Lattice $\mathcal{N} = 4$ Supersymmetric Yang–Mills

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Nonperturbative and Numerical Approaches
to Quantum Gravity, String Theory and Holography
International Centre for Theoretical Sciences, Bangalore
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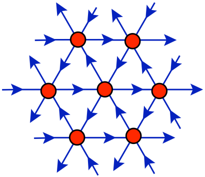
arXiv:1505.03135 arXiv:1611.06561 arXiv:1709.07025
& more to come with Simon Catterall, Raghav Jha and Toby Wiseman

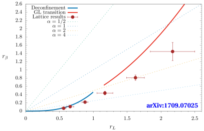
Overview and plan

Goals: Reproduce known results in
perturbative, holographic, etc. regimes

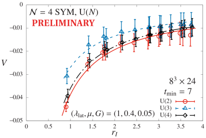
Then use lattice to access new domains



Quick lattice $\mathcal{N} = 4$ SYM recap

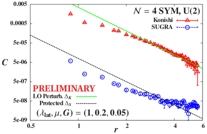


(I) Dimensionally reduced (2d) thermodynamics



(II) 4d static potential Coulomb coefficient

(III) Anomalous dimension of Konishi operator



Open questions and future directions

Lattice supersymmetry in a nutshell

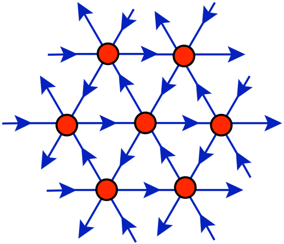
Motivation: Non-perturbative insights from first-principles lattice calcs

Obstruction: $\left\{Q^{\mathrm{I}}_{\alpha}, \overline{Q}^{\mathrm{J}}_{\dot{\alpha}}\right\} = 2\delta^{\mathrm{IJ}}\sigma^{\mu}_{\alpha\dot{\alpha}}P_{\mu}$ broken in discrete space-time

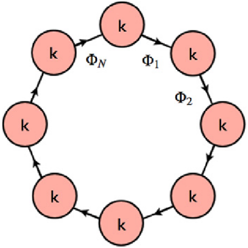
$\Longrightarrow$ Relevant susy-violating operators, typically too many to fine-tune

Solution: Preserve susy sub-algebra at non-zero lattice spacing

Equivalent constructions from topological twisting and deconstruction



Review:
arXiv:0903.4881

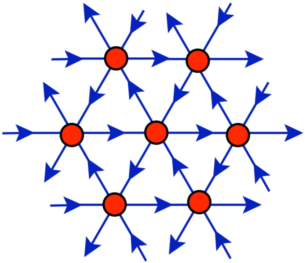


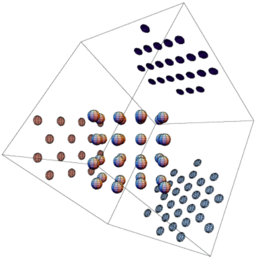
Quick review of twisted lattice $\mathcal{N} = 4$ SYM

Fields: 5 complexified links $\mathcal{U}_a$ and $\overline{\mathcal{U}}_a$ in algebra $\mathfrak{gl}(N, \mathbb{C})$

$1 + 5 + 10$ fermions on lattice sites $+$ links $+$ plaquettes

Space-time: A_4^* lattice of 5 links symmetrically spanning 4d





Complexified links $\longrightarrow$ $\mathrm{U}(N) = \mathrm{SU}(N) \otimes \mathrm{U}(1)$ gauge invariance

Must regulate both $\mathrm{SU}(N)$ and $\mathrm{U}(1)$ flat directions

Two deformations in improved lattice action

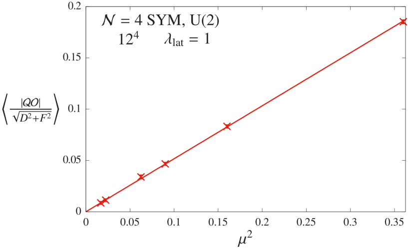
SU(N) scalar potential $\propto \mu^2 \sum_a \left(\text{Tr}\left[\mathcal{U}_a \overline{\mathcal{U}}_a\right] - N\right)^2$

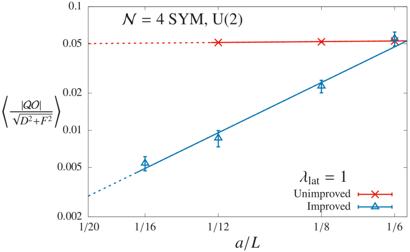
Softly breaks susy $\longrightarrow$ $\mathcal{Q}$-violating operators vanish $\propto \mu^2 \to 0$

U(1) plaquette determinant $\sim G \sum_{a<b} \left(\det \mathcal{P}_{ab} - 1\right)$

Implemented supersymmetrically as Fayet–Iliopoulos D-term potential

Test via Ward identity violations: $\mathcal{Q}\left[\eta \mathcal{U}_a \overline{\mathcal{U}}_a\right] \neq 0$





Advertisement: Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

$$
\begin{aligned}
S_{\text{imp}} &= S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \qquad (3.10)\\
S'_{\text{exact}} &= \frac{N}{2\lambda_{\text{lat}}} \sum_n \text{Tr}\Bigg[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}^{(-)}_a\psi_a(n) \\
&\qquad + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{U}_a(n) + G\sum_{a\neq b}\left(\det\mathcal{P}_{ab}(n) - 1\right)\mathbb{I}_N\right)^2\Bigg] - S_{\text{det}} \\
S_{\text{det}} &= \frac{N}{2\lambda_{\text{lat}}} G \sum_n \text{Tr}\left[\eta(n)\right] \sum_{a\neq b}\left[\det\mathcal{P}_{ab}(n)\right] \text{Tr}\left[\mathcal{U}_b^{-1}(n)\psi_b(n) + \mathcal{U}_a^{-1}(n+\widehat{\mu}_b)\psi_a(n+\widehat{\mu}_b)\right] \\
S_{\text{closed}} &= -\frac{N}{8\lambda_{\text{lat}}} \sum_n \text{Tr}\left[\epsilon_{abcde}\ \chi_{de}(n+\widehat{\mu}_a+\widehat{\mu}_b+\widehat{\mu}_c)\overline{\mathcal{D}}^{(-)}_c\chi_{ab}(n)\right], \\
S'_{\text{soft}} &= \frac{N}{2\lambda_{\text{lat}}} \mu^2 \sum_n \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - 1\right)^2
\end{aligned}
$$

$\gtrsim$100 inter-node data transfers in fermion operator — non-trivial…

To reduce barriers to entry our parallel code is publicly developed at

github.com/daschaich/susy

Evolved from MILC lattice QCD code, presented in arXiv:1410.6971

(I) Thermodynamics on a 2-torus

arXiv:1709.07025

Naive dimensional reduction $\longrightarrow$ 2d $\mathcal{N} = (8, 8)$ SYM
with four nilpotent twisted-scalar $\mathcal{Q}^2 = 0$

