$\mathcal{N} = (2, 2)$ super-Yang-Mills theory in two dimensions

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1. Continuum models in $4d$ and $2d$

2. Lattice formulation of $\mathcal{N} = (2, 2)$ SYM in 2 dimensions

3. Masses of bound states

- action

$$S=\int \mathrm{d}^4x\,\mathrm{tr}\left(-\frac{1}{4}F_{MN}F^{MN}+\frac{\mathrm{i}}{2}\bar{\lambda}\,\Gamma^M D_M\lambda\right)$$

- with $M,N\in 0,1,2,3$, $\{\Gamma_\mu,\Gamma_\nu\}=2g_{\mu\nu}\mathbb{1}_4$
- field strength tensor

$$F_{MN}=\partial_M A_N-\partial_N A_M-\mathrm{i}\,g\,[A_M,A_N]$$

- potential A_M, Majorana-field λ in adjoint representation
- on-shell susy transformations

$$\delta_\varepsilon A_\mu=\mathrm{i}\bar{\varepsilon}\,\Gamma_M\lambda,\quad \delta_\varepsilon\lambda=\mathrm{i}F^{MN}\Sigma_{MN}\,\varepsilon,\quad \delta_\varepsilon\bar{\lambda}=-\mathrm{i}\bar{\varepsilon}\,F^{MN}\Sigma_{MN}$$

- axial symmetry

$$\lambda \to e^{\mathrm{i}\alpha\,\Gamma_5}\lambda \qquad \Gamma_5 = \mathrm{i}\,\Gamma^0\Gamma^1\Gamma^2\Gamma^3$$

- broken: anomaly $\to \mathbb{Z}_{2N}$, spontaneous SBB $\to \mathbb{Z}_2$

dimensional reduction

- convenient choice

$$\Gamma_\mu = \mathbb{1} \otimes \gamma_\mu, \quad \Gamma_2 = \mathrm{i}\sigma_1 \otimes \gamma_5, \quad \Gamma_3 = \mathrm{i}\sigma_3 \otimes \gamma_5$$

- chirality and charge conjugation

$$\Gamma_5 = \sigma_2 \otimes \gamma_5, \quad \gamma_5 = \gamma_0\gamma_1, \qquad \mathcal{C}_4 = \mathbb{1} \otimes \mathcal{C}_2, \quad \mathcal{C}_2 = -\gamma^0$$

- reduction of Yang-Mills term: set $(A_M) = (A_0, A_1, \phi_1, \phi_2) \Rightarrow$

$$-\frac{1}{4}F_{MN}F^{MN} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi_m D^\mu\phi_m + \frac{g^2}{4}\left[\phi_m, \phi_n\right]\left[\phi^m, \phi^n\right]$$

- fermionic part of action

$$\bar{\lambda}\,\Gamma^M D_M\lambda = \bar{\lambda}\,\Gamma^\mu D_\mu\lambda - \mathrm{i}\,g\bar{\lambda}\,\Gamma^{m+1}\left[\phi_m, \lambda\right]$$

- $\{e_1, e_2\}$ Cartesian basis of $\mathbb{R}^2$:

$$\lambda = \sum_{r=1}^{2} e_r \otimes \chi_r \Longrightarrow \bar{\lambda} = \sum_{r=1}^{2} e_r^T \otimes \bar{\chi}_r\,,$$

- 2-component Dirac spinor, complex scalar

$$\psi = \frac{1}{\sqrt{2}}(\chi_1 + \mathrm{i}\gamma_5\chi_2)\,, \quad \bar{\psi} = \frac{1}{\sqrt{2}}(\bar{\chi}_1 + \mathrm{i}\bar{\chi}_2\gamma_5)\,, \quad \varphi = \phi_1 + \mathrm{i}\phi_2$$

- action of reduced model

$$S = \frac{1}{g^2}\int \mathrm{d}^2x \mathrm{tr}\Big\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\varphi)^\dagger(D^\mu\varphi) - \frac{1}{8}[\varphi^\dagger,\varphi]^2 + \mathrm{i}\,\bar{\psi}\gamma^\mu D_\mu\psi - \bar{\psi}P_+[\varphi,\psi] - \bar{\psi}P_-[\varphi^\dagger,\psi]\Big\}$$

