$\mathcal{O}(a)$ Improvement of 2D $\mathcal{N} = (2,2)$ Lattice SYM Theory

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Mainly based on work with M. Hanada, S. Matsuura and D. Kadoh, arXiv:1711.02319 [hep-lat]

Outline

Introduction

2D $\mathcal{N} = (2,2)$ SYM – continuum and lattice –

Improvement of the lattice theory

Summary and discussion

Introduction (1)

SUSY on lattice (Recent activities)

Bergner, Catterall, D'Adda, Damgaard, Giedt, Hanada, Jha, Joseph, Kadoh, Kamata, Kanamori, Kaplan, Kato, Katz, Kawamoto, Matsuura, Misumi, Montvay, Munster, Ohta, Poppitz, Sakamoto, Schaich, So, Suzuki, Unsal, Wenger, Wipf, Wiseman, ...

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- Possible to realize nilpotent part of the supercharges $Q^2 \sim 0$.

  Orbifolding/Deconstruction, Twisted supercharges

  [S. Catterall's lecture]

  c.f.) Conventional approach [A. Wipf's and G. Bergner's talks]

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- Examples) 2D $\mathcal{N} = (2,2), (4,4), (8,8)$ SYM with no fine tuning
- Hybrid approach to 4D $\mathcal{N} = 4$ SYM [M. Hanada's lecture]

Mass-deformed 2D $\mathcal{N} = (8,8)$ on $\mathbf{R}^2 \Rightarrow$ 4D $\mathcal{N} = 4$ on $S^2_F \times \mathbf{R}^2$

Introduction (2)

Improvement to perform precision measurements

- Improving the lattice action to achieve faster convergence

  Symanzik improvement program in lattice QCD

  In general, $S_{\mathrm{lat}} = S_{\mathrm{cont}} + \mathcal{O}(a)$.

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- We would like to consider a similar improvement for 2D $\mathcal{N} = (2, 2)$ SYM.

Introduction (3)

$\mathcal{O}(a)$ tree-level improvement

As a first step, we make the improvement

$$S_{\rm lat} = S_{\rm cont} + \mathcal{O}(a) \qquad \Rightarrow \qquad S_{\rm lat} = S_{\rm cont} + \mathcal{O}(a^2)$$

with preserving gauge symmetry, Q-SUSY and $U(1)_A$ R-symmetry on the lattice.

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  1. Write the continuum action in a Q-exact form: $S_{\rm cont} = Q\frac{1}{2g^2}\int d^2x\,\Xi(x)$.

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  1. Write the continuum action in a Q-exact form: $S_{\rm cont} = Q\frac{1}{2g^2}\int d^2x\, \Xi(x)$.
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- 3 Steps of the lattice formulation
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  2. Construct a lattice counterpart of Q-transformation, $Q_{\rm lat}$ (exact symmetry on the lattice)
  3. Define a lattice action as a $Q_{\rm lat}$-exact form: $S_{\rm lat} = Q_{\rm lat}\frac{a^2}{2g^2}\sum_x \Xi_{\rm lat}(x)$ with $\Xi_{\rm lat}(x)$ being a lattice counterpart of $\Xi(x)$.

We will consider the improvement of $Q_{\rm lat}$ and $\Xi_{\rm lat}(x)$, separately.

2D $\mathcal{N} = (2,2)$ SYM – continuum – (1)

- Action for $SU(N)$ gauge group on 2D Euclidean space:

$$
\begin{aligned}
S_{\text{cont}} \quad = \quad \frac{1}{2g^2}\int d^2x \, \mathrm{tr}\Big\{ & \frac{1}{2}F_{\mu\nu}^2 + D_\mu\phi D_\mu\bar{\phi} + \frac{1}{4}[\phi,\,\bar{\phi}]^2 - D^2 \\
& +2\bar{\lambda}_R(D_1 - iD_2)\lambda_R + 2\bar{\lambda}_L(D_1 + iD_2)\lambda_L \\
& +2\bar{\lambda}_R[\bar{\phi},\lambda_L] + 2\bar{\lambda}_L[\phi,\lambda_R]\Big\},
\end{aligned}
$$

where $\mu,\nu = 1,2$, and L, R are spinor indices.

