# 3d N = 4 Super-Yang-Mills on a Lattice

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# Introduction

- In this talk, I will describe a new formulation of 3d N=4 SYM on a lattice.

- Motivation:
  - Toy model for QCD at high temperatures (Appelquist, Pisarski)
  - Dimensional reduction of 4d N=2 SYM, described by Seiberg-Witten solution
  - Mirror symmetry (Intriligator, Seiberg)

# Lattice SUSY

Orbifolding

(Kaplan, Unsal)

Topological Twisting

(Catterall)

These constructions can describe theories in $d \leq 4$ with an integer multiple of $2^d$ supercharges in the continuum limit

# DW Twist of 4d N=2 SYM

- Original Motivation: supersymmetric QFT on general manifolds

- Spacetime curvature generically breaks susy unless you twist (Witten)

- Global symmetries: $SU(2)_l \times SU(2)_r \times SU(2)_R \times U(1)$

- Twist: $SU(2)_r \times SU(2)_R \rightarrow \mathrm{diag}\left[SU(2)_r \times SU(2)_R\right]$

# DW Twist of 4d N=2 SYM

- Twisted fields: $\{A_\mu, \phi, \bar{\phi}\}, \{\psi_\mu, \chi_{\mu\nu}, \eta\}$, where $\bar{\phi} = \phi^\dagger$ and $\chi = \star\chi$

- BRST symmetry:

$$g^2 \mathcal{L}_{4d}^{\mathcal{N}=2} = Q\, tr\left[\frac{1}{4}\chi_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}\mathcal{D}_\mu\bar{\phi}\psi^\mu + \alpha\eta\left[\phi, \bar{\phi}\right]\right] - \frac{1}{4}tr\left[\star\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right]$$

where

$$Q\,\phi = 0,\ \ Q\,\bar{\phi} = i\eta,\ \ Q\,\mathcal{A}_\mu = i\psi_\mu$$

$$Q\,\eta = \left[\bar{\phi}, \phi\right],\ \ Q\,\psi_\mu = \mathcal{D}_\mu\phi,\ \ Q\,\chi_{\mu\nu} = \mathcal{F}_{\mu\nu} + *\mathcal{F}_{\mu\nu}$$

- $Q^2$ = gauge transformation ⟶ topological theory

- Lattice formulation proposed by Sugino

# Twisting 3d N=4 SYM

- Global symmetries: $SU(2)_E \times SU(2)_N \times SU(2)_R$

- Blau-Thompson: $SU(2)_E \times SU(2)_N \to SU(2)' = \mathrm{diag}\,(SU(2)_E \times SU(2)_N)$

- Donaldson-Witten: $SU(2)_E \times SU(2)_R \to SU(2)' = \mathrm{diag}\,(SU(2)_E \times SU(2)_R)$

- Lattice formulation based on BT twist proposed by Joseph

- 3d DW twist can be obtained by dimensional reduction of 4d DW twist.

# Complexification of 4d N=2

- To make the theory amenable to geometric discretization complexify the fields:

$$g^2\mathcal{L}^* = \mathrm{Re}\left(Q\, tr\left[\frac{1}{4}\chi_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}\bar{\mathcal{D}}_\mu\bar{\phi}\psi^\mu + \alpha\eta\left[\phi,\bar{\phi}\right]\right] - \frac{1}{4}tr\left(\star\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right)\right)$$

where $\bar{\phi} \neq \phi^\dagger$ and $\chi = \star\bar{\chi}$

$$Q\,\phi = 0,\;\; Q\,\bar{\phi} = i\eta,\;\; Q\,\mathcal{A}_\mu = i\psi_\mu,\;\; Q\,\bar{\mathcal{A}}_\mu = i\bar{\psi}_\mu,$$

$$Q\,\eta = \left[\bar{\phi},\phi\right],\;\; Q\,\psi_\mu = \mathcal{D}_\mu\phi,\;\; Q\,\bar{\psi}_\mu = \bar{\mathcal{D}}_\mu\phi,\;\; Q\,\chi_{\mu\nu} = \bar{\mathcal{F}}_{\mu\nu} + *\mathcal{F}_{\mu\nu}$$

- We will find that the lattice theory lives in $d \leq 3$.

# Lattice Fields

- Let $\phi(n), \bar{\phi}(n), \eta(n)$ reside on the lattice site $n = \sum_{\mu=1}^{d} n_\mu e_\mu,\ n_\mu \in \mathbb{Z}$
- Promote $\mathcal{A}_\mu(n)$ to a Wilson line $\mathcal{U}_\mu(n)$ residing on the link $(n, n + e_\mu)$
- Let $\psi_\mu(n)$ also reside on the link $(n, n + e_\mu)$
- Let $\chi_{\mu\nu}(n)$ reside on $(n + e_\mu + e_\nu, n)$
- Conjugate fields have opposite orientation on lattice.

