Diffusive and superdiffusive energy transport in chain of oscillators with conservative noise.

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A model with different macroscopic space-time scales

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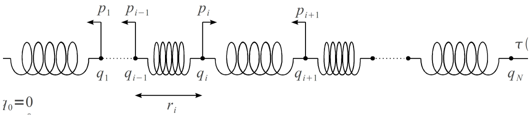
It is expected that generically for one dimensional systems with momentum conserving dynamics the energy transport (in mechanical equilibrium) is superdiffusive. We will see that this is not always true and that non-vanishing sound velocity is an essential ingredient.

Chain of oscillators with tension



Hamiltonian unpinned dynamics o FPU type: $j \in \mathbb{Z}$,

$$H = \sum_j \frac{p_j^2}{2} + V(\nabla q_j) = \sum_j \frac{p_j^2}{2} + V(r_j)$$

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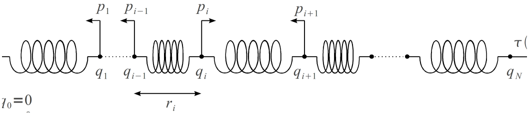

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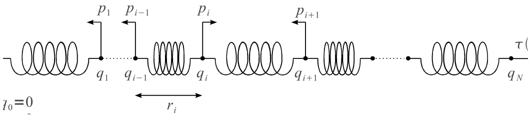

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we add a noise that conserve energy and momentum:

momentum exchange For each couple of nearest neighbor particle, we randomly exchange momentum, $(p_i, p_{i+1}) \to (p_{i+1}, p_i)$, with intensity 1.

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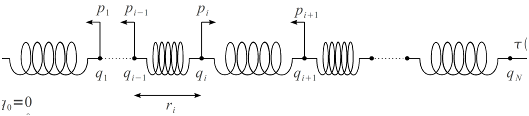

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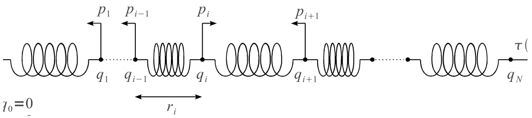

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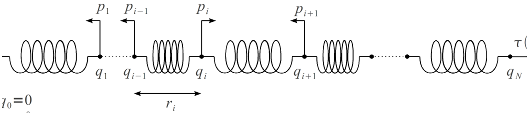

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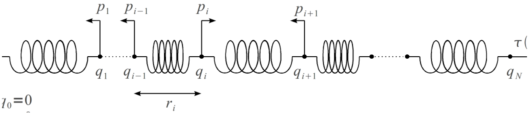

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$$\mathcal{E}_j = \frac{p_j^2}{2} + V(r_j) \qquad \text{energy of particle } j$$

Gibbs measure at temperature β^{-1}, tension τ and momentum p are:

$$d\mu_{\beta,\tau,p} = \prod_j e^{-\beta(\mathcal{E}_j - pp_j - \tau r_j) - \mathcal{G}(\beta,\tau,p)} dp_j dr_j$$

Gibbs measures and Thermodynamic Entropy

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Thermodynamic entropy is

$$S(u,r) = \inf_{\tau,\beta} \{-\beta\tau r + \beta u - \mathcal{G}(\beta,\tau,0)\}$$

$$\beta(u,r) = \partial_u S(u,r), \qquad \tau(u,r) = -\beta^{-1}\partial_r S(u,r).$$

Ergodicity (of the infinite system)

The stocastic perturbation of the dynamics is sufficient to give ergodicity:

Theorem

(Fritz, Funaki, Lebowitz, PTRF 1994)

Assume that ν is translation invariant and stationary, with finite entropy density, and furthemore that

$$\nu(dp|r)$$

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- $\nu(dp|r)$ maxwellian (Gallavotti-Verboven 1975)
- $\nu(dp|r)$ convex combination of maxwellians (Olla, Varadhan, Yau, 1993).

