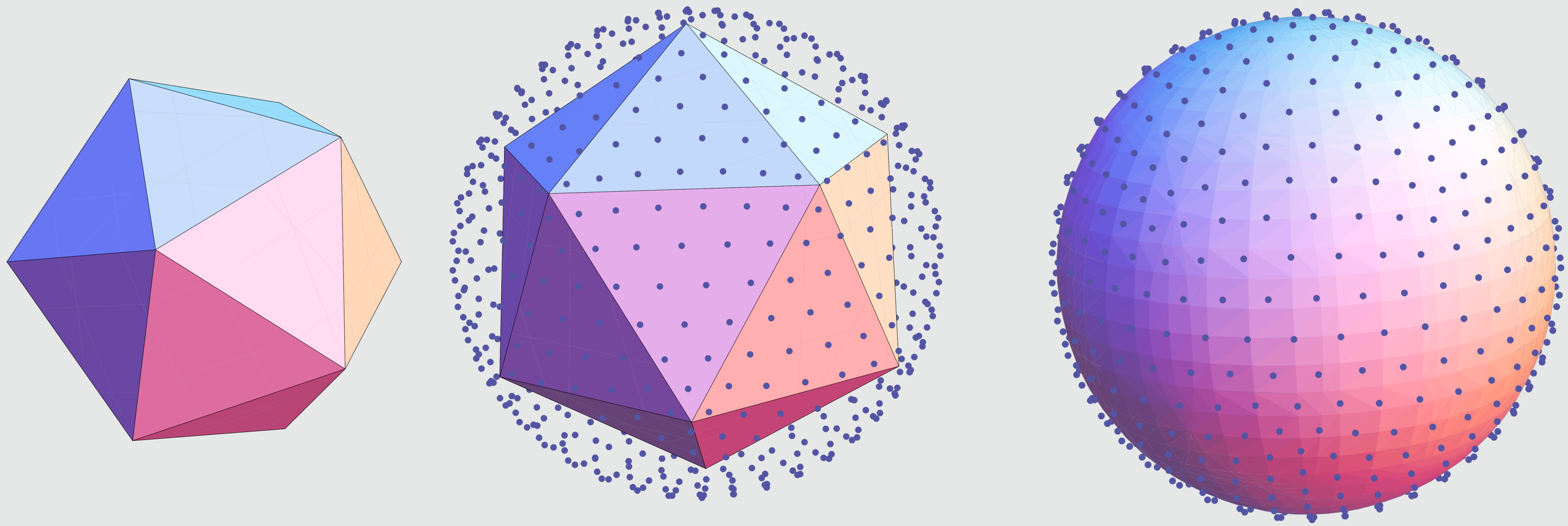
$$\frac{1}{p} + \frac{1}{q} > 1/2 \qquad \text{de Sitter} \quad \mathbb{S}^2 \qquad \text{vertex} \quad q = 3, 4, 5$$

$$\frac{1}{p} + \frac{1}{q} = 1/2 \qquad \text{flat} \quad \mathbb{T}^2 \qquad \text{vertex} \quad q = 6$$

$$\frac{1}{p} + \frac{1}{q} < 1/2 \qquad \text{Hyperbolic} \quad \mathbb{A}dS^2 \qquad \text{vertex} \quad q = 7, 8, 9, \cdots$$

Start with maximum regular Tesselation

$s = 1$ $s = 8$



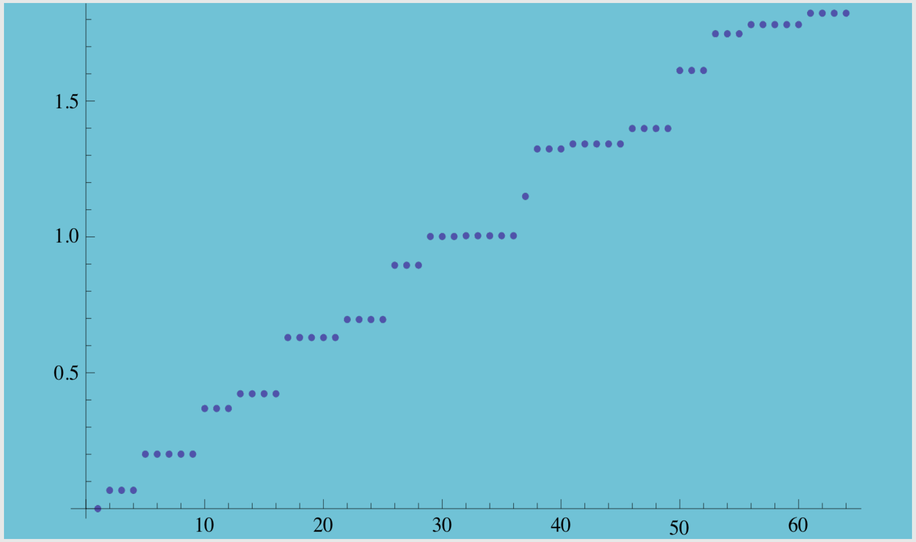
l = 0 (A),1 (T1) , 2 (H) are irreducible 120 Icosahedral subgroup of O(3)

$1/p + 1/q > 1/2$ for regular positive curvature tessellation

FEM FIXES SPECTRAL DEFECTS OF LAPLACIAN ON SPHERE

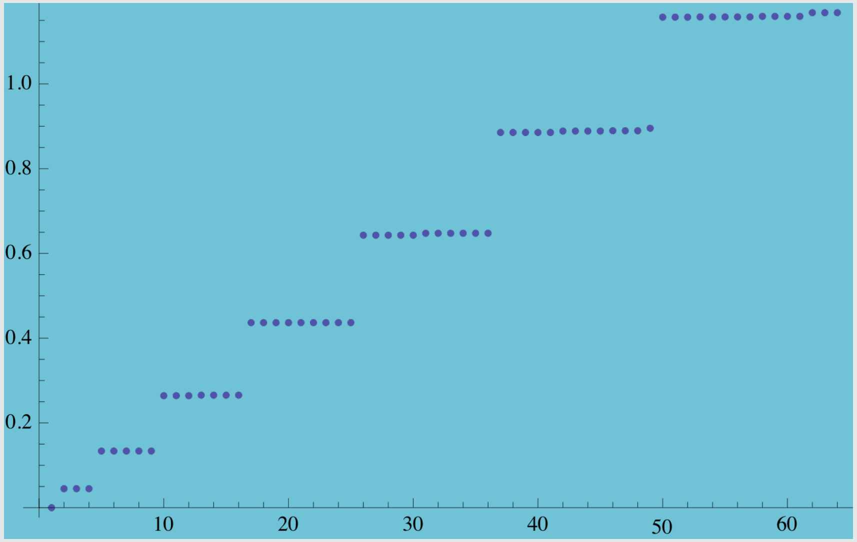
For s = 8 first (l+1)*(l +1) = 64 eigenvalues

BEFORE FEM



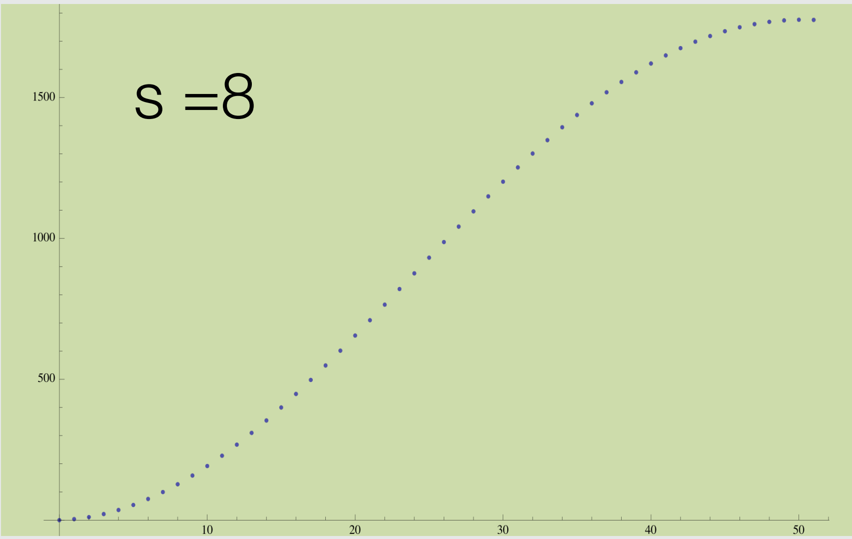
l, m

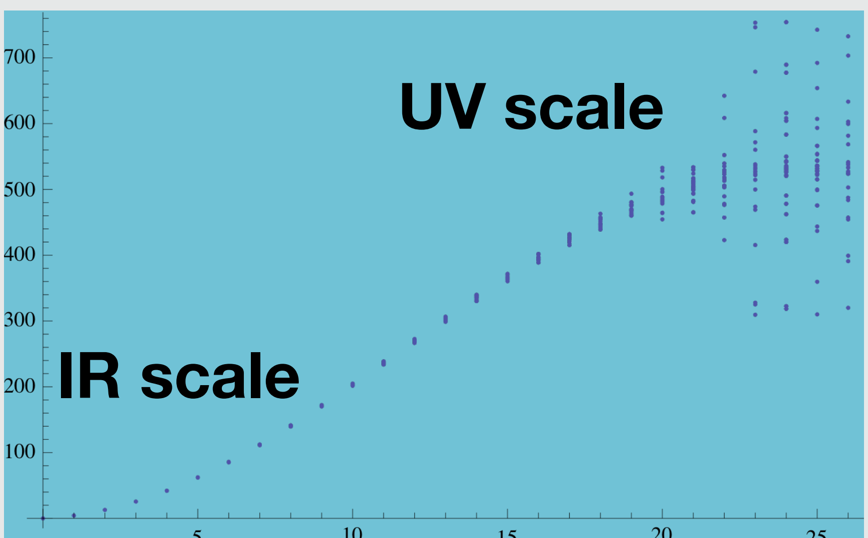
AFTER FEM

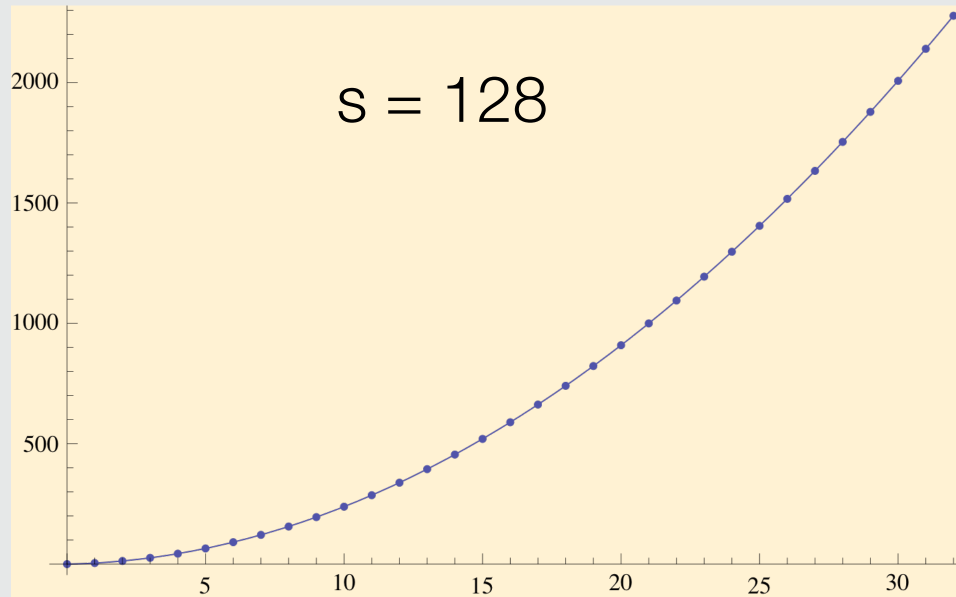


l, m

SPECTRUM OF FEM LAPLACIAN ON A SPHERE







Fit

$$l + 1.00012\, l^2 - 13.428110^{-6}\, l^3 - 5.5724410^{-6}\, l^4$$

Dirac ON SIMPLIAL MANIFOLD

$$S = \frac{1}{2}\int d^D x\sqrt{g}\bar{\psi}[\mathbf{e}^\mu(\partial_\mu - \frac{i}{4}\boldsymbol{\omega}_\mu(x)) + m]\psi(x)$$

