Introduction to Resurgence, Trans-series and Non-perturbative Physics

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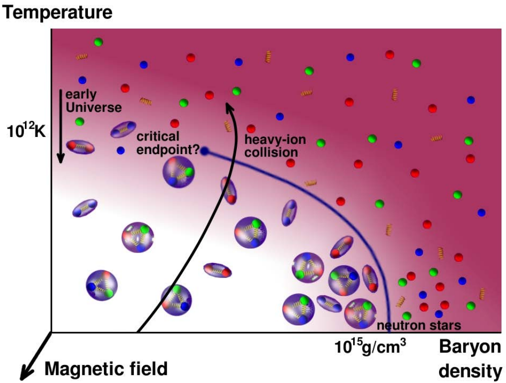
GD & Mithat Ünsal, reviews: 1511.05977, 1601.03414, 1603.04924

recent KITP Program: *Resurgent Asymptotics in Physics and Mathematics*, Fall 2017

Physical Motivation

- non-perturbative definition of non-trivial QFT, in the continuum
- analytic continuation of path integrals
- "sign problem" in finite density QFT
- dynamical & non-equilibrium physics from path integrals (strong coupling)
- uncover hidden 'magic' in perturbation theory
- new understanding of weak-strong coupling dualities
- infrared renormalon puzzle in asymptotically free QFT
- non-perturbative physics without instantons: physical meaning of non-BPS saddles
- exponentially improved asymptotics & resummation

Physical Motivation



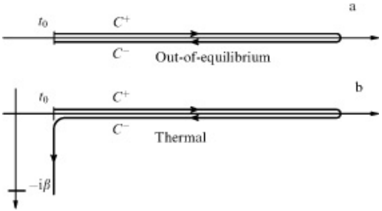
- sign problem: "complex probability" at finite baryon density?

$$\int \mathcal{D}A \, e^{-S_{YM}[A] + \ln\det(\not{D} + m + i\,\mu\gamma^0)}$$

- phase transitions and Lee-Yang & Fisher zeroes

Physical Motivation

- equilibrium thermodynamics $\leftrightarrow$ Euclidean path integral
- Kubo-Martin-Schwinger: antiperiodic b.c.'s for fermions

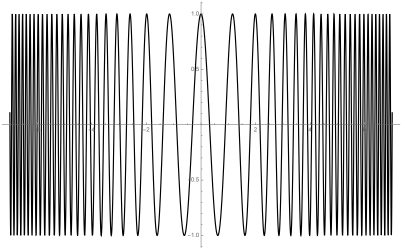


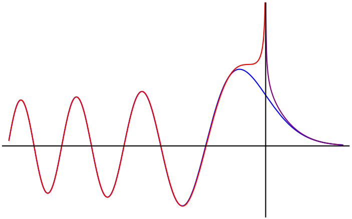
- non-equilibrium physics $\leftrightarrow$ Minkowski path integral
- Schwinger-Keldysh time contours
- quantum transport in strongly-coupled systems

Physical Motivation

what does a Minkowski path integral mean, computationally?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar}\, S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar}\, S[A]\right)$$





$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}\, t^3 + x\, t\right)}\, dt \sim \begin{cases} \dfrac{e^{-\frac{2}{3}\, x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}} & , \quad x \to +\infty \\[2em] \dfrac{\sin\left(\frac{2}{3}\,(-x)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}} & , \quad x \to -\infty \end{cases}$$

- massive cancellations $\Rightarrow \qquad \mathrm{Ai}(+5) \approx 10^{-4}$

Physical Motivation

- what does a Minkowski path integral mean?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar} S[A]\right)$$

- since we need complex analysis and contour deformation to make sense of oscillatory ordinary integrals, it is natural to expect to require similar tools also for path integrals

- an obvious idea, but how to make it work ... ?

Mathematical Motivation

Resurgence: 'new' idea in mathematics (Écalle, 1980; Stokes, 1850)

resurgence = unification of perturbation theory and non-perturbative physics

- perturbation theory generally $\Rightarrow$ divergent series
- series expansion $\longrightarrow$ *trans-series* expansion
- trans-series 'well-defined under analytic continuation'
- perturbative and non-perturbative physics entwined
- applications: ODEs, PDEs, difference equations, fluid mechanics, QM, Matrix Models, QFT, String Theory, ...
- philosophical shift:

go beyond the Gaussian approximation and view semiclassical expansions as potentially exact

1. Lecture 1: Basic Formalism of Trans-series and Resurgence
    - asymptotic series in physics; Borel summation
    - trans-series completions & resurgence
    - examples: linear and nonlinear ODEs
2. Lecture 2: Applications to Quantum Mechanics and QFT
    - instanton gas for double-well & periodic potential
    - infrared renormalon problem in QFT
    - from hyperasymptotics to Picard-Lefschetz thimbles
3. Lecture 3: Resurgence and Large N
    - parametric resurgence
    - Gross-Witten-Wadia Matrix Model
    - Mathieu equation and Nekrasov-Shatashvili limit of $\mathcal{N} = 2$ SUSY QFT

- an interesting observation by Hardy:

No function has yet presented itself in analysis, the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms

G. H. Hardy, *Divergent Series*, 1949

- deep result: “this is all we need” (J. Écalle, 1980)
- also as a closed logic system: Dahn and Göring (1980)

Resurgent Trans-Series

- Écalle: *resurgent functions* closed under all operations:

(Borel transform) + (analytic continuation) + (Laplace transform)

- basic trans-series expansion in QM & QFT applications:

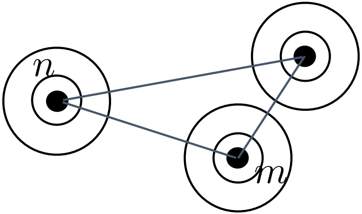
$$f(g^2) = \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zero-modes}}$$

- *trans-monomial elements*: g^2, $e^{-\frac{1}{g^2}}$, $\ln(g^2)$, are familiar
- "multi-instanton calculus" in QFT
- new: analytic continuation encoded in trans-series
- new: trans-series coefficients $c_{k,l,p}$ highly correlated
- new: exponentially improved asymptotics

Resurgence

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



resurgence = global complex analysis with divergent series

Perturbation theory

- hard problem = easy problem + "small" correction
- perturbation theory generally → divergent series

e.g. QM ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

- Zeeman: $c_n \sim (-1)^n \, (2n)!$
- Stark: $c_n \sim (2n)!$
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$
- periodic Sine-Gordon (Mathieu) potential: $c_n \sim n!$
- double-well: $c_n \sim n!$

note generic factorial growth of perturbative coefficients but it works ...

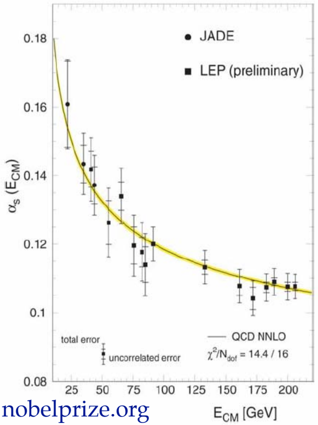
Perturbation theory works

QED perturbation theory:

$$\frac{g-2}{2} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - (0.32848\ldots)\left(\frac{\alpha}{\pi}\right)^2 + (1.18124\ldots)\left(\frac{\alpha}{\pi}\right)^3 - 1.9097(20)\left(\frac{\alpha}{\pi}\right)^4 + 9.16(58)\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

$$\left[\tfrac{1}{2}(g-2)\right]_{\rm exper} = 0.001\,159\,652\,180\,73(28)$$

$$\left[\tfrac{1}{2}(g-2)\right]_{\rm theory} = 0.001\,159\,652\,181\,78(77)$$



nobelprize.org

QCD: asymptotic freedom

$$\beta(g_s) = -\frac{g_s^3}{16\pi^2}\left(\frac{11}{3}N_C - \frac{4}{3}\frac{N_F}{2}\right)$$

Asymptotic Series vs Convergent Series

$$f(x) = \sum_{n=0}^{N-1} c_n \, (x - x_0)^n + R_N(x)$$

convergent series:

$$|R_N(x)| \to 0 \qquad , \quad N \to \infty \quad , \quad x \quad \text{fixed}$$

asymptotic series:

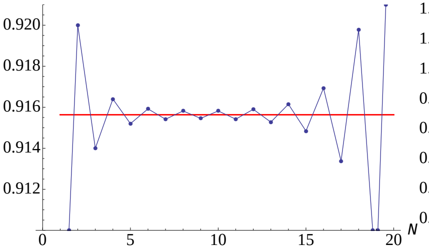
$$|R_N(x)| \ll |x - x_0|^N \qquad , \quad x \to x_0 \quad , \quad N \quad \text{fixed}$$

$\longrightarrow$ "optimal truncation":

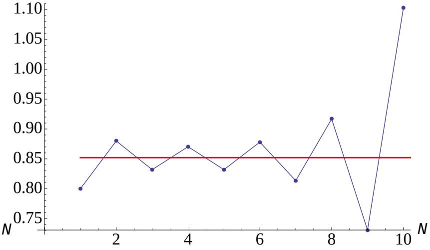
truncate just before least term (x dependent!)

Asymptotic Series vs Convergent Series

alternating asymptotic series : $$\sum_{n=0}^{\infty} (-1)^n \, n! \, x^n \sim \frac{1}{x} \, e^{\frac{1}{x}} \, E_1\left(\frac{1}{x}\right)$$



$(x = 0.1)$

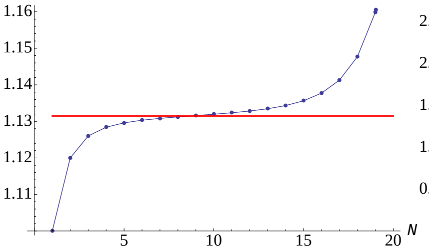


$(x = 0.2)$

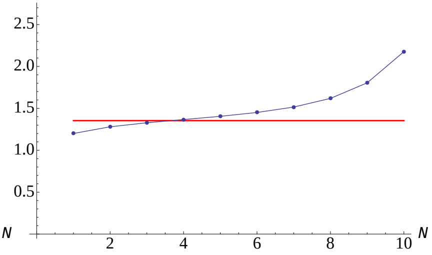
optimal truncation order depends on x: $\quad N_{\text{opt}} \approx \frac{1}{x}$

Asymptotic Series vs Convergent Series

non-alternating asymptotic series :
$$\sum_{n=0}^{\infty} n!\, x^n \sim -\frac{1}{x}\, e^{-\frac{1}{x}}\, E_1\left(-\frac{1}{x}\right)$$



$x = 0.1$



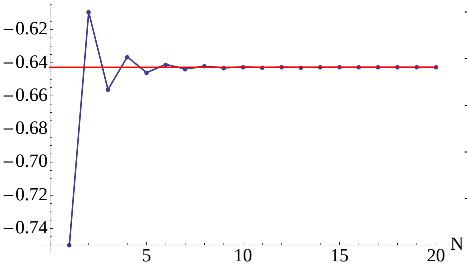
$x = 0.2$

optimal truncation order depends on x: $N_{\text{opt}} \approx \frac{1}{x}$

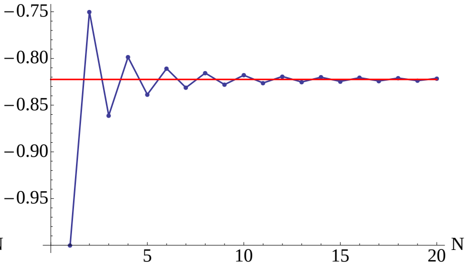
Asymptotic Series vs Convergent Series

- contrast with behavior of a convergent series, for which more terms always improves the answer, independent of x

$$\text{convergent series :} \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}\, x^n = \text{PolyLog}(2, -x)$$



