Lattice Supersymmetry I

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Plan

- Motivation for lattice SUSY. Difficulties.
- Exposing exact lattice supersymmetries. Topological twisting.
- Gauge theories. Example: $\mathcal{N} = 2$ SYM in $D = 2$.
- Exploiting topological structure.
- Lattice construction of $\mathcal{N} = 4$ SYM in $D = 4$.
- Renormalization. Additional supersymmetries. Tuning to the continuum theory.
- Simulations: problems and solutions.
- Generalizations. Quiver gauge theories. Super QCD.
- Holographic Applications.

What I won't have time for ...

- $\mathcal{N} = 1$ Yang-Mills - see Wipf, Bergner,...
- Wess-Zumino and sigma models in $D = 2$
- Details of orbifold constructions
- New ideas eg. tensor network representations (Kadoh)

See talks here and reviews:

- G. Bergner and S. Catterall, Int. J. Mod. Phys. A31 (2016) 22 1643005.
- S. Catterall, D. B Kaplan and M. Ünsal, Phys. Rept. 484 (2009) 71.
- ...

Motivation

- SUSY theories exhibit improved U.V behavior. Light scalars natural in SUSY (but moot these days ?)
- More tractable analytically – toy models for understanding confinement and chiral symmetry breaking
- Key component of string theory – remove tachyon of bosonic string.
- AdSCFT – super YM theories may tell us about gravity ..

Realistic models must break SUSY at low energy

Typically cannot occur in perturbation theory:
need non-perturbative tool – lattice

Lattice yields non-perturbative definition of SUSY theory

Problems with lattice SUSY

- Extension of Poincare symmetry: $\{Q, \overline{Q}\} = \gamma.P$. Broken by lattice.
- Equivalently: Leibniz rule does not hold for difference operators
- Fermion doubling - $n_B \neq n_F$. Wilson terms break SUSY.
- Consequence: Naively discretized classical action breaks SUSY. Effective action picks up (many) SUSY violating operators. Generically some/many of these relevant.
- Couplings to these operators must be fine tuned as $a \to 0$. Unnatural and impractical

Road block for more than twenty years!
New ideas drawn from orbifolding/TQFT led to resurgence of interest

People

Many people have contributed to the material I will talk about.

Alessandro D'Adda	Noboru Kawamoto
George Bergner	David B. Kaplan
Andy Cohen	So Matsuura
Poul Damgaard	David Schaich
Joel Giedt	Fumihiko Sugino
Anosh Joseph	Mithat Ünsal
Raghav Jha	Andreas Wipf
Daisuke Kadoh	Issaku Kanamori
Tom DeGrand	Masanori Hanada
Urs Wenger	Aarti Veernala
....	....

Simplest Example: Witten's SUSYQM

$$S = \int dt \, \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}P'(\phi)^2 + \frac{1}{2}\psi_i \frac{d\psi_i}{dt} + i\psi_1\psi_2 P''(\phi)$$

Invariant under 2 SUSYs:

$$\begin{array}{rclrcl}
\delta_A\phi & = & \psi_1\epsilon_A & \delta_B\phi & = & \psi_2\epsilon_B \\
\delta_A\psi_1 & = & \frac{d\phi}{dt}\epsilon_A & \delta_B\psi_1 & = & -iP'\epsilon_B \\
\delta_A\psi_2 & = & iP'\epsilon_A & \delta_B\psi_2 & = & \frac{d\phi}{dt}\epsilon_B
\end{array}$$

Notice

$\delta_A^2 = \delta_B^2 = \frac{d}{dt} = H$ as expected

Naive discretization

Place fields on sites of (periodic) 1D lattice. Replace $\int dt \rightarrow \sum_t a$ and replace $\frac{d}{dt}$ by symmetric difference (fermion doubling ?)

$$a\Delta^S f_x = \frac{1}{2}(f(x+a) - f(x-a))$$

Now find:

$$\delta_A S_L = \sum_t i\epsilon \left(P'\Delta^S\psi_2 + \Delta^S\phi P''\psi_2\right)$$

In continuum this is total derivative

Leibniz rule *does not hold for lattice difference ops* - susy breaking term O(a).
Naively goes away as $a \rightarrow 0$.
But radiative quantum corrections can change that ...

