Bootstrapping the Long-Range Ising Model

Connor Behan

NUMSTRINGS 2018
2018-02-03

1703.03430, 1703.05325 with L. Rastelli, S. Rychkov, B. Zan
180x.xxxxx upcoming

The model

$$H_{LRI} = -J \sum_{i \neq j} \frac{\sigma_i \sigma_j}{|i-j|^{d+s}}$$

- Known to have a second-order phase transition in $1 \leq d < 4$ [Dyson; 69].
- Possible to study with a ϕ^4 interaction [Fisher, Ma, Nickel; 72].
- Critical exponents are non-trivial functions of s for $\frac{d}{2} < s < s_*$ [Sak; 73].
- 1D and 2D estimates have been found by Monte Carlo [Angelini, Parisi, Ricci-Tersenghi; 1401.6805].
- Fixed point is known to be conformal [Paulos, Rychkov, van Rees, Zan; 1509.00008].

Continuum description

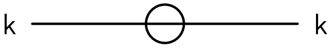
$$S = \int \int -\frac{\phi(x)\phi(y)}{|x-y|^{d+s}} dy + \frac{\lambda}{4!}\phi(x)^4 dx$$

Coupling is classically marginal for $s = \frac{d}{2} \implies$ perturb in $\epsilon = 2s - d$.

Continuum description

$$S = \int \int -\frac{\phi(x)\phi(y)}{|x-y|^{d+s}} dy + \frac{\lambda}{4!}\phi(x)^4 dx$$

Coupling is classically marginal for $s = \frac{d}{2} \implies$ perturb in $\epsilon = 2s - d$.

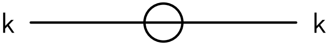


$$= -\frac{\lambda^2}{6(4\pi)^d} \frac{\Gamma\left(\frac{3\epsilon+d}{4}\right)\Gamma\left(\frac{3\epsilon-d}{4}\right)}{\Gamma\left(\frac{\epsilon+d}{4}\right)\Gamma\left(\frac{3d}{4}\right)} |k|^{-(d+3\epsilon)/2}$$

Continuum description

$$S = \int \int -\frac{\phi(x)\phi(y)}{|x-y|^{d+s}} dy + \frac{\lambda}{4!}\phi(x)^4 dx$$

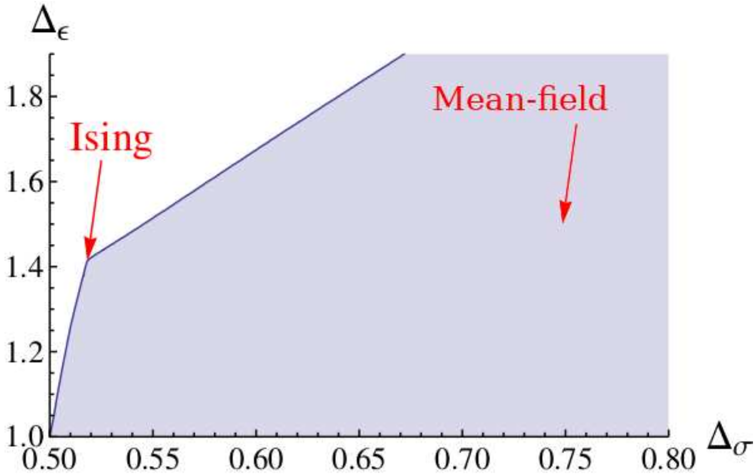
Coupling is classically marginal for $s = \frac{d}{2} \implies$ perturb in $\epsilon = 2s - d$.



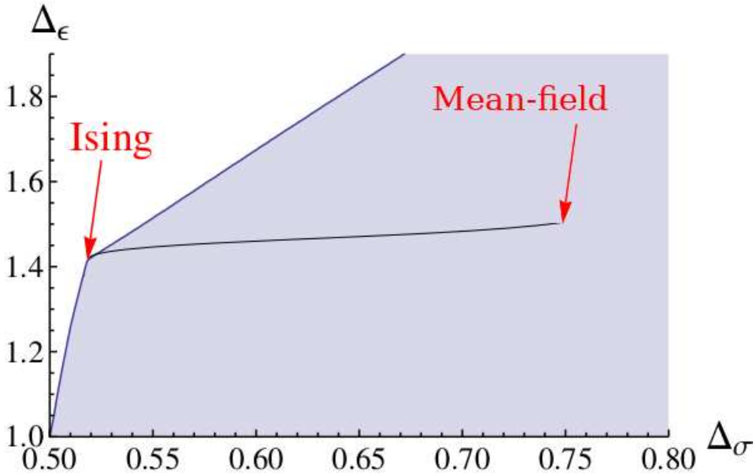
$$= -\frac{\lambda^2}{6(4\pi)^d} \frac{\Gamma\left(\frac{3\epsilon+d}{4}\right)\Gamma\left(\frac{3\epsilon-d}{4}\right)}{\Gamma\left(\frac{\epsilon+d}{4}\right)\Gamma\left(\frac{3d}{4}\right)} |k|^{-(d+3\epsilon)/2}$$

