Holographic Corrections to Meson Scattering

Adi Armoni
Swansea University
ICTS, Jan. 31, 2018

A. A., Phys.Lett.B756 (2016) 328-331
A. A., Edwin Ireson,Phys.Lett.B771 (2017) 430-434, Nucl.Phys. B919 (2017) 238-266
A. A., Edwin Ireson, Davide Vadacchino,Nucl.Phys.B, to appear.

Introduction

The 1968 Veneziano amplitude

$$\mathcal{A}(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

marks the birth of string theory.

It has some positive and phenomenological appealing properties, most importantly linear Regge trajectories of the form $J = \alpha(0) + \alpha' s$ and their daughter trajectories.

The amplitude suffers from several bad properties such as the UV behavior ($s, t \to \infty$ with s/t fixed) where the amplitude decreases exponentially $\mathcal{A}(s,t) \sim \exp -\alpha' s$.

It is then interesting to ask, what is the relation between the Veneziano amplitude and QCD?

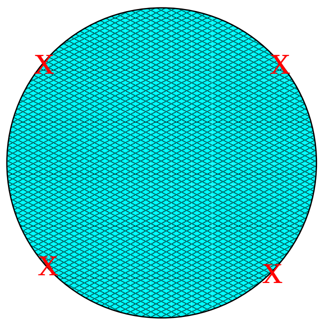
Is there a limit of QCD where meson scattering is described by the Veneziano amplitude?

The relation between the amplitude and QCD

The topology of the string amplitude, the disk, suggests that we should take the 't Hooft limit of QCD.

Indeed, in the limit $N_c \to \infty$, fixed $g^2 N_c$ and fixed N_f the QCD amplitude does not contain internal fermionic loops (windows) or handles.

$$\mathcal{A}(x_1, x_2, x_3, x_4) = \langle \bar{q}q(x_1)\bar{q}q(x_2)\bar{q}q(x_3)\bar{q}q(x_4)\rangle$$



Let us write

$$(\det(i \not{D}))^{N_f} = \exp(N_f \Gamma[A_\mu]) \, ,$$

with

$$\Gamma[A_\mu] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}$$

$$\times \int \mathcal{D}x^\mu \mathcal{D}\psi^\mu \, \exp\left\{-\int_\epsilon^T d\tau \, \left(\frac{1}{2}\dot{x}^\mu\dot{x}^\mu + \frac{1}{2}\psi^\mu\dot{\psi}^\mu\right)\right\}$$

$$\times \mathrm{Tr}\,\mathcal{P}\exp\left\{i\int_0^T d\tau \, \left(A_\mu\dot{x}^\mu - \frac{1}{2}\psi^\mu F_{\mu\nu}\psi^\nu\right)\right\}$$

expanding the exponent $\exp N_f\Gamma$ in powers of N_f/N_c yields the following expression for the scattering amplitude

$$\mathcal{A}(x_1, x_2, x_3, x_4) =$$

$$\frac{1}{2}\int_0^\infty \frac{dT}{T}\int \mathcal{D}x \, \exp\left(-\int d\tau \, \tfrac{1}{2}\dot{x}_\mu^2\right)\langle W(x_1, x_2, x_3, x_4)\rangle_{\mathrm{YM}}$$

where the worldline fermions were omitted for the brevity of writing.

Thus, in the 't Hooft limit *the scattering amplitude* $\mathcal{A}(x_1, x_2, x_3, x_4)$ *is given by sum over all sizes and shapes of Wilson loops that pass via the points* x_1, x_2, x_3, x_4.

