A Toy Model for Time Evolving QFT on a Lattice with Controllable Chaos

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Problems we/I want to solve

- Real time dynamics of QFT
- Choose initial state and evolve it
- Compute expectation values of interesting observables
- Compute response functions
- Compute entanglement entropies/patterns
- Relate entanglement to other physics

Technical problem

It is a hard problem: the sign problem

Another reason: wave functions have too many variables (one per site on a lattice) so it is hard to sample or solve the Schrödinger equation.

For ground states other techniques work (DMRG,Tensor networks)

Holographic answer

- Solve gravity in bulk of AdS (Chesler/Yaffe, …) and one can study one point functions of interesting operators in thermalization trajectories.
- Entanglement of steady states: (Ryu-Takayanagi) minimal surfaces in static spacetimes.
- Entanglement dynamics: (Hubeny+Rangamani+Takayanagi) extremal surfaces in dynamical spacetimes

Response functions

- Green’s functions on dynamical AdS spacetimes
- Response functions can encode chaos, relate Lyapunov exponents to the infalling observer problem, leads to OTOC (Shenker-Stanford).
- Analytic bound on chaos (Maldacena-Shenker-Stanford): quantum Lyapunov exponents are bounded by temperature (needs thermal ensemble).

Quantum chaos

- Quantum Lyapunov exponents match classical in semiclassical limit.
- Lyapunov exponents are ‘hard’ to compute: need to resort to numerics, sometimes very long time before convergence.
- Lyapunov exponents also control entanglement rates (Zurek-Paz), although not clear when it applies.
- Numerical quantum examples usually have very few variables (2 or 3)

MAIN IDEA

Build a class of models that can be put on a lattice where the Lyapunov exponents can be computed analytically and the entanglement dynamics is tractable numerically and semi-analytically.

Rest of the talk

- Cat maps
- Generalized cat maps
- Lattice-like models (and why they are computable)
- Some Results

Cat map

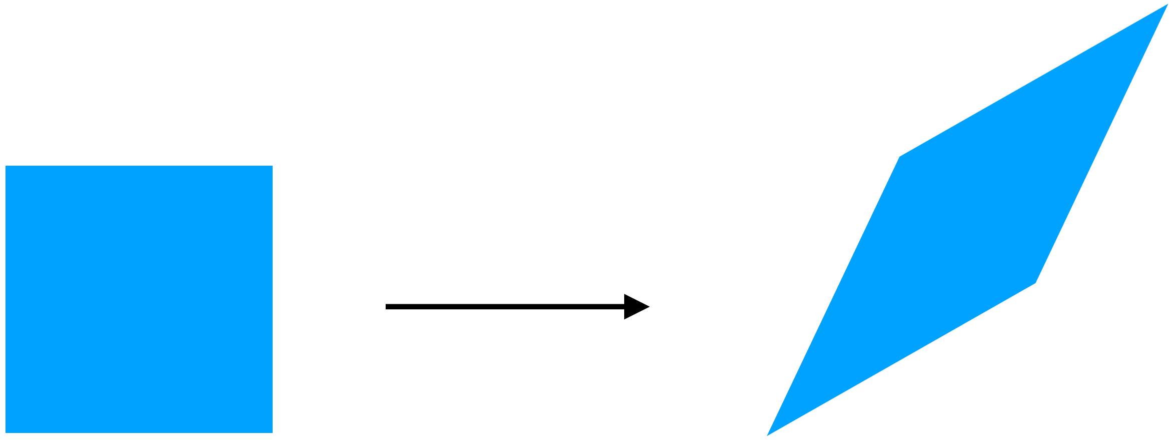
$$\begin{pmatrix} x_{New} \\ y_{New} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{Old} \\ y_{Old} \end{pmatrix}$$

It's a linear (canonical) map on a 2D phase space.

Preservation of volume makes the matrix of determinant 1

Compactify by making the coordinates periodic

Compatibility with periodicity makes the matrix transformation integer valued.



The eigenvalues of the matrix are the Lyapunov exponents.
They only depend on the trace.

The dynamics of these maps is chaotic and mixing.