At all loop orders we expect $\Delta_\phi = \frac{d-s}{2}$, proven rigorously in [Lohmann, Slade, Wallace; 1705.08540].

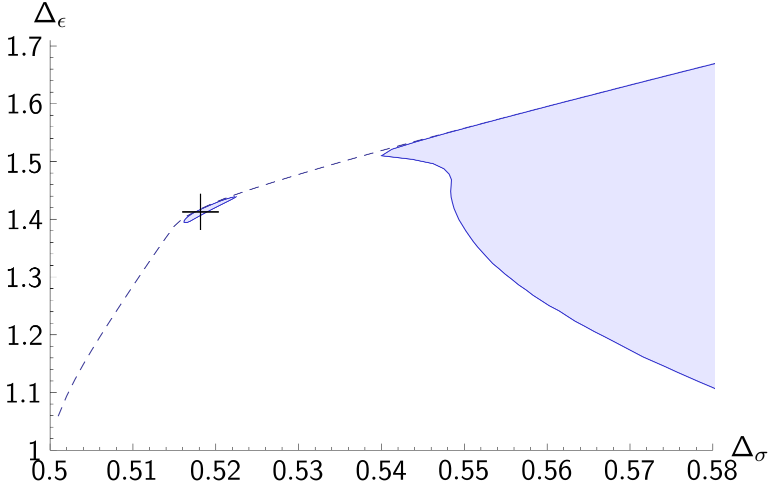
	$s = \frac{d}{2}$	$s = s_*$
Δ_ϕ	$\frac{d}{4}$	$\frac{d-s_*}{2} \equiv \Delta_\sigma^{SRI}$
Δ_{ϕ^2}	$\frac{d}{2}$	Δ_ϵ^{SRI}
Δ_T	$\frac{d+4}{2}$	d



Fixed line allowed by single correlator bound of [El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi; 12].



Fixed line allowed by single correlator bound of [El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi; 12].



Fixed line allowed by single correlator bound of [El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi; 12]. Disallowed by [Kos, Poland, Simmons-Duffin; 14].

The third relevant scalar

- In the usual Wilson-Fisher fixed point, $\partial^2\phi = \frac{\lambda}{3!}\phi^3$.
  ϕ^3 is a descendant because ∂^2 is a conformal generator.

The third relevant scalar

- In the usual Wilson-Fisher fixed point, $\partial^2 \phi = \frac{\lambda}{3!}\phi^3$.
  ϕ^3 is a descendant because ∂^2 is a conformal generator.
- Instead, our EOM is nonlocal: $\partial^s \phi = \frac{\lambda}{3!}\phi^3$.
  ϕ^3 is a primary constrained to have $\Delta_{\phi^3} = \frac{d+s}{2}$.

The third relevant scalar

- In the usual Wilson-Fisher fixed point, $\partial^2\phi = \frac{\lambda}{3!}\phi^3$. ϕ^3 is a descendant because ∂^2 is a conformal generator.
- Instead, our EOM is nonlocal: $\partial^s\phi = \frac{\lambda}{3!}\phi^3$. ϕ^3 is a primary constrained to have $\Delta_{\phi^3} = \frac{d+s}{2}$.