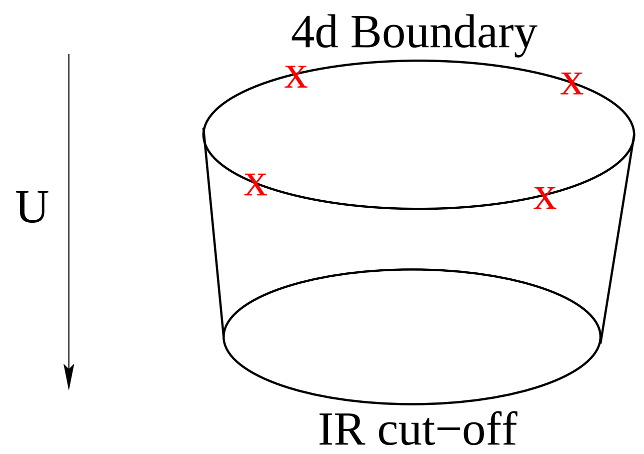
See related discussion by Makeenko and Olesen.

Holography and Wilson loops

The advantage of expressing the amplitude as a sum over Wilson loops is that since we know how to calculate expectation values of Wilson loop via holography, we can relate the field theory expression to string theory (and eventually derive the Veneziano amplitude).

The holographic prescription (Maldacena) is to find a string worldsheet which terminates on the AdS boundary. The worldsheet boundary is the contour of the Wilson loop.

The present calculation requires a contour that passes through x_1, x_2, x_3, x_4, hence a typical worldsheet looks like



Holography and Wilson loops

We propose the following expression for the amplitude

$$\frac{1}{2}\int_0^\infty \frac{dT}{T} \int \mathcal{D}x^\mu(\tau)\times$$

$$\exp\left(-\int d\tau\, \tfrac{1}{2}\dot{x}_\mu^2\right) \langle W(x_1,x_2,x_3,x_4)\rangle =$$

$$\int \mathcal{D}g^{\alpha\beta}\, \mathcal{D}x^M \times$$

$$\exp\left(-\frac{1}{2\pi\alpha'}\int d^2\sigma\, \sqrt{g} g^{\alpha\beta}\partial_\alpha x^M \partial_\beta x^N G_{MN}\right)|_{\{x_1,x_2,x_3,x_4\}}$$

The amplitude is given by a sum over all string worldsheets that terminate on the AdS space and pass through the points x_1, x_2, x_3, x_4.

The above expression holds for any gauge/gravity pair. The information about the gauge theory is encoded in the metric G_{MN}.

Witten's model

We are interested in the gravity dual of pure Yang-Mills theory. Such a dual does not exist, but Witten's model of compactified D4 branes on a thermal circle contains the essential ingredients: confinement and a mass gap.

The metric is

$$ds^2 = (U/R)^{3/2}(\eta_{\mu\nu}dx^\mu dx^\nu$$

$$+f(U)d\tau^2) + (R/U)^{3/2}(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2)$$

with

$$f(U) = 1 - U_{KK}^3/U^3$$

Due to the metric singularity at $U = U_{KK}$, large Wilson loops exhibit an area law $\langle W \rangle = \exp -(\Sigma A)$ with a string tension $\Sigma = \frac{1}{2\pi\alpha'}(\frac{U_{KK}}{R})^{\frac{3}{2}}$.

It is therefore anticipated that if Δx_i is large the sum over Wilson loops will be dominated by configurations that exhibit an area law.

String theory calculation

Let us ignore the compact directions (the four-sphere and τ). This is a reasonable assumption. These direction are more of an artefact than a feature of Yang-Mills theory. Moreover, the contribution to the path integral from compact directions is expected to be small. We therefore approximate the amplitude by

$$\int \mathcal{D}x^{\mu}\mathcal{D}U \, \exp\left(-\frac{1}{2\pi\alpha'}\int d^2\sigma \,\{(\partial_\alpha x^\mu \partial^\alpha x^\nu G_{\mu\nu})+ (\partial_\alpha U \partial^\alpha U G_{UU})\}\right)|_{\{x_1,x_2,x_3,x_4\}}$$

with the 5d metric

$$ds^2 = (U/R)^{3/2}\eta_{\mu\nu}dx^\mu dx^\nu + (R/U)^{3/2}(\frac{dU^2}{f(U)})\,.$$