Quantization

Need a single valued notion of the coordinates.

$$Q = \exp(ix) \qquad P = \exp(iy)$$

Commutation relations are those of clock-shift matrices

$$QP = PQ\exp(i\gamma)$$

Consistency requires that it is a root of unity.

$$\gamma = \frac{2\pi}{N} \simeq \hbar$$

Quantized cat map is an automorphism of the P Q algebra

$$Q_{New} = Q^a P^b$$
$$P_{New} = Q^c P^d$$

This is up to phases (normal ordering ambiguities)

One can check that quantum mechanically this has quantum Lyapunov exponents that match the classical in the large N limit.

Representation of PQ matrices is unique and of dimension N

These automorphisms act as unitary transformations on this representation:
can iterate them to get a quantum dynamics.

Problem

It’s only one degree of freedom, so there is no easy way to study entanglement.

Essentially only one (large) Lyapunov exponent

Generalize

Many P,Q each on a 'site' (each of the same size Hilbert space), consider more general "linear" automorphisms.

Simplest case: no normal ordering ambiguities Q(Q), P(P)

$$Q \to Q^M$$

$$P \to P^{(M^{-1})^T}$$

The M matrix is integer valued and of determinant one and defines a Unitary.

We suggested a slightly different set in a previous paper D.B, A. Garcia (2015)

And now we can iterate.

What to compute

- Choose initial state as a product state on the sites (disentangled)
- Evolve (choose M, and it's powers are generated by dynamics)
- Compute vevs of various monomials.
- Compute entanglement.

Why it is easy to compute vevs at sites

$$\langle Q \rangle_t \simeq \langle Q_t \rangle_0$$

$$Q_{i,t} \simeq (Q^{(M^t)_i})$$

It's always a product of Q, and since the initial state is factorized, the vev is a product of problems of matrices of size NxN, not the size of the Hilbert space!

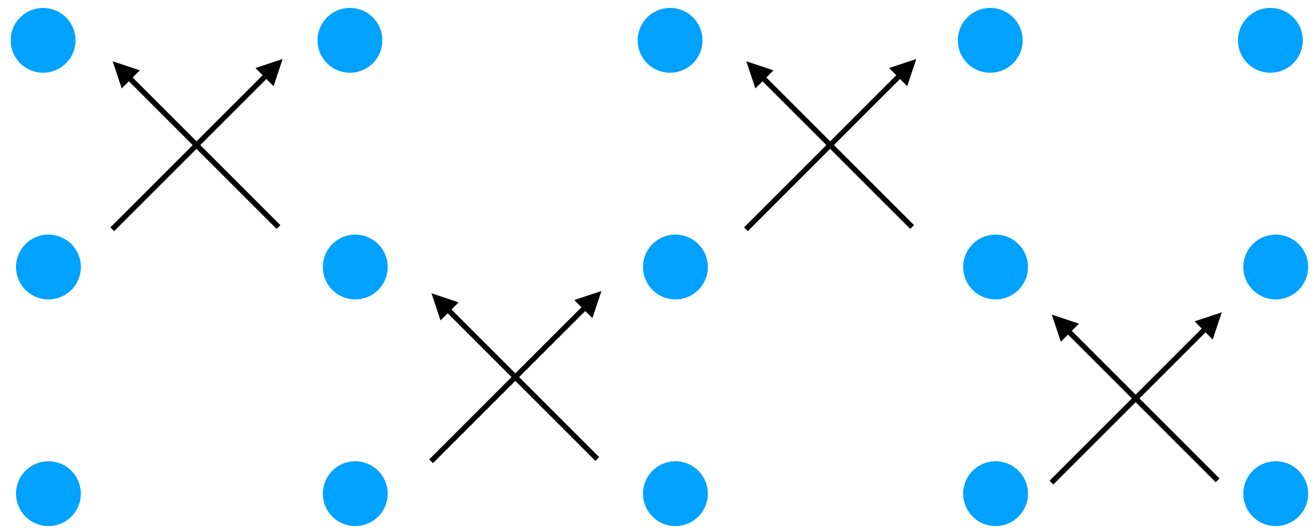
From vevs at sites one can recover the local density matrix

$$\mathrm{Tr}(\rho A)$$

Determines the density matrix if we have a complete basis for A.
This is provided by the powers of P,Q at a site (or a collection of sites).

For small number of sites it is tractable for arbitrary initial product states.

Lattices



Consider products of simple nearest neighbor entanglers to generate M, with some translation invariance

This way, both the M of Q and the M of P are sparse.

