Analytic Conformal Bootstrap in 1D

ICTS Bangalore, Feb 3 2018

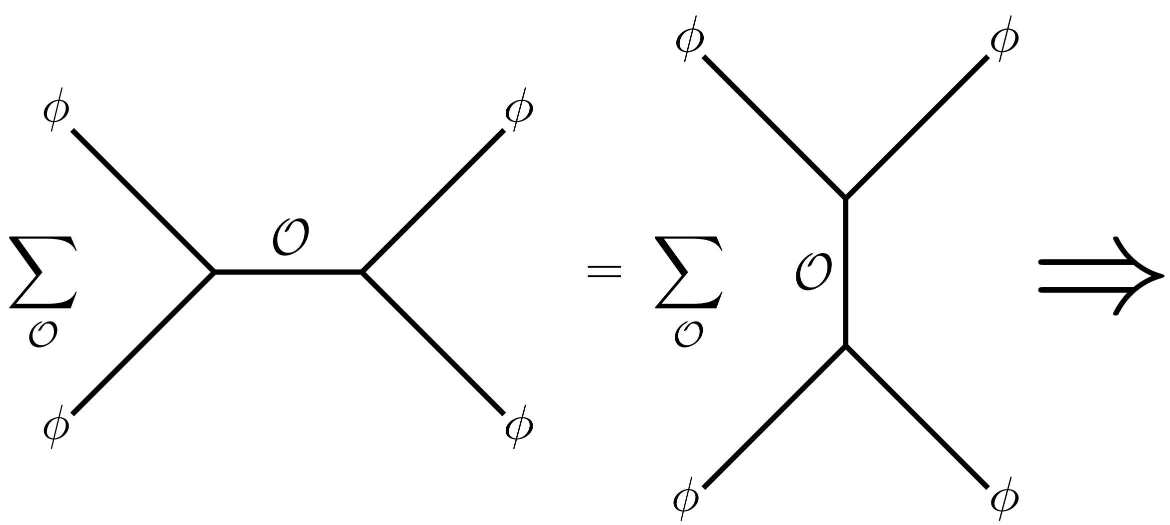
Dalimil Mazáč

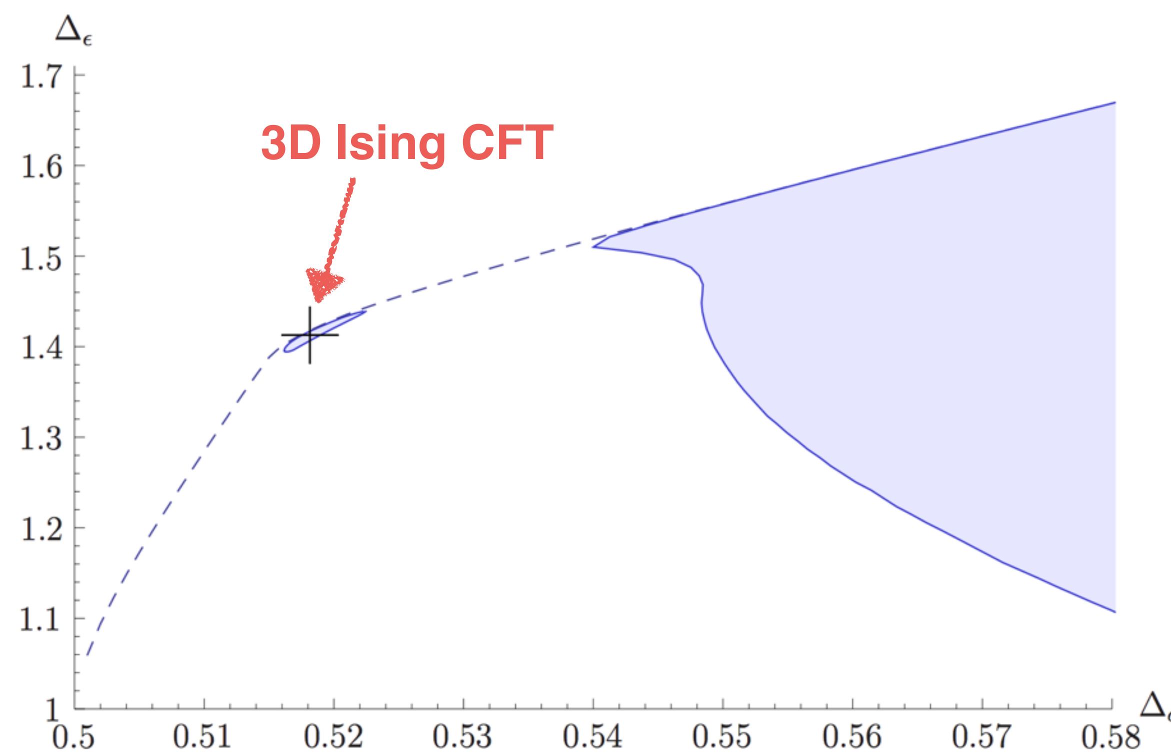
Simons Center & Yang Institute, Stony Brook