$(x = 0.75)$



$(x = 1)$

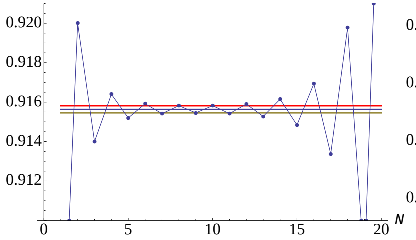
Asymptotic Series: exponential precision

$$\sum_{n=0}^{\infty}(-1)^n \, n! \, x^n \sim \frac{1}{x} \, e^{\frac{1}{x}} \, E_1\left(\frac{1}{x}\right)$$

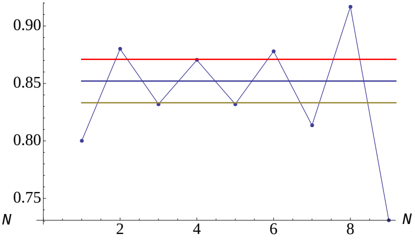
optimal truncation: error term is exponentially small

$$|R_N(x)|_{N\approx 1/x} \approx N! \, x^N\Big|_{N\approx 1/x} \approx N! N^{-N} \approx \sqrt{N} e^{-N} \approx \frac{e^{-1/x}}{\sqrt{x}}$$

- e.g. alternating exponential integral:



$(x = 0.1)$



$(x = 0.2)$

Borel summation: basic idea

alternating factorially divergent series:

$$\sum_{n=0}^{\infty} (-1)^n \, n! \, x^n = ?$$

write $n! = \int_0^\infty dt \, e^{-t} \, t^n$

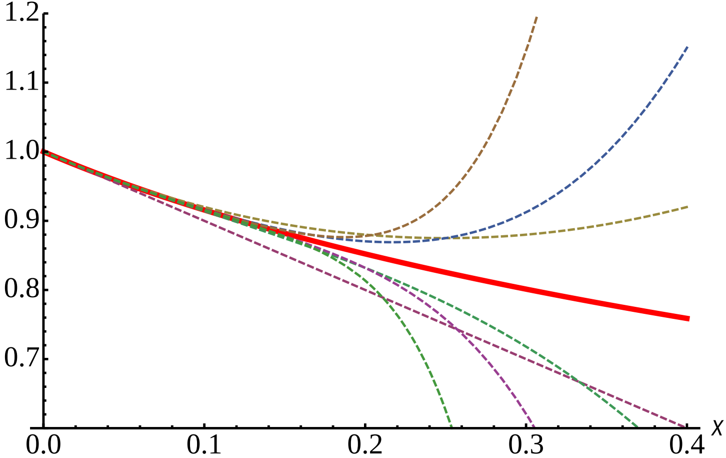
Emile Borel

$$\sum_{n=0}^{\infty} (-1)^n \, n! \, x^n = \int_0^\infty dt \, e^{-t} \, \frac{1}{1 + x\, t} \qquad (?)$$

integral is convergent for all $x > 0$: "Borel sum" of the series

Borel Summation: basic idea

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n = \int_0^{\infty} dt\, e^{-t}\, \frac{1}{1+x\,t} = \frac{1}{x}\, e^{\frac{1}{x}}\, E_1\left(\frac{1}{x}\right)$$



Borel summation: basic idea

write $n! = \int_0^\infty dt\, e^{-t}\, t^n$

non-alternating factorially divergent series:

$$\sum_{n=0}^{\infty} n!\, x^n = \int_0^\infty dt\, e^{-t}\, \frac{1}{1 - x\, t} \qquad (??)$$

pole on the (real, positive) Borel axis!

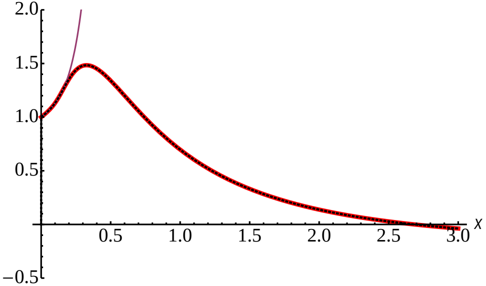
Emile Borel

$$\Rightarrow \text{non-perturbative imaginary part} = \pm \frac{i\,\pi}{x}\, e^{-\frac{1}{x}}$$

but every term in the series is real !?!

Borel Summation: basic idea

$$\text{Borel} \Rightarrow \mathcal{R}e\left[\sum_{n=0}^{\infty} n!\, x^n\right] = \mathcal{P}\int_0^\infty dt\, e^{-t}\,\frac{1}{1-x\,t} = \mathcal{R}e\left[-\frac{1}{x}\,e^{-\frac{1}{x}}\,E_1\left(-\frac{1}{x}\right)\right]$$



- note: $E_1\left(-\frac{1}{x}\right)$ also has an imaginary part $= \pm i\pi$

$$-\frac{1}{x}e^{-\frac{1}{x}}E_1\left(e^{\pm i\,\pi}\frac{1}{x}\right) = -\frac{1}{x}e^{-\frac{1}{x}}\left[Ein\left(-\frac{1}{x}\right) - \ln x - \gamma \mp i\,\pi\right]$$

- Borel encodes this non-perturbative "connection formula"

Borel summation

Borel transform of series, where $c_n \sim n! \quad , \quad n \to \infty$

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n \quad \longrightarrow \quad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

new series typically has a finite radius of convergence

Borel resummation of original asymptotic series:

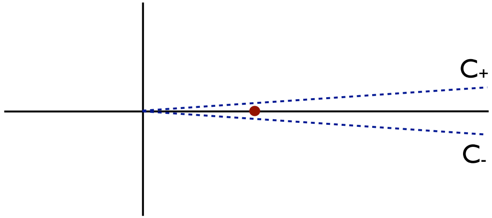
$$\mathcal{S}f(g) = \frac{1}{g} \int_0^{\infty} \mathcal{B}[f](t) e^{-t/g} dt$$

note: $\mathcal{B}[f](t)$ may have singularities in (Borel) t plane

Borel singularities

avoid singularities on $\mathbb{R}^+$: directional Borel sums:

$$\mathcal{S}_\theta f(g) = \frac{1}{g}\int_0^{e^{i\theta}\infty} \mathcal{B}[f](t) e^{-t/g} dt$$



go above/below the singularity: $\theta = 0^\pm$

$\longrightarrow$ non-perturbative ambiguity: $\pm \mathrm{Im}[\mathcal{S}_0 f(g)]$

physics challenge: use physical input to resolve ambiguity another view of resurgence:

resurgence can be viewed as a method for making formal asymptotic expansions consistent with global analytic continuation properties

resurgence = global complex analysis for divergent series

(analytic continuation, transforms, monodromy, ...)

$\Rightarrow$ "the trans-series really IS the function"

question: to what extent is this true/useful in physics?

Resurgence: canonical example = Airy function

- formal large x solution to ODE: "perturbation theory"

$$y'' = x\,y \;\Rightarrow\; \begin{Bmatrix} 2\,\mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{Bmatrix} \sim \frac{e^{\mp\frac{2}{3}x^{3/2}}}{\sqrt{\pi}\,x^{1/4}} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{n!\,\left(\frac{2}{3}\right)^n\,x^{3n/2}}$$

- non-perturbative connection formula:

$$\mathrm{Ai}\left(e^{\mp\frac{2\pi i}{3}}\,x\right) = \pm\frac{i}{2}e^{\mp\frac{\pi i}{3}}\mathrm{Bi}\left(x\right) + \frac{1}{2}e^{\mp\frac{\pi i}{3}}\mathrm{Ai}\left(x\right)$$

- Borel sum: cut along negative t axis: $t \in (-\infty, -1]$

$$Z(x) = \sum_{n=0}^{\infty} \frac{(-1)^n |a_n|}{x^{3n/2}} = \frac{4}{3}x^{3/2}\int_0^\infty dt\, e^{-\frac{4}{3}x^{3/2}t}\;{}_2F_1\left(\frac{1}{6},\frac{5}{6},1;-t\right)$$

- discontinuity across cut $\Rightarrow$ correct connection formula

$$Z\left(e^{\frac{2\pi i}{3}}\,x\right) - Z\left(e^{-\frac{2\pi i}{3}}\,x\right) = i\,e^{-\frac{4}{3}x^{3/2}}Z\left(x\right)$$

Resurgence: canonical example = Airy function

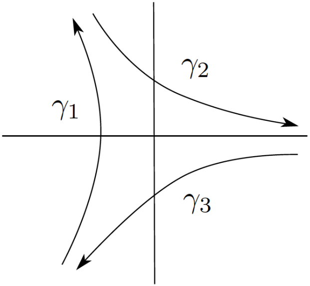
"path integral"

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\left(x\,t+\frac{t^3}{3}\right)}$$

- write $x \equiv r\, e^{i\theta}$, $t \equiv -i\sqrt{r} z$:

$$\mathrm{Ai}(x) = \frac{\sqrt{r}}{2\pi i}\int_{\gamma_k} dz\, e^{r^{3/2}\left(e^{i\theta}\, z-\frac{z^3}{3}\right)}$$

allowed z integration contours



- saddles at $z = \pm e^{i\theta/2}$
- saddle exponent (≡ "action") $= \pm\frac{2}{3}r^{3/2}e^{3i\theta/2}$

$x > 0 \Rightarrow \theta = 0 \Rightarrow$ contour through only 1 saddle $(z = -1)$
$\Rightarrow$ action $= -\frac{2}{3}r^{3/2} = -\frac{2}{3}x^{3/2}$

$x < 0 \Rightarrow \theta = \pm\pi \Rightarrow$ contour through 2 saddles $(z = \pm i)$
$\Rightarrow$ action $= \pm i\frac{2}{3}r^{3/2} = \pm i\frac{2}{3}(-x)^{3/2}$

Resurgence: canonical example = Airy function

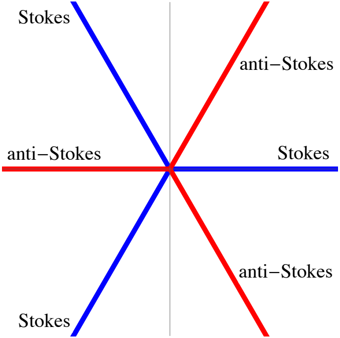
$$\mathrm{Ai}(x) = \frac{\sqrt{r}}{2\pi i}\int_{\gamma_k} dz\, e^{r^{3/2}\left(e^{i\theta}\, z - \frac{z^3}{3}\right)}$$

- saddles at $z = \pm e^{i\theta/2}$, action $= \pm\frac{2}{3} r^{3/2} e^{3i\theta/2}$
- real action when $\theta = 0, \pm\frac{2\pi}{3}$: "Stokes lines"
- imaginary action when $\theta = \pi, \pm\frac{\pi}{3}$: "anti-Stokes lines"

Stokes lines in complex x-plane

$$x = r\, e^{i\,\theta}$$

moral: keep **both** saddle contributions as we analytically continue in complex x plane



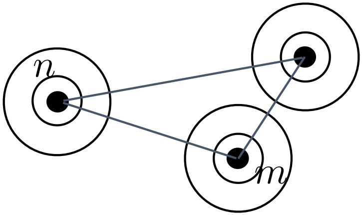
Resurgence: canonical example = Airy function

- expansions about the two saddles are explicitly related

$$a_n = \left\{ 1, -\frac{5}{72}, \frac{385}{10368}, -\frac{85085}{2239488}, \frac{37182145}{644972544}, -\frac{5391411025}{46438023168}, \dots \right\}$$

- large order/low order relation:

$$a_n \sim \frac{(n-1)!}{2^n} \left(1 - \frac{5}{72} \frac{2}{(n-1)} + \frac{385}{10368} \frac{2^2}{(n-1)(n-2)} - \dots \right)$$



- large order/low order relations are generic ! (see later)

Resurgence: Preserving Analytic Continuation

Stirling expansion for $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is divergent

$$\psi(1+z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots + \frac{174611}{6600z^{20}} - \ldots$$

- functional relation: $\psi(1+z) = \psi(z) + \frac{1}{z}$ ✓
- reflection formula: $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot(\pi z)$

$$\Rightarrow \quad \mathrm{Im}\, \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} e^{-2\pi k y}$$

- formal series only has the two "perturbative" terms

"raw" asymptotics is <u>inconsistent</u> with analytic continuation

- resurgence: add infinite series of non-perturbative terms

"non-perturbative completion"

Resurgence: Preserving Analytic Continuation

$$\text{Im}\,\psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi\sum_{k=1}^{\infty} e^{-2\pi\,k\,y}$$

• function satisfies infinite order linear ODE

$\Rightarrow$ infinitely many exponential terms in trans-series

Borel representation:

$$\psi(1+z) - \ln z = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) e^{-zt}\,dt$$

• Borel transform: poles at $t = \pm 2n\pi i$, $n = 1, 2, 3, \ldots$

• meromorphic (poles, no cuts) $\Rightarrow$ no "fluctuation factors"

• this simple example arises often in QFT: Euler-Heisenberg, finite temperature QFT, de Sitter, exact S-matrices, Chern-Simons partition functions, matrix models, ...