$\mathbf{e}^\mu(x) \equiv e^\mu_a(x)\gamma^a$ Verbein & Spin connection*

$\boldsymbol{\omega}_\mu(x) \equiv \omega^{ab}_\mu(x)\sigma_{ab}$, $\sigma_{ab} = i[\gamma_a, \gamma_a]/2$

(1) New spin structure "knows" about intrinsic geometry

(2) Need to avoid simplex curvature singularities at sites.

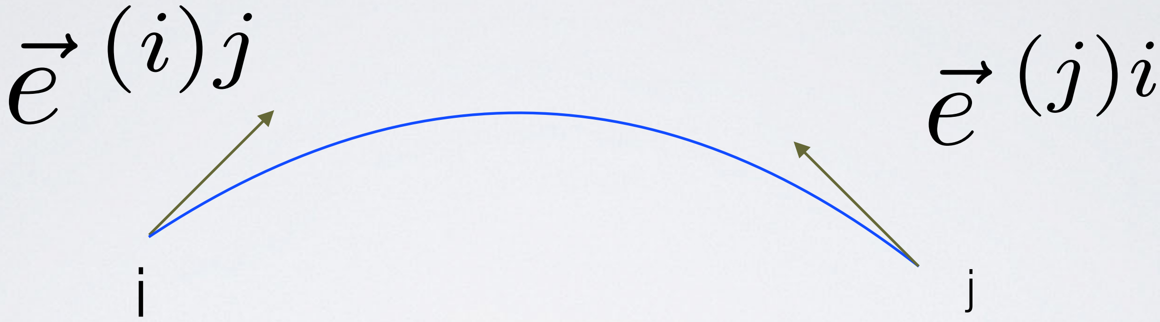
(3) Spinors rotations (Lorentz group) is double of O(D).

$$e^{i(\theta/2)\sigma_3/2} \to -1 \qquad \text{as} \quad \theta \to 2\pi$$

---

* Must satisfy the tetrad postulate! $\omega^{ab}_\mu = \frac{1}{2}e^{\nu[a}(e^{b]}_{\nu,\mu} - e^{b]}_{\mu,\nu} + e^{b]\sigma}e^c_\mu e_{\nu c,\sigma})$.

$$S_{naive} = \frac{1}{2}\sum_{\langle i,j\rangle}\frac{V_{ij}}{l_{ij}}[\bar{\psi}_i\vec{e}^{\,(i)j}\cdot\vec{\gamma}\Omega_{ij}\psi_j - \bar{\psi}_j\Omega_{ji}\vec{e}^{\,(i)j}\cdot\vec{\gamma}\psi_i] + \frac{1}{2}mV_i\bar{\psi}_i\psi_i$$



DO NOT USE PIECEWISE LINEAR REGGE CALCULUS MANIFOLD!

Simplicial Tetrad Hypothesis

$$e_a^{(i)j}\gamma^a\Omega_{ij} + \Omega_{ij}e_a^{(j)i}\gamma^a = 0$$

Gauge Invariance under Spin(D) transformations

$$\psi_i \to \Lambda_i\psi \quad , \quad \bar{\psi}_j \to \bar{\psi}_j\Lambda_j^\dagger \quad , \quad \mathbf{e}^{(i)j} \to \Lambda_i\mathbf{e}^{(i)j}\Lambda_i^\dagger \quad , \quad \Omega_{ij} \to \Lambda_i\Omega_{ij}\Lambda_j^\dagger$$

WILSON/CLOVER TERM

$$[\gamma_\mu(\partial_\mu - iA_\mu)]^2 = (\partial_\mu - iA_\mu)^2 - \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}\ ,$$

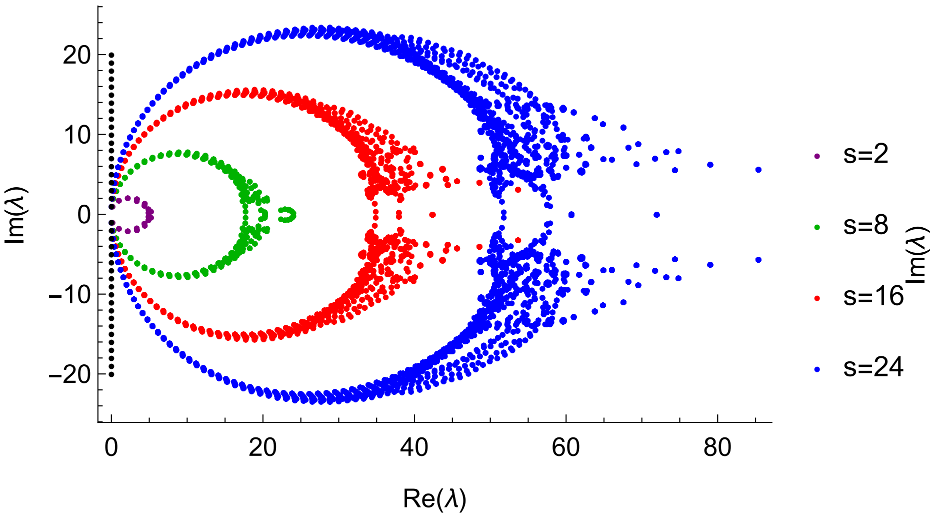
↓

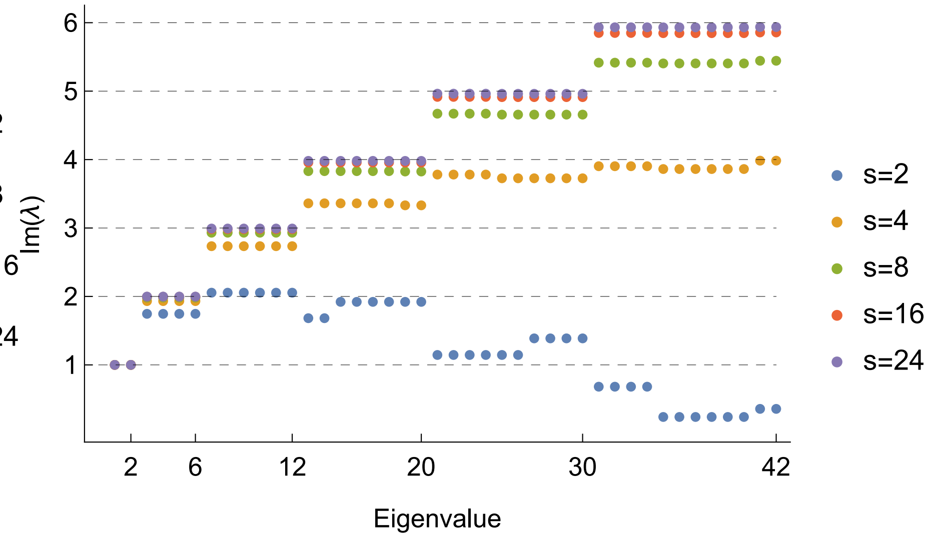
$$[\mathbf{e}^\mu_a(\partial_\mu - i\boldsymbol{\omega}_\mu)]^2 = \frac{1}{\sqrt{g}}\boldsymbol{D}_\mu\sqrt{g}g^{\mu\nu}\boldsymbol{D}_\nu - \frac{1}{2}\sigma^{ab}e^\mu_a e^\nu_b \boldsymbol{R}_{\mu\nu}$$

↓

$$S_{Wilson} = \frac{r}{2}\sum_{\langle i,j\rangle}\frac{aV_{ij}}{l_{ij}^2}(\bar{\psi}_i - \bar{\psi}_j\Omega_{ji})(\psi_i - \Omega_{ij}\psi_j)$$

2D DIRAC SPECTRA ON SPHERE





$E = j + 1/2$

Exact is integer spacing for $j = 1/2, 3/2, 5/2$ … Exact degeneracy $2j + 1$: No zero mode in chiral limit!.

https://arxiv.org/abs/1610.08587

Lattice Dirac Fermions on a Simplicial Riemannian Manifold

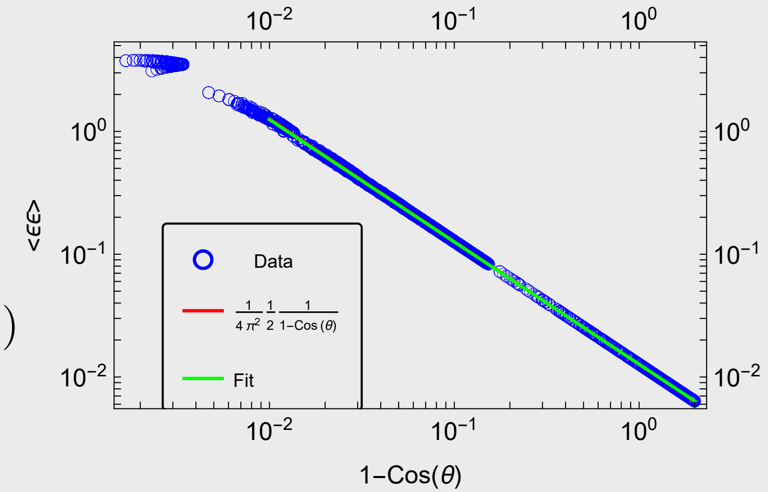
Richard C. Brower, George T. Fleming, Andrew D. Gasbarro, Timothy G. Raben, Chung-I Tan, Evan S. Weinberg