This is a good toy model for Hamiltonian dynamics on a nearest neighbor lattice.

It mimics the Lieb-Robinson bound for information transfer (notion of speed of light)

It has controllable chaos (we can choose the local entanglers to have bigger or smaller eigenvalues)

It is scalable to large lattices because many interesting questions factorize, or are sums of few things that factorize.

Can change the size of the local Hilbert space easily (change N)

Results

Two sites, with the simplest cat map.

1-D lattices with simple entanglers and various N (N=2 is really nice)

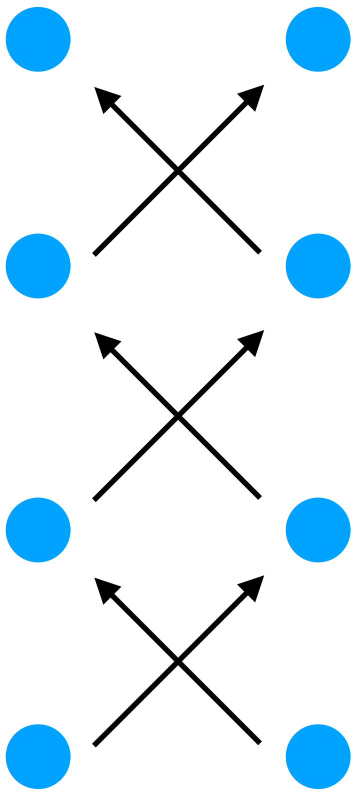
Basis choice

Diagonalize the Q for each site.

The images of Q commute with the Q, so they have to be simultaneously diagonalizable

The Unitary dynamics is a permutation dynamics on this basis, and it is the “cat map” on the rational points with denominator N.

Two sites



Basis states are factorized
Basis state -> basis state

No entanglement

Basis state x random

$$\begin{pmatrix} x_{New} \\ y_{New} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{Old} \\ y_{Old} \end{pmatrix}$$

Does the y cover the possible x and the y at time t?

Depends on arithmetic of N and the matrix. Choose N prime for simplicity.

For most times it does (up to order N times)

$$\sum_{k'} \psi_2(k') |M^t \cdot (k, k')\rangle$$

Tracing over second entry gives a diagonal density matrix, with eigenvalues

$$|\psi_2(k')|^2$$

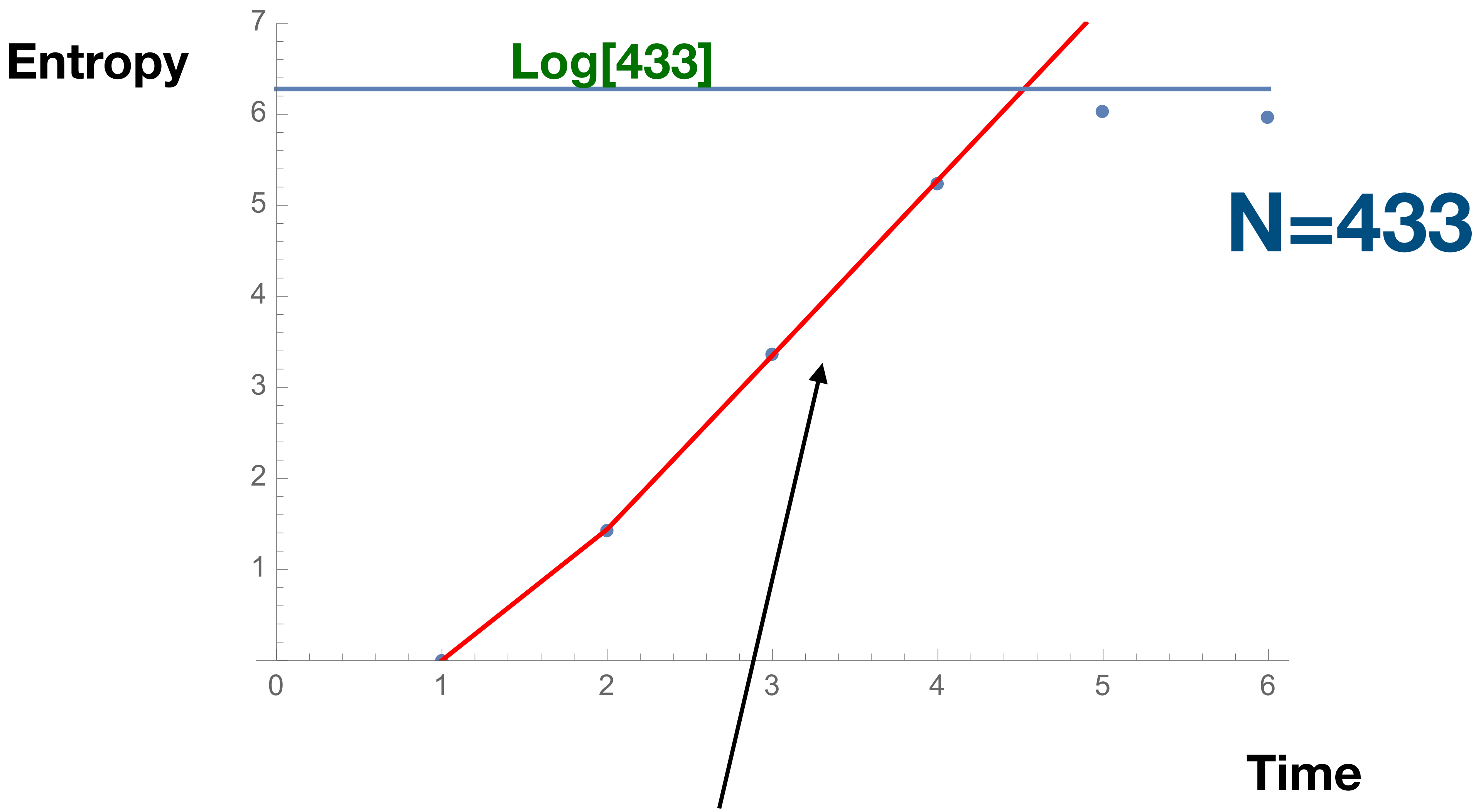
Basically, entanglement can be read from the initial probabilities for being in the basis of the second entry.

