Complex Langevin Dynamics in Large N Gauge Theories

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Nonperturbative and Numerical Approaches to Quantum Gravity, String Theory and Holography
Jan 27 - Feb 3, 2018

Complex Langevin Dynamics

This talk is about two things:

1. Complex Langevin dynamics: complex integral are generally problems. Complex Langevin strives to solve it.
2. Unitary Matrix model. One can get an unitary matrix model from gauge theory in compact spaces.
3. How complex Langevin method may be used to solve complex unitary matrix integrals? And how relatively nice matching with analytic results could be achieved.

Previous works

- There are a few works in complex langevin.**Gert Aarts,Erhard Seiler and many collaborators, recently Jun Nishimura and collaborators**
- Various models including $SU(3)$ link chain, lower dimensional bosonic model etc.

A crash course in unitary matrix models I

- We start with Gross-Witten-Wadia Model, sometimes called single plaquette model. Simplest example of a gauge theory !

$$S = a\mathrm{Tr}\,(U) + a\mathrm{Tr}\,(U^{\dagger})$$

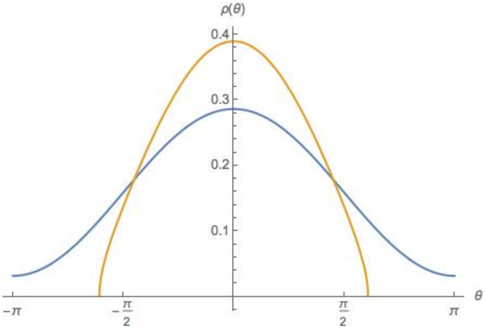
- This model is exactly solvable at large-N.
- Eigen values of the unitary matrix lies over the unit circle.
- There are two phases.
- **ungapped**

$$\rho(\theta) = \frac{1}{2\pi}\left[1 + 2a\cos(\theta)\right] \qquad a \le 0.5$$

- **gapped**

$$\rho(\theta) = \frac{2a}{\pi}\cos\left(\frac{\theta}{2}\right)\left[\frac{1}{2a} - \sin^2\left(\frac{\theta}{2}\right)\right]^{1/2} \qquad a > 0.5$$

A crash course in unitary matrix models II



- A third order GWW transition between two phases.

A crash course in unitary matrix models III

- Generalization of Gross-Witten-Wadia Model, the **ab-model**

$$S = a\mathrm{Tr}\,(U) + b\mathrm{Tr}\,(U^\dagger)$$

- U are $N \times N$ Unitary matrices
- $a, b \in \mathbf{C}$ and not necessarily same
- Can be solved analytically at large-N
- In diagonal gauge, the field configuration is given by the saddle-point equation

$$\frac{\partial S_{eff}}{\partial \theta_i} = 0$$

$$S_{eff} = \left[-\frac{1}{2} \sum_{i,j=1}^{N} \ln \sin^2 \left(\frac{\theta_i - \theta_j}{2} \right) + N \sum_{i=1}^{N} \left(i\mathcal{M}\theta_i + a e^{i\theta_i} + b e^{-i\theta_i} \right) \right]$$

- $\mathcal{M}$ is the Lagrange multiplier to enforce $\det(U) = 1$

ab-Model contd. (Ungapped phase)

In the large-N limit, approximating the sum with density function $\rho(z)$,

$$\frac{1}{N}\sum_{i=1}^{N} \to \int_{-\pi}^{\pi} \frac{ds}{2\pi} = \oint \frac{dz}{2\pi i}\rho(z)$$

Eigenvalues are in the complex plane, not on the unit circle anymore ! in ungapped phase we get,

$$\rho(z) = \frac{1}{z} - a - \frac{b}{z^2} \qquad \mathcal{M} = 0$$

The contour $r(\theta)$ is given by the transcendental equation

$$r(\theta) = \exp\left(|a| r(\theta)\cos(\theta + \phi_1) - \frac{|b|}{r(\theta)}\cos(\theta - \phi_2)\right)$$

where $\theta \in (-\pi, \pi)$
$a = |a|e^{i\phi_1}$, $b = |b|e^{i\phi_2}$

QCD on $S^3 \times S^1$ I

- The partition function of QCD at finite temperature $T = 1/\beta$, with N_f quark flavors, m_f mass coupled to μ_f chemical potential is

$$Z_{QCD} = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\int_0^\beta d\tau \int d^3x \mathscr{L}}$$

where, A is $SU(N)$ gauge field

$$\mathscr{L} = \frac{1}{4g^2}\text{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \sum_{f=1}^{N_f} \bar{\psi}_f(\not{D}A - \gamma_0\mu_f + m_f)\psi_f$$