- reduction of Lorentz invariance

$$SO(1,3) \to SO_L(1,1) \times SO_R(2)$$

- Euclidean version $SO(4) \to SO_L(2) \times SO_R(2)$

- ϕ_m and χ_m rotate under $SO_R(2) \Rightarrow$

$$\varphi \to \exp(2\mathrm{i}\,\alpha)\varphi\,, \quad \psi \to \exp(-\mathrm{i}\,\alpha\,\gamma_5)\psi, \quad \bar{\psi} \to \bar{\psi}\exp(-\mathrm{i}\,\alpha\,\gamma_5)$$

- chiral symmetry in $4d$ $\to$ phase rotation in 2d: $\psi \to \mathrm{e}^{-\mathrm{i}\alpha}\psi$

reduction of symmetries

Lorentz-symmetry $\longrightarrow$ Lorentz-symmetry $\times$ chiral symmetry in $2d$
chiral symmetry $\longrightarrow$ phase rotation in $2d$

- confinement: color-blind bound states
- reducing $3d$ supermultiplets of bound states in

particle	spin	name	particle	spin	name
$\bar{\lambda}\Gamma_5\lambda$	0	a-η	$[\phi_1,\phi_2]F_{\mu\nu}$	0	glue scalarball
$\bar{\lambda}\lambda$	0	a-f	$F_{\mu\nu}F^{\mu\nu}, [\phi_1,\phi_2]^2$	0	glueball, scalarball
$F_{\mu\nu}\Sigma^{\mu\nu}\lambda$	$\frac{1}{2}$	gluino-glueball,	$F_{\mu\nu}\Sigma^{\mu\nu}\lambda$	$\frac{1}{2}$	gluino-glueball,
$[\phi_1,\phi_2]\Sigma^{23}\lambda$	$\frac{1}{2}$	gluino-scalarball	$[\phi_1,\phi_2]\Sigma_{23}\lambda$	$\frac{1}{2}$	gluino-scalarball

Ward identities

- in continuum $\langle \mathcal{Q}\mathcal{O}_a\rangle = 0$, here

$$\mathcal{O}_a(x) = \mathrm{tr}_c\Big\{\bar{\lambda}_b(x)\big(\Gamma^{MN}\big)^b{}_a F_{MN}(x)\Big\}$$

- decompose in 3 terms $\mu = (\mu, m)$

- three Ward identities

$$W_1 = \frac{1}{2}\langle\, [\phi_1,\phi_2]^2\,\rangle - \frac{\mathrm{i}}{8}\langle \bar\lambda\Gamma_2\,[\phi_1,\lambda] + \bar\lambda\Gamma_3\,[\phi_2,\lambda]\,\rangle = 0\,,$$
$$W_2 = \frac{1}{4}\langle F_{\mu\nu}F^{\mu\nu}\rangle + \frac{\mathrm{i}}{8}\langle \bar\lambda\,\Gamma_2\,[\phi_1,\lambda] - \bar\lambda\,\Gamma_3\,[\phi_2,\lambda]\,\rangle = \frac{3}{2}\,,$$
$$W_3 = \frac{1}{2}\langle D_\mu\phi^m D^\mu\phi_m\rangle = 3.$$

- sum $W = \sum W_k$ used in 4 dimensions, $W = 9/2$

Lattice formulation

- exists nilpotent $\mathcal{Q}$ and $\mathcal{Q}$-exact deformed lattice action
- for constructions and applications see

  see Sugino, Matsuura; Suzuki, Taniguchi; Fukays, Kanamori, Suzuki, Takimi; Catteral, Joseph, ...
- direct approach: lattice breaks susy
- must introduce counterterm $m_s^2 \phi^2$
- one loop order: $m_s^2 = 0.65948255(8)$ (same as for Suginos model) Suzuki
- sufficient for super-renormalizable model
- fermion mass enters lattice Ward identities
- fine-tuning $m_f \to$ chiral and susy continuum limit

- Wilson fermions, tree-level improved Lüscher-Weisz gauge action
- fermion operator in adjoint representation U^A:

$$D_{xy} = (m_f + 2 + \Gamma_{m+1} f^a \phi_a^m)\, \delta_{x,y} \\ - \frac{1}{2} \sum_\mu (\mathbb{1} - \Gamma_\mu)\, \delta_{x+e_\mu,y} U^A_{x,\mu} + (\mathbb{1} + \Gamma_\mu)\, \delta_{x-e_\mu,y} U^{A,t}_{y,\mu}$$