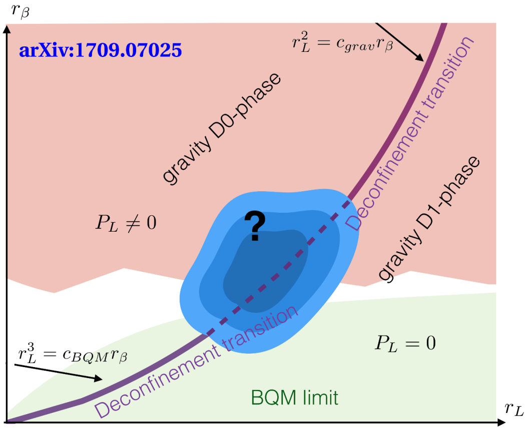
Study low temperatures $t = 1/r_\beta \longleftrightarrow$ black holes in dual supergravity

For decreasing r_L **at large *N***

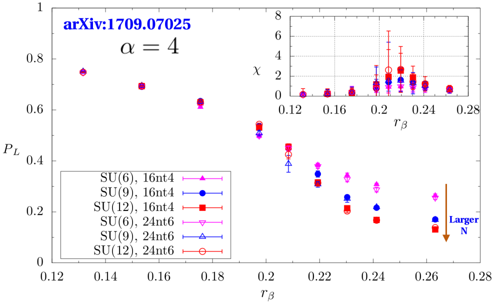
homogeneous black string (D1)
$\longrightarrow$ localized black hole (D0)

$\updownarrow$

"spatial deconfinement"
signalled by Wilson line P_L



$\mathcal{N} = (8,8)$ SYM lattice phase diagram results



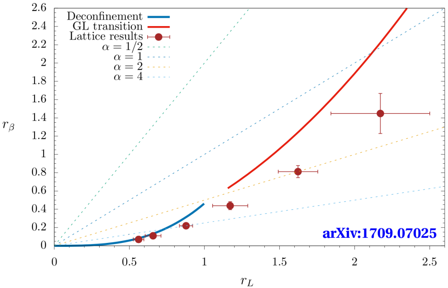
Fix aspect ratio $\alpha = r_L/r_\beta$,

scan in $r_\beta = r_L/\alpha = \beta\sqrt{\lambda}$

Inset shows susceptibility χ of Wilson line

Lower-temperature transitions at smaller $\alpha < 1 \longrightarrow$ larger errors

Results consistent with holography and high-temp. bosonic QM

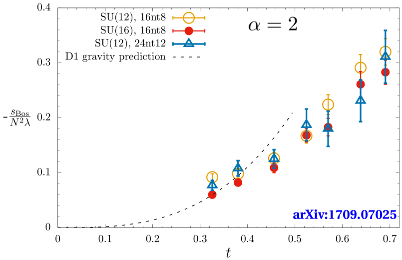


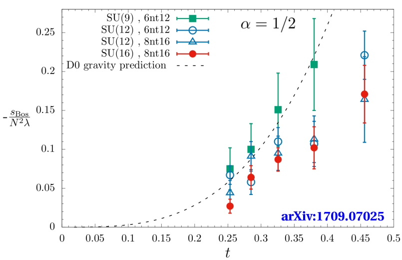
Dual black hole thermodynamics

Holography predicts bosonic action corresponding to dual black holes

$s_{\text{Bos}} \propto t^3$ for large-r_L D1 phase $\qquad$ $s_{\text{Bos}} \propto t^{3.2}$ for small-r_L D0 phase

Lattice results consistent with holography for sufficiently low $t \lesssim 0.4$



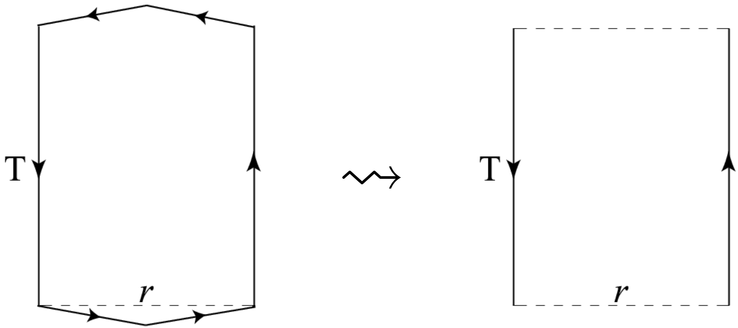


Need larger $N > 16$ to avoid instabilities at lower temperatures

(II) Static potential $V(r)$

Static probes $\longrightarrow$ $r \times T$ Wilson loops $W(r,T) \propto e^{-V(r)\,T}$

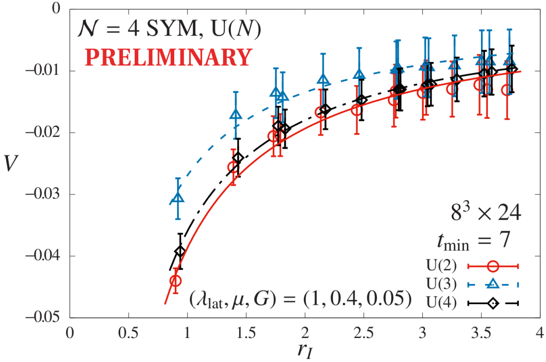
Coulomb gauge trick reduces A_4^* lattice complications



Static potential is Coulombic at all λ

Fits to confining $V(r) = A - C/r + \sigma r \longrightarrow$ vanishing string tension σ

$\Longrightarrow$ Fit to just $V(r) = A - C/r$ to extract Coulomb coefficient $C(\lambda)$

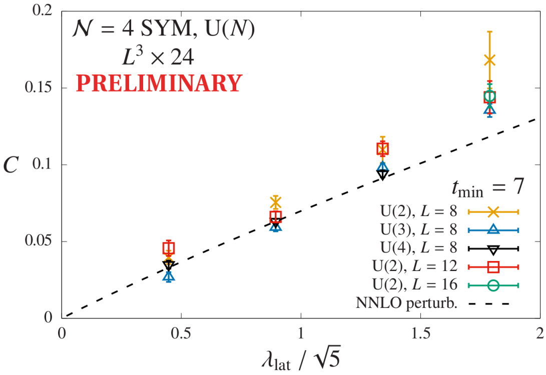


Recent progress: Incorporating tree-level improvement into analysis

Coupling dependence of Coulomb coefficient

Continuum perturbation theory predicts $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

Holography predicts $C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$ and $\lambda \to \infty$ with $\lambda \ll N$



Surprisingly good agreement with perturbation theory for $\lambda_{\text{lat}} \leq 4$

(III) Konishi operator scaling dimension

$\mathcal{O}_K(x) = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}(x)\Phi^{\mathrm{I}}(x)\right]$ is simplest conformal primary operator

Scaling dimension $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$ investigated through
perturbation theory (& S duality), holography, conformal bootstrap

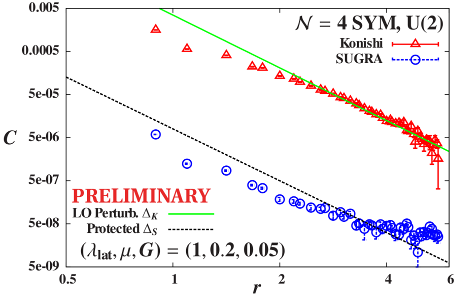
Lattice scalars $\varphi(n)$ from polar decomposition of complexified links

$$\mathcal{U}_a(n) \longrightarrow e^{\varphi_a(n)} U_a(n) \qquad \mathcal{O}_K^{\mathrm{lat}}(n) = \sum_a \mathrm{Tr}\left[\varphi_a(n)\varphi_a(n)\right] - \mathrm{vev}$$