QCD on $S^3 \times S^1$ II

- In the low temperature limit ($\beta \to \infty$), the theory reduces to $N \times N$ unitary matrix model

$$Z = \int dP e^{-N \mathrm{Tr}\, V(P)}$$

where,

$$V(P) = -\sum_{l=1}^{\infty} \sigma_l \left[\log(1 + e^{\beta(\mu - \epsilon_l)} P) + \log(1 + e^{\beta(-\mu - \epsilon_l)} P^\dagger) \right]$$

- In a double scaling limit,

$$\beta(\mu - \epsilon_l) = \xi$$
$$\beta \to \infty$$

- Phase structure as we change ξ from zero.
Ungapped $\to$ Gapped $\to$ Ungapped.

ab-Model contd. (Ungapped phase)

Complex Langevin Dynamics perfectly agrees with the analytic result.

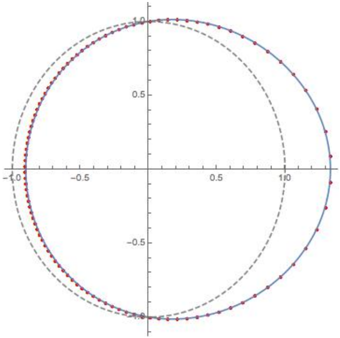


Figure: Without noise

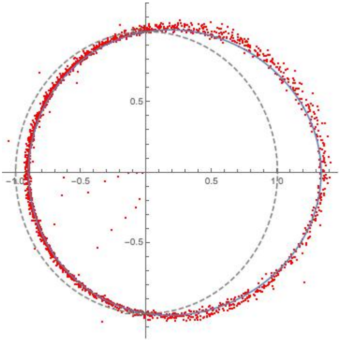


Figure: With noise

Figure: a = 0.30, b = 0.15

Complex Langevin Dynamics I

Given a partition function for real action

$$Z = \int dx e^{-S(x)}, \quad S(x) \in \mathbb{R}$$

the corresponding Langevin equation is,

$$\dot{x} = -\partial_x S + \eta$$

where η is Gaussian noise.
But when we have a complex action, the Langevin equation becomes complex,

$$\dot{z} = -\partial_z S(z) + \eta$$

where, $z \in \mathbb{C}$ and the Fokker-Planck equation becomes

$$\dot{\rho}(z,t) = \partial_z(\partial_z + S'(z))\rho(z,t)$$

$$\dot{x} = K_x + \eta_R \quad \dot{y} = K_y + \eta_I$$

Complex Langevin Dynamics II

where,

$$K_x = -\mathrm{Re}\partial_z S(z) \quad K_y = -\mathrm{Im}\partial_z S(z)$$

Complex Langevin equation (discrete)

$$\theta_i(\tau + \Delta\tau) \quad = \quad \theta_i(\tau) - \left[\frac{\partial S}{\partial \theta_i(\tau)}\right] \Delta\tau + \sqrt{\Delta\tau}\, \eta_i(\tau).$$

θ_i: complexified angle variables. $i = 1, 2, \cdots, N$
$\Delta\tau$: Langevin time step
$\eta_i(\tau)$: Gaussian random variable satisfying conditions

$$\langle \eta_i(\tau) \rangle = 0, \quad \langle \eta_i(\tau)\eta_j(\tau') \rangle = 2\delta_{ij}\delta_{\tau\tau'}.$$

$$< O >= \int dxdy O(x+iy) P(x, y; \infty)$$

Complex Langevin Dynamics III

1. Under suitable condition one may show that,

$$\int dx O(x)\rho(x,t) = \int dx dy O(x+iy) P(x,y;t) \tag{1}$$

2. A real integral is evaluated by two dimensional complex integral.
3. $P(x,y,t)$ is a probability density function and positive definite.

ab-Model contd. (Ungapped phase)

For complex values as well

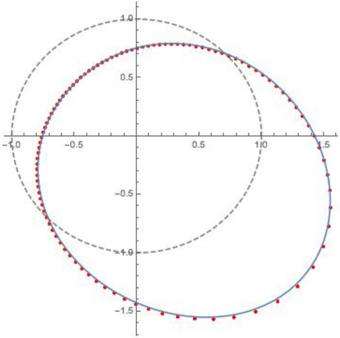


Figure: Without noise

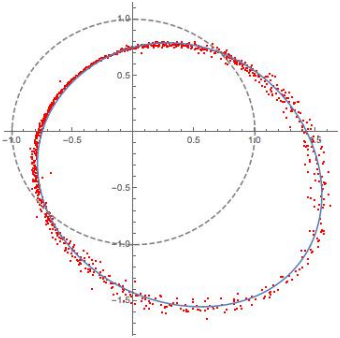


Figure: With noise

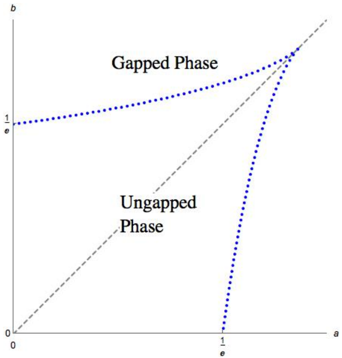
Figure: a = 0.20 + i0.20 , b = -0.10 + i0.1