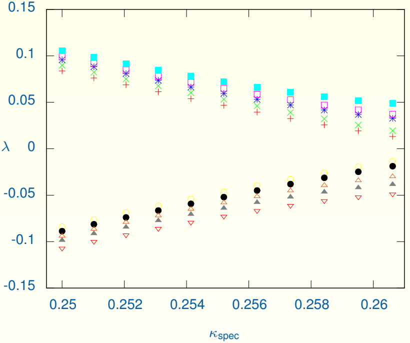
- structure constants $f^a = (f^a_{bc})$
- partition function

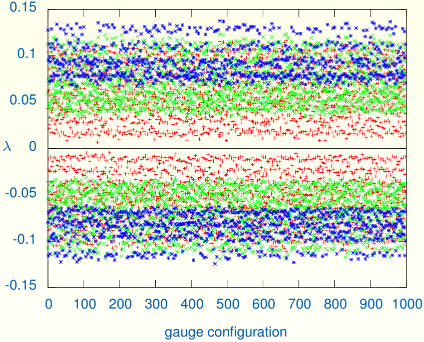
$$Z = \int \mathcal{D}U \mathcal{D}\phi \, \mathrm{sign}(\mathrm{Pf}(CD)) \det(D^\dagger D)^{\frac{1}{4}} \, e^{-S[U,\phi]}$$

- eigenvalues of $\Gamma_5 D$ real & double degenerate, continuity in κ

$$\det D = \prod_i \lambda_i^2 \quad \Longrightarrow \quad \mathrm{Pf} D[U] = \prod_i \lambda_i \, .$$

- typical configurations with (β, κ, vary κ_{spec} in D), 64×32 lattice





- 10 lowest eigenvalues for $\beta = 16, \kappa = 0.26062$ and $\kappa_{\text{spec}} \in [0, \kappa]$
- smallest eigenvalues for 1000 conf., $\kappa_{\text{spec}} \in \{\kappa, 0.25734, 0.25520\}$

- different volumes, gauge couplings, hopping parameters

sign of Pfaffian

no negative Pfaffians for $\kappa < \kappa_c$
approx. one in thousand configurations with negative Pf for $\kappa > \kappa_c$

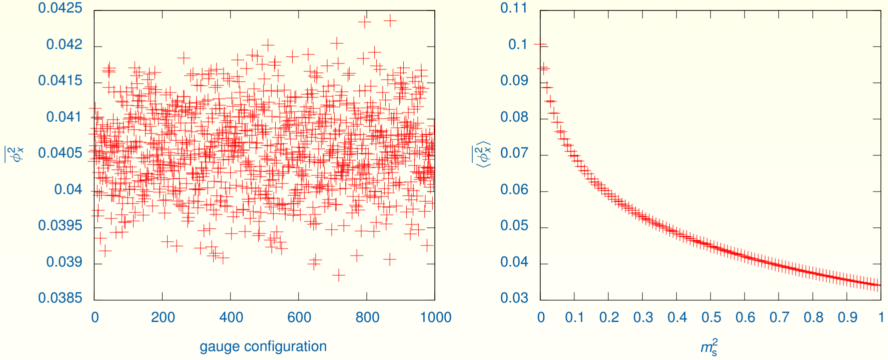
Flat directions

- scalar potential (generic to SYM with extended susy)

$$V[\phi_1, \phi_2] \propto \mathrm{tr}[\phi_1, \phi]^2 = V(\phi_1 + \alpha\phi_2, \phi_2]$$

- must introduce $m_s^2\,\mathrm{tr}\phi^2$ (even for $a \to 0$) for susy continuum limit
- monitor spatial average $\overline{\phi^2}$ at $m_s^2 = 0.66$ ($\approx$ susy point)
- expectation value $\langle\overline{\phi^2}\rangle$ dependent on m_s^2

flat directions are lifted for $m_s^2 \approx$ susy value



$$\beta = 17,\ \kappa = 0.26178,\ m_s^2 = 0.66,\ 64 \times 32$$

- already observed earlier — August, Wellegehause, AW (2016)

- $\mathrm{tr}\phi^2$ relevant $\Rightarrow$ m_s^2 fine-tuning, continuum $m_s^2 = 0.62849$
- simulations for $m_s^2 \in [0.50, 0.80] \Rightarrow$
- Ward-identities (almost) insensitive to m_s
- $d = 2 \Rightarrow \bar{\lambda}\lambda$ irrelevant, no fine tuning
- minimize susy breaking on finite lattice $\rightarrow$ tune m_f