$$n_3(s)\phi^3(x) = \int \frac{n_1(s)\phi(y)}{|x-y|^{d+s}} dy$$

Insert this into $\left\langle \phi^3(x)\Phi_2(y)\Phi_1(z) \right\rangle$ to find

$$\frac{\lambda_{12\phi^3}}{\lambda_{12\phi}} = \frac{\pi^{d/2} n_1(s)}{n_3(s)} \frac{\Gamma(\Delta_\phi - \frac{d}{2})\Gamma\left(\frac{\Delta_{\phi^3}+\Delta_{12}}{2}\right)\Gamma\left(\frac{\Delta_{\phi^3}-\Delta_{12}}{2}\right)}{\Gamma(\Delta_{\phi^3})\Gamma\left(\frac{\Delta_\phi+\Delta_{12}}{2}\right)\Gamma\left(\frac{\Delta_\phi-\Delta_{12}}{2}\right)} \equiv \frac{n_1(s)}{n_3(s)} R_{12}$$

The third relevant scalar

- In the usual Wilson-Fisher fixed point, $\partial^2\phi = \frac{\lambda}{3!}\phi^3$.
  ϕ^3 is a descendant because ∂^2 is a conformal generator.
- Instead, our EOM is nonlocal: $\partial^s\phi = \frac{\lambda}{3!}\phi^3$.
  ϕ^3 is a primary constrained to have $\Delta_{\phi^3} = \frac{d+s}{2}$.

$$n_3(s)\phi^3(x) = \int \frac{n_1(s)\phi(y)}{|x-y|^{d+s}} dy$$

Insert this into $\left\langle \phi^3(x)\Phi_2(y)\Phi_1(z) \right\rangle$ to find

$$\frac{\lambda_{12\phi^3}}{\lambda_{12\phi}} = \frac{\pi^{d/2} n_1(s)}{n_3(s)} \frac{\Gamma(\Delta_\phi - \frac{d}{2})\Gamma\left(\frac{\Delta_{\phi^3}+\Delta_{12}}{2}\right)\Gamma\left(\frac{\Delta_{\phi^3}-\Delta_{12}}{2}\right)}{\Gamma(\Delta_{\phi^3})\Gamma\left(\frac{\Delta_\phi+\Delta_{12}}{2}\right)\Gamma\left(\frac{\Delta_\phi-\Delta_{12}}{2}\right)} \equiv \frac{n_1(s)}{n_3(s)} R_{12}$$

Cancelling normalizations gives the nonperturbative ratio
$\frac{\lambda_{12\phi^3}}{\lambda_{12\phi}} / \frac{\lambda_{34\phi^3}}{\lambda_{34\phi}} = R_{12}/R_{34}$.

Dual description

$$S_1[\phi] = \int \frac{1}{2}\phi\partial^s\phi + \frac{\lambda}{4!}\phi^4 dx$$

$$S_2[\sigma,\chi] = S_{SRI}[\sigma] + \int \frac{1}{2}\chi\partial^{-s}\chi + g\sigma\chi dx$$

Instead of $\epsilon = 2s - d$, we can expand in $\delta = \frac{1}{2}(s_* - s)$.

Dual description

$$S_1[\phi] = \int \frac{1}{2}\phi\partial^s\phi + \frac{\lambda}{4!}\phi^4 dx$$
$$S_2[\sigma, \chi] = S_{SRI}[\sigma] + \int \frac{1}{2}\chi\partial^{-s}\chi + g\sigma\chi dx$$

Instead of $\epsilon = 2s - d$, we can expand in $\delta = \frac{1}{2}(s_* - s)$. Duality passes many checks [B, Rastelli, Rychkov, Zan; 1703.05325].

$$\Delta_\phi = \frac{d-s}{2} = \Delta_\sigma$$
$$\Delta_{\phi^3} = \frac{d+s}{2} = \Delta_\chi$$
$$\frac{\lambda_{12\phi^3}\lambda_{34\phi}}{\lambda_{12\phi}\lambda_{34\phi^3}} = \frac{R_{12}}{R_{34}} = \frac{\lambda_{12\chi}\lambda_{34\sigma}}{\lambda_{12\sigma}\lambda_{34\chi}}$$

Picture also resolves the loss of a stress tensor — $T_{\mu\nu}$ recombines with $\Delta_\sigma\sigma\partial_\nu\chi - \Delta_\chi\chi\partial_\nu\sigma$.