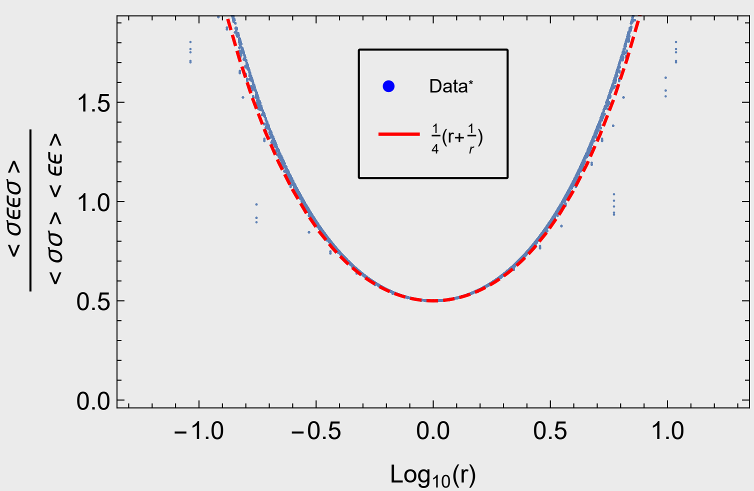
FREE MAJORANA FERMIONS ON S2

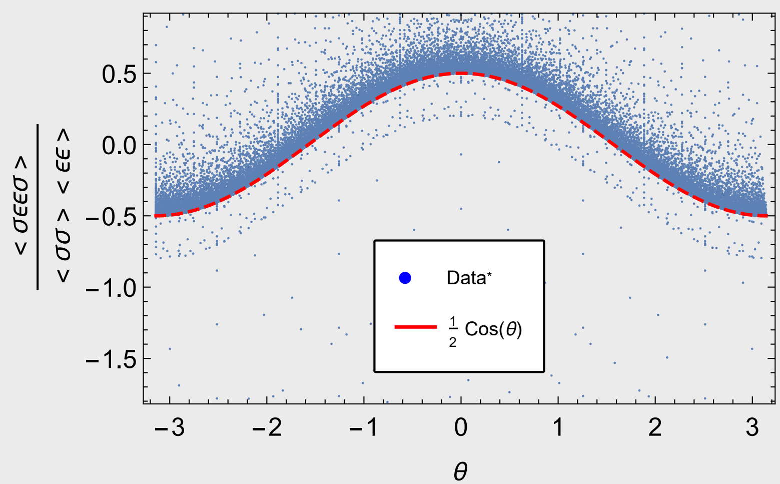
$$\langle \psi(z_1)\bar{\psi}(z_1)\bar{\psi}(z_1)\psi(z_2)\rangle = \left[\frac{1}{\partial}\right]_{z_1,z_2}\left[\frac{1}{\bar{\partial}}\right]_{z_1,z_2} = \frac{1}{4\pi^2}\frac{1}{|z_1 - z_2|^2}$$

$$\frac{\langle \sigma(0)\epsilon(z_2)\epsilon(z_3)\sigma(\infty)\rangle}{\langle \epsilon(z_2)\epsilon(z_3)\rangle}$$

$$= \frac{1}{4}|\sqrt{z_1/z_2} + \sqrt{z_2/z_1}|^2 = \frac{1}{4}(r + 1/r + 2\cos\theta)$$







Non-Abelian Gauge Fields

$$S_\sigma = \frac{1}{2g^2 N_c} \sum_{\triangle_{ijk}} \frac{V_{ijk}}{A_{ijk}^2} Tr[2 - U_{\triangle_{ijk}} - U^\dagger_{\triangle_{ijk}}]$$

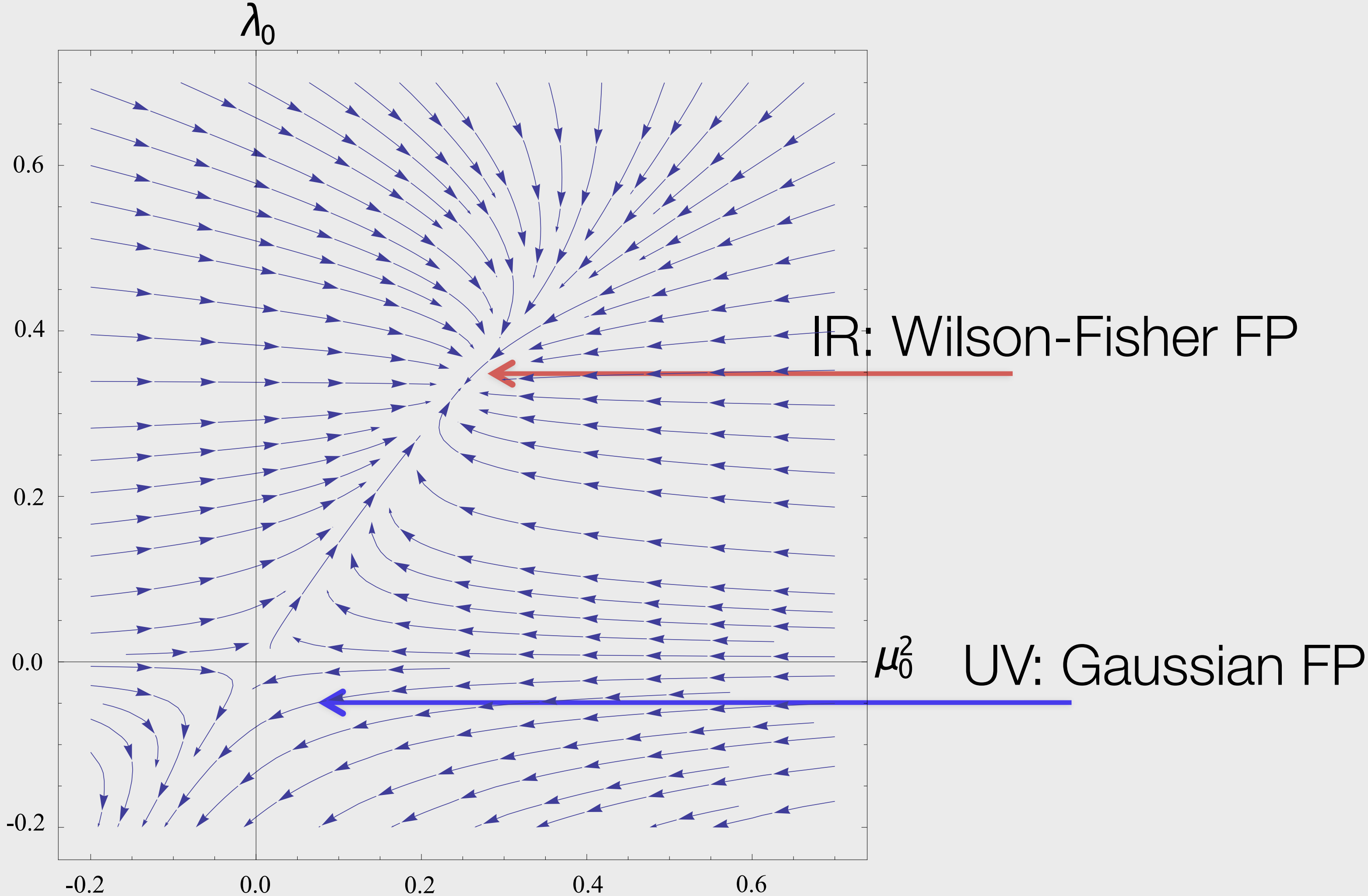
Where

$$U_{\triangle_{ijk}} = U_{ij} U_{jk} U_{ki}$$
$$A_{ijk} = |\sigma_2(ijk)|$$
$$V_{ijk} = |\sigma_2(ijk) \wedge \sigma_2^*(ijk)|$$

See N. Christ, R. Friedberg and T.D. Lee
Weights of Links and Plaquettes in a Random Lattice Nucl. Phys. B210 (1982).

PARTII: INTERACTION THEORIES

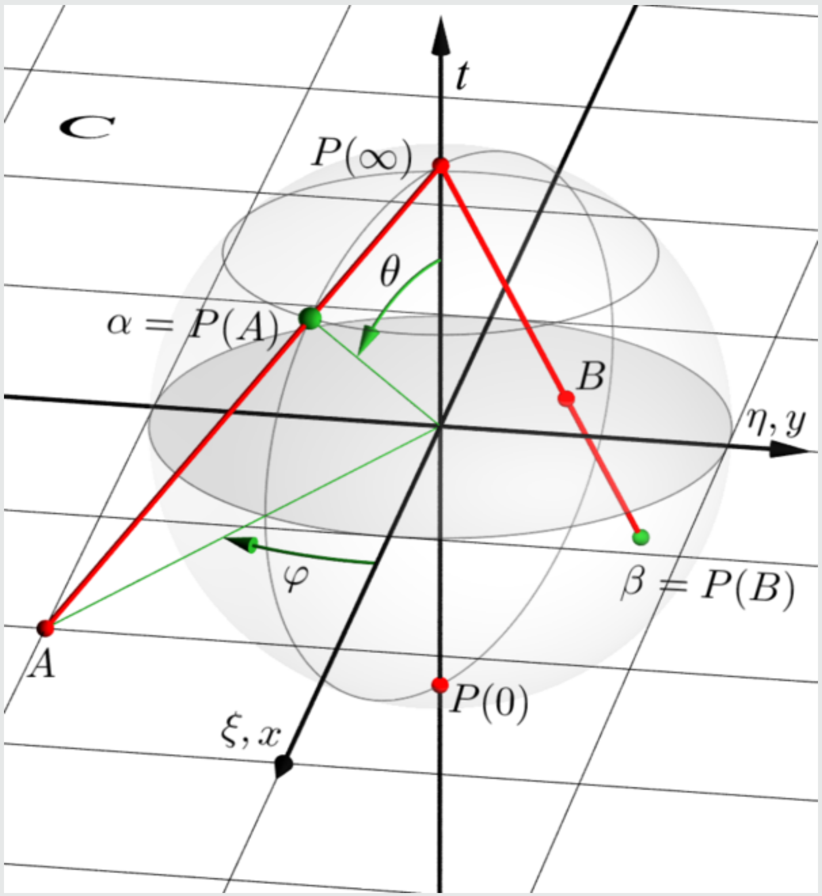
TEST CFT: PHI 4TH AT WILSON-FISHER FIXED POINT IN 2D & 3D.



$$\mathcal{L}(x) = -\frac{1}{2}(\nabla\phi)^2 + \lambda(\phi^2 - \mu^2/2\lambda)^2$$