- Four Supercharges $Q_L, Q_R, \bar{Q}_L, \bar{Q}_R$

  Nilpotent combination $Q = -\frac{1}{\sqrt{2}}\left(Q_L + \bar{Q}_R\right)$

- $U(1)_R$ symmetries: ($U(1)_V$, $U(1)_A$) charges

$$
\begin{aligned}
&\phi : (0, +2), \quad \lambda_L : (+1, +1), \quad \lambda_R : (+1, -1), \\
&\bar{\phi} : (0, -2), \quad \bar{\lambda}_L : (-1, -1), \quad \bar{\lambda}_R : (-1, +1).
\end{aligned}
$$

2D $\mathcal{N} = (2,2)$ SYM – continuum – (2)

- Q-transformation to the twisted variables

$$\psi_1 \equiv \frac{1}{\sqrt{2}}(\lambda_L + \bar{\lambda}_R), \qquad \psi_2 \equiv \frac{i}{\sqrt{2}}(\lambda_L - \bar{\lambda}_R),$$
$$\chi \equiv \frac{1}{\sqrt{2}}(\lambda_R - \bar{\lambda}_L), \qquad \eta \equiv -i\sqrt{2}(\lambda_R + \bar{\lambda}_L)$$

with $H = iD + iF_{12}$ is expressed as

$$QA_\mu = \psi_\mu, \quad Q\psi_\mu = iD_\mu\phi, \qquad Q\phi = 0,$$
$$Q\bar{\phi} = \eta, \quad Q\eta = [\phi, \bar{\phi}], \qquad Q\chi = H, \quad QH = [\phi, \chi].$$

$\Rightarrow Q^2 = -i\delta_\phi$: infinitesimal gauge transf. with parameter ϕ

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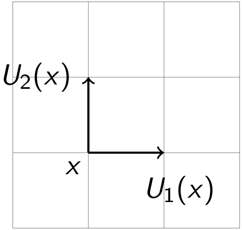
- The action is recast as a Q-exact form [Witten 1988]

$$S_{\rm cont} = Q\,\frac{1}{2g^2}\int d^2x\,\Xi_{\rm cont}(x),$$
$$\Xi_{\rm cont}(x) \equiv \mathrm{tr}\left[\frac{1}{4}\eta[\phi,\, \bar{\phi}] - i\psi_\mu D_\mu\bar{\phi} + \chi(H - 2iF_{12})\right].$$

2D $\mathcal{N} = (2,2)$ SYM – lattice – (1)

[F.S. 2003]

- 2D regular lattice (spacing a)
- Gauge fields $U_\mu(x) \simeq e^{iaA_\mu(x)} \in SU(N)$ on the link $(x, x+\hat{\mu})$
- Other fields (dimensionful) on sites.



- Q-transformation on the lattice:

$$QU_\mu(x) = ia\psi_\mu(x)\, U_\mu(x), \qquad Q\psi_\mu(x) = i\nabla_\mu\phi(x) + ia\psi_\mu(x)\psi_\mu(x),$$
$$Q\phi(x) = 0,$$
$$Q\bar{\phi}(x) = \eta(x), \qquad Q\eta(x) = [\phi(x),\, \bar{\phi}(x)],$$
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where ∇_μ (∇^*_μ) is forward (backward) covariant difference.

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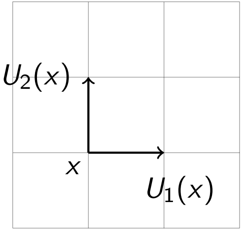

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QU_\mu(x) &= ia\psi_\mu(x)\, U_\mu(x), & Q\psi_\mu(x) &= i\nabla_\mu\phi(x) + ia\psi_\mu(x)\psi_\mu(x), \\
Q\phi(x) &= 0, & & \\
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where ∇_μ (∇^*_μ) is forward (backward) covariant difference.

