Quantum Geometry of Resurgent Perturbative/Nonperturbative Relations

Gerald Dunne

University of Connecticut

ICTS Program "Nonperturbative and Numerical Approaches to Quantum Gravity, String Theory and Holography"
January - February, 2018, ICTS-TIFR Bangalore

(work with Gökçe Başar & Mithat Ünsal: 1501.05671 [JHEP], 1701.06572 [JHEP])

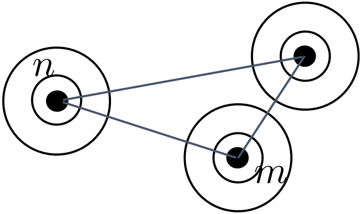
Physical Motivation:

Connecting Perturbative and Non-Perturbative Sectors

• one particular aspect of resurgence has captured the attention of physicists in quantum field theory & string theory:

all orders of multi-instanton trans-series may be encoded in perturbation theory of fluctuations about perturbative vacuum

$\Rightarrow$ possibility of "non-perturbative completion"



$$\int \mathcal{D}A\, e^{-\frac{1}{\hbar}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{\hbar}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

Physical Motivation: distinguish two types of resurgence

- resurgence generically $\to$ low order/high order relations
- cf. Berry-Howls: all-orders steepest descents:

$$I^{(n)}(\hbar) = \int_{C_n} dz\, e^{-\frac{1}{\hbar} f(z)} = \frac{1}{\sqrt{1/\hbar}}\, e^{-\frac{1}{\hbar} f_n}\, T^{(n)}(\hbar)$$

- asymptotic expansion of fluctuations about the saddle n:

$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \dots \right.$$

- *constructive* resurgent relations: low order/low order
- basic structure:

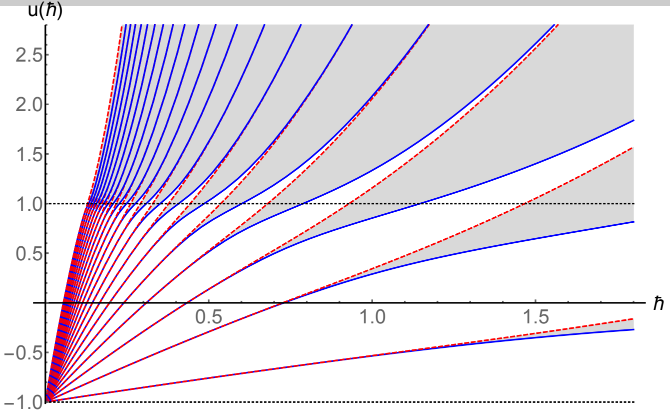
$$\Delta E_{\text{non-pert.}}(\hbar, N) \sim \frac{\partial E_{\text{pert}}}{\partial N} \exp\left[\int^{\hbar} \frac{d\hbar}{\hbar^2} \frac{\partial E_{\text{pert}}}{\partial N} \right]$$

- all non-perturbative terms are explicitly expressed in terms of perturbative data

Physical Motivation

- *constructive* resurgent relations: low order/low order
  - prefactor hint: Dykhne (1961), Connor/Marcus (1984), Weinstein/Keller (1985, 1987), ...
  - Hoe et al (1981): Stark effect
  - Álvarez & Casares (2000, 2004): cubic oscillator, double-well oscillator
  - GD & Ünsal (2013): Mathieu, DW, SUSY double-well, radial AHO, SUSY Mathieu
  - Kozçaz et al (2016): quasi-exactly-soluble systems
- note: requires $E = E(\hbar, N)$
- cf. large-N: $F(g^2, N)$

Mathieu Equation Spectrum: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = E\,\psi$



$$E_\pm(\hbar, N) = E_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}}\frac{1}{N!}\left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right] \mathcal{P}_{\rm inst}(\hbar, N) + \dots$$

$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} \exp\left[S\int_0^\hbar \frac{d\hbar}{\hbar^3}\left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\hbar^2}{S}\right)\right]$$

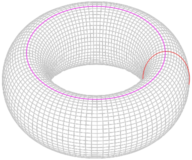
GD & Ünsal (2013); Başar & GD (2015): applies to bands & gaps

why does it happen ?