# Lattice Gauge Symmetry

- For G ϵ U(N)

$$\begin{aligned}\left\{\phi(n),\bar{\phi}(n),\eta(n)\right\} &\rightarrow G(n)\left\{\phi(n),\bar{\phi}(n),\eta(n)\right\}G^{\dagger}(n)\\ \{\mathcal{U}_{\mu}(n),\psi_{\mu}(n)\} &\rightarrow G(n)\{\mathcal{U}_{\mu}(n),\psi_{\mu}(n)\}G^{\dagger}(n+e_{\mu})\\ \chi_{\mu\nu}(n) &\rightarrow G(n+e_{\mu}+e_{\nu})\,\chi_{\mu\nu}(n)G^{\dagger}(n).\end{aligned}$$

- Lattice analogue of Hodge-duality constraint:

$$\chi_{\mu\nu}(n)=\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\bar{\chi}_{\rho\lambda}(n+\Delta)$$

- Demanding consistency with above transformations fixes $\Delta = e_{\mu}+e_{\nu}$ and

$$\sum_{\mu=1}^{4}e_{\mu}=0$$

# Base Space

- For d=3 most symmetric choice for basis vectors is $A_3^*$ lattice defined by

$$e_\mu \cdot e_\nu = \delta_{\mu\nu} - \frac{1}{4}, \quad e_1 + e_2 + e_3 + e_4 = 0, \quad \sum_{\mu=1}^{4} e_\mu{}^i e_\nu{}^j = \delta^{ij}$$

- For this choice, lattice theory has $S_4$ point symmetry group.

- The fields transform in reducible representations of the point symmetry group. For example, the 4d gauge field decomposes into a 3d gauge field and a scalar which corresponds to the $S_4$ symmetric combination

$$\sigma = \sum_{\mu=1}^{4} A_\mu$$

# Geometric Discretization

- Generalize covariant derivatives to finite differences:

$$\mathcal{D}_\mu^+ f(n) = \mathcal{U}_\mu(n) f\left(n+e_\mu\right) - f(n)\mathcal{U}_\mu(n)$$

$$\mathcal{D}_\mu^+ f_\nu(n) = \mathcal{U}_\mu(n) f_\nu\left(n+e_\mu\right) - f_\nu(n)\mathcal{U}_\mu\left(n+e_\nu\right)$$

$$\mathcal{D}_\mu^- f_\mu(n) = \mathcal{U}_\mu(n) f_\mu(n) - f_\mu\left(n-e_\mu\right)\mathcal{U}_\mu\left(n-e_\mu\right)$$

$$\mathcal{D}_\mu^- f_{\nu\lambda}(n) = \mathcal{U}_\mu\left(n-e_\mu\right) f_{\nu\lambda}(n) - f_{\nu\lambda}\left(n-e_\mu\right)\mathcal{U}_\mu\left(n+e_\nu+e_\lambda-e_\mu\right)$$

- Field strength: $\mathcal{F}_{\mu\nu}(n) = \mathcal{D}_\mu^+\mathcal{U}_\nu(n)$
- No doublers

# Lattice Theory

- Lagrangian:

$$\mathcal{L} = \mathrm{Re}\Bigg[\mathrm{tr}\Bigg(\frac{1}{4}\bar{\mathcal{F}}_{\mu\nu}(n)\mathcal{F}_{\mu\nu}(n) + \frac{1}{2}\bar{\mathcal{D}}_\mu^+\bar{\phi}(n)\mathcal{D}_\mu^+\phi(n) - \alpha\left[\phi(n), \bar{\phi}(n)\right]^2$$

$$+\frac{i}{2}\bar{\mathcal{D}}_\mu^+\eta(n)\psi_\mu(n) + i\alpha\phi(n)\left\{\eta(n), \eta(n)\right\} - \frac{i}{2}\bar{\phi}(n)\left(\psi_\mu(n)\bar{\psi}_\mu(n) + \bar{\psi}_\mu\left(n - e_\mu\right)\psi_\mu\left(n - e_\mu\right)\right)\Bigg)\Bigg] + \mathcal{L}_\chi$$

where

$$\mathcal{L}_\chi = \mathrm{tr}\Bigg[\frac{i}{8}\left(\phi(n)\bar{\chi}_{\mu\nu}(n)\chi_{\mu\nu}(n) + \phi\left(n + e_\mu + e_\nu\right)\chi_{\mu\nu}(n)\bar{\chi}_{\mu\nu}(n)\right)$$

$$-\frac{i}{2}\left(\bar{\chi}_{\mu\nu}(n)\bar{\mathcal{D}}_\mu^+\bar{\psi}_\nu(n) + \chi_{\mu\nu}(n)\mathcal{D}_\mu^+\psi_\nu(n)\right)\Bigg].$$

- $\bar{\chi}$ equation of motion (using Hodge-duality constraint):

$$2\left(\bar{\mathcal{D}}_{[\mu}^+\bar{\psi}_{\nu]}(n) + \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\mathcal{D}_\rho^+\psi_\lambda\left(n + e_\mu + e_\nu\right)\right) = \chi_{\mu\nu}(n)\phi(n) - \phi\left(n + e_\mu + e_\nu\right)\chi_{\mu\nu}(n)$$