All entropy generated in one shot, of order Log(N).

Very quantum.

Gaussian x Gaussian

Gaussian states are generated with “coherent states” for fuzzy torus (G. Ishiki)



Red line: if we do computation with linear dynamics for a Gaussian state without the periodicity imposed: semiclassical result.

Asymptotic slope in iterated true Gaussian= twice the max eigenvalue= Kolmogorov Sinai entropy.

We showed this for iterated Bogolubov transformations D.B., Asplund, 2015

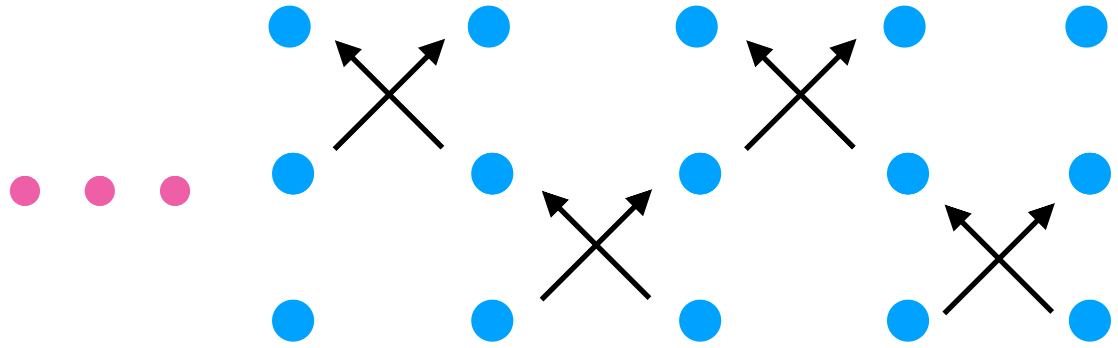
Slope is really close already in the first few steps.

Semiclassical=Good estimate all the way to saturation.

Lesson

- Entanglement rates are sensitive to choice of state.
- Very quantum initial-> Fast (one shot saturation)
- Very special-> 0
- Semiclassical -> KS entropy rate, somewhat independent of precise choices, so long as initial uncertainty is small.

Simple lattice



$$M_{2\times 2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

One each crossing.
One full step is two levels (M1.M2)

How full M looks like:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Upper triangular

All eigenvalues = 1 (no chaos) open boundary conditions
Chaos with periodic boundary conditions.