Holographic Calculation of the Amplitude

Due to the metric singularity, the path integral is dominated by U in the vicinity of U_{KK}. This is the case for large worldsheets, where the string "sits" at the horizon. Thus, let us insert $\delta(U - (U_{KK} + \epsilon))$ into the path integral, suppressing quantum fluctuations in the U directions. Later on we will discuss what is expected to happen if we omit the delta function and allow fluctuations in the U direction. The modified path integral reads

$$\int \mathcal{D}x^\mu \mathcal{D}U \, \delta(U - U_{KK} - \epsilon) \times$$

$$\times \exp\left(-\frac{1}{2\pi\alpha'} \int d^2\sigma \, \{ (\partial_\alpha x^\mu \partial^\alpha x^\nu G_{\mu\nu}) + (\partial_\alpha U \partial^\alpha U G_{UU}) \} \right) |_{\{x_1, x_2, x_3, x_4\}} =$$

$$\int \mathcal{D}x^\mu \exp\left(-\frac{1}{2\pi\alpha'} \int d^2\sigma \, \partial_\alpha x^\mu \partial^\alpha x^\nu \hat{G}_{\mu\nu} \right) |_{\{x_1, x_2, x_3, x_4\}}$$

with $\hat{G}_{\mu\nu}$ the flat 4d metric

$$ds^2 = (U_{KK}/R)^{3/2} \eta_{\mu\nu} dx^\mu dx^\nu$$

Holographic Calculation of the Amplitude

It is easier to calculate the amplitude in momenta space by inserting vertex operators of the form

$$V \sim \int dy \, \exp(ik \cdot x)$$

The resulting expression is

$$\mathcal{A}(k_1, k_2, k_3, k_4) = \int \mathcal{D}x^\mu \prod_{i=1,..,4} \int dy_i \exp\left(ik_i x(y_i)\right)$$

$$\times \exp\left(-\Sigma \int d^2\sigma \, \partial_\alpha x^\mu \partial^\alpha x_\mu\right)$$

We obtain the Koba-Nielsen expression for the scattering amplitude

$$\mathcal{A}(k_1, k_2, k_3, k_4) \sim \delta(\sum_i k_i) \times$$

$$\int \prod_i dy_i \prod_{j<l} (y_j - y_l)^{2\alpha'(R/U_{KK})^{3/2} k_j \cdot k_l}$$

which yields the Veneziano amplitude, with $\alpha'_{eff} = \alpha'(R/U_{KK})^{3/2}$.

A comment about the lattice strong coupling expansion

Interestingly the lattice strong coupling expansion, to leading order, also yields an area law for Wilson loops.

Invoking the Hopping expansion on the lattice (which is very similar to the worldline expansion), we obtain the following expression for the meson scattering amplitude

$$\langle \phi^\dagger \phi(n_1) \cdots \phi^\dagger \phi(n_4) \rangle_c \simeq \sum_l \frac{\kappa^l}{l} \sum_{n_i \in C_l} e^{-\sigma \mathcal{A}}$$

It means that on the lattice the scattering amplitude is given by the Veneziano amplitude and that to leading order in the strong coupling expansion, the lattice is equivalent to flat space string theory.

Beyond the Veneziano Amplitude

We now wish to include holographic corrections.

Let us carry out a Kruskal transformation and expand the metric near the horizon to include deviations from the flat space sigma model.

The leading correction takes the form

$$\mathcal{A}(k_1,k_2,k_3,k_4) = \int \mathcal{D}x^\mu \mathcal{D}\hat{U} \prod_{i=1,..,4} \int dy_i \exp\left(ik_i x(y_i)\right)$$

$$\times \exp(-\Sigma \int d^2\sigma \, (\partial_\alpha x^\mu \partial^\alpha x_\mu + \partial_\alpha \hat{U} \partial^\alpha \hat{U} + \lambda \hat{U}^2 \partial_\alpha x^\mu \partial^\alpha x_\mu))$$

where $\lambda \sim \frac{1}{U_{KK}^2}$ is the coupling between x^μ and the holographic coordinate $\hat{U}$.