- define m_f^c by
  - peak of chiral susceptibility
  - or minimize $m_\pi^2 \propto m_q$

β	$m_f^c(\chi_s)$	$m_f^c(m_\pi)$	$\kappa^c(m_\pi)$
14.0	-0.0983(2)	-0.1003(1)	0.2632
15.5	-0.0969(4)	-0.0931(1)	0.2622
17.0	-0.0896(22)	-0.0853(1)	0.2611
18.0	-0.0857(6)	-0.0821(1)	0.2607
19.0	-0.0819(4)	-0.0787(1)	0.2602

- extrapolation to continuum (via π-mass):

$$m_f^c(\beta) = m_0 + \frac{c}{\beta}$$

- result: $m_0 = -0.018(1), \quad c = -1.156(3)$
- c_1 encodes lattice artifacts
- consistent with expected continuum result $m_0 = 0$

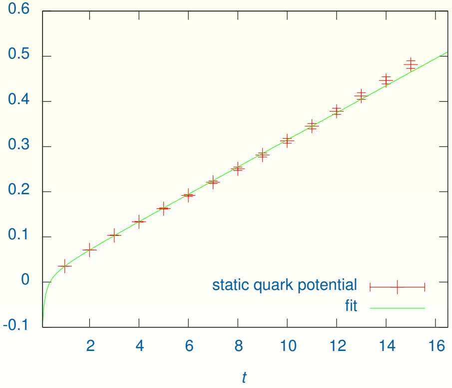
Scale setting (Sommer)

- static potential

$$V(r) = A + \frac{C}{r} + \sigma r$$

- $\beta = 17$, at critical m_f
- Sommer scale $\Rightarrow$ lattice spacing in physical units

$$\beta = 1/(ag)^2$$



- dependence on $m_\mathrm{f} \Rightarrow$ extrapolation to m_f^c

$\beta = 14$				
m_f	-0.062	-0.08	-0.09	-0.100
a[fm]	0.0734(3)	0.0700(4)	0.0678(5)	0.0650(25)
βa^2[fm]				0.0592(46)
$\beta = 15.5$				
m_f	-0.054	-0.07	-0.083	-0.093
a[fm]	0.0713(1)	0.0671(4)	0.0645(4)	0.0618(18)
βa^2[fm]				0.0592(35)
$\beta = 17$				
m_f	-0.044	-0.074	-0.084	-0.085
a[fm]	0.0676(2)	0.0620(4)	0.0589(4)	0.591(16)
βa^2[fm]				0.0595(31)

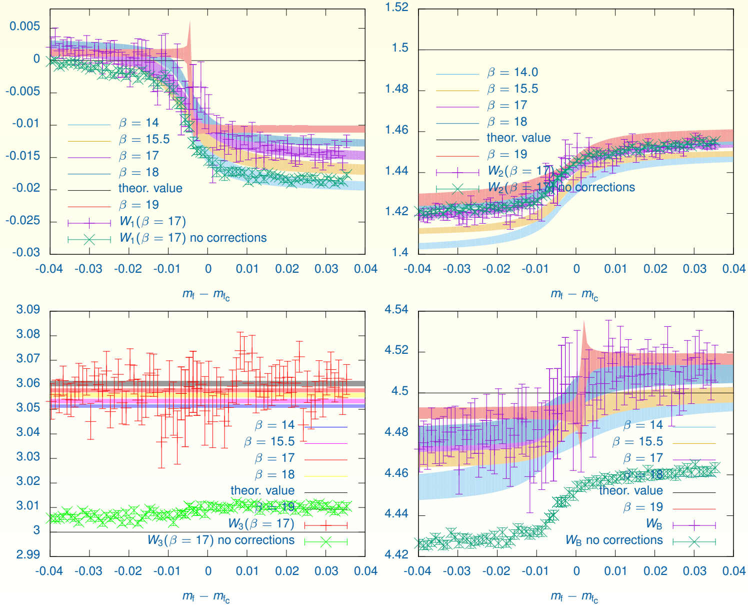
- agrees with constant value for g

- scalar fields: low pass filter
- gauge fields: Stout-smearing
- fermion sinks/sources: Jacobi smearing

Ward-identities

continuum WI $\rightarrow$ Schwinger-Dyson equations (lattice WI)