$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$

'SUGRA' is 20′ $\mathcal{O}_S \sim \varphi_{\{a}\varphi_{b\}}$
with protected $\Delta_S = 2$

To handle systemics, comparing
- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG



Scaling dimensions from MCRG stability matrix

System as (infinite) sum of operators $H = \sum_i c_i \mathcal{O}_i$

Couplings c_i flow under **symmetry-preserving** RG blocking R_b

n-times-blocked system $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Fixed point defined by $H^\star = R_b H^\star$ with couplings $c_i^\star$

Linear expansion around fixed point defines stability matrix $T_{ij}^\star$

$$c_i^{(n)} - c_i^\star = \sum_k \left. \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \right|_{H^\star} \left(c_k^{(n-1)} - c_k^\star \right) \equiv \sum_j T_{ik}^\star \left(c_k^{(n-1)} - c_k^\star \right)$$

Correlators of $\mathcal{O}_i$, $\mathcal{O}_k$ $\longrightarrow$ elements of stability matrix [Swendsen, 1979]

Eigenvalues of $T_{ik}^\star$ $\longrightarrow$ scaling dimensions of corresponding operators

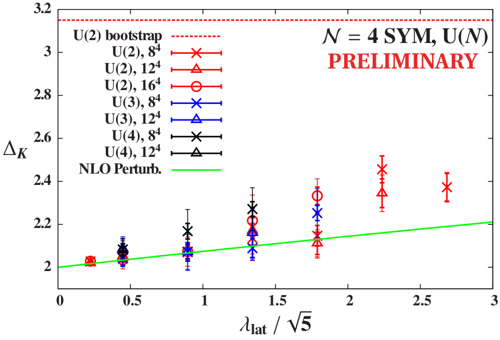
Preliminary Δ_K results from Monte Carlo RG

MCRG stability matrix

includes both $\mathcal{O}_K^{\text{lat}}$ and $\mathcal{O}_S^{\text{lat}}$

Impose protected $\Delta_S = 2$

Systematic uncertainties from different amounts of smearing



Complication: Twisted $SO(4)_{tw}$ involves only $SO(4)_R \subset SO(6)_R$

$\Longrightarrow$ Lattice Konishi operator mixes with $SO(4)_R$-singlet part of the $SO(6)_R$-nonsinglet SUGRA operator

Current work: Variational analyses to disentangle operators

Recapitulation and outlook

- Lattice promises non-perturbative insights from first principles
- Lattice $\mathcal{N} = 4$ SYM is practical thanks to exact $\mathcal{Q}$ susy
- Public code to reduce barriers to entry

Significant progress toward goals of lattice investigations

- 2d $\mathcal{N} = (8, 8)$ SYM thermodynamics consistent with holography
- 4d static potential Coulomb coefficient $C(\lambda)$ at weak coupling
- Preliminary conformal scaling dimension of Konishi operator

Many more directions are being — or can be — pursued

- Understanding the (absence of a) sign problem
- Systems with less supersymmetry, in lower dimensions, including matter fields, exhibiting spontaneous susy breaking, ...

Upcoming Workshops

Numerical approaches to holography,
quantum gravity and cosmology

21–24 May 2018

Higgs Centre for Theoretical Physics, Edinburgh

Interdisciplinary approach
to QCD-like composite dark matter

1–5 October 2018 ECT* Trento

Thank you!

Collaborators

Simon Catterall, Raghav Jha, Toby Wiseman

also Georg Bergner, Poul Damgaard, Joel Giedt, Anosh Joseph

Funding and computing resources

Supplement: Potential sign problem

Observables: $\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}][d\overline{\mathcal{U}}] \;\; \mathcal{O} \;\; e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]} \;\; \text{pf}\,\mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$

Pfaffian can be complex for lattice $\mathcal{N} = 4$ SYM, $\text{pf}\,\mathcal{D} = |\text{pf}\,\mathcal{D}| e^{i\alpha}$

Complicates interpretation of $\left\{ e^{-S_B}\,\text{pf}\,\mathcal{D} \right\}$ as Boltzmann weight

RHMC uses phase quenching, $\text{pf}\,\mathcal{D} \longrightarrow |\text{pf}\,\mathcal{D}|$, needs reweighting

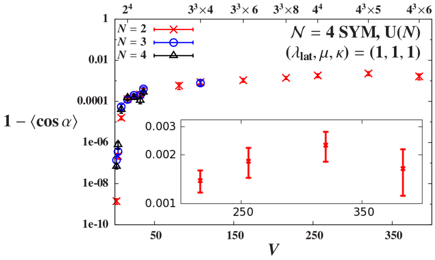
$$\langle \mathcal{O} \rangle = \frac{\left\langle \mathcal{O} e^{i\alpha} \right\rangle_{pq}}{\left\langle e^{i\alpha} \right\rangle_{pq}} \qquad \text{with} \;\; \left\langle \mathcal{O} e^{i\alpha} \right\rangle_{pq} = \frac{1}{\mathcal{Z}_{pq}} \int [d\mathcal{U}][d\overline{\mathcal{U}}]\, \mathcal{O} e^{i\alpha}\, e^{-S_B}\, |\text{pf}\,\mathcal{D}|$$

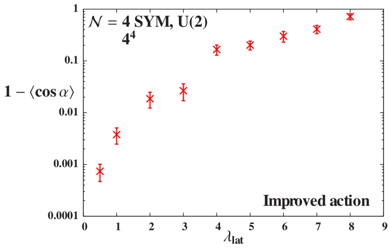
$\Longrightarrow$ Monitor $\left\langle e^{i\alpha} \right\rangle_{pq}$ as function of volume, coupling, N

Pfaffian phase dependence on volume and coupling

Left: $1 - \langle \cos(\alpha) \rangle_{pq} \ll 1$ independent of volume and N at $\lambda_{\text{lat}} = 1$

Right: Larger $\lambda_{\text{lat}} \geq 4 \longrightarrow$ much larger phase fluctuations





To do: Analyze more volumes and N with improved action

Extremely expensive $\mathcal{O}(n^3)$ computation

$\sim$50 hours $\times$ 16 cores for single U(2) 4^4 measurement

Two puzzles posed by the sign problem

Periodic temporal boundary conditions for the fermions

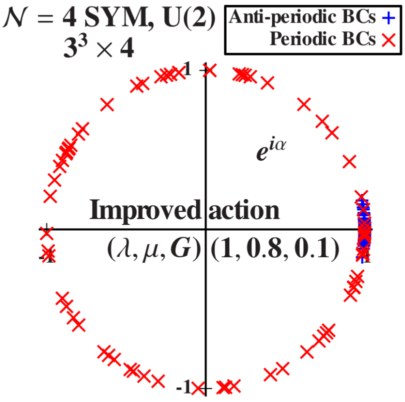
$\longrightarrow$ obvious sign problem, $\left\langle e^{i\alpha} \right\rangle_{pq} \approx 0$

Anti-periodic BCs $\longrightarrow$ $e^{i\alpha} \approx 1$, phase reweighting negligible

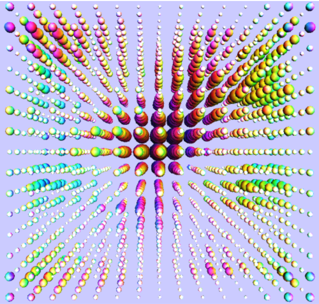
Why such sensitivity to the BCs?

