Lattice Supersymmetry III

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Generalizations ...

- Discussed formulation of supersymmetric lattice theories by discretization of twisted theory. Focus last time on $\mathcal{N} = 4$ SYM.
- All fermions were in adjoint representation and lived on fixed regular lattices

Today:

- Generalize: quiver gauge theories and super QCD. Building models with spontaneous SUSY breaking.
- $\mathcal{Q}$-exact theories on arbitrary triangulations
- Applications to holography

Starting point: 3d (twisted) super Yang-Mills

Twisted constructions work also for $\mathcal{Q} = 8$ Yang-Mills in 3d.

Vanilla lattice super YM action:

$$S = \mathcal{Q} \sum_x \mathrm{Tr} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \overline{\mathcal{D}}_a \mathcal{U}_a + \frac{1}{2} \eta d \right) - \sum_x \mathrm{Tr}\, \theta_{abc} \overline{\mathcal{D}}_{[a} \chi_{bc]}$$

with $(a, b = 1 \ldots 3)$ and cubic lattice with face/body diagonals

$$\begin{aligned} \mathcal{Q}\, \mathcal{U}_a &= \psi_a & \qquad \mathcal{Q}\, \psi_a &= 0 \\ \mathcal{Q}\, \chi_{ab} &= -\overline{\mathcal{F}}_{ab} & \mathcal{Q}\, \overline{\mathcal{U}}_a &= 0 \\ \mathcal{Q}\, \eta &= d & \mathcal{Q}\, d &= 0 \\ \mathcal{Q}\, \theta_{abc} &= 0 & & \end{aligned}$$

Lattice quiver theory

Construction: Sugino, Matsuura via orbifolding

Simple derivation:
– Take a lattice with just 2 timeslices in z-direction and free bc.
– Choose gauge groups $U(N_c)$ and $U(N_f)$ on the 2 timeslices.
– To retain gauge invariance fields on links between 2 slices *must* transform as bifundamental fields under $U(N_c) \times U(N_f)$
– Relabel fields as follows

N_c-lattice x	bifundamental fields $(x, \overline{x})$, $(\overline{x}, x)$	N_f-lattice $\overline{x}$
$\mathcal{U}_\mu(x)$	$\mathcal{U}_3 \rightarrow \phi(x, \overline{x})$	$\hat{\mathcal{U}}_\mu(\overline{x})$
$\eta(x)$	$\psi_3 \rightarrow \lambda(x, \overline{x})$	$\hat{\eta}(\overline{x})$
$\psi_\mu(x)$	$\chi_{3\mu} \rightarrow \lambda_\mu(\overline{x} + \mu, x)$	$\hat{\psi}_\mu(\overline{x})$
$\chi_{\mu\nu}(x)$	$\theta_{3\mu\nu} \rightarrow \lambda_{\mu\nu}(x, \overline{x} + \mu + \nu)$	$\hat{\chi}_{\mu\nu}(\overline{x})$

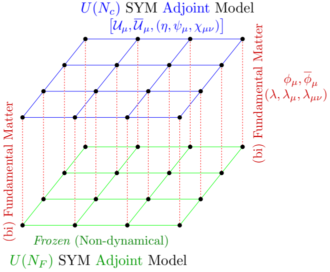
More on bifundamentals ...

$$\phi(x) \to G(x)\phi(x)H^\dagger(\overline{x})$$

$$\lambda(x) \to G(x)\lambda(x)H^\dagger(\overline{x})$$

$$\lambda_\mu(x) \to H(\overline{x}+\mu)\lambda_\mu(x)G^\dagger(x)$$

$$\lambda_{\mu\nu}(x) \to G(x)\lambda(x)H^\dagger(\overline{x}+\mu+\nu)$$



This procedure retains both gauge invariance and $\mathcal{Q}$-symmetry since action is sum of closed loops

Continuing...