work with M. Paulos (to appear)
see also DM: arXiv:1611.10060

Motivation

The conformal bootstrap equations place strong constraints on the CFT data.





[Kos, Poland, Simmons-Duffin, '14]

So far, to a certain extent, a numerical black box

Can we understand the origin of these constraints analytically?
What is the underlying structure?

Analytic conformal bootstrap to date

2D CFT with $0 < c < 1$ ✓ [Belavin, Polyakov, Zamolodchikov, '84]

The strongest result in general D: a Lorentzian inversion formula [Caron-Huot, '17]

"The CFT data of exchanged operators with spin ≥ 2
are fixed by the singularity in the crossed channel."

$$\Downarrow$$

Unification of many results obtained using the large spin expansion

[Komargodski, Zhiboedov; Fitzpatrick, Kaplan, Poland, Simmons-Duffin; Alday, Bissi, Lukowski, Aharony, Perlmutter, …]

A drawback: no straightforward constraints on the scalar spectrum

progress still can be made: [Simmons-Duffin, '17]

To understand the constraints on the spectrum of scalar operators,
we will work in 1D, since there the spin is always zero.

See also a promising approach using Polyakov's unitary blocks

[Polyakov, '74; Gopakumar, Kaviraj, Sen, Sinha, Dey, Ghosh '16,17]

Conformal symmetry in 1D

Spacetime acted on by the symmetry algebra $so(1,2) = sl(2,\mathbb{R})$

Generators $P,\ D,\ K$ (translation, dilatation, special conformal transformation)

Primary operators $\mathcal{O}_j(x)$: $[K, \mathcal{O}_j(0)] = 0,\ [D, \mathcal{O}_j(0)] = \Delta_j \mathcal{O}_j(0)$

A theory completely specified by the CFT data $\{\Delta_j\}$, $\{c_{ijk}\}$

← structure constants

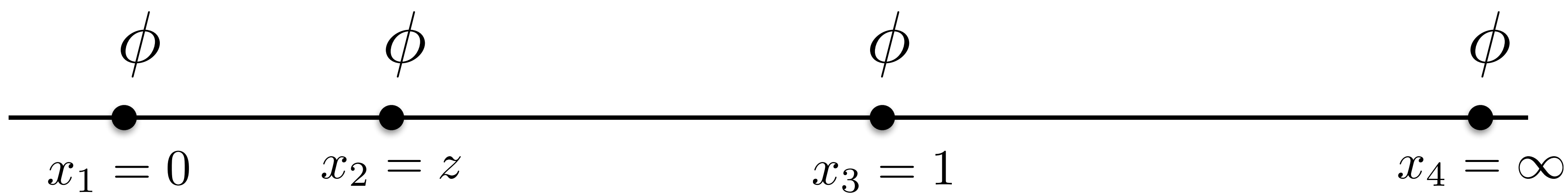
$T_{11} = 0 \Rightarrow$ no stress tensor $\Longrightarrow$ the theory must be non-local

There are many interesting examples of such theories:

- conformal boundaries, interfaces, line defects in higher-D CFTs
- cSYK model **[Gross, Rosenhaus, '17]**
- non-gravitational (1+1)D QFTs placed in AdS_2
- higher-D CFTs restricted to a line in spacetime

The 1D conformal bootstrap: Part I

Study the four-point function of a primary operator $\phi, \quad a \equiv \Delta_\phi$



Single cross-ratio $z = \dfrac{x_{12}x_{34}}{x_{13}x_{24}} \in (0,1) \qquad x_{ij} = x_i - x_j$