Other $\langle \mathcal{O} \rangle_{pq}$ are nearly identical for these two ensembles

Why doesn't sign problem affect other observables?



Backup: Essence of numerical lattice calculations



(Image credit: Claudio Rebbi)

Evaluate observables from functional integral via importance sampling Monte Carlo

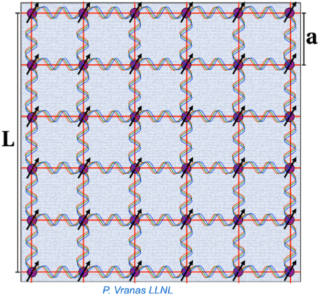
$$\langle\mathcal{O}\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}U\ \ \mathcal{O}(U)\ \ e^{-S[U]}$$

$$\longrightarrow \frac{1}{N}\sum_{i=1}^{N}\mathcal{O}(U_i)\ \ \text{with uncert.}\ \ \propto \sqrt{\frac{1}{N}}$$

U are field configurations in discretized euclidean space-time, sampled with probability $\propto e^{-S}$

$S[U]$ is lattice action, should be real and positive $\longrightarrow \frac{1}{\mathcal{Z}}e^{-S}$ as probability distribution

Backup: More features of lattice calculations



Spacing "a" between lattice sites

$\longrightarrow$ UV cutoff scale $1/a$

Removing cutoff: $a \to 0$ (with $L/a \to \infty$)

Lattice cutoff preserves hypercubic subgroup

$\longrightarrow$ restore Poincaré in continuum limit

Lattice action S defined by bare lagrangian at the UV cutoff $1/a$

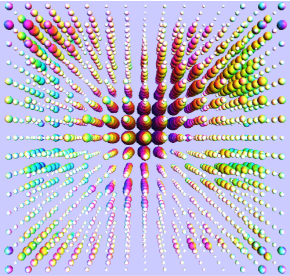
After generating and saving ensembles $\{U_n\}$ distributed $\propto e^{-S}$

often quick and easy to measure many observables $\langle \mathcal{O} \rangle$

Changing the action (generally) requires generating new ensembles

Backup: Hybrid Monte Carlo (HMC) algorithm

Goal: Sample field configurations U with probability $\frac{1}{\mathcal{Z}} e^{-S[U]}$



(Image credit: Claudio Rebbi)

HMC is Markov process based on Metropolis–Rosenbluth–Teller

Fermions $\longrightarrow$ extensive action computation

$\Longrightarrow$ Global updates using fictitious molecular dynamics

1. Introduce fictitious "MD time" τ and stochastic canonical momenta for fields
2. Inexact MD evolution along trajectory in τ $\longrightarrow$ new configuration
3. Accept/reject test on MD discretization error

Backup: Failure of Leibnitz rule in discrete space-time

$\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$ is problematic

$\longrightarrow$ try $\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2i\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu$ for a discrete translation

$$\nabla_\mu \phi(x) = \frac{1}{a}\left[\phi(x + a\widehat{\mu}) - \phi(x)\right] = \partial_\mu \phi(x) + \frac{a}{2}\partial_\mu^2 \phi(x) + \mathcal{O}(a^2)$$

Essential difference between ∂_μ and ∇_μ on the lattice, $a > 0$

$$\begin{aligned}\nabla_\mu\left[\phi(x)\eta(x)\right] &= a^{-1}\left[\phi(x + a\widehat{\mu})\eta(x + a\widehat{\mu}) - \phi(x)\eta(x)\right] \\ &= \left[\nabla_\mu \phi(x)\right]\eta(x) + \phi(x)\nabla_\mu \eta(x) + a\left[\nabla_\mu\phi(x)\right]\nabla_\mu\eta(x)\end{aligned}$$

Only recover Leibnitz rule $\partial_\mu(fg) = (\partial_\mu f)g + f\partial_\mu g$ when $a \to 0$

$\Longrightarrow$ “Discrete supersymmetry” breaks down on the lattice

(Dondi & Nicolai, “Lattice Supersymmetry”, 1977)

Backup: Basic features of $\mathcal{N} = 4$ SYM

Widely used to develop continuum QFT tools & techniques,
from scattering amplitudes to holography

Arguably simplest non-trivial 4d field theory

SU(N) gauge theory with four fermions Ψ^{I} and six scalars Φ^{IJ},
all massless and in adjoint rep.

Action consists of kinetic, Yukawa and four-scalar terms
with coefficients related by symmetries

Maximal 16 supersymmetries Q_α^{I} and $\overline{Q}_{\dot\alpha}^{\mathrm{I}}$ $\quad (\mathrm{I} = 1, \cdots, 4)$
transforming under global SU(4) $\sim$ SO(6) R symmetry

Conformal: β function is zero for any 't Hooft coupling $\lambda = g^2 N$

Backup: Topological twisting for $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}} \gamma_5$$
$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$
$$\text{with } a, b = 1, \cdots, 5$$

Kähler–Dirac muliplet of ‘twisted’ supersymmetries $\mathcal{Q}$
transforming with integer spin under ‘twisted rotation group’

$$\text{SO}(4)_{tw} \equiv \text{diag}\Big[\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R\Big] \qquad\qquad \text{SO}(4)_R \subset \text{SO}(6)_R$$

Change of variables $\longrightarrow$ closed subalgebra $\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$
that can be **exactly preserved on the lattice**

Backup: Susy subalgebra from twisted $\mathcal{N} = 4$ SYM

Fields & $\mathcal{Q}$s transform with integer spin under $\mathrm{SO}(4)_{tw}$ — no spinors

$$Q_\alpha \text{ and } \overline{Q}_{\dot\alpha} \longrightarrow \mathcal{Q},\ \mathcal{Q}_a \text{ and } \mathcal{Q}_{ab}$$

$$\Psi \text{ and } \overline{\Psi} \longrightarrow \eta,\ \psi_a \text{ and } \chi_{ab}$$

$$A_\mu \text{ and } \Phi^{\mathrm{I}} \longrightarrow \text{complexified gauge field } \mathcal{A}_a \text{ and } \overline{\mathcal{A}}_a$$

Complexification $\longrightarrow$ $\mathrm{U}(N) = \mathrm{SU}(N) \otimes \mathrm{U}(1)$ gauge theory

Schematically, under the twisted $\mathrm{SO}(d)_{tw} = \mathrm{diag}\left[\mathrm{SO}(d)_{\mathrm{euc}} \otimes \mathrm{SO}(d)_R\right]$

$$A_\mu \sim \text{vector} \otimes \text{scalar} \longrightarrow \text{vector}$$

$$\Phi^{\mathrm{I}} \sim \text{scalar} \otimes \text{vector} \longrightarrow \text{vector}$$

Easiest to see by dimensionally reducing from 5d

$$\mathcal{A}_a = A_a + i\Phi_a \longrightarrow (A_\mu, \phi) + i(\Phi_\mu, \overline{\phi})$$