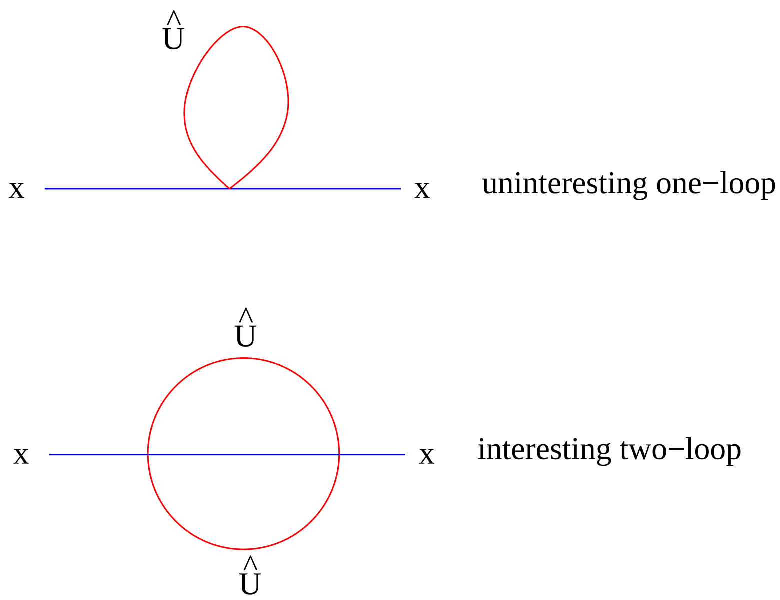
In the limit $\lambda \to 0$, the IR cut-off coincides with the UV cut-off and we recover the Veneziano amplitude.

We wish to study, by using perturbation theory, corrections due to the interaction between x^μ and $\hat{U}$.

Perturbative corrections to the x^μ propagator

Consider the following tree-level, one-loop and two-loop contributions to

$$\langle x^\mu(\sigma, \tau) x^\nu(\sigma', \tau')\rangle$$

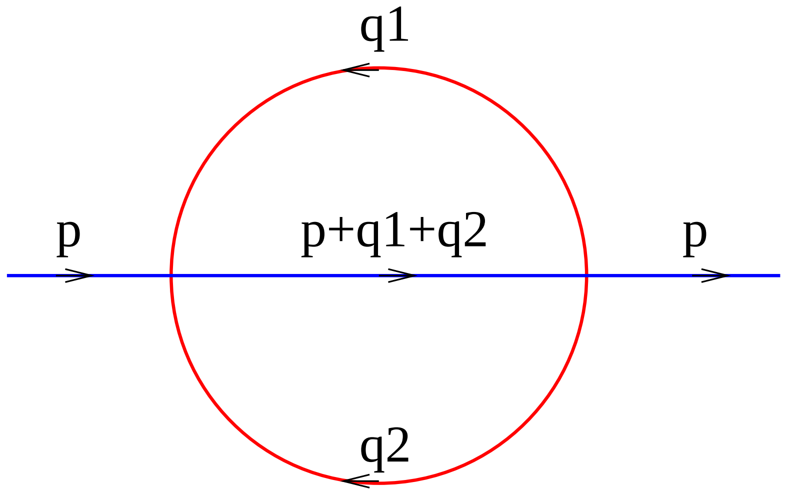


The one-loop correction renormalizes (shifts) the QCD string tension. It is not so interesting.

The two-loop correction carries some non-trivial information.

Two-loop correction to the x^μ propagator

The correction is



$$\lambda^2 \int \frac{d^2p}{2\pi^2}\frac{d^2q_1}{2\pi^2}\frac{d^2q_2}{2\pi^2}\exp(ip(\sigma-\sigma'))\times$$

$$\times\left(\frac{1}{p^2}\right)^2\frac{(p\cdot(p+q_1+q_2))^2}{q_1^2q_2^2(p+q_1+q_2)^2}$$

$$=\lambda^2\log^3(\sigma-\sigma')$$

The corrected amplitude

$$\mathcal{A}(k_1,k_2,k_3,k_4) \sim \delta(\sum_i k_i) \int \prod_i dy_i \times$$

$$\prod_{j<l} \exp 2\alpha'_{eff} k_j \cdot k_l \left(\log(y_j - y_l) - \lambda^2 \log^3(y_j - y_l)\right)$$