- trans-series from n$^{\text{th}}$ order linear ODE has n non-perturbative exponential terms
- trans-series from nonlinear ODE has infinitely many non-perturbative exponential terms
- e.g.: $y_1(x) \times y_2(x)$ satisfies 3^{rd} order **linear** ODE
  but $y_1(x)/y_2(x)$ satisfies 2^{nd} order **non-linear** ODE
- also generalizes to (some) PDE's, linear and non-linear
- Painlevé = "special functions of nonlinear ODE's"
  many physical applications: fluids, statistical physics, gravity, random matrices, matrix models, optics, QFT, strings, ...
- resurgent trans-series are the natural language for their asymptotics

Resurgence in Nonlinear ODEs: e.g. Painlevé II

Painlevé II:

$$y'' = x\, y(x) + 2\, y^3(x)$$

- Tracy-Widom law for statistics of max. eigenvalue for Gaussian random matrices
- correlators in polynuclear growth; directed polymers (KPZ)
- double-scaling limit in unitary matrix models
- double-scaling limit in 2d Yang-Mills
- double-scaling limit in 2d supergravity
- non-intersecting Brownian motions
- longest increasing subsequence in random permutations
- ... universal !

Resurgence in Nonlinear ODEs: e.g. Painlevé II

$$y'' = x\, y(x) + 2\, y^3(x)$$

- $x \to +\infty$ asymptotics: $y'' \approx x\, y(x) + \dots$

$$y \to 0 \quad \text{as} \quad x \to +\infty \quad \Rightarrow \quad y_+^{(1)}(x) \sim \sigma_+ \, \mathrm{Ai}(x) + \dots$$

- trans-series solution generated from ODE:

$$y_+(x) \sim \sum_{k=1}^{\infty} \left(\sigma_+ \frac{e^{-\frac{2}{3} x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}} \right)^{2k-1} y_+^{(k)}(x)$$

- infinite number of non-perturbative terms
- fluctuations factorially divergent & **alternating**
- σ_+ = **real** trans-series parameter (for real solution)
- large-order/low-order relations for fluc. coefficients

Resurgence in Nonlinear ODEs: e.g. Painlevé II

$$y'' = x\,y(x) + 2\,y^3(x)$$

- $x \to -\infty$ asymptotics: $0 \approx x\,y(x) + 2\,y^3(x)$

$$\text{smoothness} \quad \Rightarrow \quad y_-^{(0)}(x) \sim \sqrt{\frac{-x}{2}}\left(1 + O\left(\frac{1}{(-x)^{3/2}}\right)\right)$$

- different (!) trans-series solution generated from ODE:

$$y_-(x) \sim \sqrt{\frac{-x}{2}} \sum_{k=0}^{\infty} \left(\sigma_- \frac{e^{-\frac{2\sqrt{2}}{3}(-x)^{3/2}}}{2\sqrt{\pi}\,(-x)^{1/4}}\right)^k y_-^{(k)}(x)$$

- fluctuations:

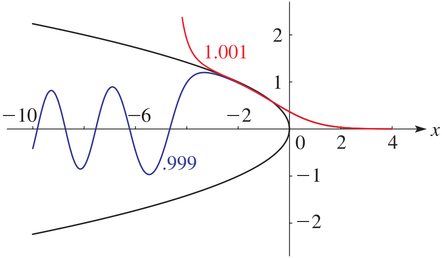
$$y_-^{(k)}(x) \sim \sum_{n=0}^{\infty} \frac{a_n^{(k)}}{(-x)^{3n/2}}$$

- fluctuations factorially divergent & **non-alternating**

- σ_- = **pure imaginary** trans-series parameter (for real solution); fixed by resurgent cancellations

Resurgence in Nonlinear ODEs: e.g. Painlevé II

$$y'' = x\, y(x) + 2\, y^3(x) \quad , \quad y(x) \sim \sigma_+ \, \mathrm{Ai}(x) \quad , \quad x \to +\infty$$



- trans-series structurally different as $x \to \pm\infty$
- note different exponents!

$$x \to +\infty \Rightarrow \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}}$$

$$x \to -\infty \Rightarrow \frac{e^{-\frac{2\sqrt{2}}{3}(-x)^{3/2}}}{2\sqrt{\pi}\, (-x)^{1/4}}$$

- Hastings-McLeod: $\sigma_+ = 1$ ($\sigma_- = i$) unique real solution on $\mathbb{R}$
  - connection formula for $\sigma_+ < 1$: $\left(d^2 \equiv -\pi^{-1} \ln\left(1 - \sigma_+^2\right)\right)$

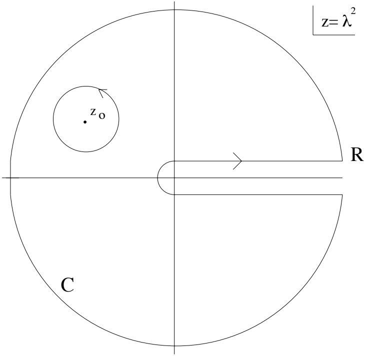
$$y_-(x) \sim \sigma_+ |x|^{-1/4} \sin\left(\frac{2}{3}|x|^{3/2} - \frac{3}{4} d^2 \ln|x| - \theta_0\right)$$

  - intricate "condensation of instantons" across transition

1. Lecture 1: Basic Formalism of Trans-series and Resurgence
   - asymptotic series in physics; Borel summation
   - trans-series completions & resurgence
   - examples: linear and nonlinear ODEs
2. Lecture 2: Applications to Quantum Mechanics and QFT
   - instanton gas for double-well & periodic potential
   - infrared renormalon problem in QFT
   - from hyperasymptotics to Picard-Lefschetz thimbles
3. Lecture 3: Resurgence and Large N
   - parametric resurgence
   - Gross-Witten-Wadia Matrix Model
   - Mathieu equation and Nekrasov-Shatashvili limit of $\mathcal{N} = 2$ SUSY QFT

Borel Summation and Dispersion Relations: QM examples

cubic oscillator: $V = x^2 + \lambda\, x^3$ A. Vainshtein, 1964



$$E(z_0) = \frac{1}{2\pi i}\oint_C dz\, \frac{E(z)}{z - z_0}$$

$$= \frac{1}{\pi}\int_0^R dz\, \frac{Im\, E(z)}{z - z_0}$$

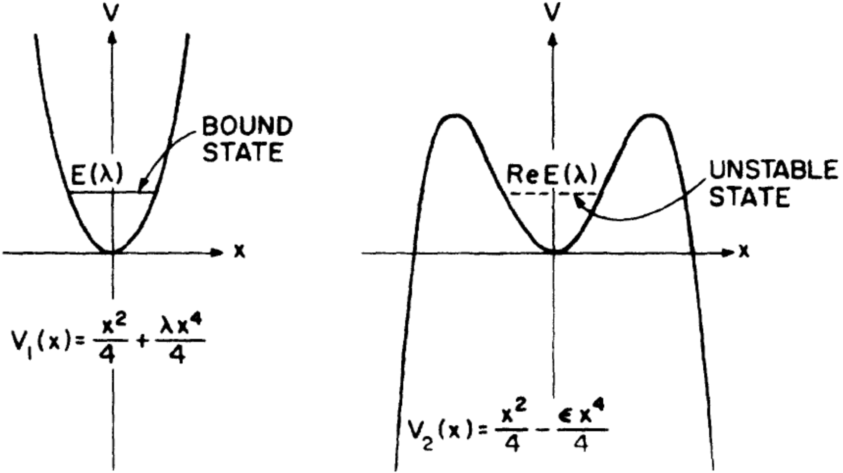
$$= \sum_{n=0}^{\infty} z_0^n \left(\frac{1}{\pi}\int_0^R dz\, \frac{\mathrm{Im}\, E(z)}{z^{n+1}} \right)$$

$$\text{WKB} \Rightarrow \mathrm{Im}\, E(z) \sim \frac{a}{\sqrt{z}}\, e^{-b/z} \quad , \quad z \to 0 \quad \leftrightarrow \quad n \to \infty$$

$$\Rightarrow \qquad c_n \sim \frac{a}{\pi}\int_0^\infty dz\, \frac{e^{-b/z}}{z^{n+3/2}} = \frac{a}{\pi}\,\frac{\Gamma(n+\frac{1}{2})}{b^{n+1/2}} \qquad \checkmark$$

Instability and Divergence of Perturbation Theory

quartic AHO: $V(x) = \frac{x^2}{4} + \lambda \frac{x^4}{4}$ Bender/Wu, 1969



Instability and Divergence of Perturbation Theory

an important part of the story ...

The majority of nontrivial theories are seemingly unstable at some phase of the coupling constant, which leads to the asymptotic nature of the perturbative series

A. Vainshtein (1964)

Borel summation in practice

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n \qquad , \qquad c_n \sim \beta^n\, \Gamma(\gamma\, n + \delta)$$

- alternating series: real Borel sum

$$f(g) \sim \frac{1}{\gamma} \int_0^\infty \frac{dt}{t} \left(\frac{1}{1+t}\right) \left(\frac{t}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{t}{\beta g}\right)^{1/\gamma}\right]$$

- nonalternating series: ambiguous imaginary part

$$\text{Re}\, f(-g) \;\sim\; \frac{1}{\gamma}\, \mathcal{P} \int_0^\infty \frac{dt}{t} \left(\frac{1}{1-t}\right) \left(\frac{t}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{t}{\beta g}\right)^{1/\gamma}\right]$$

$$\text{Im}\, f(-g) \;\sim\; \pm\frac{\pi}{\gamma} \left(\frac{1}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{1}{\beta g}\right)^{1/\gamma}\right]$$

- γ determines **power** of coupling in the exponent
- β and γ determine **coefficient** in the exponent
- β, γ and δ determine the **prefactor**

recall: divergence of perturbation theory in QM

e.g. ground state energy: $E = \sum_{n=0}^{\infty} c_n \,(\text{coupling})^n$

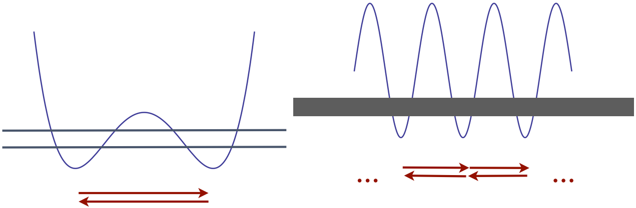
- Zeeman: $c_n \sim (-1)^n \,(2n)!$
- Stark: $c_n \sim (2n)!$
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$
- periodic Sine-Gordon potential: $c_n \sim n!$
- double-well: $c_n \sim n!$

recall: divergence of perturbation theory in QM

e.g. ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

- Zeeman: $c_n \sim (-1)^n \, (2n)!$ stable ✓
- Stark: $c_n \sim (2n)!$ unstable ✓
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$ stable ✓
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$ unstable ✓
- periodic Sine-Gordon potential: $c_n \sim n!$ stable ???
- double-well: $c_n \sim n!$ stable ???