ab-Model Phase-space

The gap opens up when

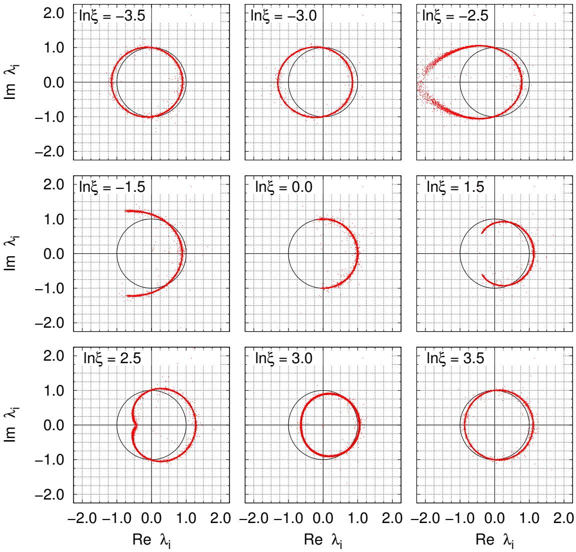
$$1 - a\cos(\theta)r(\theta) - a\sin(\theta)\frac{dr(\theta)}{d\theta} \leq 0$$

on the contour



Plots I

Plots II

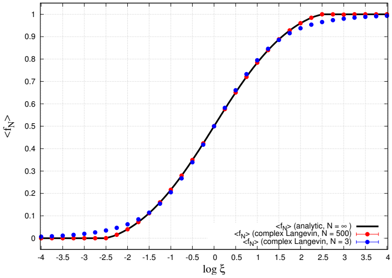


lnξ = −3.5
lnξ = −3.0
lnξ = −2.5
lnξ = −1.5
lnξ = 0.0
lnξ = 1.5
lnξ = 2.5
lnξ = 3.0
lnξ = 3.5
2.0
1.0
0.0
−1.0
−2.0
Im λ_i
−2.0 −1.0 0.0 1.0 2.0
Re λ_i

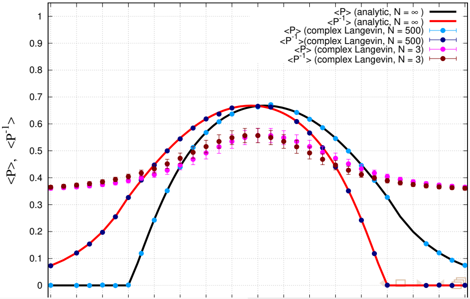
Plots III

- The eigenvalue distributions in the confined and deconfined phases as a function of the logarithm of the effective fugacity, $\log \xi$, for single-level $SU(N)$ matrix model with $N = N_f = 500$.
- Langevin time step $\Delta\tau = 0.000005$, thermalization steps $N_{\text{therm}} = 18000$, generation steps $N_{\text{gen}} = 2000$ and with measurements performed every 100 steps.

Plots



<f_N>
log ξ
<f_N> (analytic, N = ∞)
<f_N> (complex Langevin, N = 500)
<f_N> (complex Langevin, N = 3)



<P>, <P^-1>
<P> (analytic, N = ∞)
<P^-1> (analytic, N = ∞)
<P> (complex Langevin, N = 500)
<P^-1>(complex Langevin, N = 500)
<P> (complex Langevin, N = 3)
<P^-1> (complex Langevin, N = 3)

Plots

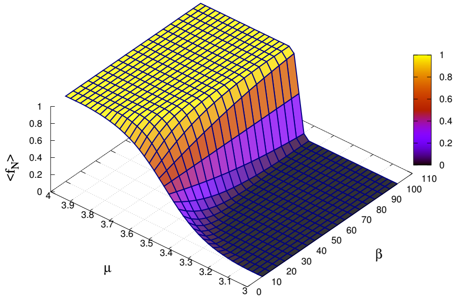


Figure: The (normalized) fermion number density $\langle f_N \rangle$ as a function of chemical potential μ and inverse temperature β for single-level matrix model with quark energy level $\epsilon = 3.5$ and quark mass $m = 0$. Here $N = N_f = 500$. The theory is in a deconfined phase when $0 < \langle f_N \rangle < 1$. The data are obtained through complex Langevin simulations with Langevin time step $\Delta\tau = 0.000005$, thermalization steps $N_{\text{therm}} = 5000$, generation steps $N_{\text{gen}} = 5000$ and with measurements performed every 50 steps.