The above expression can be represented as

$$\mathcal{A}(s,t) = \left(1 - \lambda^2 (\tilde{s}\frac{\partial^3}{\partial \tilde{s}^3} + \tilde{t}\frac{\partial^3}{\partial \tilde{t}^3})\right) B(\tilde{s},\tilde{t})$$

with $\tilde{s} = \alpha'_{eff} s$ and $\tilde{t} = \alpha'_{eff} t$.

Namely, the original Veneziano amplitude receives small holographic corrections of the form

$$\lambda^2 (\tilde{s}\frac{\partial^3}{\partial \tilde{s}^3} + \tilde{t}\frac{\partial^3}{\partial \tilde{t}^3}) B(\tilde{s},\tilde{t})$$

Regge regime

In the Regge regime of large s and fixed t

$$B(\tilde{s},\tilde{t}) \to \tilde{s}^{\tilde{t}} = \exp(\tilde{t}\log\tilde{s})$$

In this limit the corrected expression is

$$\mathcal{A}(s,t) \to \exp(\tilde{t}(\log\tilde{s} - \lambda^2\log^3\tilde{s}))$$

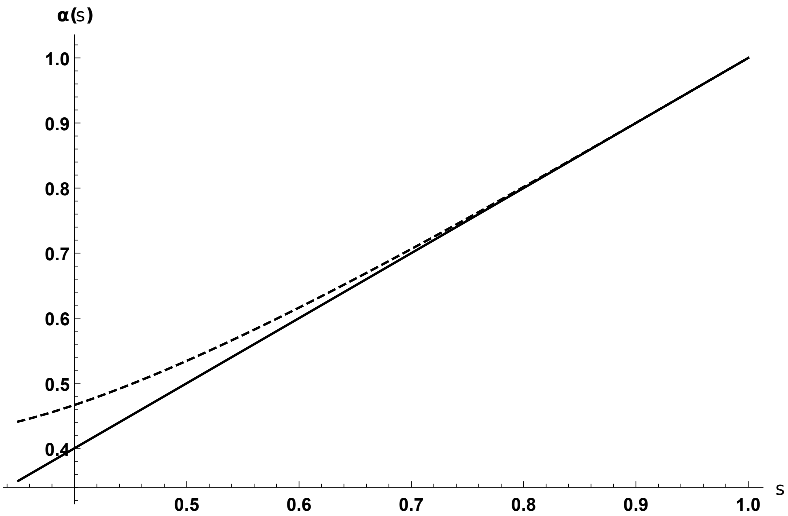
$$= \tilde{s}^{\tilde{t}(1-\lambda^2\log^2\tilde{s})}$$

We can view this correction as a small deviation from linearity of the Regge trajectory when $\lambda^2\log^2\tilde{s} \ll 1$

$$\alpha(s) = \tilde{s}^{(1-\lambda^2\log^2\tilde{s})}$$

Deviations from Linearity

The correction affects $\alpha(s)$ in the small s regime. It leads to a bending of the curve of the following form



A deviation from linearity is expected in QCD on general grounds: a perfectly linear Regge trajectory is obtained by assuming that all Wilson loops, however small, admit an area law. This is the same as $\lambda = 0$ in our approach. However, small Wilson loops are computed using perturbation theory and the result is a perimeter law. Therefore the trajectory cannot be perfectly linear.

Conclusions

- In the 't Hooft limit meson scattering can be expressed as a sum over Wilson loops.
- Using holography the sum over Wilson loops is mapped into a sum over string worldsheets.
- By using an unjustified and crude approximation where both the compact directions and the holographic direction are neglected, the path integral become gaussian. The gaussian integration yields the Veneziano amplitude.
- Incorporating a perturbative contribution due to the interaction of 4d flat space with the holographic coordinate, leads to a deviation from linearity.