- combine Seiberg-Witten, Nekrasov-Shatashvili formalism with all-orders ("exact") WKB

Classical Genus 1 Structure

- energy-momentum relation defines a Riemann surface



$$\begin{aligned} E &= \frac{p^2}{2} + V(x) \\ p^2 &= 2E - 2V(x) \end{aligned}$$

- quartic V (or less) $\Rightarrow$ genus 1: torus

- two independent cycles: $\alpha =$ "well" , $\beta =$ "barrier"

$$a_0(E) = \sqrt{2} \oint_\alpha dx \sqrt{E - V(x)} \quad , \quad \omega_0(E) = \frac{1}{\sqrt{2}} \oint_\alpha \frac{dx}{\sqrt{E - V(x)}}$$

$$a_0^D(E) = \sqrt{2} \oint_\beta dx \sqrt{E - V(x)} \quad , \quad \omega_0^D(E) = \frac{1}{\sqrt{2}} \oint_\beta \frac{dx}{\sqrt{E - V(x)}}$$

- periods and actions are elliptic functions: $\mathbb{K}$, $\mathbb{E}$, Π

- periods satisfy 2nd order ODE with respect to E

- actions satisfy 3rd order ODE (Picard-Fuchs) w.r.t. E

Quantization: "All-orders WKB", "Exact WKB"

- "quantum actions": formal expansion in $\hbar^2$

$$a(E,\hbar) = \sum_{n=0}^{\infty} \hbar^{2n}\, a_n(E) \qquad , \qquad a^D(E,\hbar) = \sum_{n=0}^{\infty} \hbar^{2n}\, a_n^D(E)$$

- explicit expansion (Dunham, 1932)

$$\begin{aligned}
a(E,\hbar) &= \sqrt{2}\left(\oint_\alpha \sqrt{E-V}\,dx - \frac{\hbar^2}{2^6}\oint_\alpha \frac{(V')^2}{(E-V)^{5/2}}dx \right.\\
&\left. -\frac{\hbar^4}{2^{13}}\oint_\alpha \left(\frac{49(V')^4}{(E-V)^{11/2}} - \frac{16V'V'''}{(E-V)^{7/2}}\right)dx - \dots\right)
\end{aligned}$$

$$\begin{aligned}
a^D(E,\hbar) &= \sqrt{2}\left(\oint_\beta \sqrt{E-V}\,dx - \frac{\hbar^2}{2^6}\oint_\beta \frac{(V')^2}{(E-V)^{5/2}}dx \right.\\
&\left. -\frac{\hbar^4}{2^{13}}\oint_\beta \left(\frac{49(V')^4}{(E-V)^{11/2}} - \frac{16V'V'''}{(E-V)^{7/2}}\right)dx - \dots\right)
\end{aligned}$$

- identical integrands !

Origin of Perturbative/Non-Perturbative Relation for Genus 1

1. classical geometry (Riemann): $a_0(E)$ determines $a_0^D(E)$
2. perturbation theory is equivalent to inversion of all-orders Bohr-Sommerfeld:

$$a(E,\hbar) = 2\pi\hbar\left(N+\frac{1}{2}\right) \qquad , \quad N = 0, 1, 2, \ldots$$

3. all higher order terms, $a_n(E)$ and $a_n^D(E)$, are generated by action of differential operators on $a_0(E)$ and $a_0^D(E)$

$$a_n(E) = \mathcal{D}_E^{(n)}\, a_0(E) \quad , \quad a_n^D(E) = \mathcal{D}_E^{(n)}\, a_0^D(E)$$

where $\mathcal{D}_E^{(n)}$ and $\mathcal{D}_E^{(n)}$ are the same! (Legendre, Weierstrass, ...)