- Nilpotency of Q up to the gauge transf. is exactly realized on the lattice.

2D $\mathcal{N} = (2,2)$ SYM – lattice – (2)

[F.S. 2004]

- Lattice action

$$S_{\rm lat} = Q\,\frac{a^2}{2g^2}\sum_x \Xi_{\rm lat}(x)$$

with

$$\begin{aligned}\Xi_{\rm lat}(x) \;\;&\equiv\;\; {\rm tr}\left\{\frac{1}{4}\eta(x)[\phi(x),\bar{\phi}(x)] - i\sum_{\mu=1}^{2}\psi_\mu(x)\nabla_\mu\bar{\phi}(x)\right.\\ &\qquad\left. +\chi(x)\left(H(x) - \frac{i}{a^2}\Phi_{\rm TL}(x)\right)\right\}.\end{aligned}$$

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We can choose lattice counterpart of the field strength $\Phi_{\rm TL} = \Phi - \left(\frac{1}{N}{\rm tr}\,\Phi\right)\mathbf{1}_N \to 2a^2F_{12}$ as

$$\Phi(x) = \begin{cases} \frac{-i(U_{12}(x)-U_{21}(x))}{1-\frac{1}{\epsilon^2}||1-U_{12}(x)||^2} & \text{for } ||1-U_{12}(x)|| < \epsilon, \\ \infty & \text{otherwise.} \end{cases}$$

$\Rightarrow$ The action has the unique minimum $U_{12}(x) = 0$.

2D $\mathcal{N} = (2,2)$ SYM – lattice – (3) [F.S. 2004]

- ϵ is a constant independent of a.
  Similar to the admissibility condition of the lattice chiral $U(1)$ gauge theory [Luscher 1999]
- Another choice of $\Phi(x)$ [Matsuura, F.S. 2014]
- No fermion doubling
- Exact symmetries (gauge symmetry, Q-SUSY, $U(1)_A$) on the lattice
  $\Rightarrow$ No radiative corrections preventing from flowing to the correct continuum theory
  - Shown to all orders in perturbation theory [F.S. 2004]
  - Shown nonperturbatively by numerical simulation
    [Kanamori, Suzuki 2009] [Hanada, Kanamori 2009]

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- A similar lattice formulation for 2D $\mathcal{N} = (4,4), (8,8)$ with preserving 2 supercharges
  Application to gauge/gravity duality [Kadoh, Kamata 2015]

Improvement of the Q-transformation (1)

- Gauge transformation   Midpoint prescription: $x_c \equiv x + \frac{a}{2}\hat{\mu}$

$$U_\mu(x) = e^{iaA_\mu(x)+\mathcal{O}(a^2)} \Rightarrow U_\mu(x) = e^{iaA_\mu(x_c)+\mathcal{O}(a^3)}$$

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$\Rightarrow \delta_\omega A_\mu = -D_\mu \omega$ becomes compatible with
$U_\mu(x) \to e^{i\omega(x)}\, U_\mu(x)\, e^{-i\omega(x+a\hat{\mu})}$ at $\mathcal{O}(a^2)$.

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- We would like to consider an improved Q-transformation like

$$Q^{imp}_{naive} U_\mu(x) = \frac{ia}{2}\left(\psi_\mu(x)\, U_\mu(x) + U_\mu(x)\psi_\mu(x+a\hat{\mu})\right)$$
$$= ia\left(\mathcal{R}^{naive}_\mu \psi_\mu(x)\right) U_\mu(x)$$

with $\mathcal{R}^{naive}_\mu = 1 + \frac{a}{2}\nabla_\mu$.