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{|x_{12}x_{34}|^{2a}}\mathcal{G}(z)$$

s-channel expansion: $z \to 0 \qquad \mathcal{G}(z) = \sum_{\mathcal{O}\in\phi\times\phi} c^2_{\phi\phi\mathcal{O}}\, G_{\Delta_\mathcal{O}}(z)$

conformal blocks

$G_\Delta(z) = z^\Delta {}_2F_1(\Delta,\Delta;2\Delta;z)$

Both convergent for $0 < z < 1$

t-channel expansion: $z \to 1 \qquad \mathcal{G}(z) = \left(\frac{z}{1-z}\right)^{2a} \sum_{\mathcal{O}\in\phi\times\phi} c^2_{\phi\phi\mathcal{O}}\, G_{\Delta_\mathcal{O}}(1-z)$

The 1D conformal bootstrap: Part II

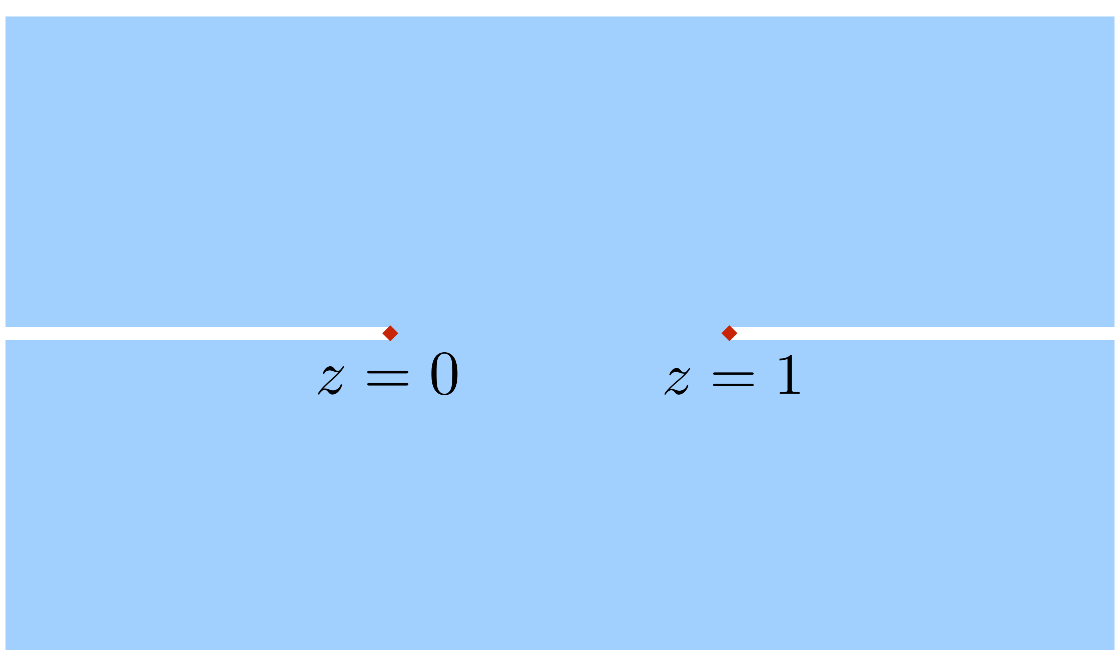
s- and t-channel expansions must be equal

$$\sum_{\mathcal{O}\in\phi\times\phi} c^2_{\phi\phi\mathcal{O}} \frac{G_{\Delta_\mathcal{O}}(z)}{z^{2a}} = \sum_{\mathcal{O}\in\phi\times\phi} c^2_{\phi\phi\mathcal{O}} \frac{G_{\Delta_\mathcal{O}}(1-z)}{(1-z)^{2a}}$$

Rewrite this as a sum of vectors with positive coefficients (unitarity)

$$\sum_{\mathcal{O}\in\phi\times\phi} c^2_{\phi\phi\mathcal{O}} F_{\Delta_\mathcal{O}}(z) = 0 \qquad \text{where} \qquad F_{\Delta_\mathcal{O}}(z) = \frac{G_{\Delta_\mathcal{O}}(z)}{z^{2a}} - (z \leftrightarrow 1-z)$$

Let z be complex. The equation holds in the common region of convergence of the s- and t-channel OPE