Bogomolny/Zinn-Justin mechanism in QM

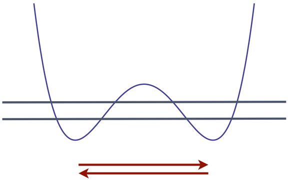


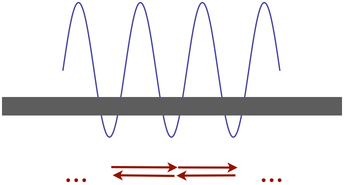
- degenerate vacua: double-well, Sine-Gordon, ...
- level splitting = real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

surprise: pert. theory non-Borel summable: $c_n \sim \frac{n!}{(2S)^n}$

- stable systems
- ambiguous imaginary part
- $\pm i\, e^{-\frac{2S}{g^2}}$, a two-instanton effect

Bogomolny/Zinn-Justin mechanism in QM





- degenerate vacua: double-well, Sine-Gordon, ...
    1. perturbation theory non-Borel summable: ill-defined/incomplete
    2. instanton gas picture ill-defined/incomplete: $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract
- regularize both by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel !

"tip of the (resurgence) iceberg"

Bogomolny/Zinn-Justin mechanism in QM

e.g., double-well: $V(x) = x^2(1 - g\,x)^2$

$$E_0 \sim \sum_n c_n\, g^{2n}$$

- perturbation theory:

$$c_n \sim -3^n\, n! \quad : \quad \text{Borel} \quad \Rightarrow \quad \text{Im}\, E_0 \sim \mp \pi\, e^{-\frac{1}{3g^2}}$$

- non-perturbative analysis: instanton: $g\,x_0(t) = \frac{1}{1+e^{-t}}$
- classical Eucidean action: $S_0 = \frac{1}{6g^2}$
- non-perturbative instanton gas:

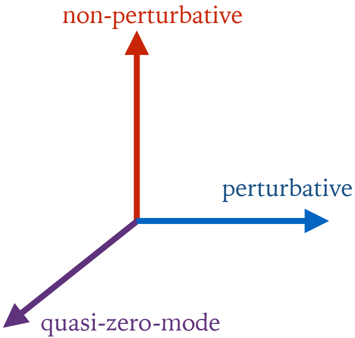
$$\Delta E_0 \sim e^{-\frac{1}{6g^2}} \quad , \quad \text{Im}\, E_0 \sim \pm \pi\, e^{-2\frac{1}{6g^2}}$$

- BZJ cancellation $\Rightarrow$ E_0 is real and unambiguous

"resurgence" $\Rightarrow$ cancellation to all orders

Decoding a Resurgent Trans-series

$$f(g^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{q=0}^{k-1} c_{n,k,q}\, g^{2n} \left[\exp\left(-\frac{S}{g^2}\right)\right]^k \left[\ln\left(-\frac{1}{g^2}\right)\right]^q$$



expansions in different directions are quantitatively related

Decoding a Resurgent Trans-series

$$f(g^2) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{q=0}^{k-1} c_{n,k,q}\, g^{2n} \left[\exp\left(-\frac{S}{g^2}\right)\right]^k \left[\ln\left(-\frac{1}{g^2}\right)\right]^q$$

- perturbative fluctuations about vacuum: $\sum_{n=0}^{\infty} c_{n,0,0}\, g^{2n}$
- divergent (non-Borel-summable): $c_{n,0,0} \sim \alpha \frac{n!}{(2S)^n}$

$\Rightarrow$ ambiguous imaginary non-pert energy $\sim \pm i\,\pi\,\alpha\, e^{-2S/g^2}$

- but $c_{0,2,1} = -\alpha$: BZJ cancellation !

pert flucs about instanton: $e^{-S/g^2}\,\left(1 + a_1 g^2 + a_2 g^4 + \dots\right)$

divergent:

$a_n \sim \frac{n!}{(2S)^n}\,(a\,\ln n + b) \Rightarrow \pm i\,\pi\, e^{-3S/g^2}\left(a \ln\frac{1}{g^2} + b\right)$

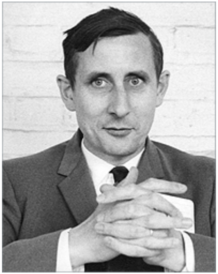
- 3-instanton: $e^{-3S/g^2}\left[\frac{a}{2}\left(\ln\left(-\frac{1}{g^2}\right)\right)^2 + b\,\ln\left(-\frac{1}{g^2}\right) + c\right]$

resurgence: *ad infinitum*, also sub-leading large-order terms

Towards Resurgence in QFT

- basic divergence due to combinatoric growth of diagrams
- new features arise in QFT due to renormalization
- asymptotically free QFT: "renormalons"

Dyson's argument (QED)



- F. J. Dyson (1952):
physical argument for divergence of QED perturbation theory

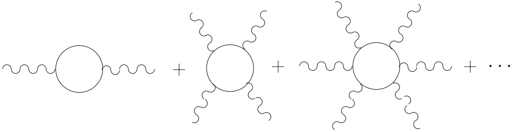
$$F(e^2) = c_0 + c_2 e^2 + c_4 e^4 + \dots$$

> *Thus [for $e^2 < 0$] every physical state is unstable against the spontaneous creation of large numbers of particles. Further, a system once in a pathological state will not remain steady; there will be a rapid creation of more and more particles, an explosive disintegration of the vacuum by spontaneous polarization.*

- *suggests* perturbative expansion cannot be convergent

Euler-Heisenberg Effective Action (1935)

review: hep-th/0406216



- 1-loop QED effective action in uniform emag field
- the birth of *effective field theory*

$$L = \frac{\vec{E}^2 - \vec{B}^2}{2} + \frac{\alpha}{90\pi}\frac{1}{E_c^2}\left[\left(\vec{E}^2 - \vec{B}^2\right)^2 + 7\left(\vec{E}\cdot\vec{B}\right)^2\right] + \dots$$

- encodes nonlinear properties of QED/QCD vacuum

QFT Application: Euler-Heisenberg

Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{hc}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2 (\mathfrak{E}\mathfrak{B}) \cdot \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) + \text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) - \text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2 - \mathfrak{E}^2)\right\}.$$

($\mathfrak{E}, \mathfrak{B}$ Kraft auf das Elektron.
$|\mathfrak{E}_k| = \frac{m^2 c^3}{e\hbar} = \frac{1}{„137“}\frac{e}{(e^2/mc^2)^2}$ = „Kritische Feldstärke“.)

- Borel transform of a (doubly) asymptotic series
- resurgent trans-series: analytic continuation $B \longleftrightarrow E$
- EH effective action $\sim$ Barnes function $\sim \int \ln\Gamma(x)$

Euler-Heisenberg Effective Action: e.g., constant B field

$$S = -\frac{B^2}{8\pi^2}\int_0^\infty \frac{ds}{s^2}\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)\,\exp\left[-\frac{m^2 s}{B}\right]$$

- perturbative (weak field) expansion:

$$S \sim -\frac{B^2}{2\pi^2}\sum_{n=0}^{\infty}\frac{\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)}\left(\frac{2B}{m^2}\right)^{2n+2}$$

- characteristic factorial divergence

$$c_n = \frac{(-1)^{n+1}}{8}\sum_{k=1}^{\infty}\frac{\Gamma(2n+2)}{(k\,\pi)^{2n+4}}$$

- instructive exercise: reconstruct correct Borel transform

$$\sum_{k=1}^{\infty}\frac{s}{k^2\pi^2(s^2+k^2\pi^2)} = -\frac{1}{2s^2}\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)$$

Euler-Heisenberg Effective Action and Schwinger Effect

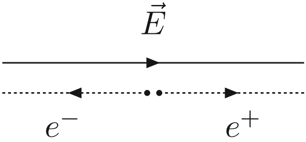
B field: QFT analogue of Zeeman effect

E field: QFT analogue of Stark effect

$B^2 \to -E^2$: series becomes non-alternating

Borel summation $\Rightarrow \mathrm{Im}\, S = \frac{e^2 E^2}{8\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{k\, m^2 \pi}{eE}\right]$

Schwinger effect:



$$2eE\frac{\hbar}{mc} \sim 2mc^2$$

$$\Rightarrow$$

$$E_c \sim \frac{m^2 c^3}{e\hbar} \approx 10^{16}\mathrm{V/cm}$$

WKB tunneling from Dirac sea

$\mathrm{Im}\, S \to$ physical pair production rate

- Euler-Heisenberg series must be divergent

- scalar QED EH in self-dual background ($F = \pm\tilde{F}$):

$$S = \frac{F^2}{16\pi^2}\int_0^\infty \frac{dt}{t}\, e^{-t/F}\left(\frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3}\right)$$

- Gaussian matrix model: $\lambda = g\,N$

$$\mathcal{F} = -\frac{1}{4}\int_0^\infty \frac{dt}{t}\, e^{-2\lambda\, t/g}\left(\frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3}\right)$$

- $c = 1$ String: $\lambda = g\,N$

$$\mathcal{F} = \frac{1}{4}\int_0^\infty \frac{dt}{t}\, e^{-2\lambda\, t/g}\left(\frac{1}{\sin^2(t)} - \frac{1}{t^2} - \frac{1}{3}\right)$$

- Chern-Simons matrix model:

$$\mathcal{F} = -\frac{1}{4}\sum_{m\in\mathbb{Z}}\int_0^\infty \frac{dt}{t}\, e^{-2(\lambda+2\pi\, i\, m)\, t/g}\left(\frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3}\right)$$

de Sitter/ anti de Sitter effective actions (Das & GD, hep-th/0607168)

- explicit expressions (multiple gamma functions)

$$\mathcal{L}_{AdS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(AdS_d)} \left(\frac{K}{m^2}\right)^n$$

$$\mathcal{L}_{dS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(dS_d)} \left(\frac{K}{m^2}\right)^n$$

- changing sign of curvature: $a_n^{(AdS_d)} = (-1)^n a_n^{(dS_d)}$
- odd dimensions: convergent
- even dimensions: divergent

$$a_n^{(AdS_d)} \sim \frac{\mathcal{B}_{2n+d}}{n(2n+d)} \sim 2(-1)^n \frac{\Gamma(2n+d-1)}{(2\pi)^{2n+d}}$$

- pair production in dS_d with d even

QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

$$c_n \sim (\pm 1)^n \frac{n!}{(2S)^n}$$

QFT: new physical effects occur, due to running of couplings with momentum

- faster source of divergence: "renormalons"

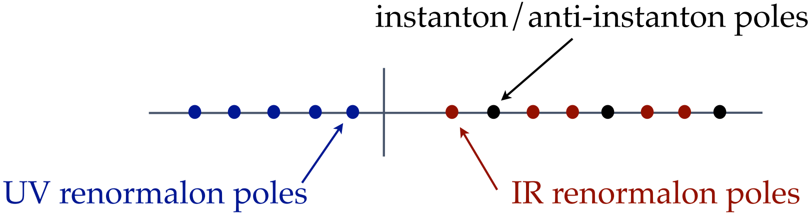
$$c_n \sim (\pm 1)^n \frac{\beta_0^n \, n!}{(2S)^n} = (\pm 1)^n \frac{n!}{(2S/\beta_0)^n}$$

- both positive and negative Borel poles

IR Renormalon Puzzle in Asymptotically Free QFT

perturbation theory: $\longrightarrow \quad \pm i\, e^{-\frac{2S}{\beta_0\, g^2}}$

instantons on $\mathbb{R}^2$ or $\mathbb{R}^4$: $\longrightarrow \quad \pm i\, e^{-\frac{2S}{g^2}}$



appears that BZJ cancellation cannot occur

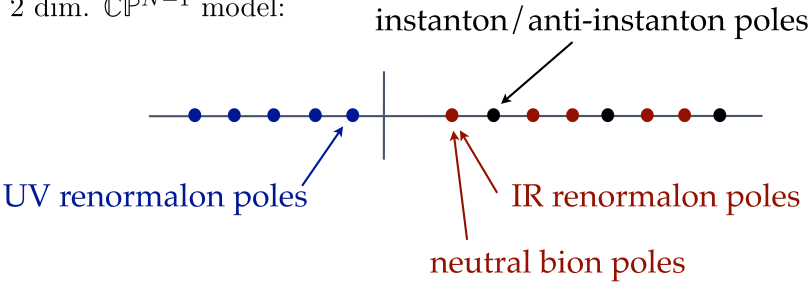
asymptotically free theories remain perturbatively inconsistent