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with $\mathcal{R}^{naive}_\mu = 1 + \frac{a}{2}\nabla_\mu$.

$$\Rightarrow \quad iaQ^{imp}_{naive} A_\mu(x_c) = ia\left(\psi_\mu(x_c) + \mathcal{O}(a^2)\right)$$

However, since $\mathcal{R}^{naive}_\mu$ has zero-mode, $Q^{imp}_{naive}\psi_\mu$ becomes ill-defined.

Improvement of the Q-transformation (2)

- Add a "Wilson term" to lift the zero-mode:

$$\mathcal{R}_\mu = 1 + \frac{a}{2}\nabla_\mu - ra^2\nabla_\mu\nabla^*_\mu \qquad (r > 0).$$

- Improved Q-transformation ($\psi'_\mu(x) \equiv \mathcal{R}_\mu\psi_\mu(x)$):

$$Q^{imp}U_\mu(x) = ia\psi'_\mu(x)U_\mu(x),$$
$$Q^{imp}\psi'_\mu(x) = i\nabla_\mu\phi(x) + ia\psi'_\mu(x)\psi'_\mu(x).$$

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  - $\nabla_\mu^{imp} \equiv \mathcal{R}_\mu^{-1}\nabla_\mu = D_\mu + \mathcal{O}(a^2) \quad \Rightarrow \quad Q^{imp}\psi_\mu$ is improved.
  - Q^{imp} acting on (U_μ, ψ'_μ) is the same as the original Q on (U_μ, ψ_μ).

Now, we achieve $Q_{\mathrm{lat}}^{imp} = Q_{\mathrm{cont}} + \mathcal{O}(a^2)$.

Improvement of Ξ

- Improvement of plaquette field

$$U_{12}(x) = \exp\left[ia^2 F_{12}(x) + i\frac{a^3}{2}(D_1 + D_2)F_{12}(x) + \mathcal{O}(a^4)\right]$$

$$\Downarrow$$

$$U^{imp}_{\mu\nu}(x) \equiv \mathcal{R}_{12} U_{\mu\nu}(x) = \exp\left[ia^2 F_{12}(x) + \mathcal{O}(a^4)\right]$$

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with $\mathcal{R}_{12} = 1 - \frac{a}{2}(\nabla_1 + \nabla_2)$.

- Then, the improved Ξ ($\Xi^{imp}_{\rm lat}(x) = \Xi_{\rm cont}(x) + \mathcal{O}(a^2)$) is given by

$$\Xi^{imp}_{\rm lat} = \mathrm{tr}\left\{\frac{1}{4}\eta[\phi, \bar{\phi}] - i\sum_{\mu=1}^{2}\psi_\mu \nabla^{imp}_\mu \bar{\phi} + \chi\left(H - \frac{i}{a^2}\Phi^{imp}_{\rm TL}\right)\right\},$$

where $\Phi^{imp}_{\rm TL}(x) = \mathcal{R}_{12}\Phi_{\rm TL}(x)$.

Improved lattice action (1)

$$S_{\rm lat}^{imp} = Q^{imp} \frac{a^2}{2g^2} \sum_x \Xi_{\rm lat}^{imp}(x)$$

- $\nabla_\mu^{imp} \equiv \mathcal{R}_\mu^{-1} \nabla_\mu$ decays with exponential of the distance on the lattice (exponential locality).
  - This is a milder locality condition than the ordinary case. It is believed sufficient to achieve the correct local continuum theory. [Hernández, Jansen, Luscher 1999]
- The nilpotency of Q is maintained: $(Q^{imp})^2 = -i\delta_\phi$.
- Gauge symmetry and $U(1)_A$ R-symmetry are also maintained.

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- Gauge symmetry and $U(1)_A$ R-symmetry are also maintained.
- Path-integral measure $\prod_x \prod_\mu [dU_\mu(x)]\, d\psi_\mu(x) \cdots$ is proven to be Q^{imp} invariant for any finite physical volume in the continuum limit.