4. knowing $a(E,\hbar)$ to some order $\Leftrightarrow$ knowledge of $\mathcal{D}_E^{(n)}$
   $\Rightarrow$ we therefore know $a^D(E,\hbar)$ to the same order

$\Rightarrow$ "perturbation theory encodes all non-perturbative physics"

Constructive Resurgence: Perturbative/Non-perturbative Connection

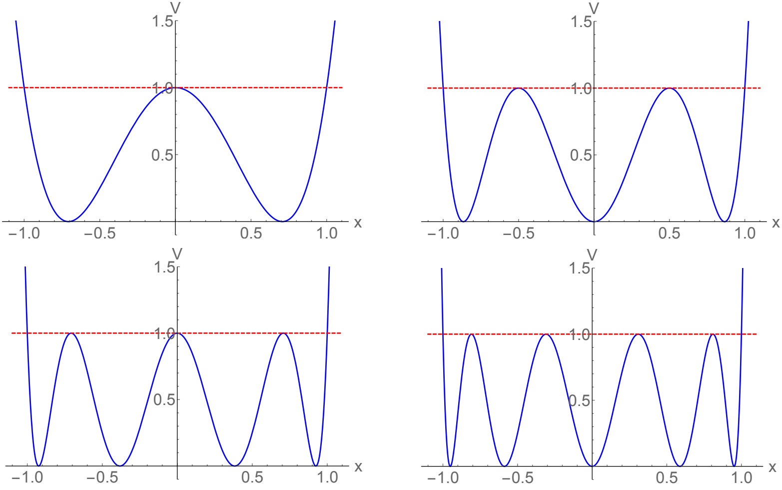
how does it happen ?

- more refined question:

for which potentials is the P/NP relation of the same form as for the Mathieu system ?

Classically: Chebyshev Potentials

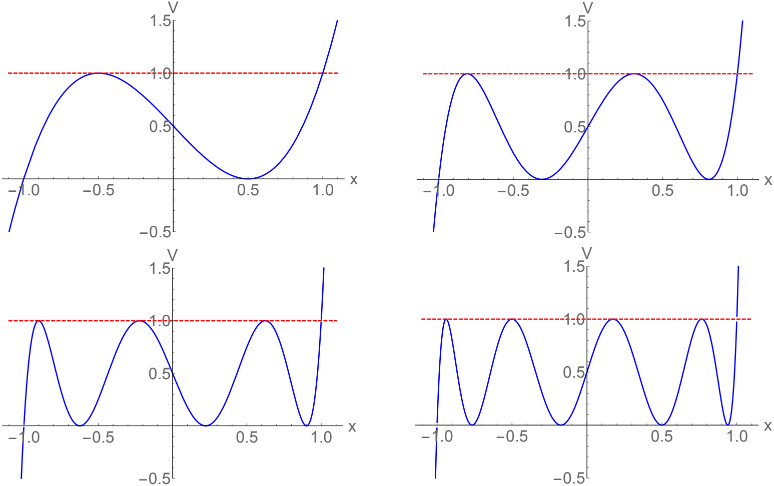
- Chebyshev potentials: $V(x) = T_m^2(x); \quad m = \text{integer}$



- classically, there is only one action and one dual action !

Classically: Chebyshev Potentials

- Chebyshev potentials: $V(x) = T_m^2(x)$; $\quad m = \text{half-integer}$



- classically, there is only one action and one dual action !

Chebyshev Potentials: "behave like genus 1, classically"

- Chebyshev potentials: $V(x) = T_m^2(x)$
- classical action and dual action (in any well or barrier)

$$
\begin{aligned}
a_0(E) &\propto E\ {}_2F_1\left(\frac{1}{2}-\frac{1}{2m}, \frac{1}{2}+\frac{1}{2m}, 2; E\right) \\
a_0^D(E) &\propto -i(1-E)\ {}_2F_1\left(\frac{1}{2}-\frac{1}{2m}, \frac{1}{2}+\frac{1}{2m}, 2; 1-E\right)
\end{aligned}
$$

- satisfy 2nd order (hypergeometric) Picard-Fuchs eq

$$
E(1-E)\frac{d^2 a_0}{dE^2} = \frac{1}{4}\left(1-\frac{1}{m^2}\right) a_0(E)
$$

- classical Wronskian relation:

$$
a_0(E)\omega_0^D(E) - a_0^D(E)\omega_0(E) = \text{constant}
$$

- classical "Matone relation":

$$
\frac{dE}{da_0} = (\text{constant}) \times \left(a_0^D - a_0 \frac{da_0^D}{da_0}\right)
$$

Classical Chebyshev Potentials

- classically behave like genus 1 systems
- (almost) modular structure (Schwarzian, triangle fns, ...)