The 1D bootstrap equation lives in the vector space $\mathcal{V}$ of functions $\mathcal{F}(z)$ holomorphic in the blue region, such that

$$\mathcal{F}(z) = -\mathcal{F}(1-z)$$

Specify behaviour as
$z \to 0, 1, \infty$

The simplest solutions of the 1D bootstrap

Free fields in AdS_2

1) Free massive scalar: $m_\phi^2 R_{\mathrm{AdS}}^2 = a(a-1)$

$$\Delta_n = 2a + 2n$$

$$\mathcal{G}_{\mathrm{b}}(z) = 1 + \left(\frac{z}{1-z}\right)^{2a} + z^{2a} = 1 + \sum_{n=0}^{\infty} \lambda_n G_{\Delta_n}(z)$$

$$\lambda_n = \frac{2\Gamma(2a+2n)^2\Gamma(4a+2n-1)}{\Gamma(2a)^2\Gamma(2n+1)\Gamma(4a+4n-1)}$$

2) Free massive fermion: $\Delta_\psi = a$

$$\tilde{\Delta}_n = 2a + 2n + 1$$

$$\mathcal{G}_{\mathrm{f}}(z) = 1 + \left(\frac{z}{1-z}\right)^{2a} - z^{2a} = 1 + \sum_{n=0}^{\infty} \tilde{\lambda}_n G_{\tilde{\Delta}_n}(z)$$

$$\tilde{\lambda}_n = \frac{2\Gamma(2a+2n+1)^2\Gamma(4a+2n)}{\Gamma(2a)^2\Gamma(2n+2)\Gamma(4a+4n+1)}$$

We will see that in these cases the bootstrap fixes OPE coefficients after we have fixed the spectrum.

The dual space

We get one constraint for every element ω of the dual space $\mathcal{V}^*$.

Define $\omega(\Delta) \equiv \omega(F_\Delta)$. Acting with ω on the bootstrap equation implies

$$\sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}\, \omega(\Delta_{\mathcal{O}}) = 0 \qquad \text{where} \quad \lambda_{\mathcal{O}} \equiv c^2_{\phi\phi\mathcal{O}}$$

An important goal of the analytic bootstrap is to identify a suitable basis $\mathcal{B} = \{\omega_j\}$ for $\mathcal{V}^*$.

But what does "suitable" mean? Simply evaluating the equation at various values of z is not very revealing. Numerical bootstrap instead typically uses odd-order derivatives evaluated at $z = 1/2$, also not too useful for analytics.

We will construct bases which manifest the crossing symmetry of the free boson and fermion theories.

This will allow us to study deformations of these theories consistent with crossing analytically.

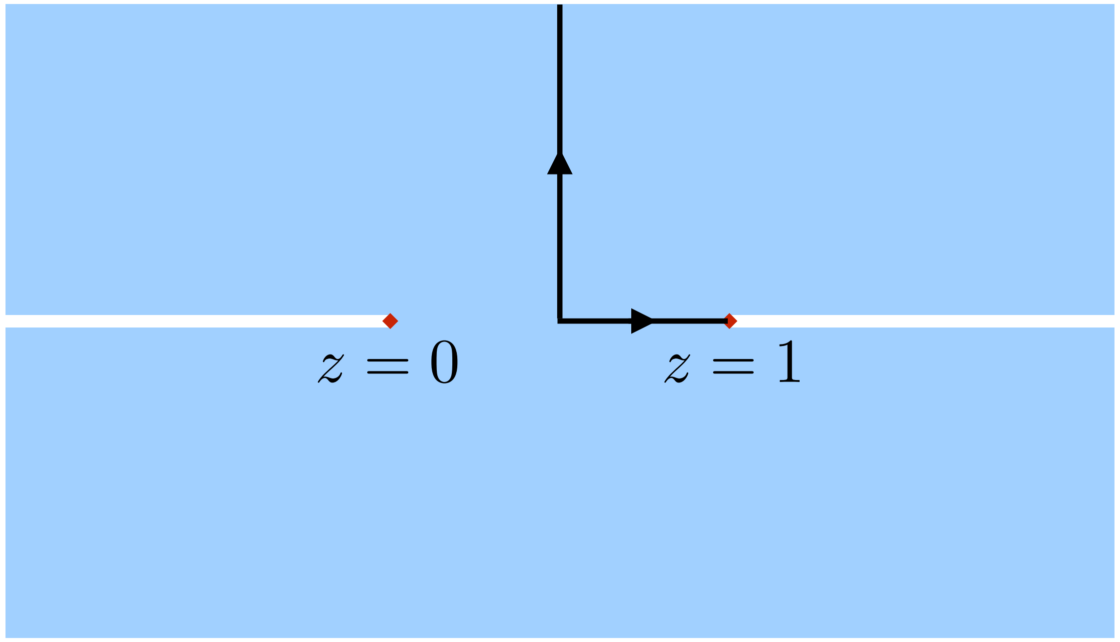
A new class of functionals

Consider functionals taking the form of an integral transform

$$\omega(\mathcal{F}) = \int_{\frac{1}{2}}^{1} dz\, h(z)\mathcal{F}(z) \pm$$

$$\pm \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} dz\, z^{2a-2}\, h(1-1/z)\, \mathcal{F}(z)$$

boson ($+$), fermion ($-$)