Special case N=2

$$M_{2\times 2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

IS a CNOT quantum gate!

$$x \to x + y$$

$$y \to y$$

If y=1, flip x.

This dynamics can be put on a quantum computer.

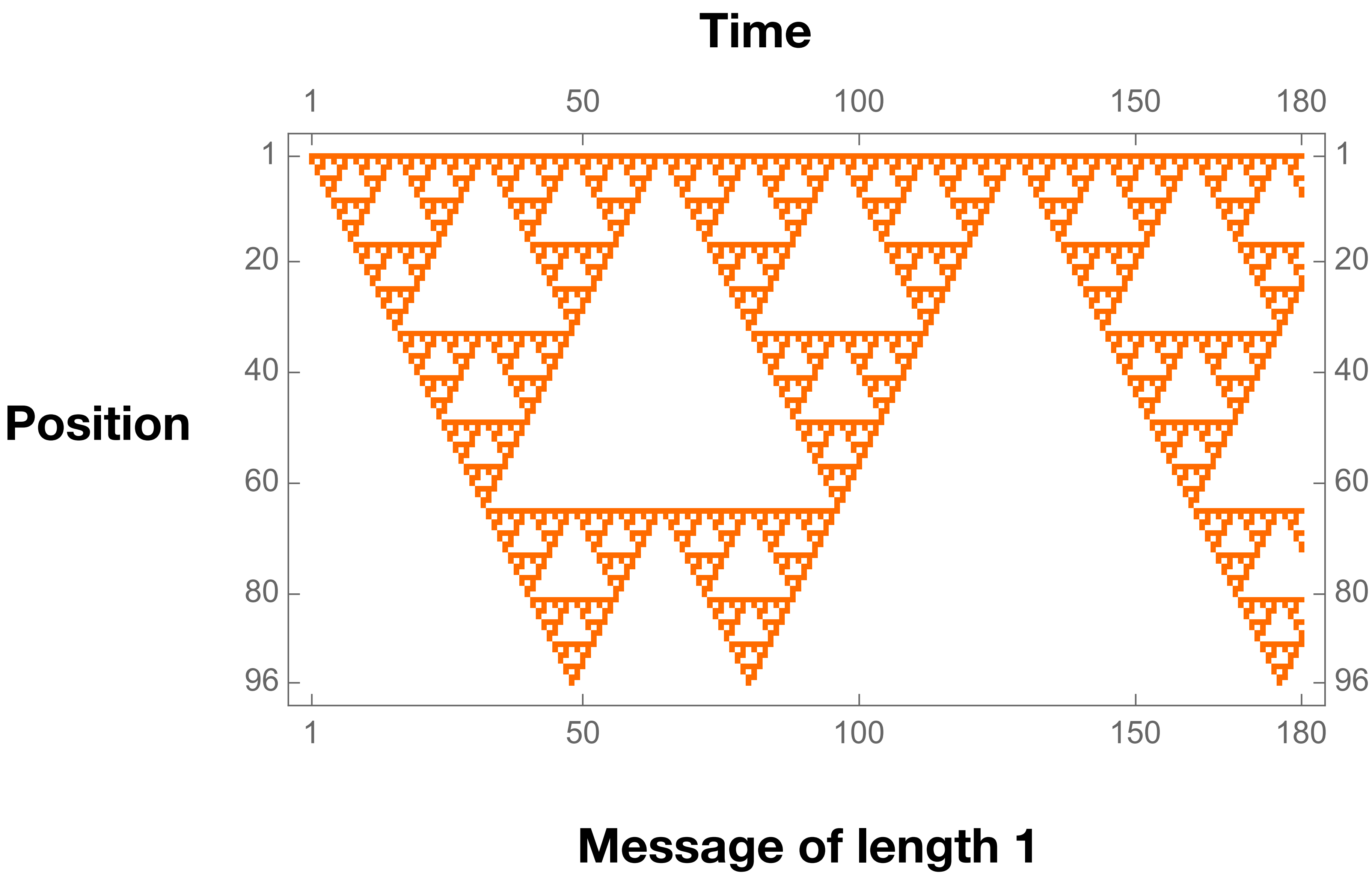
CNOT FIELD THEORY?

Features: it is a permutation dynamics where the upper entries get modified by entries below, but not the other way around.