Backup: Susy subalgebra from twisted $\mathcal{N} = 4$ SYM

Fields & $\mathcal{Q}$s transform with integer spin under SO(4)$_{tw}$ — no spinors

$$Q_\alpha \text{ and } \overline{Q}_{\dot\alpha} \longrightarrow \mathcal{Q},\ \mathcal{Q}_a \text{ and } \mathcal{Q}_{ab}$$

$$\Psi \text{ and } \overline{\Psi} \longrightarrow \eta,\ \psi_a \text{ and } \chi_{ab}$$

$$A_\mu \text{ and } \Phi^{\mathrm{I}} \longrightarrow \text{complexified gauge field } \mathcal{A}_a \text{ and } \overline{\mathcal{A}}_a$$

Twisted-scalar supersymmetry $\mathcal{Q}$

correctly interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\begin{aligned} \mathcal{Q}\,\mathcal{A}_a &= \psi_a & \qquad \mathcal{Q}\,\psi_a &= 0 \\ \mathcal{Q}\,\chi_{ab} &= -\overline{\mathcal{F}}_{ab} & \qquad \mathcal{Q}\,\overline{\mathcal{A}}_a &= 0 \\ \mathcal{Q}\,\eta &= d & \qquad \mathcal{Q}\,d &= 0 \end{aligned}$$

$\nwarrow$ bosonic auxiliary field with e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{A}_a$

Backup: Details of twisted lattice $\mathcal{N} = 4$ SYM

Lattice theory nearly a direct transcription despite breaking $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Covariant derivatives $\longrightarrow$ finite difference operators

Complexified gauge fields $\mathcal{A}_a \longrightarrow$ gauge links $\mathcal{U}_a \in \mathfrak{gl}(N, \mathbb{C})$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a \longrightarrow \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

(geometrically η on sites, ψ_a on links, etc.)

Susy lattice action $(\mathcal{Q}S = 0)$ from $\mathcal{Q}^2 \cdot = 0$ and Bianchi identity

$$S = \frac{N}{4\lambda_{\text{lat}}}\text{Tr}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de}\right]$$

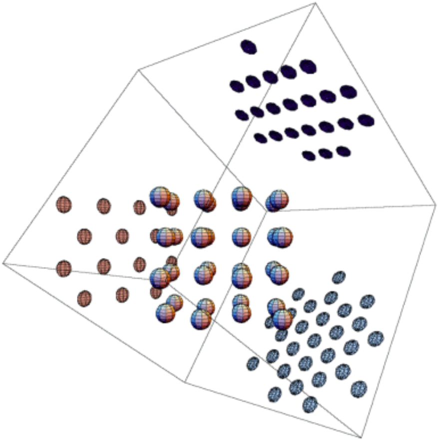
Backup: A_4^* lattice from dimensional reduction

Again easiest to dimensionally reduce from 5d,
treating all five gauge links $\mathcal{U}_a$ symmetrically

Start with hypercubic lattice
in 5d momentum space

Symmetric constraint $\sum_a \partial_a = 0$
projects to 4d momentum space

Result is A_4 lattice
$\longrightarrow$ dual A_4^* lattice in real space



Backup: Twisted SO(4) symmetry on the A_4^* lattice

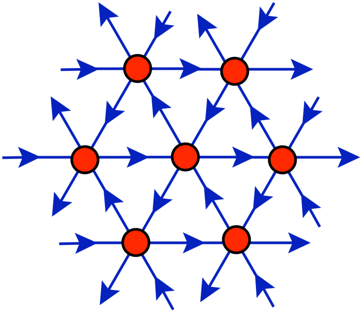
Can view A_4^* lattice

as 4d analog of 2d triangular lattice

Basis vectors linearly dependent

and non-orthogonal $\longrightarrow \lambda = \lambda_{\text{lat}}/\sqrt{5}$

Preserves S_5 point group symmetry



S_5 irreps precisely match onto irreps of twisted $\text{SO}(4)_{tw}$

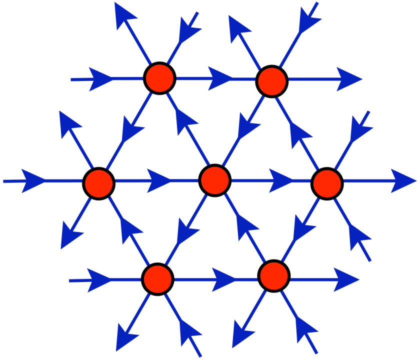
$$\mathbf{5} = \mathbf{4} \oplus \mathbf{1} : \quad \psi_a \longrightarrow \psi_\mu, \ \ \overline{\eta}$$

$$\mathbf{10} = \mathbf{6} \oplus \mathbf{4} : \quad \chi_{ab} \longrightarrow \chi_{\mu\nu}, \ \ \overline{\psi}_\mu$$

$S_5 \longrightarrow \text{SO}(4)_{tw}$ in continuum limit restores $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Backup: Hypercubic representation of A_4^* lattice

In the code it is very convenient to represent the A_4^* lattice as a hypercube plus one backwards diagonal link



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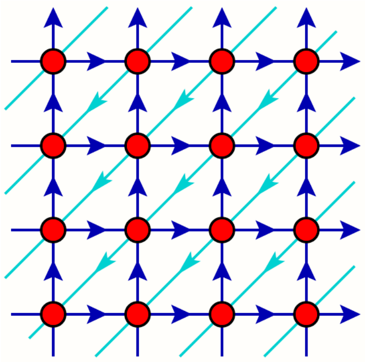


Backup: Analytic results for lattice $\mathcal{N} = 4$ SYM

U(N) gauge invariance $+$ $\mathcal{Q}$ $+$ S_5 lattice symmetries
$\longrightarrow$ several significant analytic results

Moduli space preserved to all orders of lattice perturbation theory
$\longrightarrow$ no scalar potential induced by radiative corrections

β function vanishes at one loop in lattice perturbation theory

Real-space RG blocking transformations preserving $\mathcal{Q}$ and S_5
$\longrightarrow$ no new terms in long-distance effective action

Only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

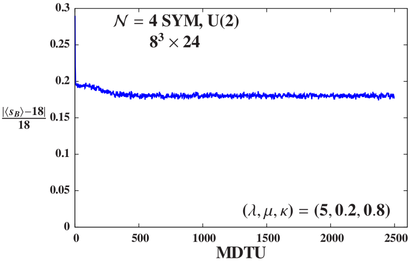
Backup: Problem with SU(N) flat directions

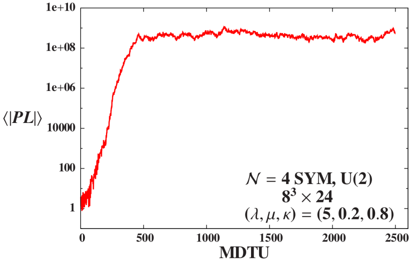
$\mu^2/\lambda_{\text{lat}}$ too small $\longrightarrow$ $\mathcal{U}_a$ can move far from continuum form $\mathbb{I}_N + \mathcal{A}_a$