$\omega \in \mathcal{V}^*$ provided $h(1-\epsilon) = O(\epsilon^{2a})$ and $h(z) = (-z)^{2a-2}h(1/z)$

If, in addition $h(z) + h(1-z) \pm \mathrm{Re}[z^{2a-2}h(1-1/z)] = 0$, a contour deformation gives

$$\omega(F_\Delta) = 2\left\{1 \mp \cos\left[\pi(\Delta - 2a)\right]\right\} \int_0^1 dz\, h(z) z^{-2a} G_\Delta(z)$$

Fermionic functionals

$$\omega(\Delta) = 4\cos^2\left[\frac{\pi}{2}(\Delta - 2a)\right]\int_0^1 dz\, h(z) z^{-2a} G_\Delta(z)$$

The solutions of the functional equations classified by the behaviour as $z \to 0$

Two infinite classes of solutions $h_m = O(z^{-2m-2})$ for $m = 0, 1, \ldots$

$\hat{h}_m = O(\log(z) z^{-2m-2})$ for $m = 0, 1, \ldots$

The singularity at $z = 0$ cancels the double pole at $\Delta = 2a + 2m + 1$

We find the following behaviour at the free fermion spectrum $\Delta_n = 2a + 2n + 1$

$$\omega_m(\Delta_n) = 0 \qquad \omega'_m(\Delta_n) = \delta_{mn}$$
$$\hat{\omega}_m(\Delta_n) = \delta_{mn} \qquad \hat{\omega}'_m(\Delta_n) = 0$$

It can be shown these functionals form a basis for the dual space.

The free fermion crossing-symmetric $\Leftrightarrow$ $\hat{\omega}_m(0) = -\tilde{\lambda}_m$