't Hooft, 1980; David, 1981

IR Renormalon Puzzle in Asymptotically Free QFT

resolution: there is another problem with the non-perturbative instanton gas analysis (Argyres, Ünsal 1206.1890; GD, Ünsal, 1210.2423)

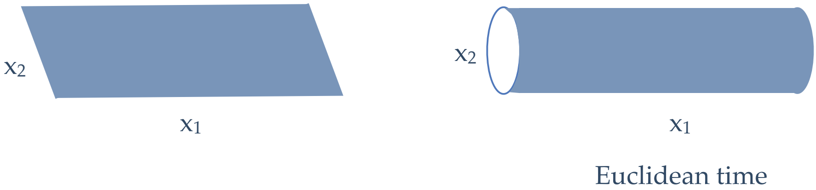
- scale modulus of instantons
- spatial compactification and principle of continuity
- 2 dim. $\mathbb{CP}^{N-1}$ model:



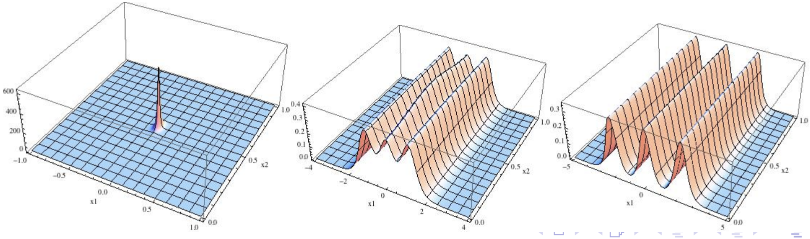
cancellation occurs ! (GD, Ünsal, 1210.2423, 1210.3646)

Topological Molecules in Spatially Compactified Theories

$\mathbb{CP}^{N-1}$: regulate scale modulus problem with (spatial) compactification: $\mathbb{R}^2 \to \mathbb{S}^1_L \times \mathbb{R}^1$



$\mathbb{Z}_N$ twist: instantons fractionalize: $S_{\rm inst} \longrightarrow \frac{S_{\rm inst}}{N} = \frac{S_{\rm inst}}{\beta_0}$



Perturbative Analysis

- weak-coupling semi-classical analysis
- perturbative $\rightarrow$ effective QM problem
- perturbation theory diverges & non-Borel summable
- perturbative sector: directional Borel summation

$$B_{\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\, B\mathcal{E}(t)\, e^{-t/g^2} = \mathrm{Re}\, B\mathcal{E}(g^2) \mp i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

- compare with non-perturbative instanton gas analysis:

$$\left[\mathcal{I}_i\bar{\mathcal{I}}_i\right]_{\pm} = \left(\ln\left(\frac{g^2\, N}{8\pi}\right) - \gamma\right)\frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}} \pm i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

exact ("BZJ") cancellation !

explicit application of resurgence to nontrivial QFT

Non-perturbative Physics Without Instantons

Dabrowski, GD, 1306.0921, Cherman, Dorigoni, GD, Ünsal, 1308.0127, 1403.1277, GD, Ünsal, 1505.07803

- 2d $O(N)$ & principal chiral model have no instantons !
- but they have finite action non-BPS saddles
- Yang-Mills, $\mathbb{CP}^{N-1}$, $O(N)$, principal chiral model, ... all have non-BPS solutions with finite action

(Din & Zakrzewski, 1980; Uhlenbeck 1985; Sibner, Sibner, Uhlenbeck, 1989)

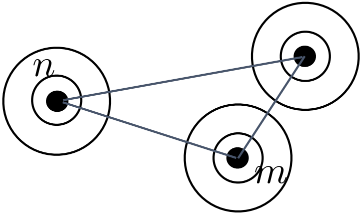
- "unstable": negative modes of fluctuation operator
- what do these mean physically ?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from directional Borel sums of perturbation theory

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

The Bigger Picture: Decoding the Path Integral

what is the origin of resurgent behavior in QM and QFT ?

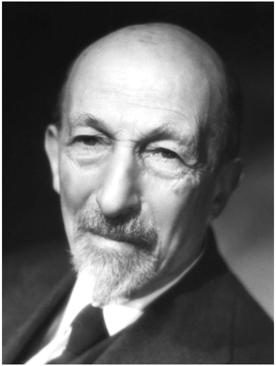


to what extent are (all?) multi-instanton effects encoded in perturbation theory? And if so, why?

- QM & QFT: basic property of all-orders steepest descents integrals
- Lefschetz thimbles: analytic continuation of path integrals

The shortest path between two truths in the real domain passes through the complex domain

Jacques Hadamard, 1865 - 1963



All-Orders Steepest Descents: Darboux Theorem

- all-orders steepest descents for contour integrals:

<u>*hyperasymptotics*</u> (Berry/Howls 1991, Howls 1992)

$$I^{(n)}(g^2) = \int_{C_n} dz\, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}} e^{-\frac{1}{g^2} f_n} T^{(n)}(g^2)$$

- $T^{(n)}(g^2)$: beyond the usual Gaussian approximation
- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)} g^{2r}$$

All-Orders Steepest Descents: Darboux Theorem

- Berry/Howls: exact resurgent relation between fluctuations about n^{th} saddle and about neighboring saddles m

$$T^{(n)}(g^2) = \frac{1}{2\pi\, i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - g^2 v/(F_{nm})} T^{(m)}\left(\frac{F_{nm}}{v}\right)$$

- proof is based on contour deformation

- universal factorial divergence of fluctuations (Darboux)

$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \dots\right.$$

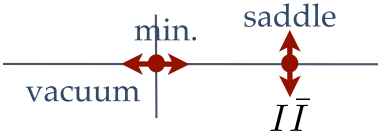
fluctuations about different saddles are explicitly related !

All-Orders Steepest Descents: Darboux Theorem

$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(g^2) = \int_0^\pi dz \, e^{-\frac{1}{g^2} \sin^2(z)}$$

- this is a Bessel function
- two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle z_0:

$$T_r^{(0)} = \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)}$$

$$\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1-\frac{\frac{1}{4}}{(r-1)}+\frac{\frac{9}{32}}{(r-1)(r-2)}-\frac{\frac{75}{128}}{(r-1)(r-2)(r-3)}+\right.$$

- low order coefficients about saddle z_1:

$$T^{(1)}(g^2) \sim i\sqrt{\pi}\left(1-\frac{1}{4}g^2+\frac{9}{32}g^4-\frac{75}{128}g^6+\dots\right)$$

- fluctuations about the two saddles are explicitly related

- simple example of a generic resurgent large-order/low-order perturbative/non-perturbative relation

Resurgence in Path Integrals: "Functional Darboux Theorem"

could something like this work for path integrals?

"functional Darboux theorem" ?

- multi-dimensional case is already non-trivial and interesting

Pham (1965); Arnold (1970); Delabaere/Howls (2002); Kontsevich (2016-)

- Picard-Lefschetz theory

- do a computation to see what happens ...

Resurgence in (Infinite Dim.) Path Integrals (GD, Ünsal, 1401.5202)

- periodic potential: $V(x) = \frac{1}{g^2}\,\sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n! \left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\, E \sim \pi\, e^{-2\frac{1}{2g^2}} \left(1 - \frac{5}{2}\cdot g^2 - \frac{13}{8}\cdot g^4 - \dots\right)$$

- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n! \left(1 - \frac{53}{6}\cdot\frac{1}{3}\cdot\frac{1}{n} - \frac{1277}{72}\cdot\frac{1}{3^2}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

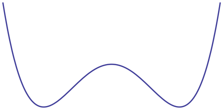
$$\mathrm{Im}\, E \sim \pi\, e^{-2\frac{1}{6g^2}} \left(1 - \frac{53}{6}\cdot g^2 - \frac{1277}{72}\cdot g^4 - \dots\right)$$

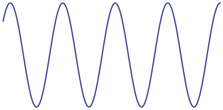
in fact, the resurgent structure is much deeper than this ...

Uniform WKB & Resurgent Trans-Series

Alvarez/Casares (2000, 2003), GD/Unsal (1306.4405, 1401.5202)

$$-\frac{d^2}{dx^2}\psi + \frac{V(g\,x)}{g^2}\psi = E\,\psi \to -g^4\,\frac{d^2}{dy^2}\psi(y) + V(y)\psi(y) = g^2\,E\,\psi(y)$$





- weak coupling: degenerate harmonic classical vacua
- non-perturbative effects: $g^2 \leftrightarrow \hbar \quad \Rightarrow \quad \exp\left(-\frac{c}{g^2}\right)$
- approximately harmonic

$\Rightarrow$ uniform WKB with parabolic cylinder functions

- ansatz (with parameter ν): $\psi(y) = \dfrac{D_\nu\left(\frac{1}{g}u(y)\right)}{\sqrt{u'(y)}}$

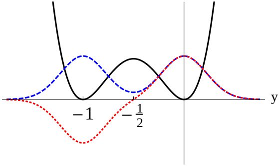
"similar looking equations have similar looking solutions"

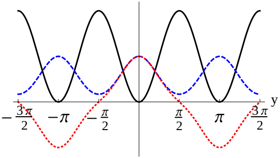
Uniform WKB & Resurgent Trans-Series

- perturbative expansion for E and $u(y)$:

$$E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)$$

- $\nu = N$: usual perturbation theory (not Borel summable)
- global analysis $\Rightarrow$ boundary conditions:





- midpoint $\sim \frac{1}{g}$; non-Borel summability $\Rightarrow$ $\quad g^2 \to e^{\pm i\,\epsilon}\, g^2$
- trans-series encodes analytic properties of D_ν

  $\Rightarrow$ generic and universal

Uniform WKB & Resurgent Trans-Series

$$D_\nu(z) \sim z^\nu \, e^{-z^2/4} \, (1 + \dots) + e^{\pm i\pi\nu} \frac{\sqrt{2\pi}}{\Gamma(-\nu)} z^{-1-\nu} \, e^{z^2/4} \, (1 + \dots)$$

$\longrightarrow$ exact quantization condition

$$\frac{1}{\Gamma(-\nu)} \left(\frac{e^{\pm i\pi} \, 2}{g^2} \right)^{-\nu} = \frac{e^{-S/g^2}}{\sqrt{\pi \, g^2}} \mathcal{P}(\nu, g^2)$$

$\Rightarrow$ ν is only exponentially close to N (here $\xi \equiv \frac{e^{-S/g^2}}{\sqrt{\pi \, g^2}}$):

$$\nu = N + \frac{\left(\frac{2}{g^2}\right)^N \mathcal{P}(N, g^2)}{N!} \xi$$

$$- \frac{\left(\frac{2}{g^2}\right)^{2N}}{(N!)^2} \left[\mathcal{P} \frac{\partial \mathcal{P}}{\partial N} + \left(\ln \left(\frac{e^{\pm i\pi} \, 2}{g^2} \right) - \psi(N+1) \right) \mathcal{P}^2 \right] \xi^2 + O(\xi^3)$$

- insert: $E = E(\nu, g^2) = \sum_{k=0}^\infty g^{2k} E_k(\nu)$ $\Rightarrow$ trans-series

Connecting Perturbative and Non-Perturbative Sector

this proves the Zinn-Justin/Jentschura conjecture:
generate *entire trans-series* from just two functions:

(i) perturbative expansion $E = E_{\rm pert}(\hbar, N)$
(ii) single-instanton fluctuation function $\mathcal{P}_{\rm inst}(\hbar, N)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

$$E(\hbar, N) = E_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} e^{-S/\hbar} \, \mathcal{P}_{\rm inst}(\hbar, N) + \dots$$

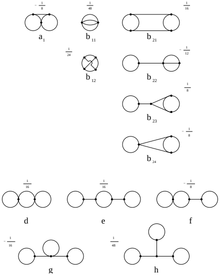
- in fact ... there is much more structure hiding here:
- instanton fluctuation factor:

$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial E_{\rm pert}}{\partial N} \exp\left[S \int_0^\hbar \frac{d\hbar}{\hbar^3} \left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N + \frac{1}{2}\right) \hbar^2}{S}\right)\right]$$

$\Rightarrow$ perturbation theory $E_{\rm pert}(\hbar, N)$ encodes everything !