Example: $\mu = 0.2$ and $\lambda_{\text{lat}} = 5$ on $8^3 \times 24$ volume

Left: Bosonic action stable $\sim$18% off its supersymmetric value

Right: Complexified Polyakov ('Maldacena') loop wanders off to $\sim 10^9$





Backup: Details of SU(N) scalar potential

$$S = \frac{N}{4\lambda_{\text{lat}}} \left[\mathcal{Q} \left(\chi_{ab}\mathcal{F}_{ab} + \eta \overline{\mathcal{D}}_a \mathcal{U}_a - \frac{1}{2}\eta d \right) - \frac{1}{4}\epsilon_{abcde}\, \chi_{ab} \overline{\mathcal{D}}_c\, \chi_{de} + \mu^2 V \right]$$

Scalar potential $V = \sum_a \left(\frac{1}{N} \text{Tr}\left[\mathcal{U}_a \overline{\mathcal{U}}_a\right] - 1 \right)^2$ lifts SU(N) flat directions

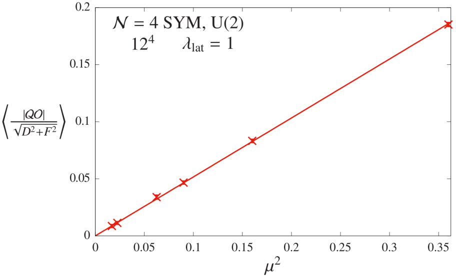
and ensures $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$ in continuum limit

Softly breaks $\mathcal{Q}$ — all susy violations $\propto \mu^2 \to 0$ in continuum limit

Ward identity violations, $\langle \mathcal{Q}\mathcal{O} \rangle \neq 0$,
show $\mathcal{Q}$ breaking and restoration

Here considering

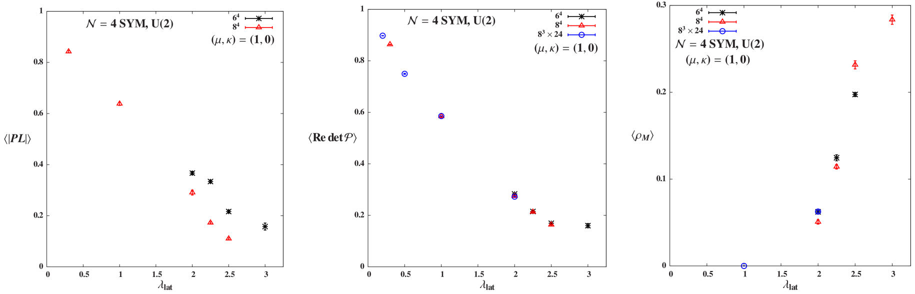
$$\mathcal{Q}\left[\eta \mathcal{U}_a \overline{\mathcal{U}}_a\right] = d\mathcal{U}_a \overline{\mathcal{U}}_a - \eta \psi_a \overline{\mathcal{U}}_a$$



Backup: Problem with U(1) flat directions

Monopole condensation $\longrightarrow$ confined lattice phase

not present in continuum $\mathcal{N} = 4$ SYM



Around the same $\lambda_{\mathrm{lat}} \approx 2\ldots$

Left: Polyakov loop falls towards zero

Center: Plaquette determinant falls towards zero

Right: Density of U(1) monopole world lines becomes non-zero

Backup: Details of U(1) plaq. determinant deformation

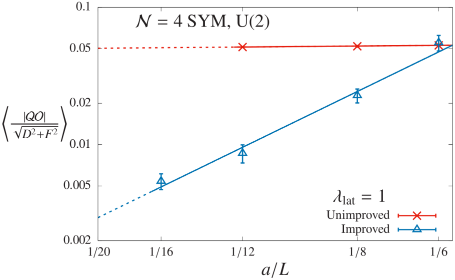
$$S = \frac{N}{4\lambda_{\text{lat}}}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \quad \downarrow \quad - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de} + \mu^2 V\right]$$

$$\eta\left\{\overline{\mathcal{D}}_a\mathcal{U}_a + G\sum_{a<b}\left[\det\mathcal{P}_{ab} - 1\right]\mathbb{I}_N\right\}$$

Modify e.o.m. for d to constrain plaquette determinant
$\longrightarrow$ lifts U(1) zero mode & flat directions without susy breaking

Much better than adding
another soft $\mathcal{Q}$-breaking term

$O(a)$ improvement, $\langle\mathcal{QO}\rangle \propto (a/L)^2$,
since $\mathcal{Q}$ forbids all dim-5 operators



Backup: More on soft supersymmetry breaking

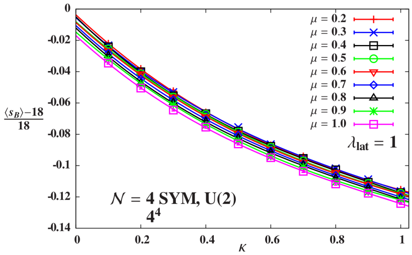
Until 2015 $(\det\mathcal{P} - 1)$ was another soft susy-breaking term

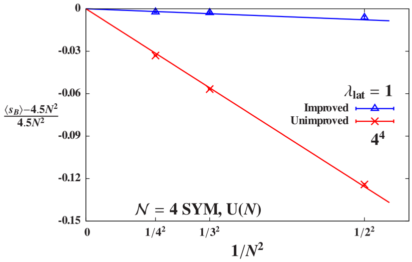
$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}}\mu^2 \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2 + \kappa \sum_{a<b} \left|\det\mathcal{P}_{ab} - 1\right|^2$$

Much larger $\mathcal{Q}$-breaking effects than scalar potential

Left: $\mathcal{Q}$ Ward identity from bosonic action $\langle s_B\rangle = 9N^2/2$

Right: Soft susy breaking suppressed $\propto 1/N^2$





Backup: Supersymmetric moduli space modification

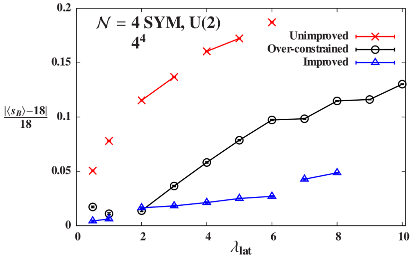
arXiv:1505.03135 introduces method to impose $\mathcal{Q}$-invariant constraints

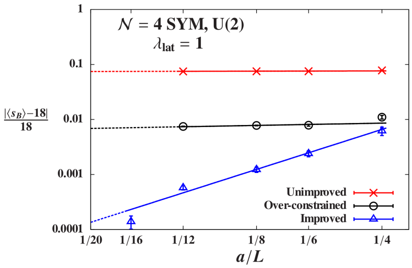
Modify auxiliary field equations of motion $\longrightarrow$ moduli space

$$d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) \qquad \longrightarrow \qquad d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) + G\mathcal{O}(n)\mathbb{I}_N$$

Including both plaquette determinant and scalar potential in $\mathcal{O}(n)$

over-constrains system $\longrightarrow$ sub-optimal Ward identity violations





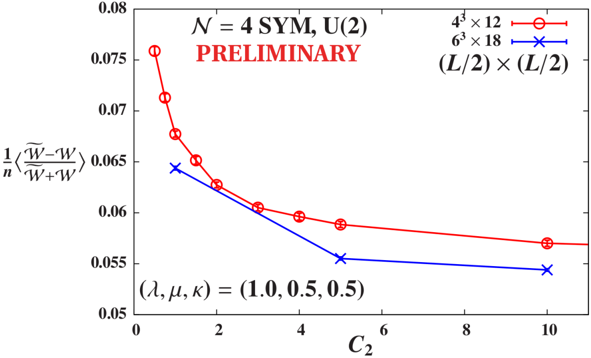
Backup: Restoration of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries

$\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ from restoration of R symmetry (motivation for A_4^* lattice)

Modified Wilson loops test R symmetries at non-zero lattice spacing

Parameter c_2 may need logarithmic tuning in continuum limit

Results from arXiv:1411.0166 to be revisited using improved action

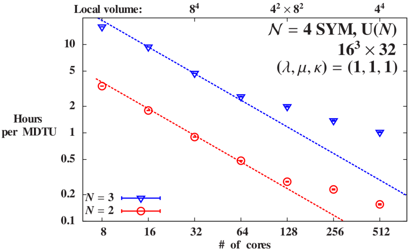


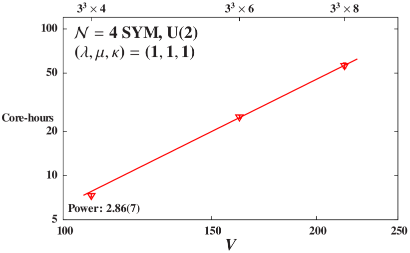
Backup: Code performance—weak and strong scaling

Results from arXiv:1410.6971 to be updated using improved action

Left: Strong scaling for U(2) and U(3) $16^3 \times 32$ RHMC

Right: Weak scaling for $\mathcal{O}(n^3)$ pfaffian calculation (fixed local volume)
$n \equiv 16N^2 V$ is number of fermion degrees of freedom





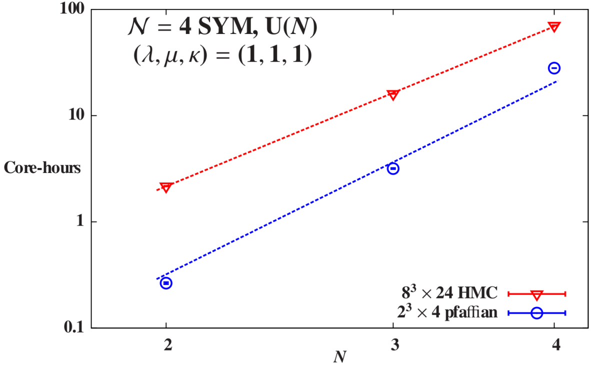
Dashed lines are optimal scaling

Solid line is power-law fit

Backup: Numerical costs for $N = 2$, 3 and 4 colors

Red: Original RHMC cost scaling $\sim N^5$ now improved to $\sim N^{3.5}$
Plot from arXiv:1410.6971 to be updated

Blue: Pfaffian cost scaling consistent with expected N^6



Backup: Dimensional reduction to $\mathcal{N} = (8, 8)$ SYM

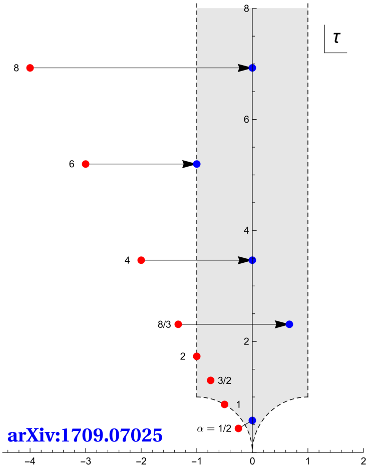
Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_y = N_z = 1$

A_4^* lattice $\longrightarrow$ A_2^* (triangular) lattice

Torus **skewed** depending on $\alpha = N_x / N_t$

Modular trans. into fundamental domain
can make skewed torus rectangular

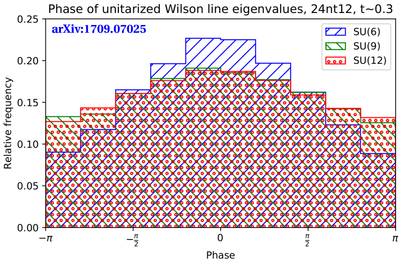
Also need to stabilize compactified links
to ensure broken center symmetries

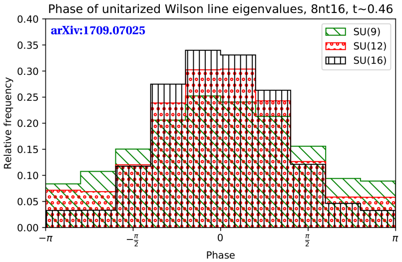


arXiv:1709.07025

Backup: $\mathcal{N} = (8, 8)$ SYM Wilson line eigenvalues

Check 'spatial deconfinement' through histograms of Wilson line eigenvalue phases



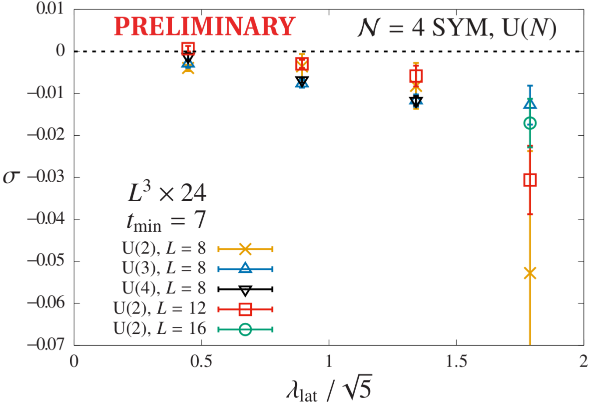


Left: $\alpha = 2$ distributions more extended as *N* increases
$\longrightarrow$ dual gravity describes homogeneous black string (D1 phase)

Right: $\alpha = 1/2$ distributions more compact as *N* increases
$\longrightarrow$ dual gravity describes localized black hole (D0 phase)

Backup: Static potential is Coulombic at all λ

String tension σ from fits to confining form $V(r) = A - C/r + \sigma r$



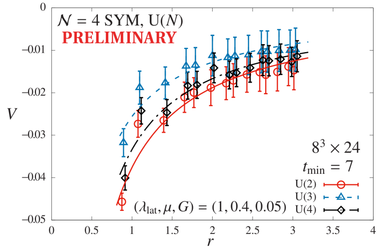
Slightly negative values flatten $V(r_I)$ for $r_I \lesssim L/2$

$\implies \sigma \to 0$ as accessible range of r_I increases on larger volumes

Backup: Discretization artifacts in static potential

Discretization artifacts visible at short distances
where Coulomb term in $V(r) = A - C/r$ is most significant

Right: Highlight artifacts by extracting fluctuations around Coulomb fit



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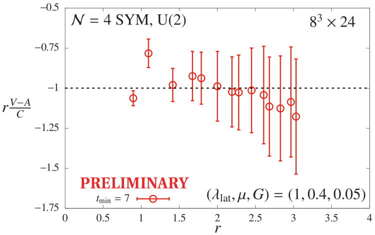


Danger of potential contamination in results for Coulomb coefficient C

Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential

(Lang & Rebbi '82; Sommer '93; Necco '03)