Crossing equations

For a correlator of scalars,

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle = \left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_{ij}} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{kl}} \frac{G(u,v)}{|x_{12}|^{\Delta_i+\Delta_j}|x_{34}|^{\Delta_k+\Delta_l}}$$

$$G(u,v) = \sum_{\mathcal{O}} \lambda_{ij\mathcal{O}}\lambda_{kl\mathcal{O}}\, g_{\mathcal{O}}^{\Delta_{ij},\Delta_{kl}}(u,v)$$

Crossing equations

For a correlator of scalars,

$$\langle\phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle = \left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_{ij}}\left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{kl}}\frac{G(u,v)}{|x_{12}|^{\Delta_i+\Delta_j}|x_{34}|^{\Delta_k+\Delta_l}}$$

$$G(u,v) = \sum_{\mathcal{O}}\lambda_{ij\mathcal{O}}\lambda_{kl\mathcal{O}}\, g_{\mathcal{O}}^{\Delta_{ij},\Delta_{kl}}(u,v)$$

crossing equations are

$$\sum_{\mathcal{O}}\left[\lambda_{ij\mathcal{O}}\lambda_{kl\mathcal{O}}F_{\mp,\mathcal{O}}^{ij;kl}(u,v)\pm\lambda_{kj\mathcal{O}}\lambda_{il\mathcal{O}}F_{\mp,\mathcal{O}}^{kj;il}(u,v)\right]=0$$

$$F_{\pm,\mathcal{O}}^{ij;kl} = v^{\frac{\Delta_k+\Delta_j}{2}}g_{\mathcal{O}}^{\Delta_{ij},\Delta_{kl}}(u,v)\pm u^{\frac{\Delta_k+\Delta_j}{2}}g_{\mathcal{O}}^{\Delta_{ij},\Delta_{kl}}(v,u)\,.$$

Crossing equations

For a correlator of scalars,

$$\langle\phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle = \left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_{ij}} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{kl}} \frac{G(u,v)}{|x_{12}|^{\Delta_i+\Delta_j}|x_{34}|^{\Delta_k+\Delta_l}}$$

$$G(u,v) = \sum_{\mathcal{O}} \lambda_{ij\mathcal{O}}\lambda_{kl\mathcal{O}}\, g_{\mathcal{O}}^{\Delta_{ij},\Delta_{kl}}(u,v)$$

crossing equations are

$$\sum_{\mathcal{O}} \left[\lambda_{ij\mathcal{O}}\lambda_{kl\mathcal{O}} F_{\mp,\mathcal{O}}^{ij;kl}(u,v) \pm \lambda_{kj\mathcal{O}}\lambda_{il\mathcal{O}} F_{\mp,\mathcal{O}}^{kj;il}(u,v)\right] = 0$$

$$F_{\pm,\mathcal{O}}^{ij;kl} = v^{\frac{\Delta_k+\Delta_j}{2}} g_{\mathcal{O}}^{\Delta_{ij},\Delta_{kl}}(u,v) \pm u^{\frac{\Delta_k+\Delta_j}{2}} g_{\mathcal{O}}^{\Delta_{ij},\Delta_{kl}}(v,u)\ .$$

We consider 6 of the 9 combinations:

$$\langle\sigma\sigma\sigma\sigma\rangle\ ,\ \langle\epsilon\epsilon\epsilon\epsilon\rangle\ ,\ \langle\chi\chi\chi\chi\rangle$$

$$\langle\sigma\sigma\epsilon\epsilon\rangle\ ,\ \langle\sigma\sigma\chi\chi\rangle\ ,\ \langle\epsilon\epsilon\chi\chi\rangle$$

Crossing equations

For each identical correlator:

$$\sum_{\mathcal{O}} \lambda_{ii\mathcal{O}}^2 F_{-,\mathcal{O}}^{ii;ii}(u,v) = 0$$

Crossing equations

For each identical correlator:

$$\sum_{\mathcal{O}} \lambda_{ii\mathcal{O}}^2 F_{-,\mathcal{O}}^{ii;ii}(u,v) = 0$$

For each mixed correlator:

$$\sum_{\mathcal{O}} \lambda_{ij\mathcal{O}}^2 F_{-,\mathcal{O}}^{ij;ij}(u,v) = 0$$

$$\sum_{\mathcal{O}} \lambda_{ii\mathcal{O}} \lambda_{jj\mathcal{O}} F_{-,\mathcal{O}}^{ii;jj}(u,v) + \sum_{\mathcal{O}} (-1)^\ell \lambda_{ij\mathcal{O}}^2 F_{-,\mathcal{O}}^{ji;ij}(u,v) = 0$$

$$\sum_{\mathcal{O}} \lambda_{ii\mathcal{O}} \lambda_{jj\mathcal{O}} F_{+,\mathcal{O}}^{ii;jj}(u,v) - \sum_{\mathcal{O}} (-1)^\ell \lambda_{ij\mathcal{O}}^2 F_{+,\mathcal{O}}^{ji;ij}(u,v) = 0$$

Gives equations labelled by $n = 1, \ldots, 12$.