Radiative corrections - (Giedt, Poppitz)

Let $P = \frac{m}{2}\phi^2 + \frac{g}{4}\phi^4$. One loop scalar self energy (superficially divergent)
Continuum:

$$\Sigma_{\rm cont} = 6g \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dp}{2\pi} \frac{-ip + m}{p^2 + m^2}$$

Actually convergent ($p \to -p$ symmetry)

$$\Sigma_{\rm cont} = 6g \left(\frac{1}{\pi} \tan^{-1} \frac{\pi}{2ma} \right) \sim 6g \left(\frac{1}{2} + \mathcal{O}(ma) \right)$$

Lattice:

$$\Sigma_{\rm latt} = \frac{6g}{L} \sum_{k=0}^{L-1} \frac{-2i \sin\left(\frac{\pi k}{L}\right) e^{i\left(\frac{\pi k}{L}\right)} + Ma}{\sin^2\left(\frac{\pi k}{L}\right) + (Ma)^2} \to 6g!$$

with $Ma = ma + 2\sin^2\left(\frac{\pi k}{2L}\right)$ - Wilson term

Radiative corrections II

- If take $a \to 0$ after doing sum get twice the result!
- Would be doublers have mass $O(1/a)$ and make an additional contribution to integral (don't decouple from small loops)
- Quantum corrections wreck naive continuum limit
- Restore SUSY need to add counterterm

$$S_L \to S_L - \sum_t 3g\phi^2$$

  SUSY broken but regained now as $a \to 0$.
- Example of general approach: naive discretizations of SUSY actions require fine tuning of counterterms to achieve SUSY continuum limit.
  For superrenormalizable theories can be done in (lattice) p theory.

Exact SUSY

Actually can do better. Find combination of SUSY's that can be preserved on lattice.
Notice that:

$$\delta_A S_L = -i\delta_B \sum_x P' \Delta^S \phi \qquad \delta_B S_L = i\delta_A \sum_x P' \Delta^S \phi$$

Thus

$$\left(\delta_A + i\delta_B\right) S_L = -\left(\delta_A + i\delta_B\right) O \qquad \text{where } O = \sum_t P' \Delta^S \phi$$

So can find δS_{exact} of form

$$S_L^{\text{exact}} = \sum_t \frac{1}{2}(\Delta^S \phi)^2 + \frac{1}{2} P'^2 + P' \Delta^S \phi + \overline{\psi}(\Delta^S + P'')\psi$$

Twisted fields ...

Where

$$\begin{aligned}\psi &= \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)\\ \overline{\psi} &= \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2)\end{aligned}$$

and the new supersymmetry acts:

$$\begin{aligned}\delta\phi &= \psi\epsilon\\ \delta\psi &= 0\\ \delta\overline{\psi} &= (\Delta^S\phi + P'(\phi))\epsilon\end{aligned}$$

Notice: $\delta^2 = 0$ on shell now. No translations.

$$S_L^b = \sum_x \left(\Delta^S\phi + P'(\phi)\right)^2$$

Supersymmetric action inherits counterterm automatically...

Connection to topological field theory ...

Notice that $\delta^2 = 0$ for all fields using EOM.
Can render symmetry nilpotent off-shell by introducing auxiliary field

$$\begin{aligned} Q\phi &= \psi \\ Q\psi &= 0 \\ Q\overline{\psi} &= B \\ QB &= 0 \end{aligned}$$

Note: absorbed ϵ into variation δ and renamed it Q. Also

$$S_L^b = \sum_x B(\Delta^+\phi + P') - \frac{1}{2}B^2$$

note: $\Delta^+ = \Delta^S + m_W$

More on connection to TQFT

Remarkably:

$$S_L = Q \sum_x \overline{\psi}(\Delta^+ \phi + P' - \frac{1}{2} B)$$

The action is Q-exact. Like BRST ?
Consider bosonic model with $S(\phi) = 0$.
Invariant under $\phi \to \phi + \epsilon$ – topological symmetry.