Associate $V(r)$ data with r from Fourier transform of gluon propagator

Recall $\frac{1}{4\pi^2 r} = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{e^{ir\cdot k}}{k^2}$ where $\frac{1}{k^2} = G(k)$ in continuum

$$\text{On } A_4^* \text{ lattice} \longrightarrow \frac{1}{r_I^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4 \widehat{k}}{(2\pi)^4} \frac{\cos\left(ir_I \cdot \widehat{k}\right)}{4\sum_{\mu=1}^{4} \sin^2\left(\widehat{k} \cdot \widehat{e}_\mu \;/\; 2\right)}$$

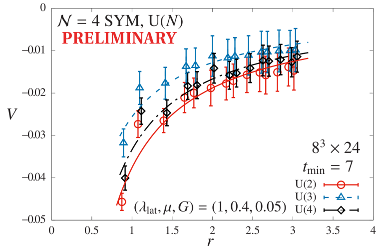
Tree-level perturbative lattice propagator from arXiv:1102.1725

$\widehat{e}_\mu$ are A_4^* lattice basis vectors

while momenta $\widehat{k} = \frac{2\pi}{L}\sum_{\mu=1}^{4} n_\mu \widehat{g}_\mu$ depend on dual basis vectors

Backup: Tree-level-improved static potential

Tree-level improvement significantly reduces discretization artifacts



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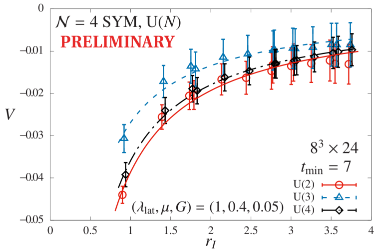

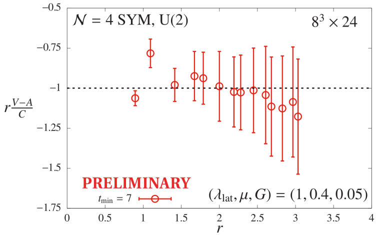

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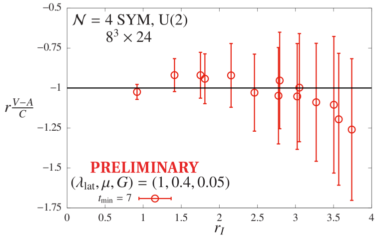
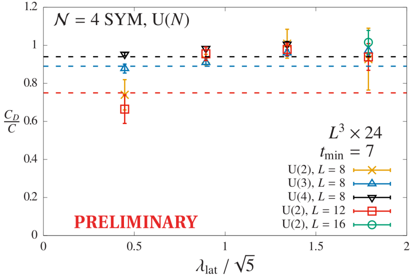
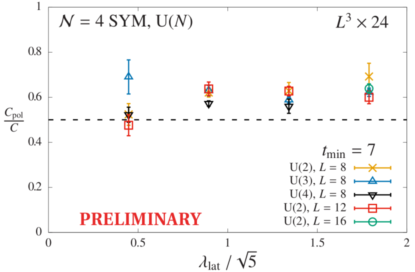


Backup: More $\mathcal{N} = 4$ SYM static potential tests

Left: Projecting Wilson loops from U(N) $\longrightarrow$ SU(N) $\Longrightarrow$ factor of $\frac{N^2-1}{N^2}$

Right: Unitarizing links removes scalars $\Longrightarrow$ factor of $1/2$





Several ratios end up above expected values

Cause not clear — seems insensitive to lattice volume and μ

Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve $\mathcal{Q}$ and S_5 symmetries $\longleftrightarrow$ geometric structure

Simple transformation constructed in arXiv:1408.7067

$$\mathcal{U}_a'(n') = \xi\,\mathcal{U}_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) \qquad \eta'(n') = \eta(n)$$

$$\psi_a'(n') = \xi\left[\psi_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) + \mathcal{U}_a(n)\psi_a(n+\widehat{\mu}_a)\right] \qquad \text{etc.}$$

Doubles lattice spacing $a \longrightarrow a' = 2a$, with ξ a tunable rescaling factor

Scalar fields from polar decomposition $\mathcal{U}(n) = e^{\varphi(n)}U(n)$
are shifted, $\varphi \longrightarrow \varphi + \log\xi$, since blocked U must remain unitary

$\mathcal{Q}$-preserving RG blocking needed
to show only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Backup: Smearing for Konishi analyses

As for glueballs, smear to enlarge operator basis

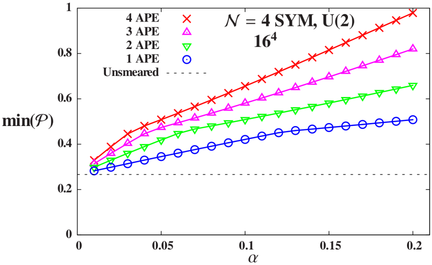
APE-like smearing: $— \longrightarrow (1-\alpha)— + \frac{\alpha}{8}\sum \sqcap$,

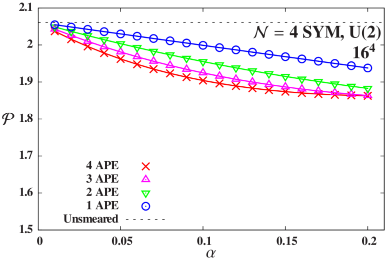
staples built from unitary parts of links but no final unitarization

(unitarized smearing — e.g. stout — doesn't affect Konishi)

Average plaquette stable upon smearing (**right**)

while minimum plaquette steadily increases (**left**)





Backup: Lattice superQCD in 2d & 3d

Add fundamental matter multiplets without breaking $\mathcal{Q}^2 = 0$

Proposed by Matsuura [arXiv:0805.4491] and Sugino [arXiv:0807.2683], first numerical study by Catterall & Veernala [arXiv:1505.00467]

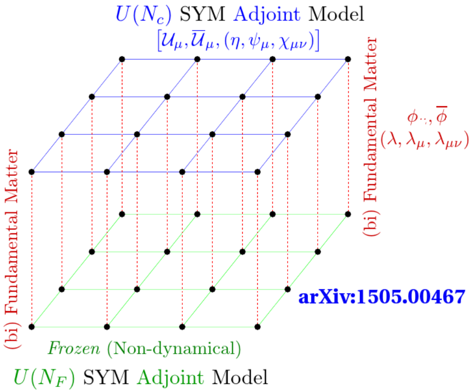
2-slice lattice SYM with U(N) $\times$ U(F) gauge group

Adj. fields on each slice

Bi-fundamental in between

Set U(F) gauge coupling to zero

$\longrightarrow$ U(N) in $d-1$ dims. with F fund. hypermultiplets



Backup: Spontaneous supersymmetry breaking

Auxiliary field e.o.m. $\longrightarrow$ Fayet–Iliopoulos D-term potential

$$d = \overline{\mathcal{D}}_a \mathcal{U}_a + \sum_{i=1}^{F} \phi_i \overline{\phi}_i + r\mathbb{I}_N \qquad \longrightarrow \qquad S_D \propto \sum_{i=1}^{F} \text{Tr}\left[\phi_i \overline{\phi}_i + r\mathbb{I}_N\right]^2$$

$\langle \mathcal{Q}\eta \rangle = \langle d \rangle \neq 0 \Longrightarrow \langle 0 \,| H |\, 0 \rangle > 0$ (spontaneous susy breaking)

$\longrightarrow N \times N$ conditions vs. $N \times F$ degrees of freedom