TEST 2D ISING/PHI 4TH ON THE RIEMANN SPHERE

Stereographic project of Complex Plane:



$$\xi = \tan(\theta/2)e^{-i\phi} = \frac{x+iy}{1+z}$$

$$|\xi| = \sqrt{\xi_1^2 + \xi_2^2} \qquad \xi = \xi_1 + i\xi_2$$

$$\vec{r} = (x, y, z) \qquad \vec{r}\cdot\vec{r} = 1$$

$$|\vec{r}_1 - \vec{r}_2| = 2 - 2\cos(\theta_{12})$$

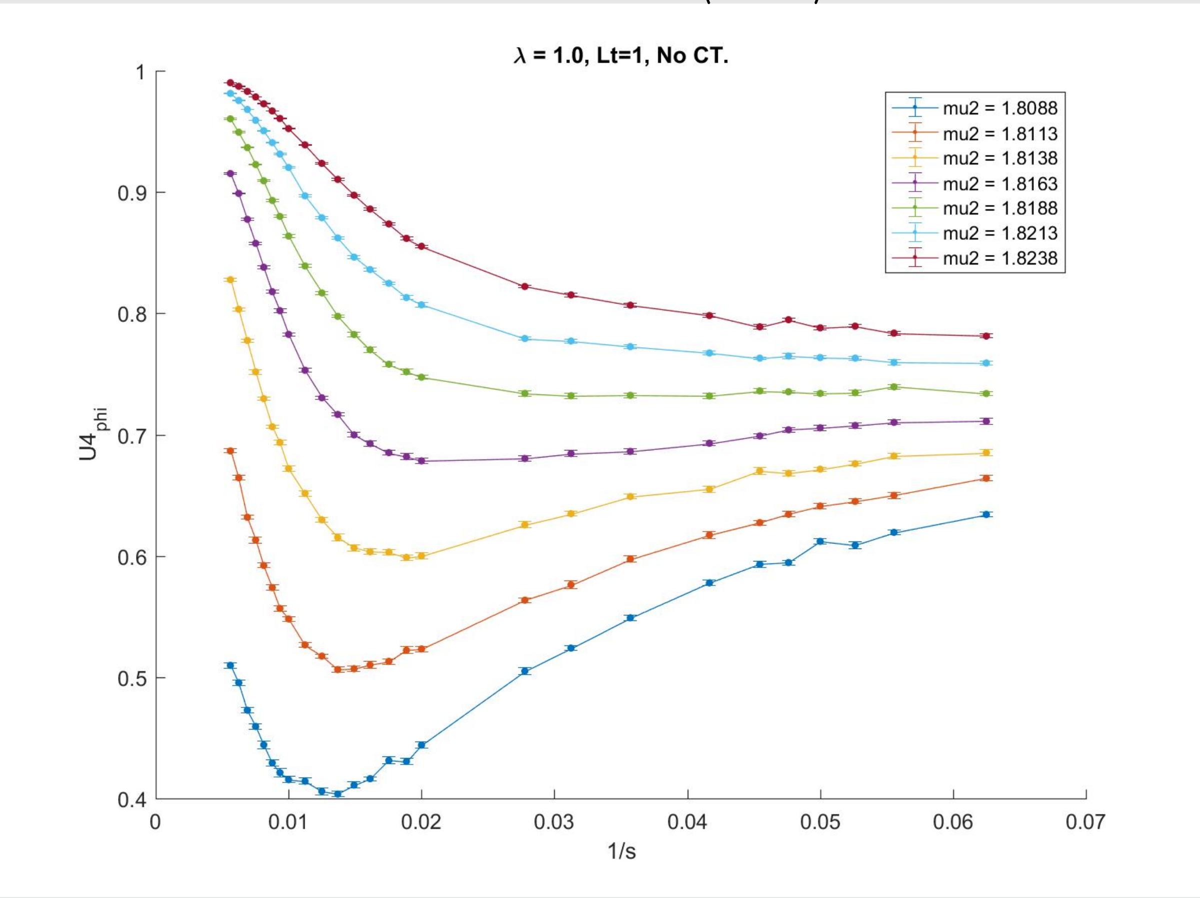
Conformally Invariant
Cross Ratios are “Preserved”

$$\frac{|\xi_1 - \xi_2||\xi_3 - \xi_4|}{|\xi_1 - \xi_3||\xi_1 - \xi_4|} = \frac{|\vec{r}_1 - \vec{r}_2||\vec{r}_1 - \vec{r}_2|}{|\vec{r}_1 - \vec{r}_3||\vec{r}_1 - \vec{r}_4|}$$

BINDER CUMULANT NEVER CONVERGES

$$U_4 = \frac{3}{2}\left[1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}\right]$$

0.85102



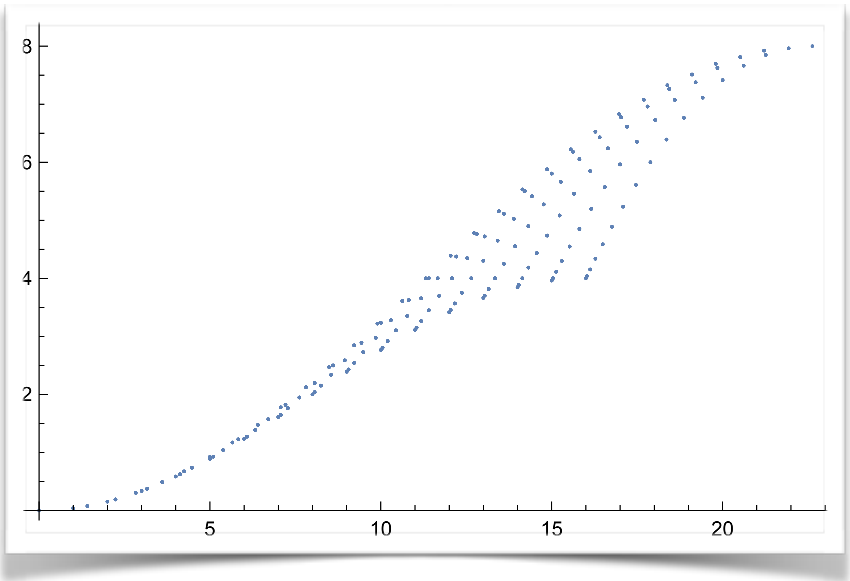
Very fast cluster algorithm:

Brower,Tamayo ‘Embedded Dynamics for phi 4th Theory” PRL 1989.

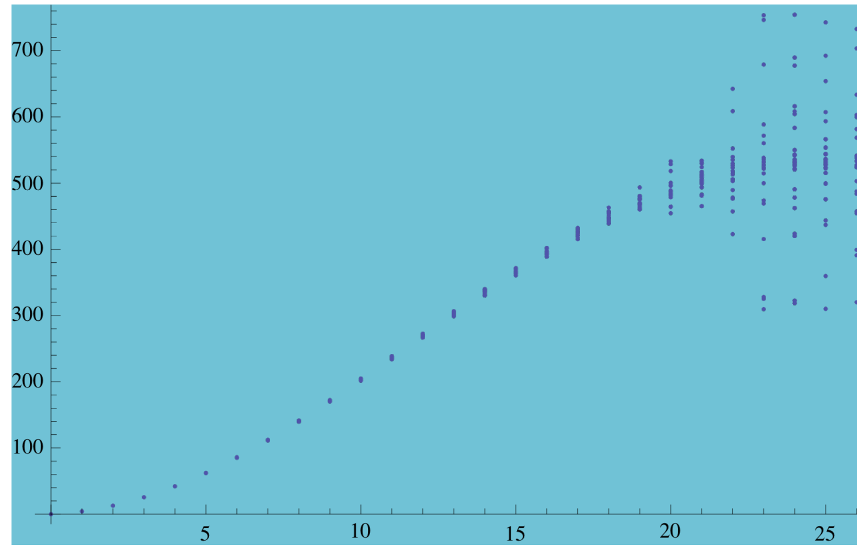
Wolff single cluster + plus Improved Estimators etc

Restoring Isometries for ON A SIMPLICIAL COMPLEX

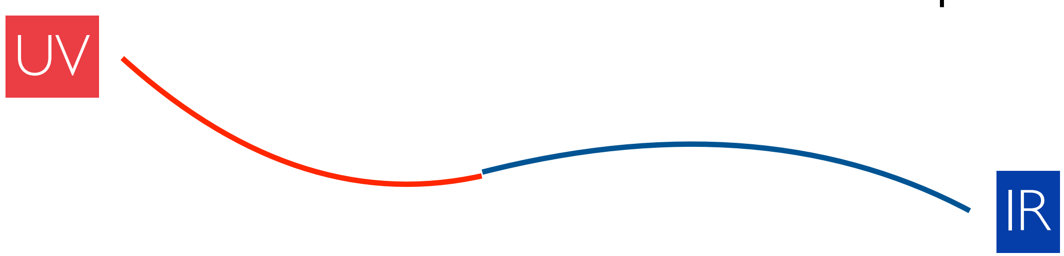
How much help do you need from FEM ?



Hypercubic Lattice

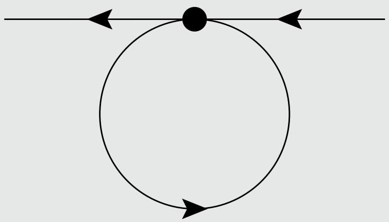


Simplicial Sphere

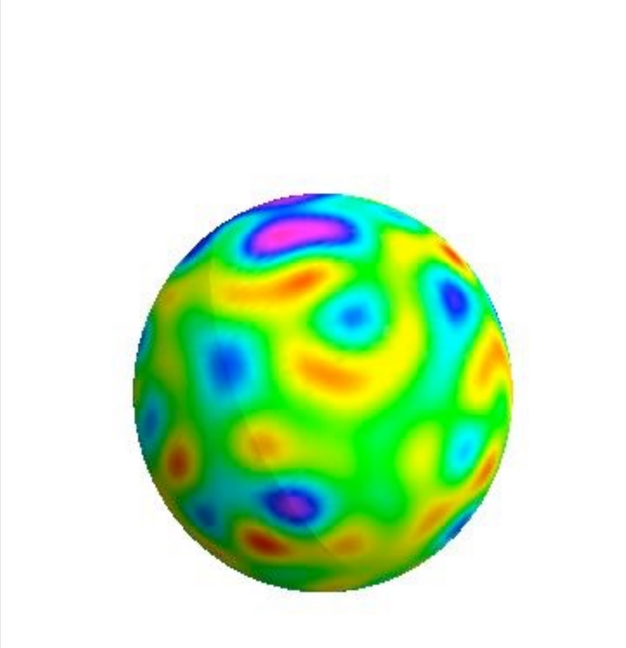


Is Renormalized perturbation theory at the UV fixed point enough to uniquely/correctly define the IR Quantum Field Theory?