Quantize:

pick gauge function $\mathcal{N} = 0$ and introduce Fadeev-Popov factor

$$Z = \int D\phi \,\det\left(\frac{\partial \mathcal{N}}{\partial \phi}\right) e^{-\frac{1}{2\alpha}\mathcal{N}^2(\phi)}$$

If $\mathcal{N} = \Delta^+ \phi + P'(\phi)$ and ψ,$\overline{\psi}$ as ghost fields ($\alpha = 1$ gauge) recover SQM !
(gauge function sometimes called Nicolai map)

Important!

Q looks like BRST charge

Action can be formally derived by gauge fixing a topological theory. But lattice theory we construct is NOT topological

Use twisted/TQFT construction to expose scalar charge that can be transferred to lattice.
But do NOT restrict to topological sector $Q|state>= 0$

Think of twisting as exotic change of variables ...

Options for lattice SUSY

Just do it ...

- Certain simple cases eg. $\mathcal{N} = 1$ SYM in 4D with Wilson fermions – single counterterm - gluino mass m_g. (DWF more expensive but $m_g \to 0$ as $L_s \to \infty$)
- For $D < 4$ finite number of divergences occurring at small numbers of loops - calculate using (lattice) perturbation theory and subtract with counter terms

Better but not always possible ..

In some cases can find a SUSY subalgebra that is compatible with lattice and protects the theory from most of dangerous SUSY violating operators.

Preserving (some) supersymmetry

Key Idea:

- Try to preserve a subset of SUSY algebra on lattice
- Hope that this helps to reduce number of *relevant* operators breaking full SUSY

Topological twisting

Reformulate theory in new *twisted* variables that exposes a scalar supercharge $\mathcal{Q}$

Typically:

- $\mathcal{Q}^2 = 0$ up to gauge transformations
- $S = \mathcal{Q}\Lambda(\text{fields})$

Concrete example: $\mathcal{N}=2$ SYM in $D=2$

1 gauge field A, 2 real scalars B^a and 2 Majorana fermions ψ^a:

$$S=\int d^2x\,\mathrm{tr}\,\left(\frac{1}{4}F_{\mu\nu}^2+\frac{1}{2}D_\mu B^a D_\mu B^a\right)+S_F$$

and

$$S_F=\int d^2x\,\sum_{a=1}^{2}\overline{\psi}^a\gamma_\mu D_\mu\psi^a+\mathrm{Yukawas}$$

Global symmetries: $SO_R(2)\times SO_{\mathrm{Lorenz}}(2)\,(\times\mathrm{additional}\;SO(2))$

$$\psi\to L^{\alpha\beta}R^{ab}\psi_\beta^b=L^{\alpha\beta}\psi_\beta^b R^{T\,ba}$$

Twisted rotation group

Decompose fields under diagonal subgroup $R=L$

Twisted fermions

Fermions as matrices $\psi^a_\alpha \to \Psi_{a\alpha}$!

$$S_F = \int d^2x \, \mathrm{Tr} \left[\overline{\Psi} \gamma_\mu D_\mu \Psi \right]$$

Expanding

$$\Psi = \frac{\eta}{2} I + \psi_a \gamma_a + \chi_{12} \gamma_1 \gamma_2$$

Doing the trace yields Kähler-Dirac action

$$S_F = \int d^2x \, \mathrm{tr} \left(\frac{\eta}{2} D_\mu \psi_\mu + \chi_{\mu\nu} D_{[\mu} \psi_{\nu]} \right)$$

$\frac{1}{2} \times \frac{1}{2} = 1 + 0$

Twisted fermions appear as p forms

Twisted bosons

After twisting

$A_\mu \rightarrow A_\mu$ while scalars transform as vector $B^a \rightarrow B_\mu$
Twisted bosons packed into single complex gauge field $\mathcal{A}_\mu = A_\mu + iB_\mu$.