Plots

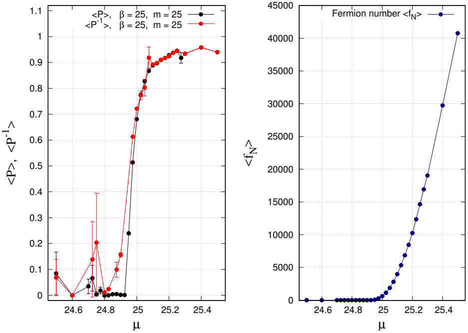


Figure: The Silver Blaze behavior of observables $\langle f_N \rangle$, $\langle P \rangle$ and $\langle P^{-1} \rangle$ at non-zero quark mass m. (Left) Polyakov line $\langle P \rangle$ and inverse Polyakov line $\langle P^{-1} \rangle$ and (Right) fermion number $\langle f_N \rangle$ as a function of chemical potential for large quark mass near onset at $\mu = m = 25$. Here $N = N_f = 500$ and $\beta = 25$ (low T). The data are obtained through complex Langevin simulations with Langevin time step $\Delta\tau = 0.00005$, thermalization steps $N_{\text{therm}} = 5000$, generation steps $N_{\text{gen}} = 5000$ and with measurements performed every 50 steps.

Plots

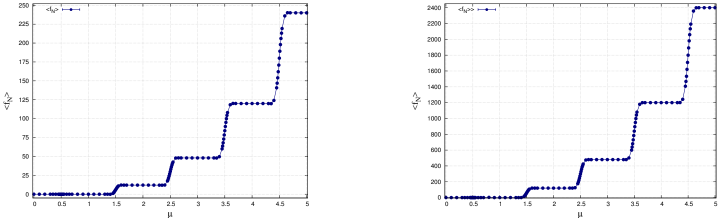


Figure: Expectation values of the effective fermion number $\langle f_N \rangle$ as a function of the quark chemical potential for QCD on $S^1 \times S^3$. Here $m = 0$, inverse temperature $\beta = 30$, $N = N_f = 3$ (Left) and $N = N_f = 30$ (Right). The data are obtained through complex Langevin simulations with Langevin time step $\Delta\tau = 0.00005$, thermalization steps $N_{\text{therm}} = 10000$, generation steps $N_{\text{gen}} = 50000$ and with measurements performed every 100 steps. The solid lines are to guide the eye.

Plots

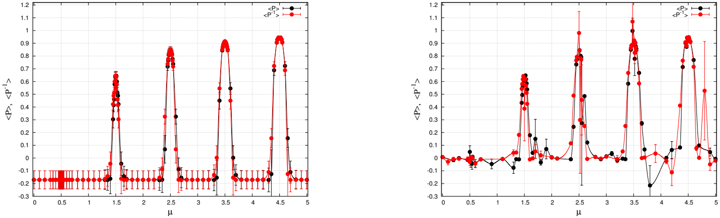
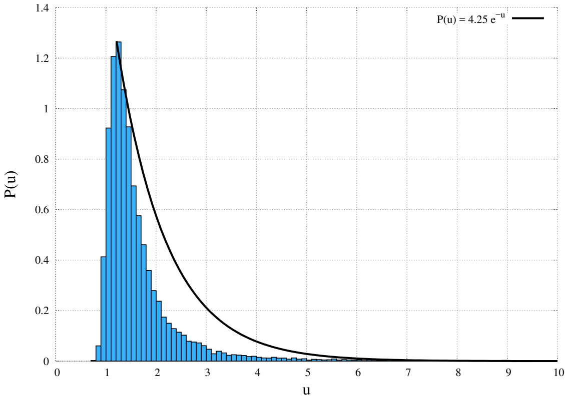


Figure: Expectation values of the Polyakov line $\langle P \rangle$ and inverse Polyakov line $\langle P^{-1} \rangle$ as a function of the quark chemical potential μ for QCD on $S^1 \times S^3$. Here $m = 0$, inverse temperature $\beta = 30$, $N = N_f = 3$ (Left) and $N = N_f = 30$ (Right). The data are obtained through complex Langevin simulations with Langevin time step $\Delta\tau = 0.00005$, thermalization steps $N_{\text{therm}} = 10000$, generation steps $N_{\text{gen}} = 50000$ and with measurements performed every 100 steps. The solid lines are to guide the eye.

Asymptotic behavior



Conclusion

- We have looked at various form of interactons. It does not change the picture.
- We are working on $N \times N$ matrices. Like a 0+1 dim bosonic gauge theory with R-charge.
- Possibly we need gauge cooling.
- Exciting new avenues.