Prescription for lattice derivatives generalizes:

$$
\begin{aligned}
eg.\ \mathrm{Tr}\,\chi_{ab}\mathcal{D}_a\psi_b(x) \quad & \overset{3d}{=} \quad \mathrm{Tr}\,\chi_{ab}\left(\mathcal{U}_a(x)\psi_b(x+a) - \psi_b(x)\mathcal{U}_a(x+b)\right) \\
& \xrightarrow{b=3,a=\mu} \quad \mathrm{Tr}\,\lambda_\mu(x)\left(\mathcal{U}_\mu(x)\lambda(x+\mu) - \lambda(x)\hat{U}_\mu(\overline{x})\right)
\end{aligned}
$$

$$U(N_c) \qquad\qquad\qquad U(N_f)$$

Similarly:

$$\mathrm{Tr}\,\chi_{ab}\mathcal{D}_a\psi_b(x) \xrightarrow{b=\mu,a=3} \mathrm{Tr}\,\lambda_\mu(x)\left(\phi(x)\hat{\psi}_\mu(\overline{x}) - \psi_\mu(x)\phi(x+\mu)\right)$$

Check gauge invariance:

$$\lambda_\mu(x)\phi(x)\psi_\mu(x)\phi(x{+}\mu) \to H(x{+}\mu)\lambda_\mu(x)G^\dagger(x)G(x)\psi_\mu(x)G^\dagger(x{+}\mu)G(x{+}\mu$$

Super QCD

Just set $\hat{\eta}, \hat{\chi}_{\mu\nu}, \hat{\psi}_\mu, \hat{d}, \hat{\phi} = 0$ and $\hat{U}_\mu = I_{N_f \times N_f}$ i.e $g_{Nf} \to 0$

Previous expressions become:

$$\mathrm{Tr}\, \lambda_\mu(x) \left(\mathcal{U}_\mu(x)\lambda(x+\mu) - \lambda(x)\right)$$

Kinetic op. for N_f flavors of fermion in fundamental rep. of $U(N_c)$ and

$$-\mathrm{Tr}\, \lambda_\mu(x)\psi_\mu(x)\phi(x+\mu)$$

Yukawa interaction.

Result:

$\mathcal{Q}$-invariant lattice theory of 2D super QCD with N_c gauge symmetry and global N_f flavor symmetry

Fayet-Illopoulous (FI) term

(Aarti Veernala, SC)
Since $U(N)$ we can additionally add a new $\mathcal{Q}$ exact term

$$\Delta S = r\mathcal{Q} \sum_x \mathrm{Tr}\ \left(\eta(x) I_{N_c \times N_c}\right)$$

–Yields new e.o.m for auxiliary d-field (and F.I D term in action)

$$d = \overline{\mathcal{D}}_\mu \mathcal{U}_\mu + \phi\overline{\phi} - rI_{N_c \times N_c}$$

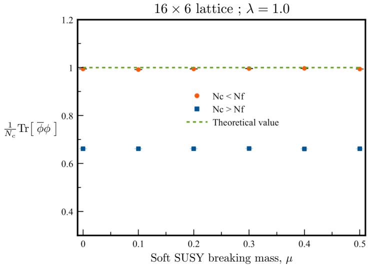
Integrating out the d field yields a scalar potential:

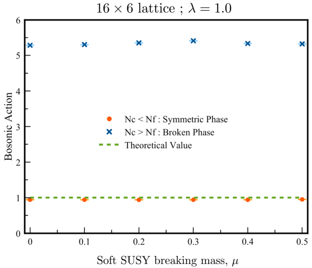
$$V = \sum_x \mathrm{Tr}\ \left(\sum_{f=1}^{Nf} \phi^f \overline{\phi}^f - rI\right)^2$$

Dynamical $\mathcal{Q}$ breaking

- Spontaneous breaking indicated by $< d > \neq 0$. Depends on N_c, N_f.
- Consider $\sum_x \mathrm{Tr}\, d(x) = \sum_x \mathrm{Tr} \left(\sum_f^N {}_f \phi^f(x) \overline{\phi}^f(x) - r I_{N_c} \right)$
- Setting $r = 1$ this depends on rank of $N_c \times N_c$ matrix $\sum_{f=1}^{N_f} \phi^f \overline{\phi}^f$.

