Topology and Strongly Interacting Fermions

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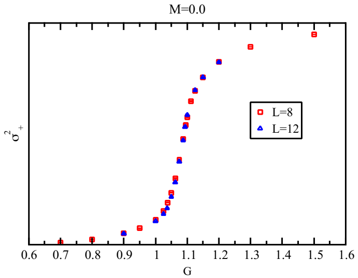
Goal

Describe a continuum theory of strongly interacting fermions which exhibits an exotic phase structure:

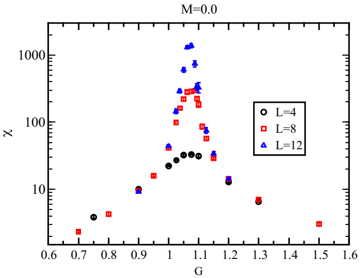
- For strong coupling vacuum populated by four fermion condensate.
- This breaks no symmetries but nevertheless fermions acquire masses
- Argue that this phase is separated from a massless symmetric phase by one or more phase transitions.
- One of these transitions can be understood as arising from the condensation of topological defects in an auxiliary field.
- Show that the model discretizes to a staggered fermion model that has been examined recently using numerical simulation

A staggered fermion model

Four *reduced* staggered fermions in 4D coupled thru $G^2\chi^1\chi^2\chi^3\chi^4$
(model proposed by Chandrasekharan)



Four fermion condensate



$\sum_x < \chi^1(x)\chi^2(x)\chi^1(0)\chi^2(0) >$

Massless phase G small. *Massive* phase $G \to \infty$
Both phases $< \chi^1\chi^2 >= 0$

Questions ..

- Is massive symmetric phase lattice artifact ? Is it bordered by 1st order phase transition ?
- Is there an intermediate symmetry broken phase ? (NJL scenario)
- If phase transition(s) continuous what is continuum theory ?

In this talk:

- Construct a continuum theory which yields this lattice model
- Show how topological defects can explain the four fermion condensate phase and the presence of a fermion mass at strong coupling.
- Argue that possible to modify model so that a single BKT-like transition separates massless from massive phases. New CFT for strongly interacting fermions in 4D?

Starting Point

Consider 4 Majorana spinors with action

$$S = \int d^4x \sum_{A=1}^{4} \overline{\psi}^A \gamma.\partial \, \psi^A$$

The global symmetry is $G = SO_F(4) \times SO_{Lorenz}(4)$ under which

$$\psi \to L_{\alpha\beta} \psi^B_\beta F^T_{BA}$$

Focus on diagonal subgroup D where $L = F$ under which $\psi \to \Psi$ with Ψ a 4×4 matrix

$$S = \int d^4x \, \mathrm{Tr} \left(\Psi \, \gamma.\partial \, \Psi \right) \quad \text{with } \overline{\Psi} = C \Psi^T C^{-1} = \Psi$$

take 4 copies and add $SO(4)$ invariant interaction

$$\delta S = G^2 \int d^4x \, \mathrm{Tr} \left(\Psi^a \Psi^b \right) \mathrm{Tr} \left(\Psi^c \Psi^d \right) \epsilon^{abcd}$$

Notice ..

Interaction breaks global symmetry G down to diagonal subgroup D (twisted Lorentz symmetry)
Additional $SO(4)$ symmetry prevents bilinear mass terms

Introduce lattice

$$\Psi^a(y) = \sum_b^{16} \gamma^{x+b} \chi^a(x+b) \quad \text{with } b_\mu = 0, 1 \qquad \gamma^{x+b} = \prod_{\mu=1}^{4} \gamma_\mu^{x_\mu + b_\mu}$$

Let $\partial_\mu \to \Delta_\mu^S$ and do traces
(note: assign 16 components of continuum fermion to unit hypercube of lattice)

Find staggered four fermion model

Auxiliary field and symmetry breaking

$$S = S_0 + \int d^4x \left[G\sigma_+^{ab} \mathrm{Tr}\left(\Psi^a \Psi^b\right) + \frac{1}{2}\left(\sigma_+^{ab}\right)^2 \right]$$

where $\sigma_+^{ab} = \frac{1}{2}\left(\sigma^{ab} + \frac{1}{2}\epsilon^{abcd}\sigma^{cd}\right)$ transforms in adj rep of $SU_+(2)$ with $SO(4) = SU_+(2) \times SU_-(2)$.