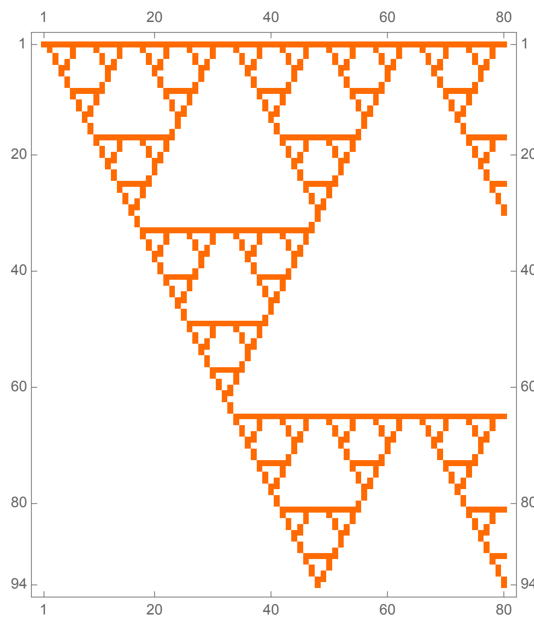
Can think of it as sending (classical) messages to the left only

Question: when do the last m bits get mapped 1-1 into a collection of m bits at position s?

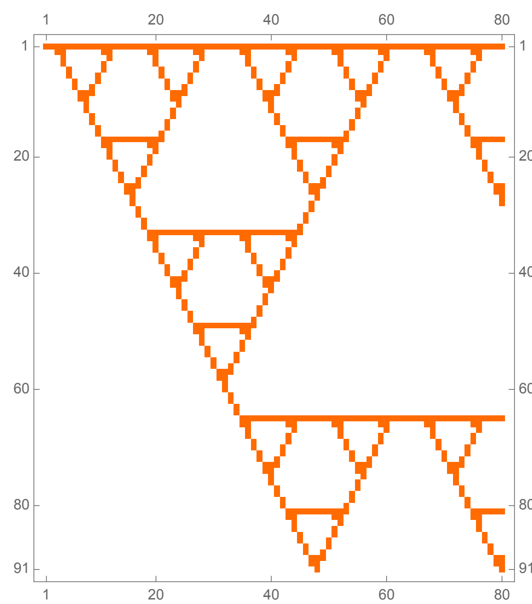
This depends on the determinant of a sub-block of the matrix being invertible or not.



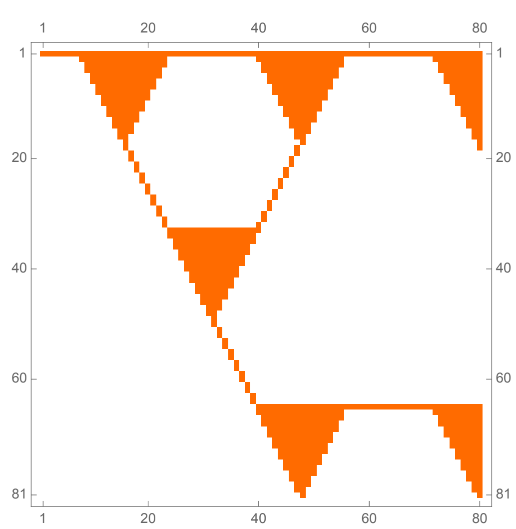
Pattern looks self-similar.



Length 3 message



Length 6 message



Length 16 message

Quantum evolution of basis states is classical

$$|A\rangle \to |f(A)g(A)\rangle \to |f(f(A))g(f(A))h(A)\rangle$$

Equivalent to repeating the message.

$$|A\rangle \to |AA\rangle \to |AAA\rangle$$

Velocity of first message is same regardless of length.