Crossing equations

$$\sum_{\mathcal{O}+,2|\ell} [\lambda_{\sigma\sigma\mathcal{O}} \; \lambda_{\epsilon\epsilon\mathcal{O}} \; \lambda_{\chi\chi\mathcal{O}}] A^{n}_{\Delta,\ell} \begin{bmatrix} \lambda_{\sigma\sigma\mathcal{O}} \\ \lambda_{\epsilon\epsilon\mathcal{O}} \\ \lambda_{\chi\chi\mathcal{O}} \end{bmatrix}$$

$$+ \sum_{\mathcal{O}-} \lambda^2_{\sigma\epsilon\mathcal{O}} B^{n}_{\Delta,\ell} + \sum_{\mathcal{O}-} \lambda^2_{\epsilon\chi\mathcal{O}} C^{n}_{\Delta,\ell} + \sum_{\mathcal{O}+} \lambda^2_{\sigma\chi\mathcal{O}} D^{n}_{\Delta,\ell} = 0$$

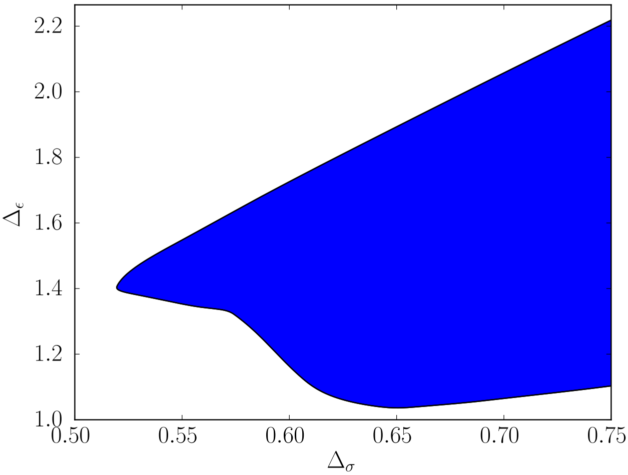
Crossing equations

$$\sum_{\mathcal{O}+,2|\ell} [\lambda_{\sigma\sigma\mathcal{O}}\ \lambda_{\epsilon\epsilon\mathcal{O}}\ \lambda_{\chi\chi\mathcal{O}}] A^n_{\Delta,\ell} \begin{bmatrix} \lambda_{\sigma\sigma\mathcal{O}} \\ \lambda_{\epsilon\epsilon\mathcal{O}} \\ \lambda_{\chi\chi\mathcal{O}} \end{bmatrix}$$
$$+ \sum_{\mathcal{O}-} \lambda^2_{\sigma\epsilon\mathcal{O}} B^n_{\Delta,\ell} + \sum_{\mathcal{O}-} \lambda^2_{\epsilon\chi\mathcal{O}} C^n_{\Delta,\ell} + \sum_{\mathcal{O}+} \lambda^2_{\sigma\chi\mathcal{O}} D^n_{\Delta,\ell} = 0$$

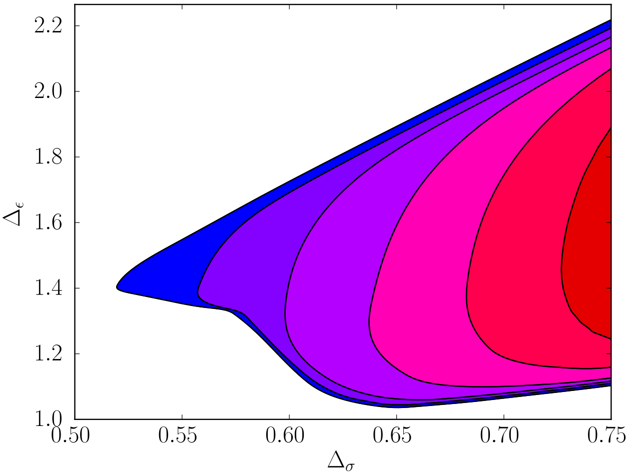
Search for functional α satisfying:

$$\begin{aligned} \alpha(A^n_{\Delta,\ell}) &\succeq 0 \\ \alpha(B^n_{\Delta,\ell}) &\geq 0 \\ \alpha(C^n_{\Delta,\ell}) &\geq 0 \\ \alpha(D^n_{\Delta,\ell}) &\geq 0 \end{aligned}$$