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- Path-integral measure $\prod_x \prod_\mu [dU_\mu(x)]\, d\psi_\mu(x) \cdots$ is proven to be Q^{imp} invariant for any finite physical volume in the continuum limit.

$\Rightarrow$ No fined tuning needed to obtain the target continuum theory

Improved lattice action (2)

At $r = \frac{1}{2}$

- Choices of $\mathcal{R}_\mu$ (with $r > 0$):

$$\mathcal{R}_\mu^{(+)} \equiv 1 + \frac{a}{2}\nabla_\mu - ra^2\nabla_\mu^*\nabla_\mu \quad \rightarrow \quad 1 + \frac{a}{2}\nabla_\mu^*$$

$$\mathcal{R}_\mu^{(-)} \equiv \left(1 - \frac{a}{2}\nabla_\mu^* - ra^2\nabla_\mu^*\nabla_\mu\right)^{-1} \quad \rightarrow \quad \left(1 - \frac{a}{2}\nabla_\mu\right)^{-1}$$

$$\mathcal{R}_\mu^{(\mathrm{e})} \equiv \exp\left[\frac{a}{4}\left(\nabla_\mu + \nabla_\mu^*\right)\right]$$

- Choices of $\mathcal{R}_{12}$ (with $r > \frac{1}{4}$):

$$\mathcal{R}_{12}^{(+)} \equiv \left(1 + \frac{a}{2}\sum_{\mu=1,2}\nabla_\mu - ra^2\sum_{\mu=1,2}\nabla_\mu^*\nabla_\mu\right)^{-1} \quad \rightarrow \quad \left(1 + \sum_{\mu=1,2}\frac{a}{2}\nabla_\mu^*\right)^{-1}$$

$$\mathcal{R}_{12}^{(-)} \equiv 1 - \frac{a}{2}\sum_{\mu=1,2}\nabla_\mu^* - ra^2\sum_{\mu=1,2}\nabla_\mu^*\nabla_\mu \quad \rightarrow \quad 1 - \sum_{\mu=1,2}\frac{a}{2}\nabla_\mu$$

$$\mathcal{R}_{12}^{(\mathrm{e})} \equiv \exp\left[-\frac{a}{4}\sum_{\mu=1,2}\left(\nabla_\mu + \nabla_\mu^*\right)\right]$$

Improved lattice action (3)

Note: Any combination of $\mathcal{R}_\mu$ and $\mathcal{R}_{12}$ is allowed.

Choice of $\mathcal{R}_\mu^{(-)}$ and $\mathcal{R}_{12}^{(-)}$

- Makes the action in a ultra-local form in terms of ψ'_μ:

$$\Xi^{imp}_{\mathrm{lat}} = \mathrm{tr}\left\{\frac{1}{4}\eta[\phi,\bar{\phi}] - i\sum_{\mu=1}^{2}(\mathcal{R}_\mu^{-1}\psi'_\mu)\,(\mathcal{R}_\mu^{-1}\nabla_\mu\bar{\phi}) + \chi\left(H - \frac{i}{a^2}\mathcal{R}_{12}\Phi_{\mathrm{TL}})\right)\right\},$$

Note that Q^{imp}-transformation is ultra-local in terms of ψ'_μ.

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  Note that Q^{imp}-transformation is ultra-local in terms of ψ'_μ.

- If we use ψ'_μ as fundamental variables, the improved action is written as a ultra-local form.
  Then, observables $\mathcal{O}(\psi_\mu)$ should be expressed as $\mathcal{O}(\mathcal{R}_\mu^{-1}\psi'_\mu)$.

$\Rightarrow$ The choice gives convenient expressions in numerical simulations.

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Future directions

- Improvement at the loop level
- Application to 2D $\mathcal{N} = (4, 4), (8, 8)$ SYM (with exact two supercharges on the lattice)
- Confirmation of the improvement in numerical simulations

Thank you very much for your attention!