- lattice Ward-Identities W_1, W_2, W_3 :
  discretized continuum WI + extra terms from $\delta S \neq 0$
- $d = 2$: no SSB $\rightarrow$ correlators smooth functions of m_f



0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
-0.03
$\beta = 14$
$\beta = 15.5$
$\beta = 17$
$\beta = 18$
theor. value
$\beta = 19$
$W_1(\beta = 17)$
$W_1(\beta = 17)$ no corrections
-0.04 -0.03 -0.02 -0.01 0 0.01 0.02 0.03 0.04
$m_f - m_{f_c}$
1.52
1.5
1.48
1.46
1.44
1.42
1.4
$\beta = 14.0$
$\beta = 15.5$
$\beta = 17$
$\beta = 18$
theor. value
$\beta = 19$
$W_2(\beta = 17)$
$W_2(\beta = 17)$ no corrections
-0.04 -0.03 -0.02 -0.01 0 0.01 0.02 0.03 0.04
$m_f - m_{f_c}$
3.09
3.08
3.07
3.06
3.05
3.04
3.03
3.02
3.01
3
2.99
$\beta = 14$
$\beta = 15.5$
$\beta = 17$
$\beta = 18$
theor. value
$\beta = 19$
$W_3(\beta = 17)$
$W_3(\beta = 17)$ no corrections
-0.04 -0.03 -0.02 -0.01 0 0.01 0.02 0.03 0.04
$m_f - m_{f_c}$
4.54
4.52
4.5
4.48
4.46
4.44
4.42
$\beta = 14$
$\beta = 15.5$
$\beta = 17$
$\beta = 18$
theor. value
$\beta = 19$
W_B
W_B no corrections
-0.04 -0.03 -0.02 -0.01 0 0.01 0.02 0.03 0.04
$m_f - m_{f_c}$

- correction term from δS relevant for W_3, W

Ward identity	$\beta = 14$	$\beta = 15.5$	$\beta = 17$	$\beta = 18$	theor. value
W_1	$-0.0093(2)$	$-0.0084(3)$	$-0.0069(2)$	$-0.0059(2)$	0
W_2	1.4264(4)	1.4315(4)	1.4370(3)	1.4409(4)	$\frac{3}{2}$
W_3	3.0514(8)	3.0534(6)	3.0578(8)	3.0558(9)	3
W_B	4.477(1)	4.4842(9)	4.4942(8)	4.494(2)	$\frac{9}{2}$

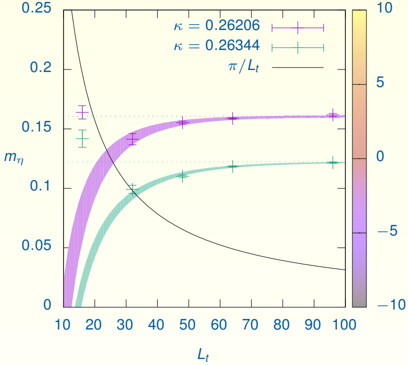
Ward identities point to the restoration of susy in continuum limit

spectrum of bound states

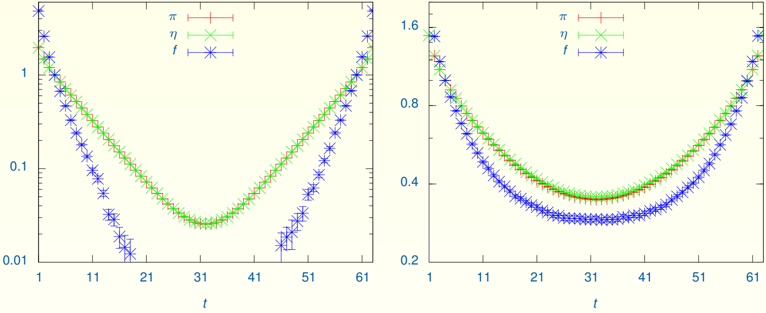
- extrapolations $\lim_{a\to 0}\lim_{m_f\to m_f^c}\lim_{L\to\infty}$
- volume dependence — Münster, Lüscher

$$m_L = m_0 - \frac{c}{L}\mathrm{e}^{-L/L_0}$$

- L_0: scale, where finite size effects become relevant
- η-meson lightest particles: $L_0 = \pi/m_\eta$
- only consider masses $m_L \geq \pi/L_t$
- κ-values slightly below κ^c (no sign problem)