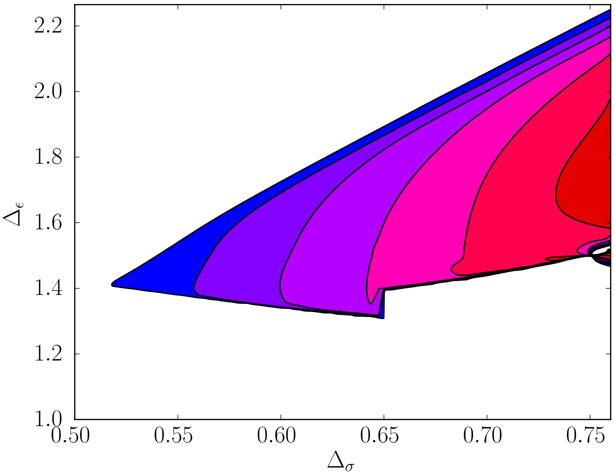
Demand these for $\Delta \in [\Delta_{unitary}, \infty)$ when $\ell = 1, 2, 3, \ldots$ or $\Delta \in \{\Delta_\sigma, \Delta_\epsilon, \Delta_\chi\} \cup [3, \infty)$ when $\ell = 0$.



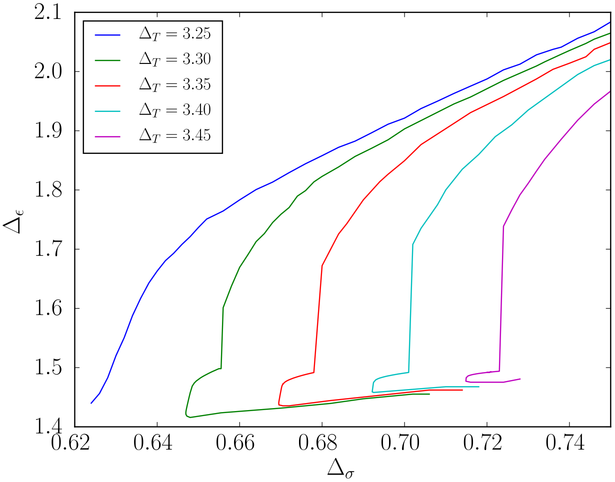
Bound should become more restrictive as the minimum dimension for spin-2 operators goes from 3 to 3.5.



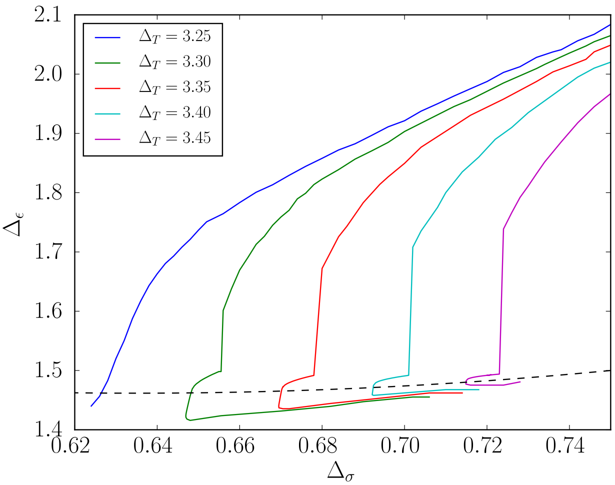
No interesting features but we have not yet imposed $\lambda^2_{\sigma\epsilon\chi} = \frac{R_{\chi\epsilon}}{R_{\sigma\epsilon}} \lambda_{\sigma\sigma\epsilon}\lambda_{\chi\chi\epsilon}$.