Quantum Entanglement

Prepare quantum message in last 2k sites, leave the others in simple basis state (0).
Wait time k, each classical message gets copied to a
different ket on the second argument. This happens again and again.

$$\rho = \mathrm{diag}(\mathrm{Basis\ states})$$

density matrix is probabilities of measuring in basis states.

$$S \simeq (2k)\log(N)$$

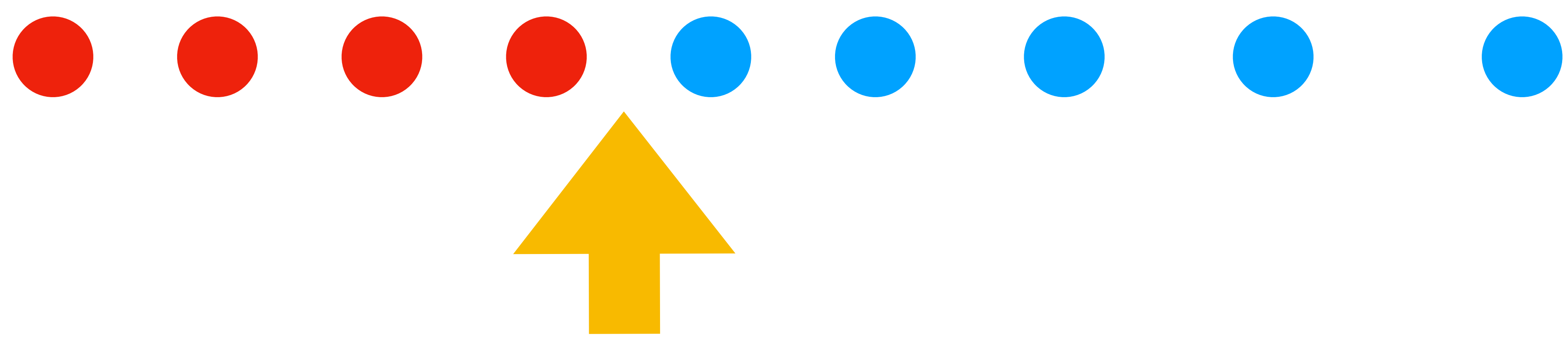
$$\dot{s} \simeq 2\log(N)$$

Dual P basis?

The matrix is inverted and transposed: two by two blocks are lower triangular.
This only sends classical P messages to the right.

Quantum can subvert classical intuition.

Eigenstates of P also go to eigenstates of P: dual basis dynamics.



One can stop the entangling: no entropy generated to the left.

Can produce weird patterns of entanglement: contained in an interval.

Conclusion

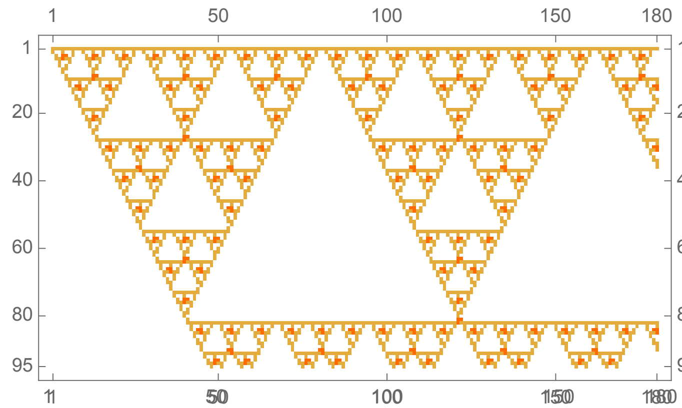
- Toy model where both chaos and entanglement can be studied

- Highly computable (many problems factorize)

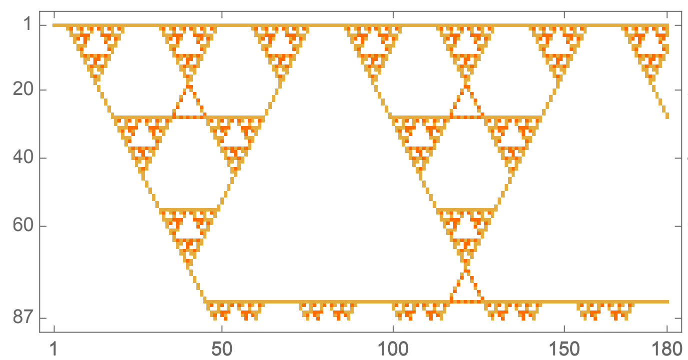
- Can compute density matrices and even do some things analytically for some classes of states.

- Details of entanglement depend on state and N: bound on entanglement controlled by Log(N), but Lyapunov in semiclassical regime.