$$S_B = \int d^2x \,\mathrm{tr} \left(\mathcal{F}_{\mu\nu} \overline{\mathcal{F}}_{\mu\nu} + \frac{1}{2} [\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 \right)$$

where

$$\mathcal{F}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu] \quad \overline{\mathcal{F}}_{\mu\nu} = [\overline{\mathcal{D}}_\mu, \overline{\mathcal{D}}_\nu]$$

and $\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu$

Note:

Only $U(N)$ gauge invariance

Twisted supersymmetries

As for fermions:

$$Q^a_\alpha \rightarrow \mathcal{Q} + \mathcal{Q}_\mu + \mathcal{Q}_{12}$$

with

Original SUSY algebra: $\{\mathcal{Q}, \mathcal{Q}_\mu\} = p_\mu$ and $\mathcal{Q}^2 = 0$ and ...

Hence p is $\mathcal{Q}$-exact

Plausible that $T_{\mu\nu} = p_\mu p_\nu + \ldots$ and hence also action $\mathcal{Q}$-exact.

Here:

$$S = \mathcal{Q}\, \mathrm{tr} \int d^2x \left(\chi_{12}\mathcal{F}_{12} + \eta[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] + \frac{1}{2}\eta d \right)$$

with

$$\mathcal{Q}\mathcal{A}_\mu = \psi_\mu \quad \mathcal{Q}\psi_\mu = 0 \quad \mathcal{Q}\chi_{\mu\nu} = \overline{\mathcal{F}}_{\mu\nu} \quad \mathcal{Q}\eta = d \quad \mathcal{Q}d = 0 \quad \mathcal{Q}\overline{\mathcal{A}}_\mu = 0$$

Just a change of variables ...

$\mathcal{Q}$-variation, integrate d:

$$S = \int \mathrm{Tr}\left(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu} - \frac{\eta}{2}\overline{\mathcal{D}}_\mu\psi_\mu\right)$$

Rewrite using $\mathcal{F}_{\mu\nu} = F_{\mu\nu} - [B_\mu, B_\nu] + iD_{[\mu}B_{\nu]}$

$$S = \int \mathrm{Tr}\left(-F_{\mu\nu}^2 + 2B_\mu D_\nu D_\nu B_\mu - [B_\mu, B_\nu]^2 + L_F\right)$$

where

$$L_F = \begin{pmatrix} \chi_{12} & \frac{\eta}{2} \end{pmatrix} \begin{pmatrix} -D_2 - iB_2 & D_1 + iB_1 \\ D_1 - iB_1 & D_2 - iB_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Just $\mathcal{N} = 2$ action !

Supersymmetric lattice gauge theories

Strategy

If action is $\mathcal{Q}$-exact can maintain $\mathcal{Q}$-invariance if we maintain subalgebra $\mathcal{Q}^2 = 0$ - possible even on lattice

But lattice theory must also retain gauge invariance and avoid fermion doubling - provide very strong constraints on construction ..

Orbifold actions

Lattice SYM with exact SUSY were first constructed by Kaplan and Ünsal using orbifold/deconstruction techniques.
Remarkably it is possible to reach same lattice theories by careful discretization of the twisted theory I have described.

Many people contributed to understanding these interconnections

Sugino, Kawamoto, d'Adda, Damgaard, Matsuura, Hanada, SC ...

One wrinkle ....

$\mathcal{N} = 2$ has an A twist and a B twist ...

Here I have described the B twist.

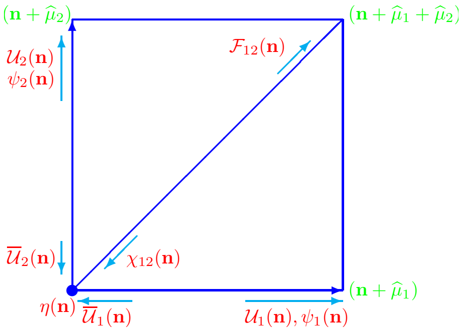
Sugino was the first to discretize this theory via the A twist.

Key difference is that the exact supersymmetry is not nilpotent

$$\mathcal{Q}_A^2 = \text{gauge transformation}$$

The A twist lattice theories differ in some important respects

Continuum to lattice for B twist ..