$$N_f \geq N_c \quad \text{supersymmetric vacuum}$$
$$N_f < N_c \quad \text{supersymmetry broken}$$





Goldstino

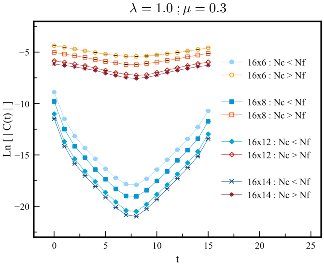
If susy breaks expect a massless fermion

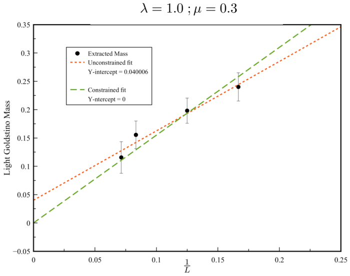
Measure

$$C(t) = \sum_{x,y} \langle O'(y,t) O(x,0) \rangle$$

where

$$\begin{aligned} O(x,0) &= \psi_\mu(x,0)\overline{\mathcal{U}}_\mu(x,0)\left[\phi(x,0)\overline{\phi}(x,0) - rI_{N_c}\right] \\ O'(y,t) &= \eta(y,t)\left[\phi(y,t)\overline{\phi}(y,t) - rI_{N_c}\right] \end{aligned}$$





Super QCD summary

- Can construct 2D super QCD with N_f flavors and gauge group $U(N_c)$ using generalizations of the twisted formalism
- Can include $\mathcal{Q}$ invariant F.I term.
- See clear signals for spontaneous susy breaking depending on N_c/N_f in accord with expectations. Notice that $< \phi\overline{\phi} > \neq 0$ also implies Higgsing of gauge symmetries.
- Approach can be used to build $\mathcal{Q}$-exact 3D super QCD (A. Joseph)

$\mathcal{Q}$ symmetry on curved manifold

Consider $\mathcal{N} = 2$ SYM in 2D:

Twisted formulation geometrical in flavor – in continuum scalar SUSY can be realized on any smooth manifold

$$S = \mathcal{Q} \int d^2 \sqrt{g} \mathrm{Tr} \left[\frac{1}{4} \eta [\phi, \overline{\phi}] - g^{\mu\nu} \psi_\nu D_\mu \overline{\phi} + \chi_{12} \left(B_{12} - \frac{2}{\sqrt{g}} F_{12} \right) \right]$$

Note: A twist employed

$$\mathcal{Q}\,\phi = 0 \qquad \mathcal{Q}\,\overline{\phi} = \eta \qquad \mathcal{Q}\,A_\mu = \psi_\mu \quad \mathcal{Q}\,\psi_\mu = D_\mu \phi$$
$$\mathcal{Q}\,B_{12} = [\phi, \chi_{12}] \qquad \mathcal{Q}\,\eta = [\phi, \overline{\phi}] \qquad \mathcal{Q}\,\chi_{12} = B_{12}$$

Lattice version ?

$\mathcal{Q}$-exact theory on random lattice (Matsuura et al)

$$
\begin{array}{llll}
\mathcal{Q}\,\phi_s = 0 & \mathcal{Q}\,\overline{\phi}_s = \eta_s & \mathcal{Q}\,U_l = \psi_l U_l & \mathcal{Q}\,\psi_l = U_l\phi_t U_l^\dagger - \phi_s \\
\mathcal{Q}\,B_f = [\phi_f, B_f] & \mathcal{Q}\,\eta_s = [\phi_s\overline{\phi}_s] & \mathcal{Q}\,\chi_f = B_f &
\end{array}
$$

and

$$
\begin{aligned}
S = {} & \alpha_s \mathcal{Q} \sum_s \mathrm{Tr}\left(\frac{1}{4}\eta_s[\phi_s, \overline{\phi}_s]\right) \\
& + \alpha_l \mathcal{Q} \sum_l \mathrm{Tr}\left(\psi_l \left(U_l\overline{\phi}_t U_l^\dagger - \overline{\phi}_s\right)\right) \\
& + \alpha_f \mathcal{Q} \sum_f \mathrm{Tr}\left(\chi_f \left(B_f - \beta_f F(U_f)\right)\right)
\end{aligned}
$$

As expected scalars like η are assigned to sites, vectors like ψ_l assigned to endpoint of a link and 2-forms like χ_f assigned to one representative site on face.