Resurgence at work

- fluctuations about $\mathcal{I}$ (or $\bar{\mathcal{I}}$) saddle are determined by those about the vacuum saddle, to all fluctuation orders



- "QFT computation": 3-loop fluctuation about $\mathcal{I}$ for double-well and Sine-Gordon:

Escobar-Ruiz/Shuryak/Turbiner 1501.03993, 1505.05115

$$\text{DW}: \quad e^{-\frac{S_0}{\hbar}}\left[1-\frac{71}{72}\,\hbar-0.607535\,\hbar^2-\ldots\right]$$

$$\text{resurgence}: e^{-\frac{S_0}{\hbar}}\left[1+\frac{1}{72}\,\hbar\left(-102N^2-174N-71\right)\right.$$

$$\left.+\frac{1}{10368}\hbar^2\left(10404N^4+17496N^3-2112N^2-14172N-6299\right)+\ldots\right]$$

- known for all N and to essentially any loop order, directly from perturbation theory !

- diagramatically mysterious ...

Deconstructing Zero: P/NP Resurgence for SUSY QM

GD & Ünsal: 1609.05770

- SUSY: $E^{\text{perturbative}}_{\text{ground state}}(\hbar) = 0$ to all orders !
- how can it encode non-perturbative effects ?
- broken SUSY: $E^{\text{nonpert.}}_{\text{g.s.}}(\hbar, N) \sim \hbar^{\beta} e^{-S/\hbar} \mathcal{P}_{\text{fluc}}(\hbar, N) > 0$

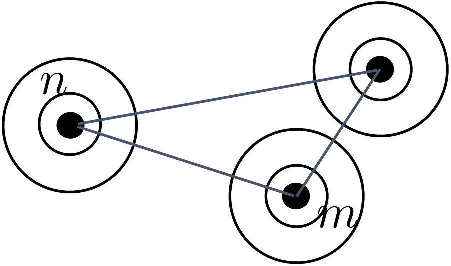
$$\mathcal{P}_{\text{fluc}}(\hbar, N) = \frac{\partial E^{\text{pert}}}{\partial N} \exp\left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3}\left(\frac{\partial E^{\text{pert}}(\hbar, N)}{\partial N} - \hbar + \frac{N\,\hbar^2}{S}\right)\right]$$

- note that $\left[E^{\text{pert}}\right]_{N=0} = 0$, but $\left[\frac{\partial E^{\text{pert}}}{\partial N}\right]_{N=0} \neq 0$
- unbroken SUSY: $E^{\text{non-pert.}}_{\text{g.s.}}(\hbar) = 0$, due to cancellations between two saddles

$\Rightarrow$ resurgence explains SUSY breaking or non-breaking

Connecting Perturbative and Non-Perturbative Sector

all orders of multi-instanton trans-series are encoded in perturbation theory of fluctuations about perturbative vacuum



why ? turn to path integrals again
... look for a semiclassical explanation

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{thimbles}\, k} \mathcal{N}_k\, e^{-\frac{i}{g^2}S_{\text{imag}}[A_k]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2}S_{\text{real}}[A]}$$

Lefschetz thimble = "functional steepest descents contour"

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ...

Stokes phenomenon: intersection numbers $\mathcal{N}_k$ can change with phase of parameters

Thimbles from Gradient Flow

gradient flow to generate steepest descent thimble:

$$\frac{\partial}{\partial \tau} A(x;\tau) = -\overline{\frac{\delta S}{\delta A(x;\tau)}}$$

- keeps $Im[S]$ constant, and $Re[S]$ is monotonic

$$\frac{\partial}{\partial \tau}\left(\frac{S-\bar{S}}{2i}\right) = -\frac{1}{2i}\int \left(\frac{\delta S}{\delta A}\frac{\partial A}{\partial \tau} - \overline{\frac{\delta S}{\delta A}}\,\overline{\frac{\partial A}{\partial \tau}}\right) = 0$$

$$\frac{\partial}{\partial \tau}\left(\frac{S+\bar{S}}{2}\right) = -\int \left|\frac{\delta S}{\delta A}\right|^2$$

- Chern-Simons theory (Witten 2010)
- comparison with complex Langevin (Aarts 2013, ...)
- lattice (Aurora, 2013; Tokyo/RIKEN): Bose-gas ✓
- generalized thimble method: (Alexandru, Başar, Bedaque et al, 2016)

- puzzle 1: how do approximate bion solutions yield exact SUSY answers?
- puzzle 2: how to explain SUSY breaking for DW semiclassically?
- puzzle 3: how to explain SUSY non-breaking for SG semiclassically?

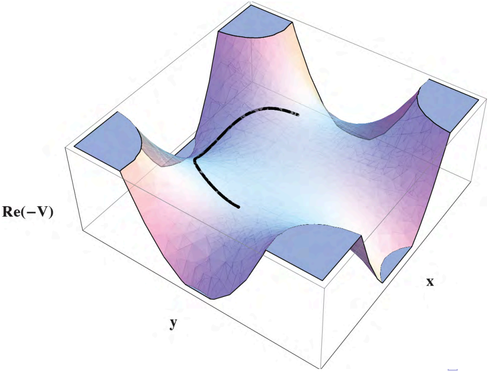
Complex Saddles in Path Integrals

- complex classical equations of motion

$$\frac{d^2 z}{dt^2} = \frac{\partial V}{\partial z} \qquad \text{or equivalently} \qquad \frac{d^2 x}{dt^2} = +\frac{\partial V_r}{\partial x}$$

$$\frac{d^2 y}{dt^2} = -\frac{\partial V_r}{\partial y}$$

- very different from 2d motion !

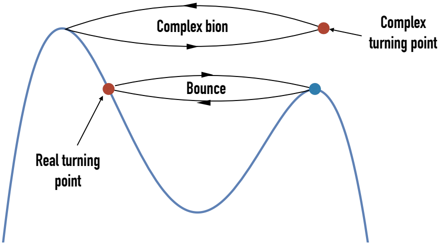


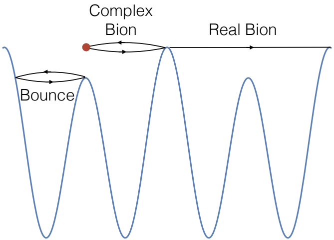
Complex Saddles in Path Integrals

- complex classical saddles from effective potential

$$V_{\text{eff}} = \left(W'\right)^2 \pm g\, W''$$

- arises from integrating out the fermions





Necessity of Complex Saddles

SUSY QM: $g\,\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\left(W'\right)^2 \pm \frac{g}{2}W''$

- complex saddles have complex action:

$$S_{\text{complex bion}} \sim 2S_I + i\pi$$

- $W = \cos\frac{x}{2} \to$ double Sine-Gordon

$$E_{\text{ground state}} \sim 0 - 2\,e^{-2S_I} - 2\,e^{-i\pi}e^{-2S_I} = 0 \qquad \checkmark$$

- $W = \frac{1}{3}x^3 - x \to$ tilted double-well

$$E_{\text{ground state}} \sim 0 - 2\,e^{-i\pi}e^{-2S_I} > 0 \qquad \checkmark$$

semiclassics $\Rightarrow$ complex saddles required for SUSY algebra

Resurgence, Trans-series and Non-perturbative Physics

1. Lecture 1: Basic Formalism of Trans-series and Resurgence
   - asymptotic series in physics; Borel summation
   - trans-series completions & resurgence
   - examples: linear and nonlinear ODEs
2. Lecture 2: Applications to Quantum Mechanics and QFT
   - instanton gas for double-well & periodic potential
   - infrared renormalon problem in QFT
   - from hyperasymptotics to Picard-Lefschetz thimbles
3. Lecture 3: Resurgence and Large N
   - parametric resurgence
   - Gross-Witten-Wadia Matrix Model
   - Mathieu equation and Nekrasov-Shatashvili limit of $\mathcal{N} = 2$ SUSY QFT

Connecting weak and strong coupling

physics question:

does weak coupling analysis contain enough information to extrapolate to strong coupling ?

…even if the degrees of freedom re-organize themselves in a very non-trivial way?

classical asymptotics is clearly not enough:

is resurgent asymptotics (= resurgent semiclassics) enough?

"Parametric Resurgence": Both N and g^2

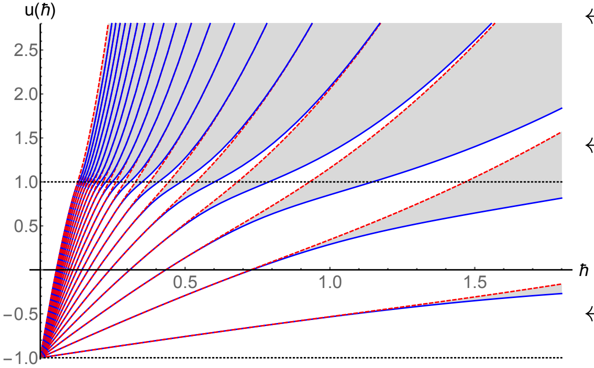
- trans-series expansion is a double-expansion: can be organized in different ways

$$
\begin{aligned}
F(N, g^2) &\sim \sum_n g^{2n} p_n^{(0)}(N) + e^{-\frac{S}{g^2}} \sum_n g^{2n} p_n^{(1)}(N) + \dots \\
&\sim \sum_k \frac{1}{g^{2k}} c_k(N) \quad + \quad ??? \\
&\sim \sum_h \frac{1}{N^{2h-2}} f_h^{(0)}(N\, g^2) + e^{-S\, N} \sum_h \frac{1}{N^{2h-2}} f_h^{(1)}(N\, g^2) + \dots
\end{aligned}
$$

- how does a divergent trans-series at weak coupling turn into a convergent series at strong-coupling?
- what happens to the resurgent structure?
- what about the 't Hooft limit? $N \to \infty; g^2 \to 0; N g^2 = t$
- separated by a phase transition: "instantons condense"

Resurgence in $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ Theories (Başar, GD, 1501.05671)

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$$



⟵ electric sector (convergent)

⟵ magnetic sector

⟵ dyonic sector (divergent)

- energy: $u = u(N, \hbar)$; 't Hooft coupling: $\lambda \equiv N\,\hbar$
- very different physics for $\lambda \gg 1$, $\lambda \sim 1$, $\lambda \ll 1$

Resurgence of $\mathcal{N}=2$ SUSY SU(2): Mathieu Eqn Spectrum

- moduli parameter: $u = \langle \mathrm{tr}\, \Phi^2 \rangle$
- electric: $u \gg 1$; magnetic: $u \sim 1$; dyonic: $u \sim -1$
- $a = \langle \text{scalar} \rangle$, $a_D = \langle \text{dual scalar} \rangle$, $a_D = \frac{\partial \mathcal{W}}{\partial a}$
- Nekrasov twisted superpotential $\mathcal{W}(a, \hbar, \Lambda)$:
- Mathieu equation: (Mironov/Morozov)

$$-\frac{\hbar^2}{2} \frac{d^2\psi}{dx^2} + \Lambda^2 \cos(x)\, \psi = u\, \psi \quad , \quad a \equiv \frac{N\hbar}{2}$$