More on the fermionic functionals

Closed formula for $h_m(z)$ when $a = 1/2$

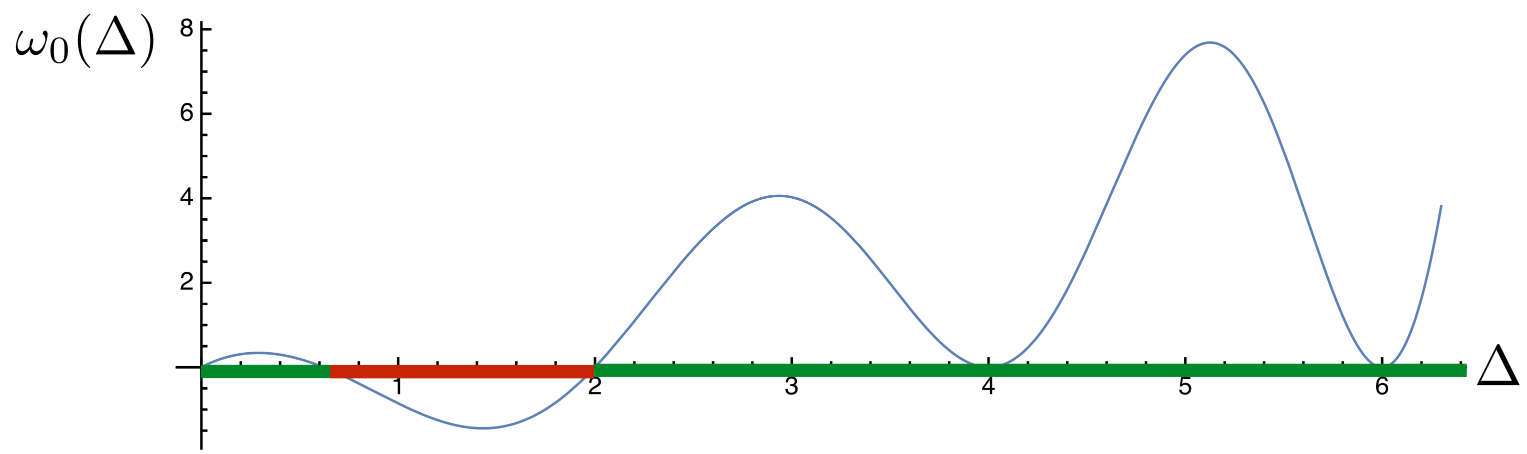
Legendre polynomials

$$h_m(z) = z^{-1} P_{2m+1}(2/z - 1) - P_{2m+1}(2z - 1)$$

Closed formula for the action of ω_0 for $a = 1/2$

Trigamma function

$$\omega_0(\Delta) = \frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2} \sin^2\left(\frac{\pi\Delta}{2}\right) \left\{ \frac{1}{(\Delta-2)(\Delta+1)} - \left[\Delta(\Delta-1) + \frac{1}{2}\right] \left[\psi'\left(\frac{\Delta+1}{2}\right) - \psi'\left(\frac{\Delta}{2}\right)\right] - 2 \right\}$$



For any a, functional ω_0 proves that the free fermion maximizes the gap among all unitary 1D solutions to crossing: $\Delta_{\text{gap}} \leq 2a + 1$

Dream: are there analytic formulas for the functionals of interacting extremal CFTs?

Deformations of the free fermion solution

A deformation of the free fermion, such that no new operators appear in the OPE:

$$\Delta_n(g) = 2a + 2n + 1 + \gamma_n^{(1)} g + O(g^2)$$

$$\lambda_n(g) = \lambda_n^{(0)} + \lambda_n^{(1)} g + O(g^2)$$

Impose crossing symmetry

$$F_0(z) + \sum_{n=0}^{\infty} \lambda_n(g) F_{\Delta_n(g)}(z) = 0$$

Apply ω_m and expand to the first order in g $\quad \Rightarrow \quad \gamma_m^{(1)} = 0$

Apply $\hat{\omega}_m$ and expand to the first order in g $\quad \Rightarrow \quad \lambda_m^{(1)} = 0$

The free fermion admits no deformations
unless we introduce new states in the OPE!

Bosonic functionals

$$\omega(\Delta) = 4\sin^2\left[\frac{\pi}{2}(\Delta - 2a)\right]\int_0^1 dz\, h(z) z^{-2a} G_\Delta(z)$$

The structure of solutions to the functional equations is similar to the fermionic case except one functional is "missing".

$\Delta_n = 2a + 2n,\ n = 0, 1, \ldots$

Specifically, every functional vanishing on the whole spectrum has at least two simple zeros in the spectrum, rather than one.

The complete basis now consists of ω_m for $m = 1, 2, \ldots$

and $\hat{\omega}_m$ for $m = 0, 1, \ldots$

such that

$\omega_m(\Delta_n) = 0 \text{ for } m \geq 1, n \geq 0$	$\omega'_m(\Delta_n) = \delta_{mn} \text{ for } m, n \geq 1$
$\hat{\omega}_m(\Delta_n) = \delta_{mn} \text{ for } m, n \geq 0$	$\hat{\omega}'_m(\Delta_n) = 0 \text{ for } m \geq 0, n \geq 1$

but $\omega'_m(\Delta_0),\ \hat{\omega}'_m(\Delta_0) \neq 0$

Legendre polynomials

For example, for $a = 1$ $\quad h_m(z) = P_{2m+1}(2/z - 1) + P_{2m+1}(2z - 1) - 2\left(\frac{1}{z} + z - 1\right)$

Hence $h_0(z) = 0$ (missing functional)

Deformations of the free boson solution

A deformation of the free boson, such that no new operators appear in the OPE:

$$\Delta_n(g) = 2a + 2n + \gamma_n^{(1)} g + \gamma_n^{(2)} g^2 + O(g^3)$$

$$\lambda_n(g) = \lambda_n^{(0)} + \lambda_n^{(1)} g + \lambda_n^{(2)} g^2 + O(g^3)$$