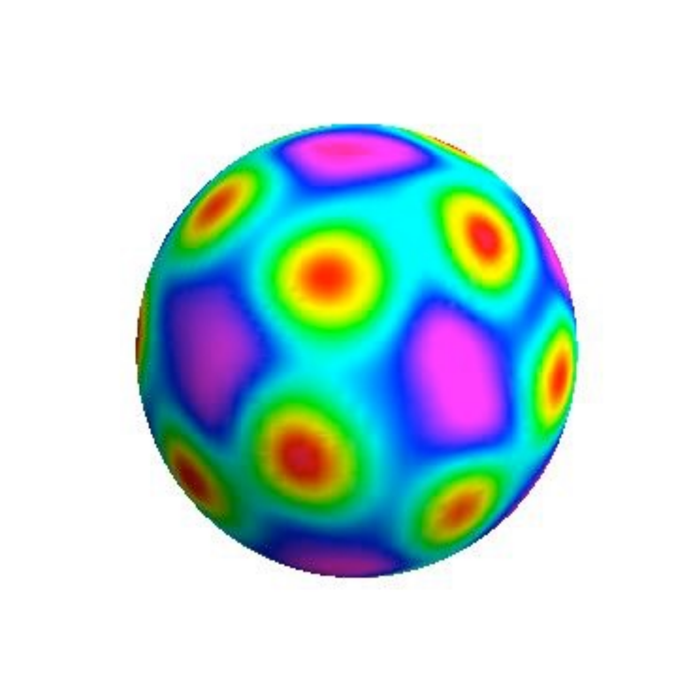
UV DIVERGENCE BREAKS ROTATIONS



$$\delta m^2 = \lambda \langle \phi(x)\phi(x) \rangle \to \frac{1}{K_{xx}}$$



one configuration



average of config.

One LOOP Counter Term

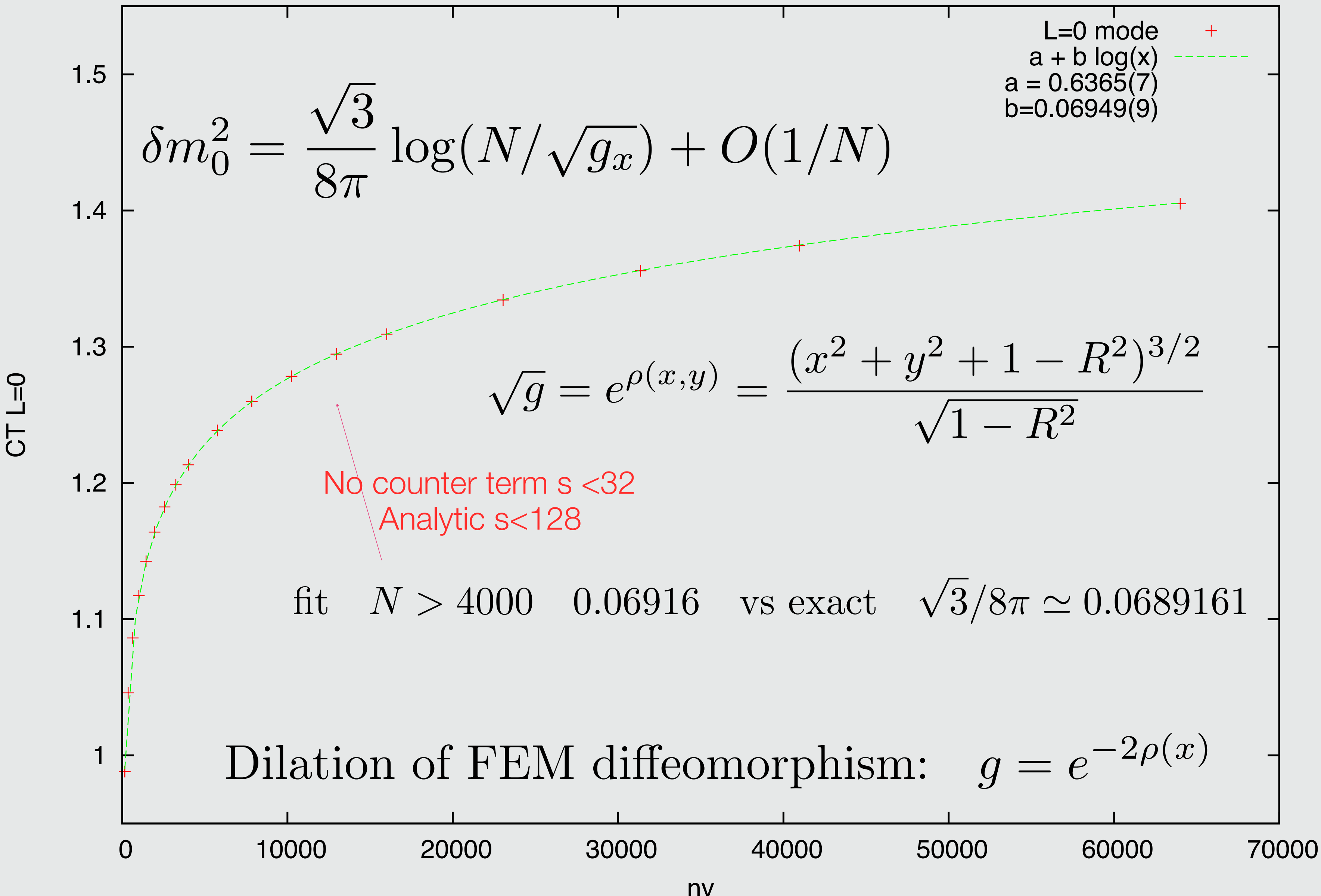
$$\Delta m_i^2 = 6\lambda \left[K^{-1}\right]_{ii} \simeq \frac{\sqrt{3}}{8\pi}\lambda \log(1/m_0^2 a_i^2) = \frac{\sqrt{3}}{8\pi}\lambda \log(N_s) + \frac{\sqrt{3}}{8\pi}\lambda \log(a^2/a_i^2)$$

Exact Continuum Divergence

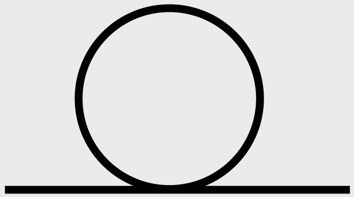
Local Cut-off Scheme Dependence

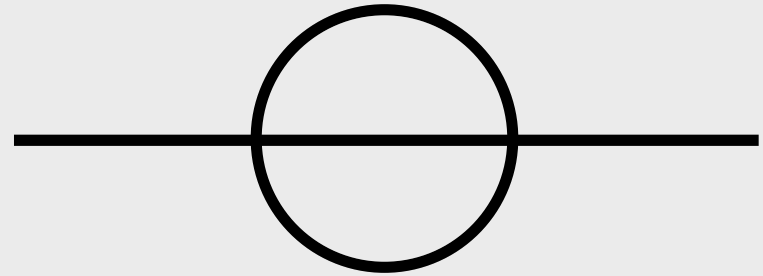
$$\delta\mu_i^2 = -6\lambda(\left[K^{-1}\right]_{ii} - \frac{1}{N_s}\sum_{j=1}^{N_s}\left[K^{-1}\right]_{jj})$$

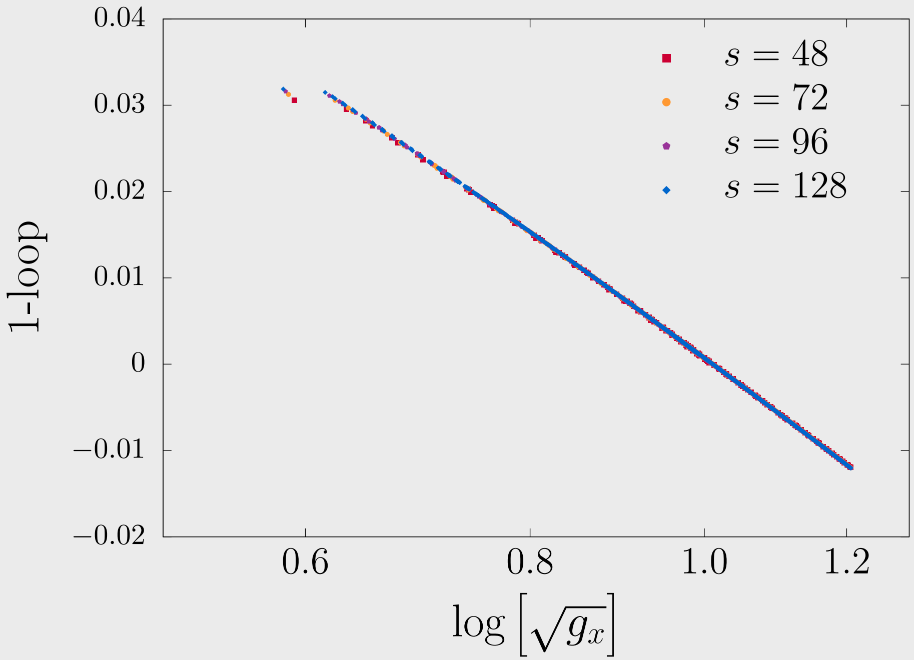
MODEL OF COUNTER TERM

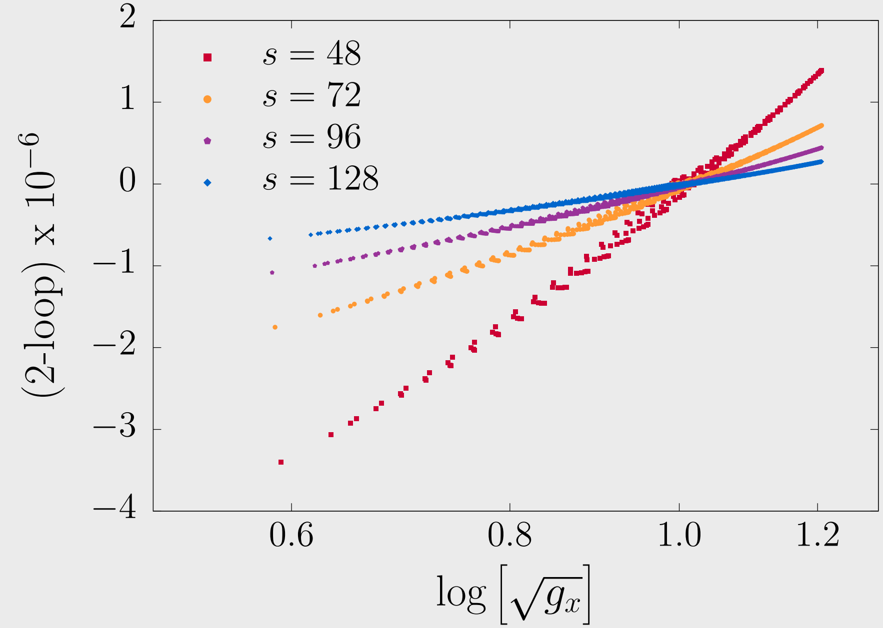


One Loop Counter Term vs Two Loop Convergence









“RG Proof” Of UNIVERSAL UV LOgs

$$G_{xx}(m) \simeq c_x log(1/m^2 a_x^2) + O(a^2 m^2)$$

$$\Longrightarrow \gamma(m_0^2) = -m_0 \frac{\partial}{\partial m_0} G_{xx}(m_0^2) \simeq 2c_x + O(m_0^2)$$

$$m_0 = am \to 0$$

FEM Spectral Fidelity

IR: Region.