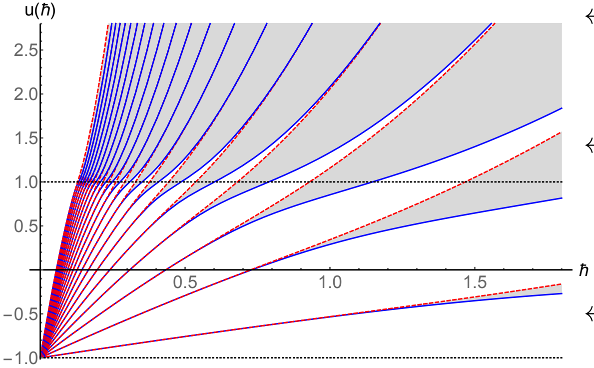
- (quantum) Matone relation:

$$u(a, \hbar) = \frac{i\pi}{2} \Lambda \frac{\partial \mathcal{W}(a, \hbar, \Lambda)}{\partial \Lambda} - \frac{\hbar^2}{48}$$

- $\mathcal{N} = 2^*$ $\leftrightarrow$ Lamé equation

Resurgence in $\mathcal{N}=2$ and $\mathcal{N}=2^*$ Theories (Başar, GD, 1501.05671)

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$



$\longleftarrow$ electric sector (convergent)

$\longleftarrow$ magnetic sector

$\longleftarrow$ dyonic sector (divergent)

- energy: $u=u(N,\hbar)$; 't Hooft coupling: $\lambda\equiv N\,\hbar$
- very different physics for $\lambda\gg 1$, $\lambda\sim 1$, $\lambda\ll 1$

Mathieu Equation Spectrum

$$-\frac{\hbar^2}{2}\,\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$

- small $\hbar$: divergent, non-Borel-summable $\rightarrow$ trans-series

$$u(N,\hbar) \quad \sim \quad -1+\hbar\left[N+\frac{1}{2}\right]-\frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2+\frac{1}{4}\right] -\frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3+\frac{3}{4}\left(N+\frac{1}{2}\right)\right]-\dots$$

- large $\hbar$: convergent expansion: $\longrightarrow$?? trans-series ??

$$u(N,\hbar)\sim\frac{\hbar^2}{8}\left(N^2+\frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4+\frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 +\frac{9N^4+58N^2+29}{64(N^2-1)^5(N^2-4)(N^2-9)}\left(\frac{2}{\hbar}\right)^{12}+\dots\right)$$

- note: poles in coefficients

Mathieu Equation Spectrum: far above the barrier

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$$

- narrow gaps high in the spectrum: complex instantons
- Dyhkne: same formula for band/gap splittings

$$\Delta u \sim \frac{2}{\pi}\frac{\partial u}{\partial N}\, e^{-\frac{2\pi}{\hbar}\mathrm{Im}\, a_0^D}$$

$$\begin{aligned}\Delta u_N^{\rm gap} \quad &\sim \quad \frac{\hbar^2}{4}\frac{1}{\left(2^{N-1}(N-1)!\right)^2}\left(\frac{2}{\hbar}\right)^{2N}\left[1+O\left(\left(\frac{2}{\hbar}\right)^4\right)\right] \\ &\sim \quad \frac{N\,\hbar^2}{2\pi}\left(\frac{e}{N\,\hbar}\right)^{2N} \qquad , \quad N \gg 1\end{aligned}$$

Keldysh Approach in QED

Brézin/Itzykson, 1970; Popov, 1971

- Schwinger effect in $E(t) = \mathcal{E}\cos(\omega t)$
- adiabaticity parameter: $\gamma \equiv \frac{m\,\omega}{\mathcal{E}}$
- WKB $\Rightarrow$ $P_{\rm QED} \sim \exp\left[-\pi\,\frac{m^2}{\hbar\,\mathcal{E}}\,g(\gamma)\right]$

$$P_{\rm QED} \sim \begin{cases} \exp\left[-\pi\,\frac{m^2}{\hbar\,\mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(non-perturbative)} \\ \left(\frac{\mathcal{E}}{\omega\,m}\right)^{4m/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(perturbative)} \end{cases}$$

- semi-classical instanton (saddle) interpolates between non-perturbative 'tunneling pair-production" and perturbative "multi-photon pair production"

- exact mapping $\Rightarrow$ physical interpretation of different non-pert expressions

$$\hbar \leftrightarrow \frac{4\omega^2}{\mathcal{E}} \quad ; \quad N \leftrightarrow \frac{m}{\omega} \quad ; \quad u = 1 + 2\gamma^2$$

Beyond Large N: Multi-instantons at strong coupling

$$u(N,\hbar) \sim \frac{\hbar^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 + \ldots\right.$$

- re-organize as a multi-instanton expansion

$$u_N^{(\pm)}(\hbar) = \frac{\hbar^2 N^2}{8}\sum_{n=0}^{N-1}\frac{\alpha_n(N)}{\hbar^{4n}} \pm \frac{\hbar^2}{8}\frac{1}{(2^{N-1}(N-1)!)^2}\left(\frac{2}{\hbar}\right)^{2N}\sum_{n=0}^{N-1}\frac{\beta_n(N)}{\hbar^{4n}} + \ldots$$

- fluctuation series are polynomials !
- 1-instanton gap splitting: (Basar, GD, Unsal, 2014)

$$\Delta u_N \equiv \frac{1}{(2^{N-1}(N-1)!)^2}\frac{\partial u}{\partial N}e^{A(N,\hbar)} \quad \Rightarrow \quad \frac{\partial A}{\partial \hbar^2} = -\frac{4}{\hbar^4}\frac{\partial u}{\partial N}$$

- 1-inst. flucts. determined by pert. expansion

- resurgent multi-instanton structure in convergent region

Resurgence in Matrix Models: Mariño: 0805.3033, Ahmed & GD: 1710.01812

Gross-Witten-Wadia Unitary Matrix Model

- resurgent Borel-Écalle analysis of partition functions, Wilson loops, etc ... in matrix models

$$Z(g^2, N) = \int_{U(N)} DU \, \exp\left[\frac{1}{g^2}\text{tr}\left(U + U^\dagger\right)\right]$$

- matrix model for 2d lattice Yang-Mills
- two variables: g^2 and N ('t Hooft coupling: $t \equiv g^2 N/2$)
- "parametric resurgence"
- 3rd order phase transition at $N = \infty$, $t = 1$ (universal!)
- double-scaling limit: Painlevé II
- 3rd order phase transition: condensation of instantons
- similar in 2d Yang-Mills on sphere and disc

GWW Phase Transition in 2d Gauge Theory

"... one can attempt to expand the partition function $Z(\epsilon)$ of two dimensional Yang-Mills in powers of the gauge coupling constant ϵ. In doing so (in a suitable topological sector), one finds a remarkable result: the perturbation series in ϵ stops after finitely many terms, yet $Z(\epsilon)$ is not a polynomial. $Z(\epsilon)$ contains exponentially small terms which can be identified as contributions of *unstable* classical solutions to the functional integral."

E. Witten (*Two Dimensional Gauge Theories Revisited*, 1992)

... resurgence approach to non-perturbative effects in large N

Gross-Witten-Wadia Model: Trans-series Structure

$$Z(g^2, N) = \int_{U(N)} DU \, \exp\left[\frac{1}{g^2} \mathrm{tr}\left(U + U^\dagger\right)\right]$$

- transseries structure: $\ln Z(g^2, N)$, as fn of both g^2 & N
- "parametric resurgence"

	Weak coupling	Strong coupling
Fixed N; expansion in coupling g^2	$g^2 \ll N$	$g^2 \gg N$
	• divergent (non-alternating)	• convergent
	• trans-series completion	• trans-series completion
	• imaginary trans-series parameter	• real trans-series parameter
Large N 't Hooft limit: $N \to \infty$, $t \equiv Ng^2/2$ fixed; expansion in $1/N$	$t \ll 1$	$t \gg 1$
	• divergent (non-alternating)	• divergent (alternating)
	• trans-series completion	• trans-series completion
	• imaginary trans-series parameter	• real trans-series parameter
Double-scaling limit: $N \to \infty$, $t \sim 1 + \kappa/N^{2/3}$; expansion in κ	$\kappa \leq 0$	$\kappa \geq 0$
	• divergent (non-alternating)	• divergent (alternating)
	• trans-series completion	• trans-series completion
	• imaginary trans-series parameter	• real trans-series parameter

Resurgence in Gross-Witten-Wadia Model

- partition function = $N \times N$ Toeplitz determinant

$$Z(g^2, N) = \det \left(I_{j-k}(x) \right)_{j,k=1,\dots N} \qquad , \quad x \equiv \frac{2}{g^2}$$

- so $Z(g^2, N)$ satisfies $(N+1)^{\text{th}}$ order **linear** ODE, $\forall N$

$\Rightarrow$ weak-coupling resurgent trans-series "guaranteed"

$$Z(x,N) \sim Z_0(x,N) \left[\sum_{n=0}^{\infty} \frac{a_n^{(0)}(N)}{x^n} + i \frac{(4x)^{N-1}}{\Gamma(N)} e^{-2x} \sum_{n=0}^{\infty} \frac{a_n^{(1)}(N)}{x^n} + \right.$$
$$\left. \dots + \frac{G(N+1)}{\prod_{i=0}^{N-1} \Gamma(N-i)} e^{-2Nx} \sum_{n=0}^{\infty} \frac{a_n^{(N)}(N)}{x^n} \right]$$

- but strong-coupling expansion is **convergent!**

$$Z(x,N) \sim e^{x^2/4} \left[1 - \left(\frac{(x/2)^{N+1}}{(N+1)!} \right)^2 \left(1 - \frac{1}{2} \frac{(N+1)\, x^2}{(N+2)^2} + \dots \right) + \dots \right]$$

- idea: map it to a Painlevé function (Painlevé III)

$$\Delta(x,N) \equiv \langle \det U \rangle = \frac{\det\left[I_{j-k+1}(x)\right]_{j,k=1,\ldots,N}}{\det\left[I_{j-k}(x)\right]_{j,k=1,\ldots,N}}$$

- for any N, $\Delta(x,N)$ satisfies a PIII-type equation:

$$\Delta'' + \frac{1}{x}\Delta' + \Delta\left(1-\Delta^2\right) + \frac{\Delta}{(1-\Delta^2)}\left[\left(\Delta'\right)^2 - \frac{N^2}{x^2}\right] = 0$$

$\Rightarrow$ generate trans-series solutions: weak- & strong-coupling

- N is a parameter ! $\Rightarrow$ large N limit by rescaling
- direct relation to the partition function:

$$\Delta^2(x,N) = 1 - \frac{Z(x,N-1)\,Z(x,N+1)}{Z^2(x,N)}$$

$$Z(x,N) = \exp\left[\frac{1}{2}\int_0^x x\,dx\left(1-\Delta^2(x,N)\right)\left(1+\Delta(x,N-1)\Delta(x,N+1)\right)\right]$$

Resurgence in Gross-Witten-Wadia Model

- weak-coupling expansion is a divergent series:
  $\rightarrow$ trans-series non-perturbative completion

- strong-coupling expansion is a convergent series:
  but it still has a non-perturbative completion !