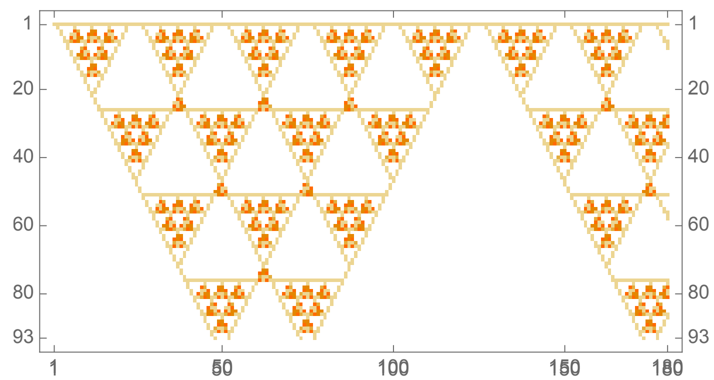
- Can block the entanglement production (stop the message).



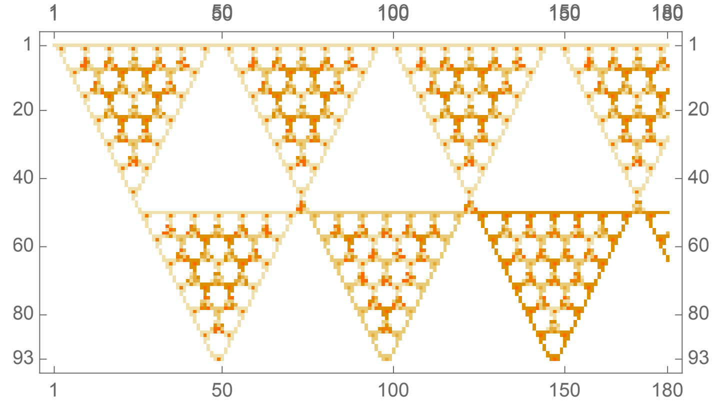
N=3, L=2



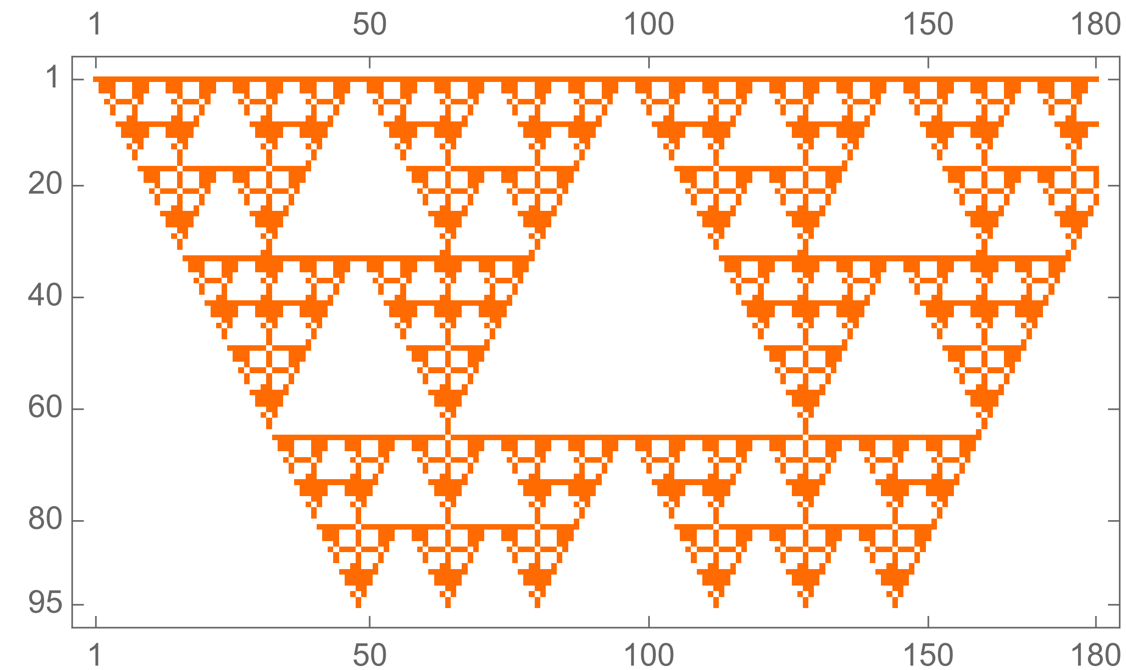
N=3, L=10



N=5, L= 8



N=7, L=4



Different M, N=2, L=2