- Natural to map p-form fields to lattice fields living on p-cells.
- In practice replace plaquette fields by additional body diagonal links

Fields transform according to their end points eg.

$$\mathcal{U}_\mu(x) \rightarrow G(x)\mathcal{U}_\mu(x)G^\dagger(x+\mu) \quad \eta(x) \rightarrow G(x)\eta(x)G^\dagger(x)$$

$$\psi_\mu(x) \rightarrow G(x)\psi_\mu(x)G^\dagger(x+\mu) \quad \chi_{12}(x) \rightarrow G(x+1+2)\chi_{12}(x)G^\dagger(x)$$

Derivatives and Gauge Invariance

Need a prescription to replace continuum derivatives with (covariant) difference operators

- Lattice expressions should all have a link character and transform accordingly
- Continuum gauge invariant expressions should correspond to closed loops
- Lattice expressions should reduce to continuum form as $a \to 0$
- Should not lead to fermion doubling

$$\begin{aligned} \mathcal{D}_\mu \psi_\nu &\to \mathcal{D}^{(+)}_\mu \psi_\nu = \mathcal{U}_\mu(x)\psi_\nu(x+\mu) - \psi_\nu(x)\mathcal{U}_\mu(x+\nu) \\ \mathcal{D}_\mu \psi_\mu &\to \overline{\mathcal{D}}^{(-)}_\mu \psi_\mu = \psi_\mu(x)\overline{\mathcal{U}}_\mu(x) - \overline{\mathcal{U}}(x-\mu)\psi_\mu(x-\mu) \end{aligned}$$

More on derivatives

Each term transforms like the appropriate link under GTs eg.

$$\mathcal{U}_\mu(x)\psi_\nu(x+\mu) \rightarrow G(x)\mathcal{U}_\mu(x)G^\dagger(x+\mu)G(x+\mu)\psi_\nu(x+\mu)G^\dagger(x+\mu+\nu)$$

i.e link $x \rightarrow x+\mu+\nu$

Note this is to be contracted with $\chi_{\mu\nu}(x)$ which runs from $x+\mu+\nu \rightarrow x$. Hence orientation choice of χ.

After trace this a closed loop

Setting $\mathcal{U} = I + a\mathcal{A}_\mu + \ldots$ easy to see that continuum covariant derivative obtained.

- Note: crucial that $\mathcal{Q}$ commutes with gauge transformations. Is ensured by mapping of fermions to links

Lattice Action

After $\mathcal{Q}$ variation and integration over d

$$g^2 S = \sum_x \mathrm{Tr}\left(\mathcal{F}_{\mu\nu}\overline{\mathcal{F}}_{\mu\nu} + \frac{1}{2}\left(\overline{\mathcal{D}}_\mu \mathcal{U}_\mu\right)^2\right) + S_F$$

where

$$\overline{\mathcal{F}}_{\mu\nu} = \mathcal{D}_\mu^{(+)}\mathcal{U}_\nu = \mathcal{U}_\mu(x)\mathcal{U}_\nu(x+\mu) - \mathcal{U}_\nu(x)\mathcal{U}_\mu(x+\nu)$$

is antisymmetric in its indices and

$$S_F = \sum_x \mathrm{Tr}\left(\chi_{\mu\nu}\mathcal{D}_\mu^{(+)}\psi_\nu + \frac{\eta}{2}\overline{\mathcal{D}}_\mu^{(-)}\psi_\mu\right)$$

Notice $\mathcal{F}\overline{\mathcal{F}}$ contains Wilson plaquette op. (for complex links)

(No) fermion doubling

You may have noticed ...

curl-like ops $D \to D^+$ while divergence-like ops $D \to D^-$

Guarantees no doubling in discrete theory

(Rabin using homology theory)

Explicitly (free theory):

$$S_F = \begin{pmatrix} \eta/2 & \chi_{12} \end{pmatrix} \begin{pmatrix} \Delta_1^- & \Delta_2^- \\ -\Delta_2^+ & \Delta_1^+ \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$\det M_{\rm fermion} = \det \Delta^+ \Delta^-$ – single zero at $k = 0$