Generalized Wilson plaquette

$$F(U_f) = \left(U_f - U_f^\dagger\right)^{-1}\left(2 - U_f - U_f^\dagger\right) + \left(2 - U_f - U_f^\dagger\right)\left(U_f - U_f^\dagger\right)^{-1}$$

where $U_f = \prod_m^{face} U_m$.

Form of F chosen so that unique vacuum $U = I$ selected

Action with $\alpha_s = \alpha_l = \alpha_f = \beta_f = 1$ on cubic lattice reduces to Sugino's lattice action for the A twist of $\mathcal{N} = 2$ YM in $D = 2$.

Notice:

- No doublers: fermion action is of Kähler-Dirac form. But fermions are all effectively located at sites.
- Nothing depends on representative point chosen for face f.
- Gauge invariant and $\mathcal{Q}$-symmetric for random lattice

Prospects ...

- Can show that classical continuum limit recovers continuum theory.
- Radiative corrections can be handled by mild tuning of couplings .. True ?
- Topological observables in continuum independent of metric. Expect lattice theory similar - example of topological lattice field theory !
- What about non topological observables; expect metric dependence. Suppose I sum over lattices as a way of incorporating 2d metric fluctuations - is this a toy model for twisted supergravity ?
- Can this construction be generalized to $D > 2$ and $n(\mathcal{Q}) > 4$?

Applications: Holography

According to Maldacena:

Maximally supersymmetric SU(N) Yang-Mills theory in $1 + p$ dimensions describes N Dp-branes in the decoupling limit

In more detail:

Defining dimensionless temperature $t = T/\lambda^{3-p}$ and taking $N \to \infty$ with

$$N^{-\left[\frac{4}{(4-p)(3-p)}\right]} << t << 1$$

SUGRA predicts SYM energy

$$\epsilon \sim N^2 t^{\frac{2(7-p)}{(5-p)}} \lambda^{\frac{(1+p)}{(3-p)}}$$

and

$$\phi^2 \sim t^{\frac{2}{5-p}} \lambda^{\frac{1}{3-p}}$$

Thermal instabilities

All $p < 3$ theories suffer from instabilities for small *t*/strong coupling

- Scalar eigenvalues and complex Polyakov/Wilson lines diverge - typically one or more scalar eigenvalues gets large.
- Simulation algorithms fail: lattice theories evolve to points in field space where Monte Carlo is unable to update configuration.
- Strong metastability visible; it may take thousands of Monte Carlo updates before this divergence is manifested ...
- This instability sets in at temperatures comparable to the regime where the leading SUGRA result holds...

Can get an understanding of why this happens in the SYM system by expanding about the moduli space of the theory

Effective potential for moduli (with T. Wiseman)

Expand scalars about classical moduli space $\Phi_\mu = \phi_\mu + \hat{\Phi}_\mu$ with ϕ_μ diagonal constant matrices and $\hat{\Phi}_\mu$ off-diagonal matrices $(\mu = 1 \ldots 9 - p)$
Scalar action becomes:

$$S = \frac{N}{\lambda} \int^\beta \sum_{ab} \hat{\Phi}^{*\,ab}_\mu \left(\delta_{\mu\nu} |\Delta\phi^{ab}_\mu|^2 - \Delta\phi^{ab}_\mu \Delta\phi^{ab}_\nu \right) \hat{\Phi}^{ab}_\nu + \text{interactions}$$

where $\Delta\phi^{ab}_\mu = \phi^a_\mu - \phi^b_\mu$ yield mass terms for the off-diagonal fields.