$$\tau_0(E) \equiv \frac{\omega_0^D(E)}{\omega_0(E)}$$

- actions given by differentiation, not integration:

$$a_0(E) = (\text{constant}) \frac{d}{d\tau_0} \left(\frac{1}{\omega_0} \right) \quad , \quad a_0^D(E) = -(\text{constant}) \frac{d}{d\tau_0^D} \left(\frac{1}{\omega_0^D} \right)$$

- $a_0^D(E)$ is completely determined by $a_0(E)$

$$a_0^D(E) = \tau_0(E)\, a_0(E) - i \frac{(\text{constant})}{\omega_0(E)}$$

Quantization of Chebyshev Systems

- what happens to this classical geometric structure after quantization?

"quantum geometry"

- "mirror symmetry" and Calabi-Yau manifolds
- "quantum modular forms"
- [some interesting number theory enters here]

Classical Detour: Ramanujan's Generalized Elliptic Functions

- recall Mathieu system and elliptic functions:

$$
\begin{aligned}
\omega_0(E) &= \pi\sqrt{2}\;{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;E\right) = 2\sqrt{2}\,\mathbb{K}(E) \\
a_0(E) &= \pi\sqrt{2}E\;{}_2F_1\left(\frac{1}{2},\frac{1}{2},2;E\right) = 4\sqrt{2}\left(-(1-E)\mathbb{K}(E)+\mathbb{E}(E)\right) \\
\omega_0^D(E) &= i\,\pi\sqrt{2}\;{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;1-E\right) = i\,2\sqrt{2}\,\mathbb{K}(1-E) \\
a_0^D(E) &= -i\,\pi\sqrt{2}(1-E)\;{}_2F_1\left(\frac{1}{2},\frac{1}{2},2;1-E\right) \\
&\qquad = -i\,4\sqrt{2}\left(-E\,\mathbb{K}(1-E)+\mathbb{E}(1-E)\right)
\end{aligned}
$$

- classical Wronskian condition $\equiv$ Legendre identity

$$
\mathbb{E}(E)\mathbb{K}(1-E)+\mathbb{K}(E)\mathbb{E}(1-E)-\mathbb{K}(E)\mathbb{K}(1-E)=\frac{\pi}{2}
$$

- familiar in QFT from Seiberg-Witten

Classical Detour: Ramanujan's Generalized Elliptic Functions

- modular structure of Mathieu ($\mathcal{N} = 2$ SUSY QFT):

$$\tau_0 \equiv \frac{\omega_0^D(E)}{2\,\omega_0(E)} = \frac{i\,\mathbb{K}(1-E)}{2\,\mathbb{K}(E)}$$

- Jacobi's inversion formula:

$$E(\tau_0) = \frac{\vartheta_2^4(2\tau_0)}{\vartheta_3^4(2\tau_0)}$$

- classical period & action in terms of τ_0:

$$\begin{aligned} \omega_0(\tau_0) &= \vartheta_3^2(2\tau_0) \\ a_0(\tau_0) &= \frac{2\sqrt{2}\pi}{3\vartheta_3^2(2\tau_0)}\left(E_2(2\tau_0) + \vartheta_2^4(2\tau_0) - \vartheta_4^4(2\tau_0)\right) \end{aligned}$$

- E_2 is the classical "Eisenstein series"
- $\mathrm{S}: \tau_0 \to -\frac{1}{4\tau_0}$, $\mathrm{T}: \tau_0 \to \tau_0 + 1$
- $S: E \to 1-E$, $T: E \to E$

Classical Detour: Ramanujan's Generalized Elliptic Functions

- Ramanujan (1914) generalized this; also Fricke (1916)

$$\mathbb{K}_M(E) = \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{M}, 1-\frac{1}{M}, 1; E\right) \quad , \quad M = 2, 3, 4, 6$$

- identify with Chebyshev periods: $M = \frac{2m}{m-1}$
- Chebyshev Wronskian rel. $\equiv$ generalized Legendre rel.

$$\mathbb{E}_M(E)\mathbb{K}_M(1-E)+\mathbb{K}_M(E)\mathbb{E}_M(1-E)-\mathbb{K}_M(E)\mathbb{K}_M(1-E) = \frac{\pi}{4}\frac{M}{M-1}\sin\left(\frac{\pi}{M}\right)$$