Integrate over fermions $\to \mathrm{Pf}\left(\gamma.\partial + G\sigma_+\right)$. Invariance under $SU_-(2)$ makes all eigenvalues doubly degenerate - no sign problem.
Effective action:

$$S_{\mathrm{eff}}(\sigma_+) = \frac{1}{2}\sigma_+^2 - \frac{1}{4}\mathrm{tr}\ln\left(-\partial^2 + G^2\sigma_+^2 + G\gamma.\partial\sigma_+\right)$$

Set σ_+ constant – for $G > G_c$ symmetry breaks $SU_+(2) \to U(1)$

$$\sigma_+^a = \mu n^a \tau^a \quad \text{with } n^a n^a = 1 \quad \text{vacuum manifold } S^2$$

Effective Action $S_{\text{eff}}(\sigma_+)$

Assume broken phase and expand in powers of $\frac{1}{m}$ with $m = \mu G$

$$S_{\text{eff}} = \frac{1}{m^2} \int d^4x \, (\partial_\mu n^a)^2 + \frac{1}{m^4} \int d^4x \, \left(\epsilon^{abc} \partial_\mu n^a \times \partial_\nu n^b\right)^2 + \ldots$$

Fadeev-Skyrme term

Analysis easier if change variables:

$$n^a(x)\sigma^a = U^\dagger(x)\sigma_3 U(x)$$

Mapping invariant under

Local $U(1)$

$$U(x) \to e^{i\sigma_3 \beta(x)} U(x)$$

Also global $SU(2)$ symmetry: $U(x) \to U(x)G$

Hopf defects

$$S_{\text{eff}} = \frac{1}{m^2}\int d^4x\, \text{tr}\left[(D_\mu U)^\dagger (D_\mu U)\right] + \frac{1}{m^4}\int d^4x\, F_{\mu\nu}^2$$

where $D_\mu = \partial_\mu + iA_\mu\sigma_3$.

Topological defects

Set $D_\mu U = 0$ for large r to avoid action $S \sim L^2$

$$\rightarrow A_\mu = \frac{i}{2}\text{tr}\left(\partial_\mu U U^\dagger \sigma_3\right) \qquad \text{and}$$

$$S = \frac{1}{m^4}\int d^4x\, \frac{1}{4}\text{tr}\left(\partial_\mu U \partial_\nu U^\dagger \sigma_3\right)^2 \qquad \text{with}$$

$$U = \begin{pmatrix} \alpha_1 + i\alpha_2 & -\alpha_3 + i\alpha_4 \\ \alpha_3 + i\alpha_4 & \alpha_1 - i\alpha_2 \end{pmatrix} \quad \alpha_i = \frac{x_i}{r}$$

Hopf map: $\Pi_3(S^2) = Z$

Role of defects ?

Topological charge gotten from $\int \epsilon_{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}$.

Action $\int F_{\mu\nu}^2$ of single defect $S_{\rm Hopf} \sim \frac{b}{m^4} \log V$

Recall 2D XY model

Single vortex has log divergent action. Pair of vortices has finite action and logarithmic interaction

Conjecture something similar here - pairs of Hopf-antiHopf defects bind in pairs. Play no role for small G but unbind to populate vacuum as $G \to \infty$.

Transition when entropy $\log V$ of single defect outweighs action cost $1 > \frac{b}{m^4}$ i.e $\mu(G_c) G_c \sim b^{\frac{1}{4}}$

Fermion acquires mass in background of defect

$$\begin{aligned} G_F(x,y) &= \langle \Psi^a(x)\Psi^a(y)\rangle \\ &= \mathrm{tr}\left[\frac{-\gamma_\mu\partial_\mu + m\, n^a\sigma^a}{\left(-\partial_\mu^2 + m^2 + mP\right)}\right] \end{aligned} \tag{1}$$

where

$$P = \gamma_\mu\left(\partial_\mu U^\dagger(x)\sigma_3 U(x) + U^\dagger(x)\sigma_3\partial_\mu U(x)\right)$$

Far from core $P = 0$ and

$$G_F(x,y) = \frac{-\gamma_\mu\partial_\mu}{-\Box + m^2}$$

Unusual fermion propagator.. Fermions acquire mass without breaking symmetries !

Phase structure

As G increased generically expect 3 phases

- $G < G_c^1$ 16 free massless Majorana fermions. Trivial IR fixed point. Lorentz and flavor symmetries restored.
- $G_c^1 < G < G_c^2$ Phase with broken $SO(4)$ symmetry. Fermion mass determined by bilinear condensate. Conventional NJL scenario.
- $G > G_c^2$ Four fermion condensate. Auxiliary field picture - proliferation of topological defects. Fermions acquire masses propagating in this background. IR behavior depends on whether phase transition at G_c^2 continuous ..?

BKT transition

Suppose I add a bare gradient term $\int DUDU^\dagger$ term. Tune its coefficient so only Maxwell term in $S_{\rm eff}$

Setting $U = e^{i\theta^a\sigma^a}$

$$S_{\rm eff} = \frac{1}{m^4}\int d^4x \left(\theta^3\Box^2\theta^3\right)$$

$$\begin{aligned} < n^3(x) > &\sim \exp < \theta^3(x)\theta^3(x) > \\ &= \exp\left(m^4\int_{1/L}\frac{d^4k}{k^4}\right) = \exp -m^4\ln L = L^{-m^4} \end{aligned}$$

Broken phase removed. Single BKT-like phase transition between symmetric massless and massive phases ...