UV

$$G_{xy}(m^2) = \sum_n \frac{\phi_n^*(x)\phi_n(x)}{E_n^{(0)} + m^2}$$

$$\simeq \frac{\sqrt{3}}{8\pi} \sum_{l=0}^{L_0} \frac{(2l+1)P_l(r_x \cdot r_y)}{l(l+1) + \mu_0^2} + \sum_{n=(L_0+1)^2}^{N} \frac{\phi_n^*(x)\phi_n(y)}{E_n^{(0)} + m^2}$$

Insensitive to UV defects

$$\gamma(m_0^2) = -m_0 \frac{\partial}{\partial m_0} G_{xx}(m_0^2)$$

$$\simeq \frac{\sqrt{3}}{8\pi} \int_0^{\Lambda_0^2} dE^{(0)} \frac{m_0^2}{(E^{(0)} + m_0^2)^2} = \frac{\sqrt{3}}{8\pi} \frac{1}{1 + m_0^2/\Lambda_0^2}$$

EXACT C = 1/2 CFT ON 2D SPHERE

Exact Two point function

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \to \frac{1}{|1 - cos\theta_{12}|^{\Delta}}$$

$$\Delta = \eta/2 = 1/8$$

$$x^2 + y^2 + z^2 = 1$$

4 pt function

$$(x_1, x_2, x_3, x_4) = (0, z, 1, \infty)$$

$$g(0, z, 1, \infty) = \frac{1}{2|z|^{1/4}|1 - z|^{1/4}}[|1 + \sqrt{1 - z}| + |1 - \sqrt{1 - z}|]$$

Critical Binder Cumulant

$$U_B^* = 1 - \frac{\langle M^4\rangle}{3\langle M^2\rangle\langle M^2\rangle} = 0.567336$$

Dual to Free Fermion

Now Binder Cumulant Converges

$$U_{2n}(\mu^2, \lambda, s) = U_{2n,cr} + a_{2n}(\lambda)[\mu^2 - \mu^2_{cr}]s^{1/\nu} + b_{2n}(\lambda)s^{-\omega}$$

FIT $U_{4,\mathrm{cr}} = 0.85020(58)(90)$

THEORY $U_4^* = 0.8510207(63)$

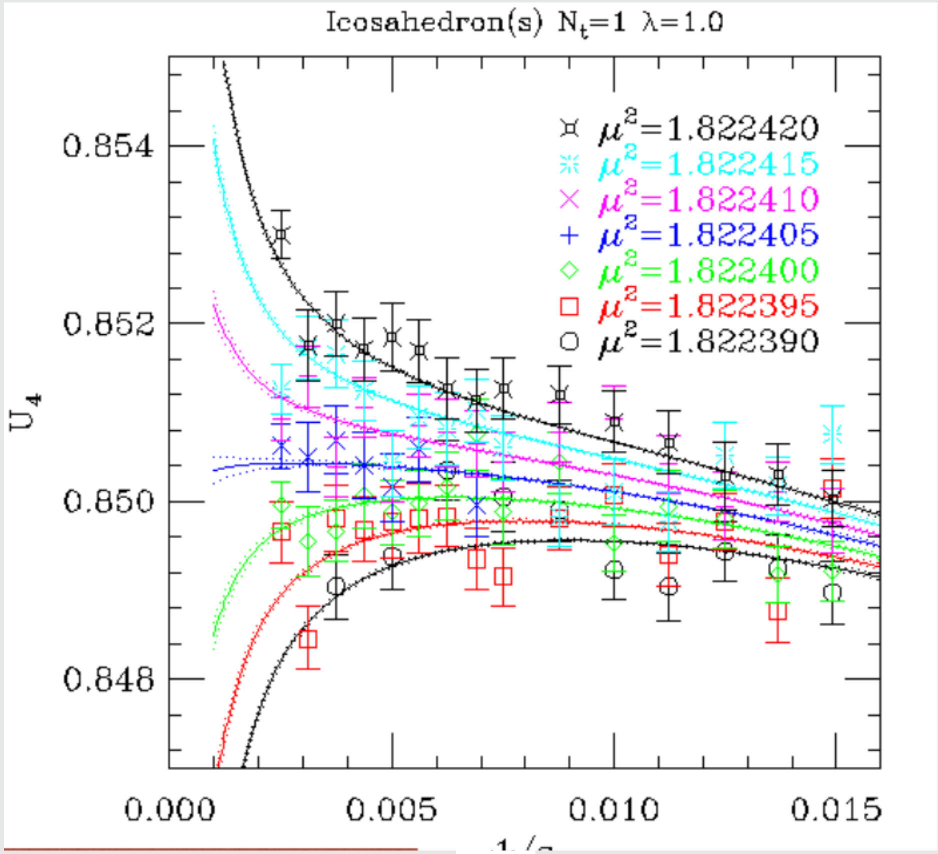
FIT $U_{6,\mathrm{cr}} = 0.77193(37)(90)$

THEORY $U_6^* = 0.773144(21)$

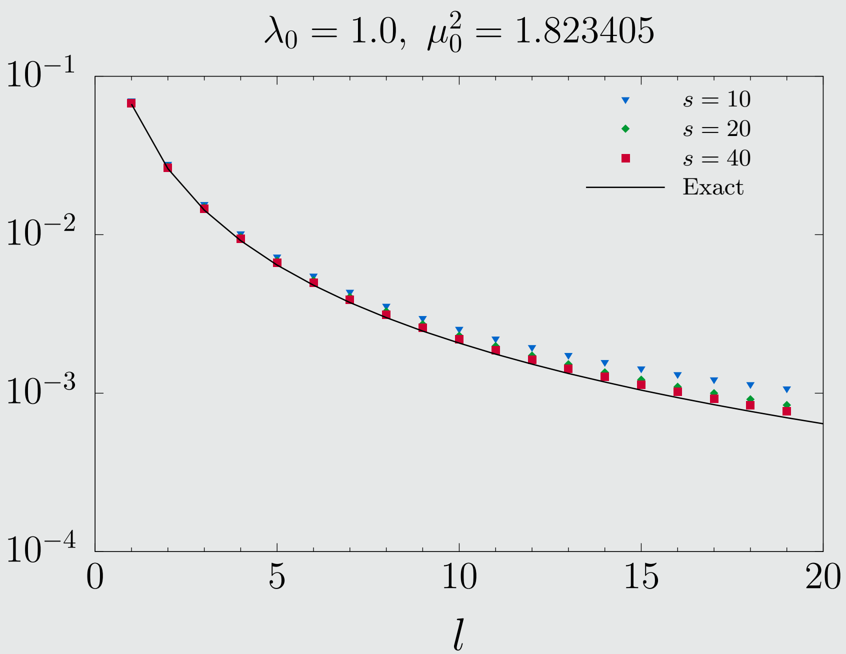
$$U_4 = \frac{3}{2}[1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle \langle M^2 \rangle}]$$

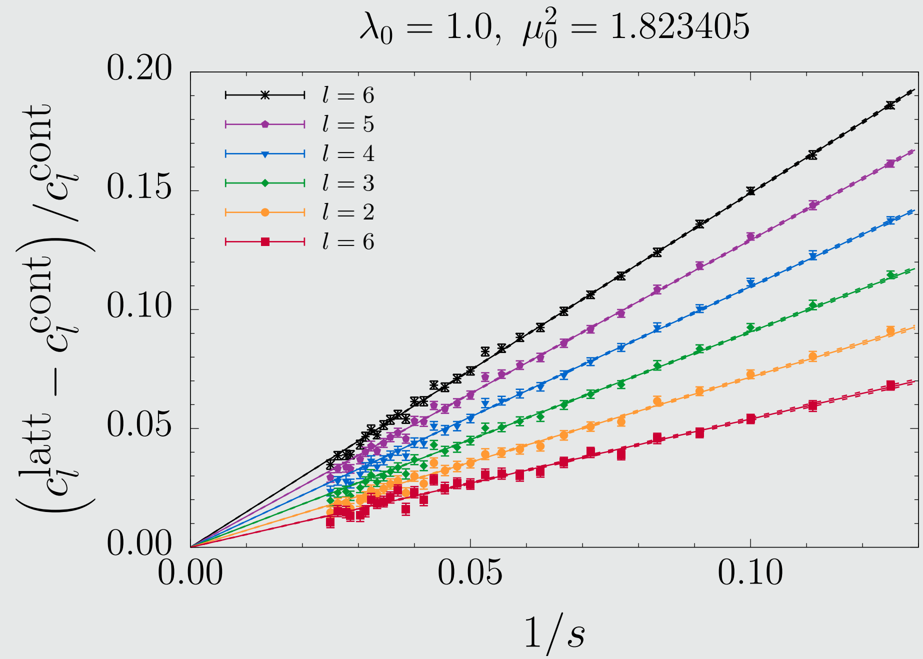
$$\mu^2_{cr} = 1.82240070(34)$$

$$dof = 1701 \quad , \quad \chi^2/dof = 1.026$$



Simultaneous fit for s up 800:
E.G. **6,400,002** Sites on Sphere





$$\int_{-1}^{1} dz \left(\frac{2}{1-z}\right)^{1/8} P_l(z)$$

$$\Delta_\sigma = \eta/2 = 1/8 \simeq 0.128$$

$$\Longrightarrow \frac{16}{7}, \frac{16}{35}, \frac{48}{161}, \frac{816}{3565}, \frac{12240}{64883}, \frac{493680}{3049501}, \frac{33456}{234577}, \frac{55760}{435643}, \frac{3602096}{30930653}, \frac{20129360}{187963199}, \frac{541373840}{5450932771} \cdots$$