One functional missing $\Leftrightarrow$ A one-parameter deformation allowed

Define the coupling as the anomalous dimension of the first operator $g = \Delta_0(g) - 2a$

$$a = 1$$

$$\Delta_0 = 2 + g$$

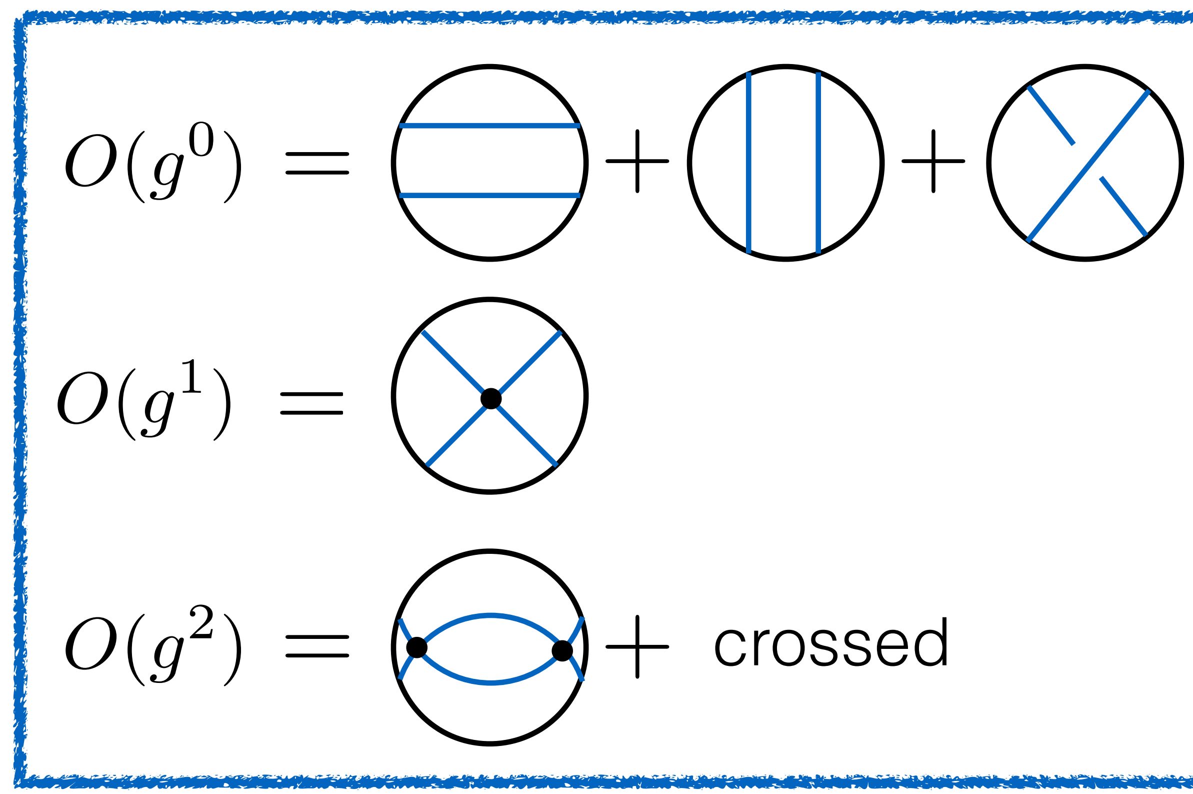
$$\Delta_1 = 4 + \frac{1}{6} g + \left(\frac{317}{144} - \frac{5}{3}\zeta(3)\right) g^2 + O(g^3)$$

$$\Delta_2 = 6 + \frac{1}{15} g + \left(\frac{25127}{10800} - \frac{28}{15}\zeta(3)\right) g^2 + O(g^3)$$

$$\lambda_0 = 2 - 2g + \left(\frac{5}{2} - 4\zeta(3) + \frac{\pi^4}{15}\right) g^2 + O(g^3)$$

$$\lambda_1 = \frac{6}{5} - \frac{37}{150} g + \left(-\frac{612119}{108000} + \frac{19\zeta(3)}{15} + \frac{\pi^4}{25}\right) g^2 + O(g^3)$$

$$\lambda_2 = \frac{5}{21} - \frac{1627}{79380} g + \left(-\frac{3889170127}{3000564000} + \frac{1177\zeta(3)}{2835} + \frac{\pi^4}{126}\right) g^2 + O(g^3)$$



Reproduce field theory in AdS_2

d>1:

[Heemskerk, Penedones, Polchinski, Sully, '09]
[Aharony, Alday, Bissi, Perlmutter, '16]

Outlook

Does the deformation of the free boson exist for a finite coupling?

What is its relationship to the sine-Gordon theory in AdS_2?

The physical origin of the functional kernels?

The relationship to the Caron-Huot formula and Polyakov bootstrap? ✓

$d > 1$

Thank you!