Now if moduli well separated $\Delta\phi^{ab}_\mu >> 1$ can integrate out the off-diagonal modes to generate effective potential for moduli. Including the fermions yields:

$$V_{\text{eff}}(\phi_\mu) \sim e^{-\beta |\Delta^{ab}_\mu|} + \ldots$$

Divergences

- Thermally induced potential vanishes for widely spaced moduli; thus integration over moduli yields divergence of thermal partition function. Note: 1 loop exact for widely spaced moduli.
- The potential vanishes faster at low temperature and is $\mathcal{O}(1/N)$ suppressed - as observed.
- Dual picture: black p-brane system unstable to radiation of D0 branes.

Take home message

To check holography is not easy! Must include $\mathcal{Q}$-breaking regulator to control this divergence. Removal of regulator can be delicate – need to send $N \to \infty$ first and understand the extrapolation.

Several talks at this meeting on tests of holography for $p = 0, 1$.

Lattice Supersymmetry

- Rapid development over last decade. New actions exist preserving one or more SUSYs. SYM and super QCD.
- $\mathcal{Q} = 4$ model generalized to random lattice using Sugino-like construction
- Orbifold lattice actions are exotic - fermions on links, complex bosons in the algebra, $U(N)$ gauge symmetry etc etc
- $\mathcal{N} = 4$ SYM can be done. Renormalization understood. Sign problem seems tame .. Lots of applications to holography once one understands potential pitfalls ..
- Understand $\mathcal{N} = 4$ away from planar limit eg. anomalous dimension of Konishi ? What about tests of S duality ? and the bootstrap ?
- What about non-leading corrections in $1/g^2$ and $1/N$ corresponding to quantum SUGRA and string corrections ...
- Nature of QG: how is the geometry encoded in YM fields ?

Thank You !

Backup: Scaling dimensions from Monte Carlo RG

Write system as (infinite) sum of operators $\mathcal{O}_i$ with couplings c_i

Couplings c_i flow under RG blocking transformation R_b

n-times-blocked system is $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Consider linear expansion around fixed point $H^\star$ with couplings $c_i^\star$

$$c_i^{(n)} - c_i^\star = \sum_j \left. \frac{\partial c_i^{(n)}}{\partial c_j^{(n-1)}} \right|_{H^\star} \left(c_j^{(n-1)} - c_j^\star \right) \equiv \sum_j T_{ij}^\star \left(c_j^{(n-1)} - c_j^\star \right)$$

$T_{ij}^\star$ is the stability matrix

Obtained from measured correlators of $\mathcal{O}_i$

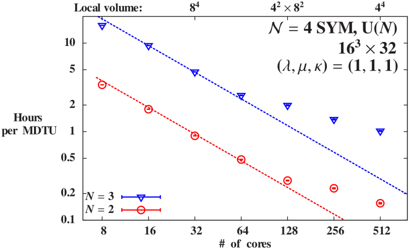
Eigenvalues of $T_{ij}^\star \longrightarrow$ scaling dimensions of corresponding operators

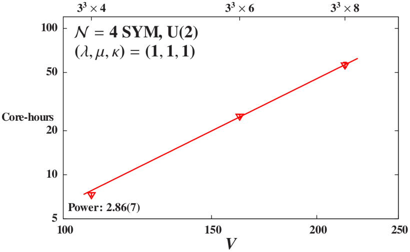
Backup: Code performance—weak and strong scaling

Results from arXiv:1410.6971 using the unimproved action

Left: Strong scaling for U(2) and U(3) $16^3 \times 32$ RHMC

Right: Weak scaling for $\mathcal{O}(n^3)$ pfaffian calculation (fixed local volume)
$n \equiv 16N^2L^3N_T$ is number of fermion degrees of freedom





Both plots on log–log axes with power-law fits

Backup: Numerical costs for 2, 3 and 4 colors

Red: Find RHMC cost scaling $\sim N^5$ (recall adjoint fermion d.o.f. $\propto N^2$)

Blue: Pfaffian cost scaling consistent with expected N^6

Additional factor of $\sim 2\times$ from improved action, but same scaling