# Lattice SUSY

- After plugging $\bar{\chi}$ eom back into Lagrangian, it can be expressed as follows:

$$\mathcal{L} = \mathrm{Re}\left[Q\,\mathrm{tr}\left(\frac{1}{4}\chi_{\mu\nu}(n)\mathcal{F}_{\mu\nu}(n) + \frac{1}{2}\bar{\mathcal{D}}^{+}_{\mu}\bar{\phi}(n)\psi_{\mu}(n) + \alpha\eta(n)\left[\phi(n),\bar{\phi}(n)\right]\right)\right.$$

$$\left. -\frac{1}{8}\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}\left(\mathcal{F}_{\mu\nu}(n)\mathcal{F}_{\rho\lambda}\left(n+e_{\mu}+e_{\nu}\right)\right)\right],$$

where

$$\begin{aligned}
&Q\,\phi(n) = 0,\;\; Q\,\bar{\phi}(n) = i\eta(n),\\
&Q\,\eta(n) = \left[\bar{\phi}(n),\phi(n)\right],\\
&Q\,\mathcal{U}_{\mu}(n) = i\psi_{\mu}(n),\;\; Q\,\bar{\mathcal{U}}_{\mu}(n) = -i\bar{\psi}_{\mu}(n)\\
&Q\,\psi_{\mu}(n) = \mathcal{D}^{+}_{\mu}\phi(n),\;\; Q\,\bar{\psi}_{\mu}(n) = \bar{\mathcal{D}}^{+}_{\mu}\phi(n)\\
&Q\,\chi_{\mu\nu}(n) = \bar{\mathcal{F}}_{\mu\nu}(n) + \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\mathcal{F}_{\rho\lambda}\left(n+e_{\mu}+e_{\nu}\right)
\end{aligned}$$

# Lattice SUSY

- Hodge-duality constraint and $\bar{\chi}$ eom are Q-invariant.
- $Q^2$ generates a lattice gauge transformation:

$$Q^2\phi(n) = 0, \;\; Q^2\bar{\phi}(n) = i\left[\bar{\phi}(n), \phi(n)\right],$$

$$Q^2\eta(n) = i\left[\eta(n), \phi(n)\right],$$

$$Q^2\mathcal{U}_\mu(n) = i\mathcal{D}^+_\mu\phi(n), \;\; Q^2\bar{\mathcal{U}}_\mu(n) = -i\bar{\mathcal{D}}^+_\mu\phi(n).$$

$$Q^2\psi_\mu(n) = i\left(\psi_\mu(n)\phi\left(n+e_\mu\right) - \phi(n)\psi_\mu(n)\right)$$

$$Q^2\bar{\psi}_\mu(n) = i\left(\bar{\psi}_\mu(n)\phi\left(n\right) - \phi\left(n+e_\mu\right)\bar{\psi}_\mu(n)\right).$$

$$Q^2\chi_{\mu\nu}(n) = 2i\left(\bar{\mathcal{D}}^+_{[\mu}\bar{\psi}_{\nu]}(n) + \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\mathcal{D}^+_\rho\psi_\lambda\left(n+e_\mu+e_\nu\right)\right)$$

- Hence first term in Lagrangian is Q-exact.
- Second term is Q-closed by lattice Bianchi identity $\epsilon_{\mu\nu\rho\lambda}\mathcal{D}^-_\mu\mathcal{F}_{\rho\lambda}\left(n+e_\mu\right) = 0$

# Renormalization

- In 3d, $g_{YM}$ has mass dimension ½ so the theory is super-renormalizable.

- Marginal operators induced by radiative corrections are suppressed by $g_{YM}^2 a$ so vanish as the lattice spacing a $\rightarrow$ 0. Hence, we only need to consider relevant operators.

- After rescaling fields by $g_{YM}$ to obtain canonical kinetic terms, there are only four relevant operators consistent with lattice gauge invariance, Q-invariance, $S_4$ point symmetry, and $\eta \to \eta + b\mathbb{I}$

- All these operators have mass dimension 2, so radiative corrections can only occur at 1-loop.

# Continuum Limit

- Classical continuum limit reduces to complexified 3d N=4 SYM. In order to obtain the desired spectrum, add mass terms for unwanted fields:

$$\mathcal{L}_U = m_U^2 \left( \frac{1}{N} \sum_{\mu} \mathrm{tr}\left( \mathcal{U}_\mu(n) \overline{\mathcal{U}}_\mu(n) \right) - 1 \right)^2$$

$$\mathcal{L}_\phi = m_\phi^2 \mathrm{tr} \left| \phi(n)^\dagger - \bar{\phi}(n) \right|^2$$

along with analogous mass terms for other fields.

- To decouple unwanted fields, choose $m \gg g_{YM}^2$
- Mass terms break lattice susy, but renormalization still manageable.

# Conclusion

- Proposed lattice formulation of 3d N=4 SYM based on DW twist
- Strategy: Complexify DW twist of 4d N=2 SYM and apply geometric discretization
- Compatibility of Hodge-duality constraint with lattice gauge invariance $\longrightarrow$ $d \leq 3$
- Add mass terms to reach uncomplexified continuum theory

- **Future directions**:
  - Perturbative renormalization and numerical simulation
  - Incorporate matter multiplets, investigate mirror symmetry
  - Holography