Summary

- Introduced a model based on Kähler-Dirac fermions that discretizes to staggered four fermion model studied recently.
- In broken phase topological Hopf defects are possible. Must occur in pairs for finite action. Fermions acquire mass at strong coupling in background of these defects. No symmetry breaking needed.
- Condensation of defects equivalent to four fermion condensate.
- Argued that by tuning out leading term in $S_{\rm eff}$ can remove any symmetry broken intermediate phase.
- In this case expect a single BKT-like transition between two symmetric phases. New 4D CFT!
- Question: is this theory Lorentz invariant – naturally only invariant under twisted Lorentz symmetry...

What (else) might this be good for ?

- Lattice models that use carefully chosen quartic interactions to gap out fermions without breaking symmetries receiving a lot of attention in CM literature (Kitaev ...). SPT phases, topological insulators ...
- In lattice gauge theory there was an old proposal to build chiral gauge theories by using quartic interactions to gap out (doubler) mirror states (Eichten-Preskill 1986). In the old days this failed because always generated symmetry broken phase associated with a bilinear condensate. Can this model evade that problem ?
- Caution: one cannot simply trade staggered fermions for naive Weyl fermions

$$\epsilon_{abcd}\psi^a_\alpha\psi^b_\beta\psi^c_\gamma\psi^d_\delta = 0$$

Eichten+Preskill

Basic idea (recast a la Golterman)

Naive Weyl lattice fermions transforming in complex rep of gauge group
eg **16** of SO(10) and $S_0 = \sum_{x,\mu} \psi \sigma_\mu \Delta_\mu \psi$
Additional states - $\Delta_\mu \to \sin ak_\mu = 0$ for $k_\mu \sim 0$ or $k_\mu \sim \frac{\pi}{a}$

doublers with $ka \sim O(1)$ render theory vector-like

How to remove them ? Cannot use a Wilson term since theory is chiral

Four fermi terms

Generalized Wilson

For Weyl ψ gauge invariant four fermi terms possible. Use to construct generalized Wilson term that couples strongly to doublers

$$S = S_0 - \lambda \sum_x \Delta \left(\psi^T \sigma_2 T^a \psi\right)^2 \text{ where}$$

$$\Delta \psi_i \psi_j \psi_k \psi_l = \sum_{\pm\mu} \left(S_\mu \psi_i \psi_j \psi_k \psi_l + \psi_i S_{\pm\mu} \psi_j \psi_k \psi_l + \ldots - 4\psi_i \psi_j \psi_k \psi_l\right)$$

with $S_\mu \psi(x) = \psi(x+\mu) + \psi(x-\mu)$ and T^a a 16×16 matrix that projects $\mathbf{16} \times \mathbf{16} \rightarrow \mathbf{10}$ (real rep)

Phases ..

λ small

Massless primary Weyl field plus doublers - vectorlike free theory

$\lambda \to \infty$

Expect a massive phase in which would be doublers pair with composite $\Psi = \psi\psi\psi$ fermions in complex conjugate rep to form massive Dirac fermions.

Old hope ..

Single undoubled Weyl field survives after appropriate tuning in the continuum limit. Crucial that $SO(10)$ not broken spontaneously ...

In the old days this never worked – massive four fermion phase separated by phase of broken $SO(10)$ symmetry. Latter phase ensures all fermions acquire mass.

Can this be avoided?

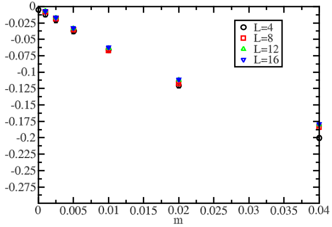
Perhaps the new model can be tuned to gap the doublers without breaking symmetries ?

Simplest idea won't work:

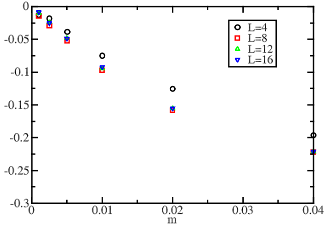
$$\epsilon_{abcd}\psi^a_\alpha\psi^b_\beta\psi^c_\gamma\psi^d_\delta = 0$$

We need to think harder ...
Domain wall setup ?

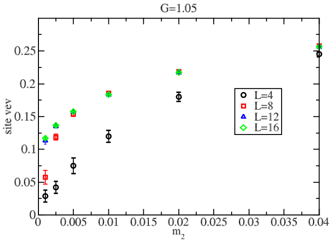
Backup: mass scans in staggered fermion model



$G = 0.95$



$G = 1.15$



$G = 1.05$