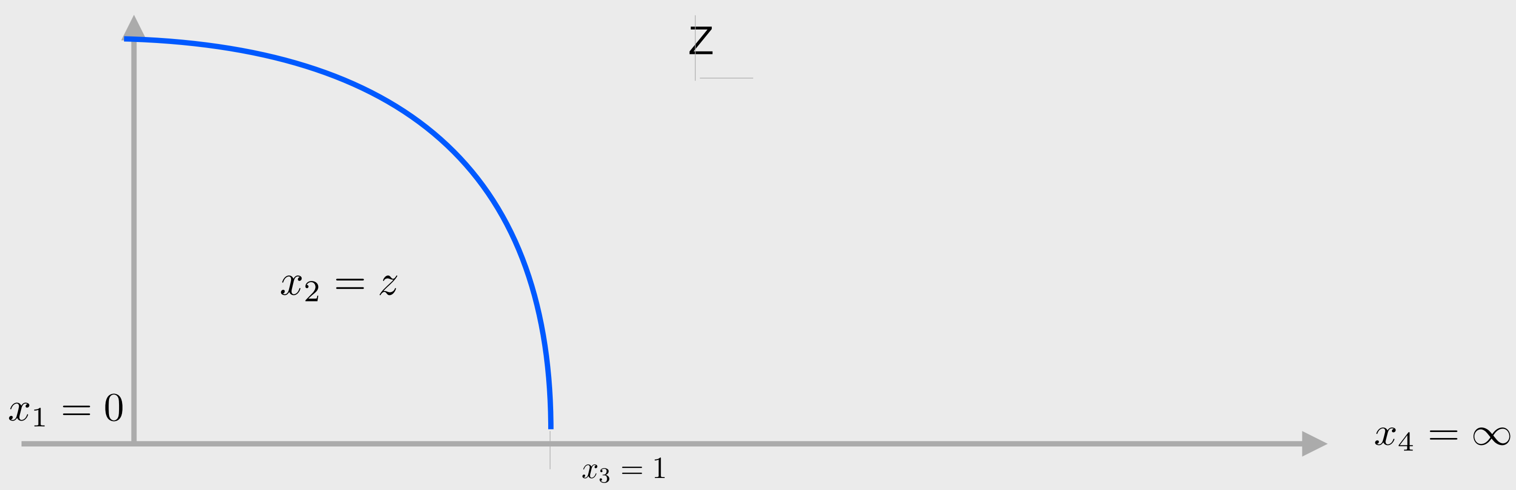
EXACT FOUR POINT FUNCTION

$u = z\bar{z} \quad , \quad v = (1-z)(1-\bar{z})$

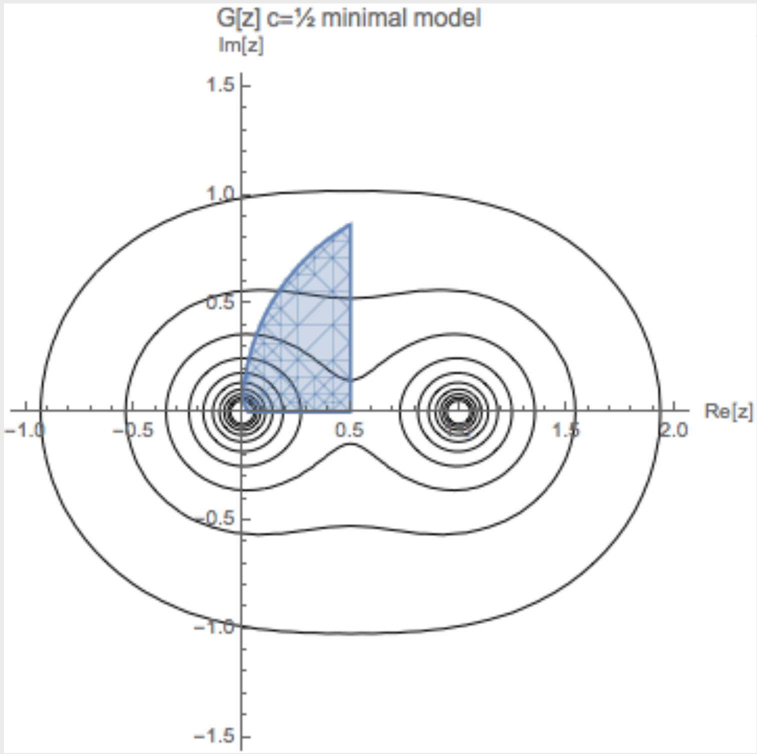
$$g(u,v) = \frac{\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle}{\langle \phi_1 \phi_3 \rangle \, \langle \phi_2 \phi_4 \rangle}$$

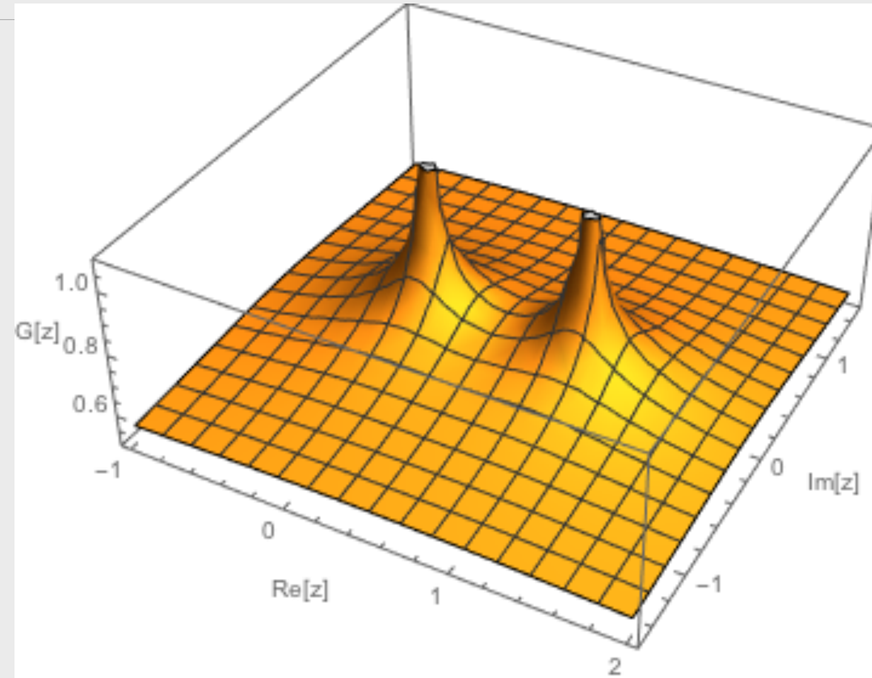
$$= \frac{1}{\sqrt{2}|z|^{1/4}|1-z|^{1/4}} \left[|1+\sqrt{1-z}| + |1-\sqrt{1-z}| \right]$$

Crossing Sym: $|1+\sqrt{1-z}| + |1-\sqrt{1-z}| = \sqrt{2 + 2\sqrt{(1-z)(1-\bar{z})} + 2\sqrt{z\bar{z}}}$



OPE Expansion: $\phi \times \phi = \mathbf{1} + \phi^2$ or $\sigma \times \sigma = \mathbf{1}$ -





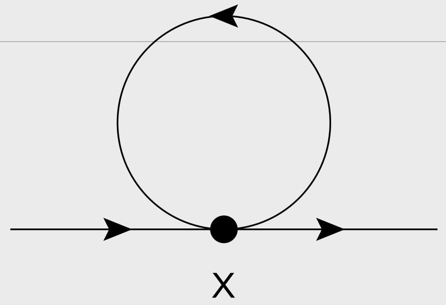
$$G_s(r,\theta) \propto 1 + \lambda_\epsilon^2\, g_{\epsilon,0}(r,\theta) + \lambda_T^2\, g_{T,2}(r,\theta)$$

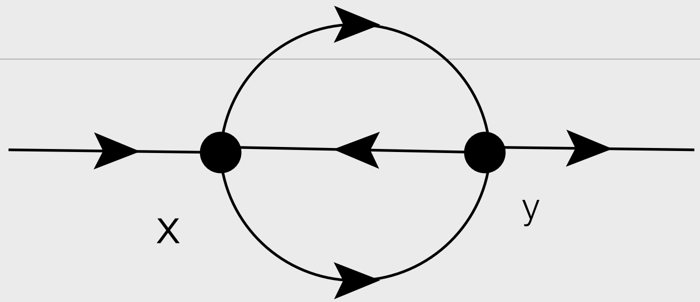
$$\lambda_T^2 = \frac{\Delta_\sigma^2 d^2 |z|^{d-2}}{C_T(d-1)^2} \to \frac{1}{16 C_T} \qquad \text{for } d = 2\,, \qquad g_{T,2}(z) = -3\left(1 + \frac{1}{z}\left(1 - \frac{z}{2}\right)\log(1-z)\right) + \text{c.c.}$$

Fit TO OPE EXPANSION

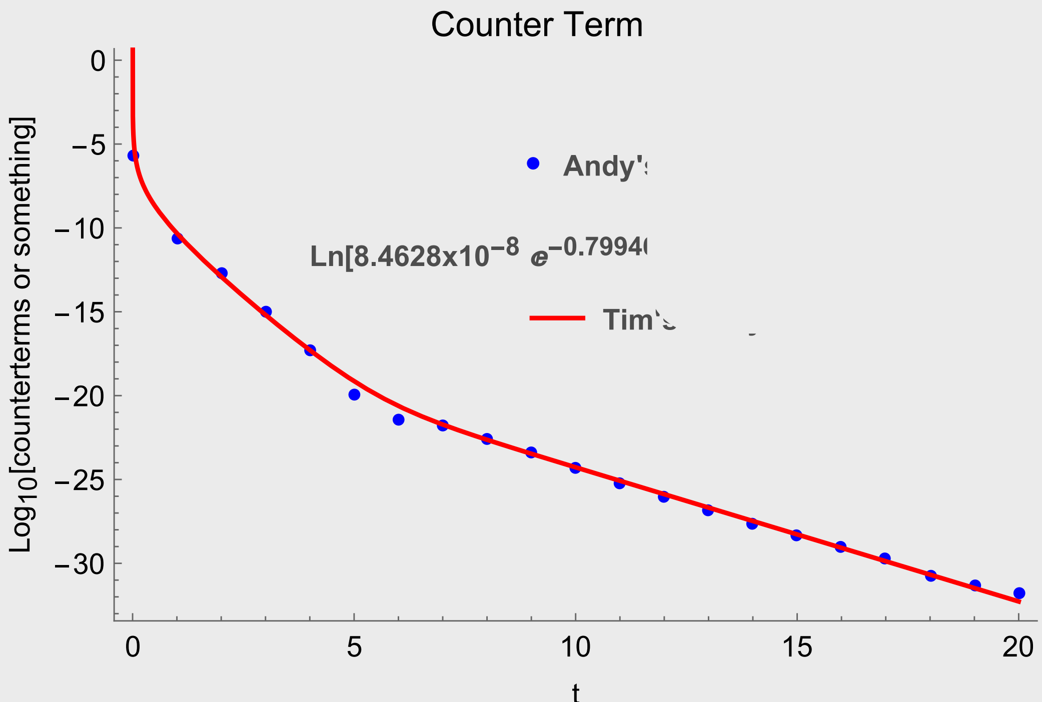
μ^2	s	$r_{\min} \leq r \leq r_{\max}$	norm	Δ_ϵ	λ_ϵ^2	c
1.82241	9	$0.25 \leq r \leq 0.75$	0.2900	1.075	0.2536	0.4668
1.82241	9	$0.30 \leq r \leq 0.70$	0.2901	1.075	0.2533	0.4704
1.82241	9	$0.35 \leq r \leq 0.65$	0.2902	1.077	0.2533	0.4738
1.82241	9	$0.40 \leq r \leq 0.60$	0.2902	1.016	0.2427	0.4747
1.82241	18	$0.25 \leq r \leq 0.75$	0.2051	1.068	0.2563	0.4866
1.82241	18	$0.30 \leq r \leq 0.70$	0.2051	1.056	0.2544	0.4878
1.82241	18	$0.35 \leq r \leq 0.65$	0.2051	1.050	0.2535	0.4904
1.82241	18	$0.40 \leq r \leq 0.60$	0.2051	1.046	0.2526	0.4884
1.82241	36	$0.25 \leq r \leq 0.75$	0.1457	1.031	0.2528	0.4926
1.82241	36	$0.30 \leq r \leq 0.70$	0.1458	1.026	0.2519	0.4932
1.82241	36	$0.35 \leq r \leq 0.65$	0.1458	1.018	0.2508	0.4931
1.82241	36	$0.40 \leq r \leq 0.60$	0.1458	1.007	0.2486	0.4933