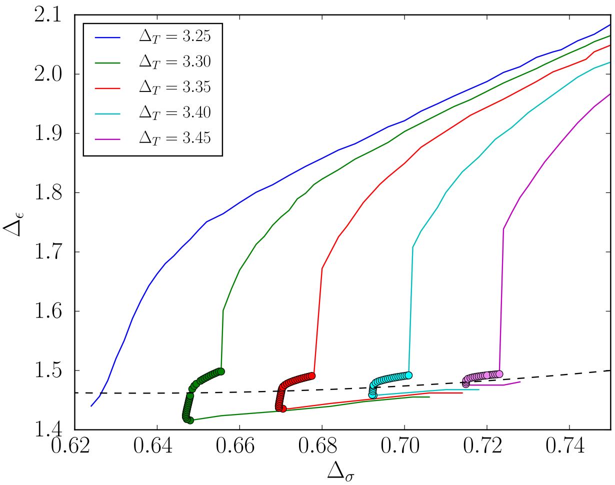
No interesting features for $\Delta_T^{min} = 3.1, 3.2, 3.3$ but there is a kink for 3.4!



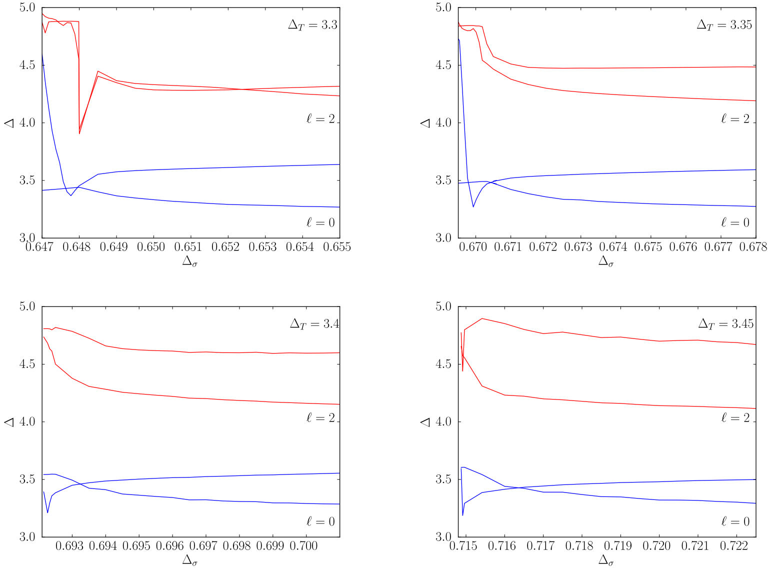
With less truncation, there are kinks at $\Delta_T^{min} \leq 3.3$ having good agreement with the ε-expansion.



With less truncation, there are kinks at $\Delta_T^{min} \leq 3.3$ having good agreement with the ε-expansion.



With less truncation, there are kinks at $\Delta_T^{min} \leq 3.3$ having good agreement with the ε-expansion.


$\Delta_T = 3.3$
$\ell = 2$
$\ell = 0$
Δ
Δ_σ
5.0
4.5
4.0
3.5
3.0
0.647 0.648 0.649 0.650 0.651 0.652 0.653 0.654 0.655
$\Delta_T = 3.35$
0.670 0.671 0.672 0.673 0.674 0.675 0.676 0.677 0.678
$\Delta_T = 3.4$
0.693 0.694 0.695 0.696 0.697 0.698 0.699 0.700
$\Delta_T = 3.45$
0.715 0.716 0.717 0.718 0.719 0.720 0.721 0.722

Conclusions

- It is easy for nonlocal CFTs to exist in continuous families.
- 3D long-range Ising models occupy special points in the regions allowed by six four-point functions.
- Extension to long-range $O(N)$ models should be straightforward.
- Some features of a full solution are still missing.

Conclusions

- It is easy for nonlocal CFTs to exist in continuous families.
- 3D long-range Ising models occupy special points in the regions allowed by six four-point functions.
- Extension to long-range $O(N)$ models should be straightforward.
- Some features of a full solution are still missing.

Spin-2 operator could be added to the system of correlators [Dymarsky, Kos, Kravchuk, Poland, Simmons-Duffin; 1708.05718].
Finding kinks could still be possible in 2D [Paulos, Penedones, Toledo, van Rees, Vieira; 1708.06765].
Analytic bootstrap techniques might accomodate these theories [Fitzpatrick, Kaplan, Poland, Simmons-Duffin; Komargodski, Zhiboedov; Gopakumar, Kaviraj, Sen, Sinha; Alday, Caron-Huot].