- modular parameter (here $r := 4\sin^2\left(\frac{\pi}{M}\right) = 4\cos^2\left(\frac{\pi}{2m}\right)$)

$$\tau_0(E) = \frac{1}{r}\frac{\omega_0^D(E)}{\omega_0(E)} = \frac{-i}{\sqrt{r}}\frac{{}_2F_1\left(\frac{1}{M}, 1-\frac{1}{M}, 1; 1-E\right)}{{}_2F_1\left(\frac{1}{M}, 1-\frac{1}{M}, 1; E\right)}$$

- periods: modular forms; actions: quasi-modular forms
- modular group = Hecke group for M, r integer

$$\mathrm{S} : \tau_0 \to -\frac{1}{r\tau_0} \quad , \quad \mathrm{T} : \tau_0 \to \tau_0 + 1$$

Example 1

$$\exp\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right)=\frac{x}{16}\left(1+\frac{1}{2}x+\frac{21}{64}x^2+\cdots\right)$$

Example 2

$$\exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-x)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;x)}\right)=\frac{x}{27}\left(1+\frac{5}{9}x+\cdots\right)$$

Example 3

$$\exp\left(-\sqrt{2}\pi\frac{{}_2F_1(\frac{1}{4},\frac{3}{4};1;1-x)}{{}_2F_1(\frac{1}{4},\frac{3}{4};1;x)}\right)=\frac{x}{64}\left(1+\frac{5}{8}x+\cdots\right)$$

Example 4

$$\exp\left(-2\pi\frac{{}_2F_1(\frac{1}{6},\frac{5}{6};1;1-x)}{{}_2F_1(\frac{1}{6},\frac{5}{6};1;x)}\right)=\frac{x}{432}\left(1+\frac{13}{18}x+\cdots\right)$$

We do not know Ramanujan's intention in giving Examples 1–4.

(B. Berndt, Ramanujan's Notebook 3)

Classical Detour: Ramanujan's Generalized Elliptic Functions

- Jacobi's inversion formula:

(Borwein/Borwein, 1991; Berndt, Bhargava, Garvan, 1995)

$$E = \frac{1}{2}\left(1 - \frac{E_6(\tau_0)}{(E_4(\tau_0))^{3/2}}\right) \quad , \quad r = 1$$

$$\frac{E}{1-E} = \left(\frac{r^{\frac{1}{4}}\,\eta(r\,\tau_0)}{\eta(\tau_0)}\right)^{\frac{24}{r-1}} \quad , \quad r = 2,3,4$$

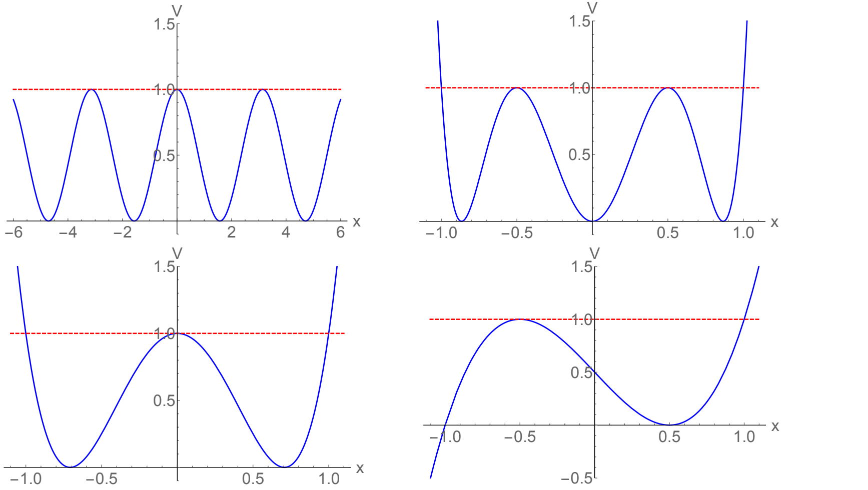
- η: Dedekind eta function
- E_4 & E_6: Eisenstein series of weight 4 and 6
- classical period & action in terms of τ_0:

$$\omega_0(\tau_0) = \sqrt{\frac{2}{3}}\pi\,(E_4(\tau_0))^{1/4} \quad , \quad r = 1$$

$$\omega_0(\tau_0) = \frac{\sqrt{2}\,\pi}{m}\sin\left(\frac{\pi}{2m}\right)\sqrt{\frac{rE_2(r\,\tau_0) - E_2(\tau_0)}{(r-1)}} \quad , \quad r = 2,3,4$$