Counter term in 3D





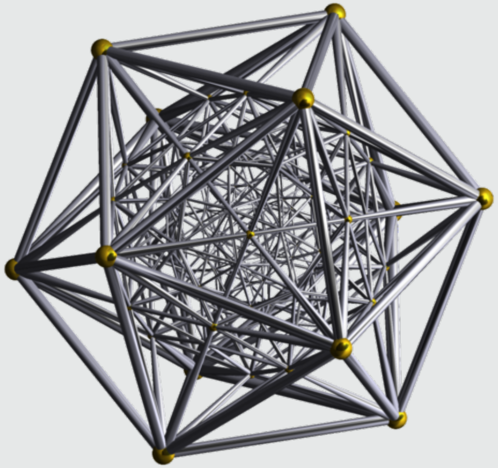
$$\delta\mu^2_{CT}(x,y) \ \sim\ \lambda_0 c_x \delta_{xy} + \lambda_0^2 e^{-6|x-y|/a}$$

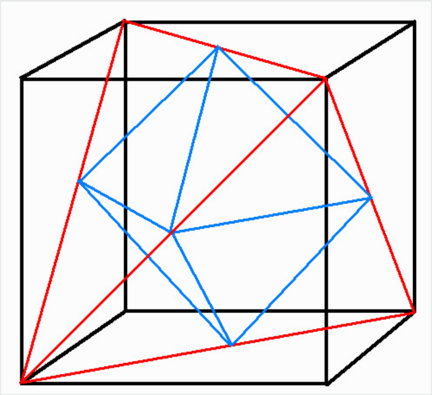


PART III: THE FUTURE

Sphere Constant Positive Curvature vs Cylinders

$$\mathbb{S}^d \quad \to \quad \mathbb{R} \times S^d$$





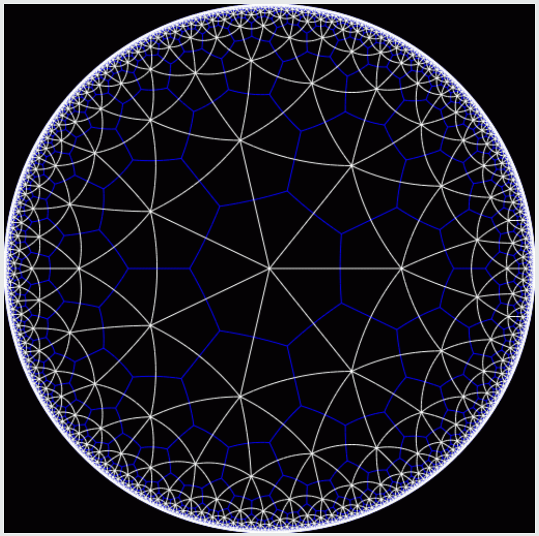
Aristotle' s 2% Error!

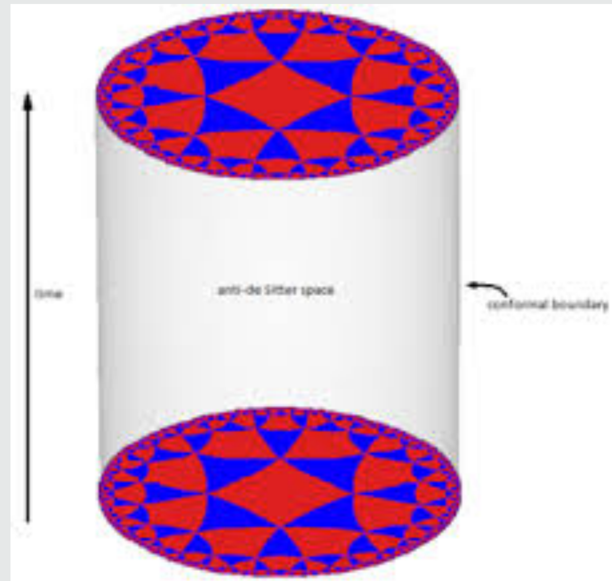
$$(2\pi - 5ArcCos[1/3])/(2\pi) = 0.0204336$$

Fast Code Domains of
Regular 3D Grids on Refinement

Hyperbolic (e.g. Poincare Disk) and Global AdS

$$\mathbb{H}^d \quad \rightarrow \quad \mathbb{A}dS^d$$





1/p + 1/q < 1/2

Triangle Group Tesselation: Preserve Finite subgroup of the Modular Group

These Hyperbolic Tesselatoin are “Tensor Networks” :
What can we do them as lattice Field Theories?

CONCLUSION

1. ***Simplicial Method for Scalars, Fermion and Gauge fields YES!***

2. ***Quantum Corrections (QFE): Super renormalizable YES ?***

3. ***4D UV asymptotical free: Local RG approach/Wilson Flow ??***

4. ***BSM Multi-flavor Gauge Theories with IR fixed point. WHY NOT***

5. ***Simplicial Lattice in Anti de Sitter space YES! WHY?***

6. ***General Mathematical "Proofs" DIFFICULT!***

NEED COLLABORATORS & SUPPORT

EXTRAS

Using Binder Cumulants

$$U_4 = \frac{3}{2}\left(1 - \frac{m_4}{3\,m_2^2}\right)$$

$$U_6 = \frac{15}{8}\left(1 + \frac{m_6}{30\,m_2^3} - \frac{m_4}{2\,m_2^2}\right)$$

$$U_8 = \frac{315}{136}\left(1 - \frac{m_8}{630\,m_2^4} + \frac{2\,m_6}{45\,m_2^3} + \frac{m_4^2}{18\,m_2^4} - \frac{2\,m_4}{3\,m_2^2}\right)$$

$$U_{10} = \frac{2835}{992}\left(1 + \frac{m_{10}}{22680\,m_2^5} - \frac{m_8}{504\,m_2^4} - \frac{m_6\,m_4}{108 m_2^5} + \frac{m_6}{18\,m_2^3} + \frac{5\,m_4^2}{36\,m_2^4} - \frac{5\,m_4}{6\,m_2^2}\right)$$

$$U_{12} = \frac{155925}{44224}\left(1 - \frac{m_{12}}{1247400\,m_2^6} + \frac{m_{10}}{18900\,m_2^5} + \frac{m_8\,m_4}{2520\,m_2^6} - \frac{m_8}{420\,m_2^4} + \frac{m_6^2}{2700 m_2^6} - \frac{m_6 m_4}{45\,m_2^5} + \frac{m_6}{15\,m_2^3} - \frac{m_4^3}{108\,m_2^6} + \frac{m_4^2}{4\,m_2^4} - \frac{m_4}{m_2^2}\right)$$

$$m_n = \langle \phi^n \rangle$$

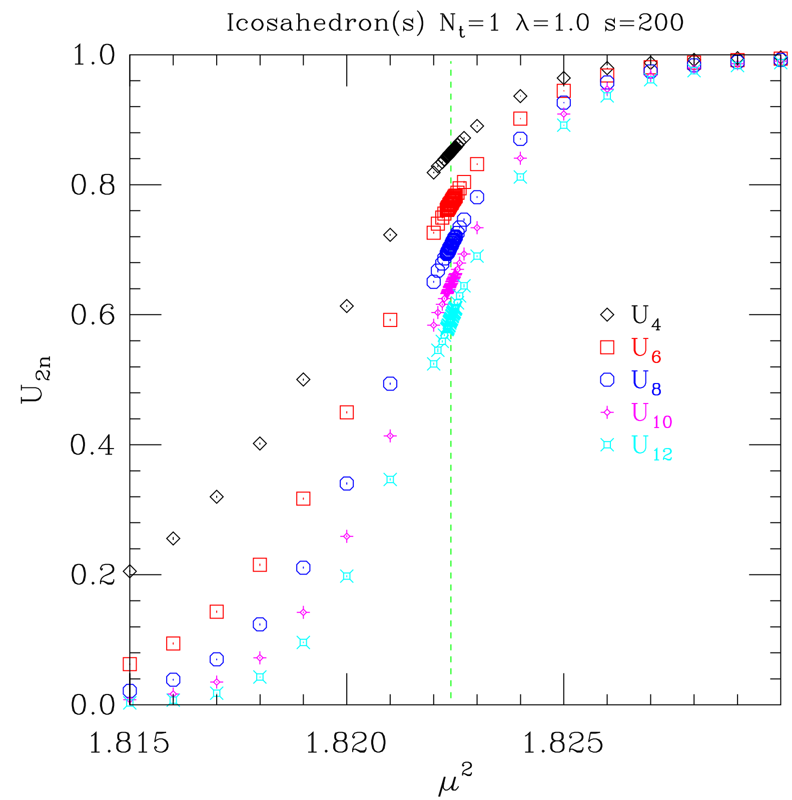
In infinite volume

$U_{2n}=0$ in disordered phase

$U_{2n}=1$ in ordered phase

$0<U_{2n}<1$ on critical surface

- $U_{2n,cr}$ are universal quantities.
- Deng and Blöte (2003): $U_{4,cr}=0.851001$
- Higher critical cumulants computable using conformal $2n$-point functions:
  Luther and Peschel (1975)
  Dotsenko and Fateev (1984)



EXAMPLE SCALAR THEORY

$$S = \frac{1}{2}\int_{\mathcal{M}} d^D x\sqrt{g}\,\left[g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) + m^2\phi^2(x) + \lambda\phi^4(x)\right]$$

$$\vec{y} = \xi_1\vec{r}_1 + \cdots + \xi_{D+1}\vec{r}_{D+1}$$
$$\text{with} \quad \xi_1 + \cdots + \xi_{D+1} = 1$$

$$I_\sigma = \frac{1}{2}\int_\sigma d^D y[\vec{\nabla}\phi(y)\cdot\vec{\nabla}\phi(y) + m^2\phi^2(y) + \lambda\phi^4(y)]$$
$$= \frac{1}{2}\int_\sigma d^D\xi\sqrt{g}\,[g^{ij}\partial_i\phi(\xi)\partial_j\phi^2(\xi) + m^2\phi^2(\xi) + \lambda\phi^4(\xi)]$$

$$I_\sigma \simeq \sqrt{g_0}\left[g_0^{ij}\frac{(\phi_i - \phi_0)(\phi_j - \phi_0)}{l_{i0}\, l_{j0}} + m^2\phi_0^2 + \lambda\phi_0^4\right]$$

DESCENDANTS

ZERO PARAMETER FIT

s= 10 Run for 1/2 hour

$$g(u,v) = \sum_l g_l(|z|)\cos(l\theta)$$

$$z = |z|e^{il\theta}$$