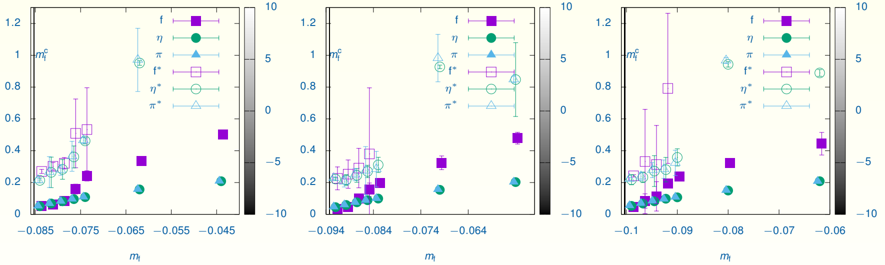
- mass of η-meson for $\beta = 12$
- even close to κ^c: extrapolation works
- all mass extraction for 64×32 and $\kappa < 0.264$



- masses above lattice-cutoff
- π-correlator $=$ η-correlator (for all κ considered)
  checked: disconnected part of η-correlator $\leq 10^{-2}$ connected part

mass of η-meson vanishes in chiral limit

- chiral limit $\kappa \to \kappa^c$: f- and η-mesons degenerate
- excited meson-states η^*, f^*, π^* (larger errors)
- $\beta = 17$ (left), $\beta = 15.5$ (middle), $\beta = 14$ (right)
- critical m_f: $-0.85, -0.93, -0.100$
- all β and κ : $m_\eta = m_\pi$; $f - \eta$ degeneracy for $m_f \to m_f^c$



a-f and a-η in same supermultiplet, VY-multiplet lightest one

- interpolating operator (4*d* notation)

$$O_{GG} = \Sigma_{\mu\nu} F^{\mu\nu} \lambda$$

- smearing on larger lattices necessary
- $\Rightarrow$ periodic and antiperiodic parts (parity projections $(\mathbb{1} \pm \Gamma_0)/2$)

$$C_A(t) = \langle O_{GG}(t) O^\dagger(0) \rangle, \quad C_S(t) = \langle O_{GG}(t) \Gamma_0 O^\dagger(0) \rangle$$

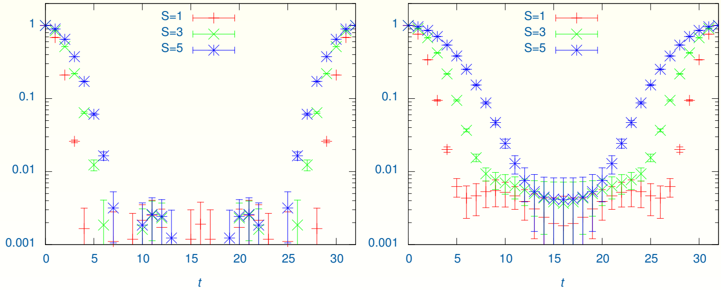
- m_A and m_S depending on parameter S ($\beta = 17,\ m_f = -0.084,\ 64 \times 32$)

S	0	6	12	18	24	30	36
m_A	0.324(13)	0.531(29)	0.404(16)	0.358(13)	0.333(11)	0.315(10)	0.302(10)
m_S	0.391(12)	0.633(14)	0.517(7)	0.469(5)	0.441(4)	0.421(4)	0.406(4)
S	48	60	80	160	240	320	400
m_A	0.282(9)	0.269(8)					0.222(1)
m_S	0.384(3)	0.369(3)					0.252(2)

- indication for $m_A, m_S \to m$ (parity not broken)
- m depend weakly on β and m_f
- extrapolation to infinite volume: $m_{GG} \approx 0.251$
- almost degenerate with excited mesons???
- ground state of m_{GG} not seen yet?

Glue- and scalarballs

- glueball, scalarball, glue-scalarball decouples (as for pure YM) Bralic
- left: glueball YM, right: glueball SYM



- no sign problem, similar as $\mathcal{Q}$-exact formulation
- results insensitive to m_s near one-loop value
- susy restoration in chiral limit (m_π)
- spectrum related but different to $4d$ mother-theory
- massless dimensionally reduced VY multiplet
- Farrer-Gabadadze-Schwetz multiplet decouples
- problem with missing light gluino-glueball N_t to small?