Chebyshev and Ramanujan

potential	$V(x)$	Chebyshev index m	modular signature M	modular level r	modular group
Mathieu	$\cos^2(x)$	∞	2	4	$\Gamma_0(4)$
triple-well	$x^2(3-4x^2)^2$	3	3	3	$\Gamma_0(3)$
double-well	$(1-2x^2)^2$	2	4	2	$\Gamma_0(2)$
cubic oscill.	$\frac{1}{2}(1+x)(1-2x)^2$	$3/2$	6	1	$\Gamma_0(1)$



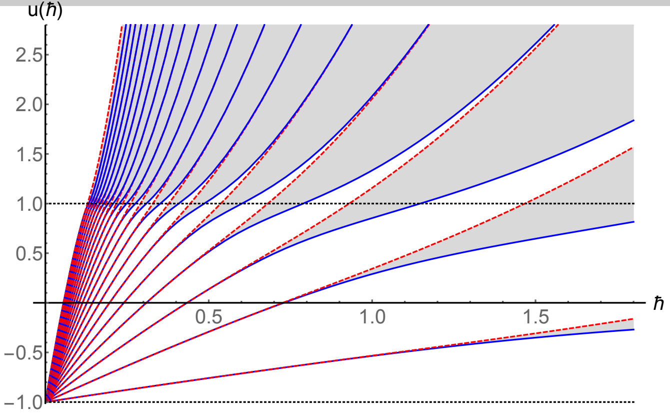
- special "arithmetic" class of Chebyshev potentials

Quantization?

so far all this Chebyshev and Ramanujan discussion is <u>**classical**</u>

what about quantization ?

recall: Mathieu Equation: $-\frac{\hbar^2}{2}\,\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = E\,\psi$



$$E_\pm(\hbar, N) = E_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}}\,\frac{1}{N!}\left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right]\,\mathcal{P}_{\rm inst}(\hbar, N) + \dots$$

$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial E_{\rm pert}(\hbar, N)}{\partial N}\,\exp\left[S\int_0^{\hbar}\frac{d\hbar}{\hbar^3}\left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\,\hbar^2}{S}\right.\right.$$

Quantization of Chebyshev Systems: recall Mathieu equation

- P/NP relations in terms of (all-orders) quantum actions
- Mathieu has all-orders quantum Matone relation:

$$\frac{\partial E(a,\hbar)}{\partial a} = \frac{i\pi}{2}\left(a^D(a,\hbar) - a\frac{\partial a^D(a,\hbar)}{\partial a} - \hbar\frac{\partial a^D(a,\hbar)}{\partial \hbar}\right)$$

Flume et al (2004)

- Mathieu has all-orders quantum Wronskian relation:

$$\left[a(E,\hbar) - \hbar\frac{\partial a(E,\hbar)}{\partial \hbar}\right]\frac{\partial a^D(E,\hbar)}{\partial E} - \left[a^D(E,\hbar) - \hbar\frac{\partial a^D(E,\hbar)}{\partial \hbar}\right]\frac{\partial a(E,\hbar)}{\partial E} = \frac{2i}{\pi}$$

Başar & GD (2015)

Quantization of Chebyshev Systems Başar, GD, Ünsal (2017)

- quantum result: all the "arithmetic" Chebyshev systems have identical all-orders quantum Wronskian relation:

$$\left[a(E,\hbar)-\hbar\frac{\partial a(E,\hbar)}{\partial\hbar}\right]\frac{\partial a^D(E,\hbar)}{\partial E}-\left[a^D(E,\hbar)-\hbar\frac{\partial a^D(E,\hbar)}{\partial\hbar}\right]\frac{\partial a(E,\hbar)}{\partial E}=\frac{2ic_M}{\pi}$$

- result: identical P/NP relation satisfied by the "arithmetic" Chebyshev potentials corresponding to Ramanujan's generalized elliptic functions; just a different constant c_M on RHS