- Δ small $\Rightarrow$ linearize $\rightarrow$ Bessel equation

$$\Delta'' + \frac{1}{x}\Delta' + \Delta\left(1-\Delta^2\right) + \frac{\Delta}{(1-\Delta^2)}\left[\left(\Delta'\right)^2 - \frac{N^2}{x^2}\right] = 0$$

$$\Rightarrow \Delta(x,N)\big]_{\text{strong}} \approx \sigma\, J_N(x)$$

- strong-coupling expansion $(x \equiv \frac{2}{g^2})$ is clearly convergent

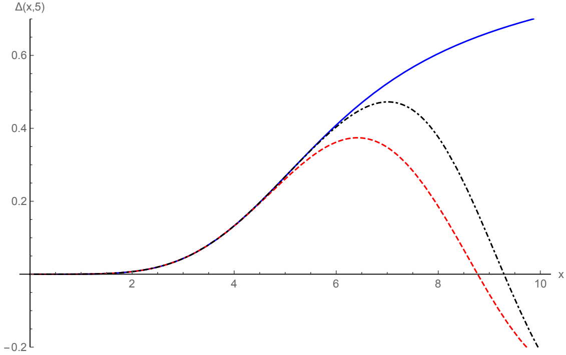
- but full solution is a non-perturbative trans-series:

$$\Delta(x,N) = \sum_{k=1,3,5,\dots}^{\infty} \left(\sigma_{\text{strong}}\right)^k \Delta_{(k)}(x,N)$$

- all higher terms are Bessel kernels with lower terms

- strong-coupling trans-series (convergent !!!):

$$\Delta(x,N) = \sum_{k=1,3,5,\dots}^{\infty} (\sigma_{\text{strong}})^k \Delta_{(k)}(x,N)$$

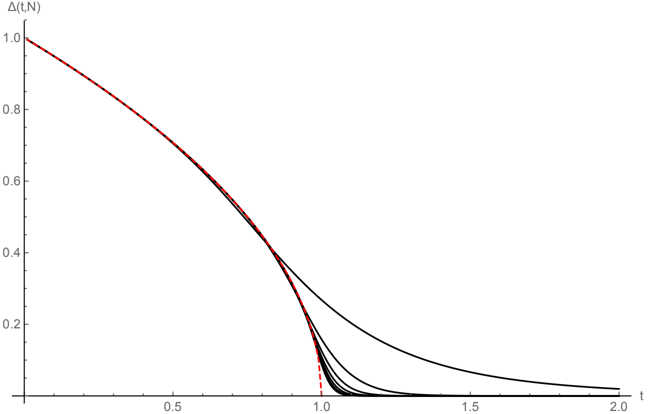


blue: exact , red: $\Delta_{(1)} = J_5(x)$, black: includes $\Delta_{(3)}$

Resurgence in GWW: 't Hooft limit and phase transition

- Gross-Witten-Wadia $N = \infty$ phase transition:

$$\Delta(t, N) \xrightarrow{N \to \infty} \begin{cases} 0 & , \quad t \geq 1 \quad \text{(strong coupling)} \\ \sqrt{1-t} & , \quad t \leq 1 \quad \text{(weak coupling)} \end{cases}$$



$t \equiv \frac{N}{x} \equiv \frac{N g^2}{2}$

black lines:
$N = 5, 25, 50, \ldots 150$

red dashed line:
$\Delta = \sqrt{1-t}$

Resurgence in GWW: 't Hooft limit and phase transition

- rescaled PIII equation: $t \equiv N g^2/2 \equiv \frac{N}{x}$

$$t^2\Delta'' + t\Delta' + \frac{N^2\Delta}{t^2}\left(1-\Delta^2\right) = \frac{\Delta}{1-\Delta^2}\left(N^2 - t^2\left(\Delta'\right)^2\right)$$

- GWW $N=\infty$ phase transition:

$$\Delta(t,N) \xrightarrow{N\to\infty} \begin{cases} 0 & , \quad t \geq 1 \quad \text{(strong coupling)} \\ \sqrt{1-t} & , \quad t \leq 1 \quad \text{(weak coupling)} \end{cases}$$

- large N at weak coupling:

$$\frac{\Delta}{t^2}\left(1-\Delta^2\right) = \frac{\Delta}{1-\Delta^2} \qquad \Rightarrow \qquad 1-\Delta^2 = t$$

Resurgence in GWW: 't Hooft limit and phase transition

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- large N at weak coupling:

$$\frac{\Delta}{t^2}\left(1-\Delta^2\right) = \frac{\Delta}{1-\Delta^2} \qquad \Rightarrow \qquad 1 - \Delta^2 = t$$

Resurgence in GWW: 't Hooft limit and phase transition

- full large N trans-series at weak-coupling:

$$\Delta(t,N) \sim \sqrt{1-t}\sum_{n=0}^{\infty}\frac{d_n^{(0)}(t)}{N^{2n}} - \frac{i}{2\sqrt{2\pi N}}\,\sigma_{\rm weak}\,\frac{t\,e^{-N S_{\rm weak}(t)}}{(1-t)^{1/4}}\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} + \dots$$

- large N weak-coupling action

$$S_{\rm weak}(t) = \frac{2\sqrt{1-t}}{t} - 2\,\text{arctanh}\left(\sqrt{1-t}\right)$$

- confirm (parametric!) resurgence relations, for all t:

$$\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{1}{N} + \dots$$

- large-order growth of perturbative coefficients ($\forall t<1$):

$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}}\frac{\Gamma(2n-\frac{5}{2})}{(S_{\rm weak}(t))^{2n-\frac{5}{2}}}\left[1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{S_{\rm weak}(t)}{(2n-\frac{7}{2})} + \dots\right.$$

Resurgence in GWW: 't Hooft limit and phase transition

- large N transseries at strong-coupling: $\Delta(t,N) \approx \sigma J_N\left(\frac{N}{t}\right)$

$$\Delta(t,N) = \sum_{k=1,3,5,\ldots}^{\infty} (\sigma_{\text{strong}})^k \Delta_{(k)}(t,N)$$

- Debye expansion for Bessel function: $J_N(N/t)$

$$\Delta(t,N) \sim \frac{\sqrt{t}\, e^{-N S_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n} + \frac{1}{4(t^2-1)} \left(\frac{\sqrt{t} e^{-N S_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \right)^3 \sum_{n=0}^{\infty} \frac{U_n^{(1)}(t)}{N^n} + \ldots$$

- large N strong-coupling action:
$S_{\text{strong}}(t) = \text{arccosh(t)} - \sqrt{1-1/t^2}$

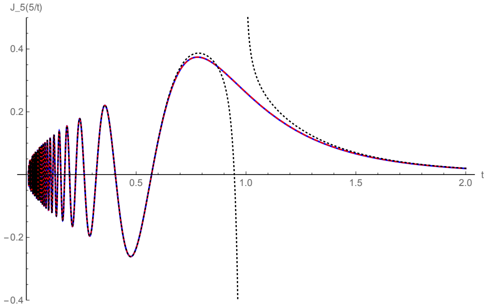
- large-order/low-order (parametric) resurgence relations:

$$U_n(t) \sim \frac{(-1)^n (n-1)!}{2\pi (2S_{\text{strong}}(t))^n} \left(1 + U_1(t) \frac{(2S_{\text{strong}}(t))}{(n-1)} + U_2(t) \frac{(2S_{\text{strong}}(t))^2}{(n-1)(n-2)} + \right.$$

Resurgence in GWW: 't Hooft limit and phase transition

- Debye expansion has unphysical divergence at $t = 1$
- uniform asymptotic expansion:

$$J_N\left(\frac{N}{t}\right) \sim \left(\frac{4\left(\frac{3}{2}S_{\rm strong}(t)\right)^{2/3}}{1-1/t^2}\right)^{\frac{1}{4}} \frac{{\rm Ai}\left(N^{\frac{2}{3}}\left(\frac{3}{2}S_{\rm strong}(t)\right)^{2/3}\right)}{N^{\frac{1}{3}}}$$



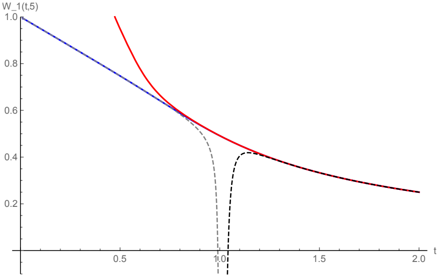
- nonlinear analogue of uniform WKB (coalescing saddles)

- Wilson loop: $\mathcal{W} \equiv \frac{1}{N}\frac{\partial \ln Z}{\partial x}$

$$\mathcal{W}(t,N) = \frac{1}{2t}\left(1-\Delta^2(t,N)\right)\left(1+\Delta(t,N-1)\Delta(t,N+1)\right)$$

- uniform large N approximation at strong-coupling:

$$\mathcal{W}(t,N)\Big|^{\text{strong}} \approx \frac{1}{2t}\left(1-J_N^2(N/t)\right)\left(1+J_{N-1}(N/t)J_{N+1}(N/t)\right)$$



blue: exact
red: uniform large N
dashed: usual large N

uniform resummation of instantons & fluctuations

Resurgence in GWW: 't Hooft limit and phase transition

- uniform asymptotic expansion:

$$\Delta_{\rm strong}(N,t) \sim \left(\frac{4\left(\frac{3}{2}S_{\rm strong}(t)\right)^{2/3}}{1-1/t^2}\right)^{\frac{1}{4}} \frac{\mathrm{Ai}\left(N^{\frac{2}{3}}\left(\frac{3}{2}S_{\rm strong}(t)\right)^{2/3}\right)}{N^{\frac{1}{3}}}$$

- physical meaning of "uniform large-N instantons" ?
- nonlinear anaolgue of "uniform WKB"
- technically: coalescence of two saddles $\longrightarrow$ "bion"
- expect similar phenomena in QFT

Resurgence in GWW: double-scaling limit = Painlevé II

- reduction cascade of Painlevé equations
- "zoom in" on vicinity of phase transition:

$$\kappa \equiv N^{2/3}(t-1) \quad ; \quad \Delta(t,N) = \frac{t^{1/3}}{N^{1/3}} y(\kappa)$$

- $N \to \infty$ with κ fixed:

$$\Delta \quad \text{PIII equation} \quad \longrightarrow \quad \frac{d^2 y}{d\kappa^2} = 2\, y^3(\kappa) + 2\kappa\, y(\kappa) \quad \text{(PII)}$$

- e.g. on strong-coupling side:

$$\lim_{N\to\infty} J_N(N - N^{1/3}\kappa) = \left(\frac{2}{N}\right)^{1/3} \text{Ai}\left(2^{1/3}\kappa\right)$$

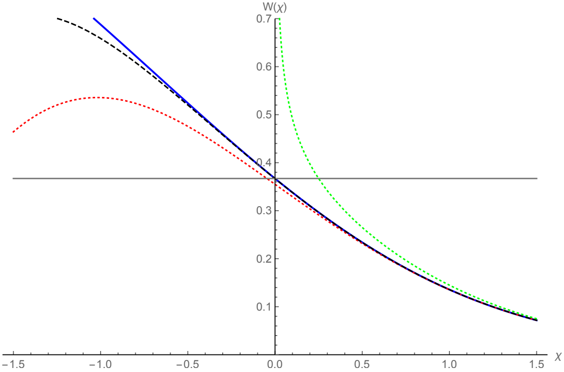
- integral equation form of PII:

$$y(\chi) = \sigma\, \text{Ai}(\chi) + 2\pi \int_\chi^\infty \left[\text{Ai}(\chi)\text{Bi}(\chi') - \text{Ai}(\chi')\text{Bi}(\chi)\right] y^3(\chi')\, d\chi'$$

Resurgence in GWW: double-scaling limit = Painlevé II

- "zoom in" on vicinity of phase transition:
- integral equation form of PII:

$$y(\chi) = \sigma\,\mathrm{Ai}(\chi) + 2\pi \int_{\chi}^{\infty} \left[\mathrm{Ai}(\chi)\mathrm{Bi}(\chi') - \mathrm{Ai}(\chi')\mathrm{Bi}(\chi)\right] y^3(\chi')\, d\chi'$$



iterate $\longrightarrow$ resummed trans-series instanton expansion

blue: exact

red: leading uniform large N

dashed: sub-leading uniform large N

green dashed: usual large N

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative analysis, via trans-series
- trans-series 'encode' analytic continuation information
- expansions about different saddles are intimately related
- there is extra un-tapped 'magic' in perturbation theory
- QM, matrix models, large N, strings, SUSY QFT
- IR renormalon puzzle in asymptotically free QFT
- multi-instanton physics from perturbation theory
- $\mathcal{N}=2$ and $\mathcal{N}=2^*$ SUSY gauge theory
- appliactions to sign problem and non-equil. path integrals
- moral: go complex and consider all saddles, not just minima

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