- Mathieu, double-well, cubic oscillator, triple-well

- only cases where Hecke group & modular group match

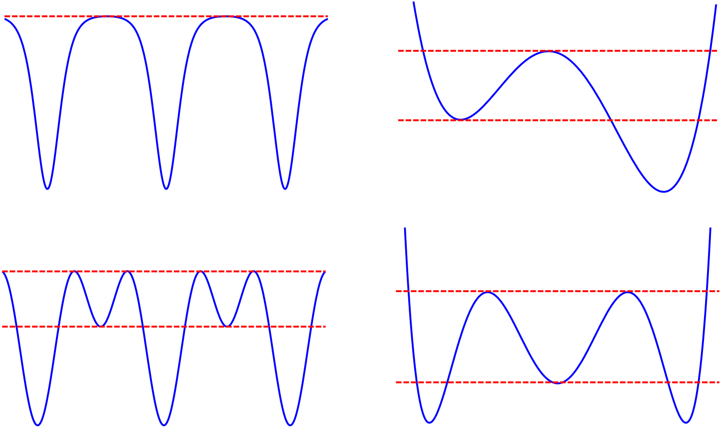
- the only Chebyshev cases that are truly genus 1

- Ramanujan's classical modular structure is crucial for quantization

- same structure in super-conformal $SU\left(\frac{2m}{m-1}\right)$ QFT

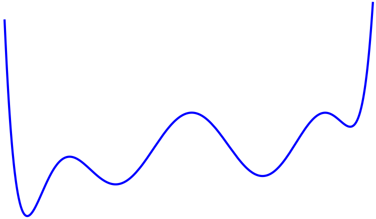
(Argyres-Douglas; Ashok et al, 2016)

Other Genus 1 Systems?



- amazingly: still only one action, and one dual action !
- genus 1: with full 3rd order classical Picard-Fuchs eqn
- P/NP relation has different form compared to the quantum Wronskian condition of the arithmetic Chebyshev examples

Higher Genus Systems?



- example: 3 barriers; 4 wells (one redundant); genus 3
- higher order classical Picard-Fuchs equation
- actions and periods are hyper-elliptic functions
- *conjecture:* $\exists$ (more complicated) P/NP relations
- Sibuya (1967): Stokes properties for general polynomial potentials
- *conjecture:* best possible outcome: only need $g(=\text{genus})$ perturbative actions, not the g non-perturbative actions

Conclusions

- Constructive Resurgence: very explicit "encoding" of non-perturbative information in perturbation theory
- proof of P/NP relations for genus 1 systems
- explicit results for Chebyshev potentials
- unique subclass (Mathieu, double-well, triple-well, cubic oscillator) has identical form of P/NP relation
- related to Ramanujan's elliptic & modular functions
- QFT perspective leads to new spectral results
- diagrammatically mysterious ...
- higher genus: examples and results? EBK quantization?
- resurgence in two-parameter trans-series ?
- applications to [super-conformal] QFT ?

Other Genus 1 Systems: Example: Lamé potential

- $V(x) = \mathcal{P}(x;\tau)$ doubly periodic!

  $\Rightarrow$ 3 saddles: dominant contribution depends on elliptic parameter τ

- periods & actions involve all 3 elliptic functions: $\mathbb{K}$, $\mathbb{E}$, $\mathbb{\Pi}$
- classical "Wronskian-like" relation:

$$\frac{\partial a_0(E;\tau)}{\partial E}\frac{\partial a_0^D(E;\tau)}{\partial \tau} - \frac{\partial a_0^D(E;\tau)}{\partial E}\frac{\partial a_0(E;\tau)}{\partial \tau} = \text{constant}$$

- all-orders quantum "Wronskian-like" relation:

$$\frac{\partial a(E,\hbar;\tau)}{\partial E}\frac{\partial a^D(E,\hbar;\tau)}{\partial \tau} - \frac{\partial a^D(E,\hbar;\tau)}{\partial E}\frac{\partial a(E,\hbar;\tau)}{\partial \tau} = \text{constant}$$

Başar, GD, Ünsal (2017)

- related to $\mathcal{N} = 2^*$ SUSY QFT