Hyperbolic Scaling: Euler equations

3 conserved quantities:

volume stretch $\mathcal{R}_N(t)[G] = \frac{1}{N}\sum_i G(i/N) r_i(Nt)$

momentum $\pi_N(t)[G] = \frac{1}{N}\sum_i G(i/N) p_i(Nt)$

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$$(\mathcal{R}_N(t), \pi_N(t), \mathfrak{e}_N(t)) \longrightarrow (r(x,t)dx, \pi(x,t)dx, \mathfrak{e}(x,t)dx)$$

$$\boxed{\begin{array}{ccc} \partial_t r = \partial_x \pi & \partial_t \pi = \partial_x \tau & \partial_t \mathfrak{e} = \partial_x(\tau\pi) \\ \pi(0,t) = 0, & \multicolumn{2}{c}{\tau(r(1,t), U(1,t)) = \tau_1(t)} \end{array}}$$

$U = \mathfrak{e} - \pi^2/2$: internal energy. For smooth solutions this is proven in:

- N. Even, S.O., ARMA (2014)
- S.O., SRS Varadhan, HT Yau, CMP (1993)

entropy evolution

$$\partial_t r = \partial_x \pi \qquad \partial_t \pi = \partial_x \tau \qquad \partial_t \mathfrak{e} = \partial_x (\tau \pi)$$
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When shocks will appear, we will have dissipation and

$$\frac{d}{dt}\int_0^1 S(r(x,t), U(x,t))dx > 0$$

and (eventually) *convergence* to a **mechanical** equilibrium, characterized by constant tension and momentum.

Mechanical Equilibrium

In particular starting with smooth initial profiles such that

$$p_0(x) = 0, \qquad \tau(u_0(x), r_0(x)) = \tau_0 = \text{cost}$$

$u_0(x) = e_0(x) - \frac{p^2(x)}{2} = e_0(x)$
and non constant entropy profile $S(u_0(x), r_o(x))$
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This means that they evolve towards *thermal equilibrium* at a larger time scale:

- diffusive scaling $(N^2 t, Nx)$, if thermal conductivity is finite, with heat equation governing the evolution of the temperature profile,
- superdiffusive scaling $(N^\alpha t, Nx), 1 < \alpha < 2$, if thermal conductivity is infinite.

If $\partial_r \tau(u, r) > 0$ (acoustic chains), we expect superdiffusion of heat, confirmed by many numerical experiments.

Energy Superdiffusion

Consider the Harmonic chain with noise conserving energy and momentum.

$$V(r) = \frac{r^2}{2}, \qquad \tau(r, U) = r, \qquad S = 1 + \log(\pi\beta^{-1})$$

$$\mathfrak{e}(x,0) = \frac{1}{2} r(x,0)^2 + \frac{1}{2}\pi(x,0)^2 + \beta^{-1}(x,0)$$

In the hyperbolic space-time scale limit we have

$$\boxed{\partial_t r = \partial_x \pi \qquad \partial_t \pi = \partial_x r \qquad \partial_t \mathfrak{e} = \partial_x (r\pi)}$$

linear wave equation: no shocks! no dissipation!
Profiles of **temperature** and **entropy** do not change in time:

$$\beta^{-1}(x,t) = \mathfrak{e}(x,t) - \frac{1}{2} r(x,t)^2 - \frac{1}{2}\pi(x,t)^2 = \beta^{-1}(x,0)$$

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Since microscopically the noise made the dynamic 'ergodic', the convergence to equilibrium should happen in another space-time scale.

Temperature profile

In the infinite space case, in the hyperbolic scale, after $t \to \infty$, all the phonon modes go to infinity, and the *phonon energy*

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This temperature profile will start to evolve at a larger time scale, supediffusive, shorter than the diffusive one.

What superdiffusion?

Assume $V(r) = \frac{\alpha r^2}{2}$, and centered initial conditions:

$$< p_j(0) >= 0, \qquad < r_j(0) >= 0$$

This is maintained at any positive time t.

Theorem (Jara-Komorowski-Olla, CMP 2015)

Assume $\frac{1}{N}\sum_x \langle \mathcal{E}_i(0) \rangle < C$, $G \in \mathcal{C}_0(\mathbb{R})$:

$$\frac{1}{N}\sum_i G(i/N)\left\langle \mathcal{E}_i(N^{3/2}t)\right\rangle \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} G(x)\; T(t,x)dx$$

$$\boxed{\partial_t T(x,t) \; = \; -\hat{c}|\Delta_x|^{3/4} T(x,t)} \qquad \hat{c} = \frac{\alpha^{3/4}}{2^{9/4}(3\gamma)^{1/2}}$$

$T(x,0) = \beta^{-1}(x,0)$.

In pinned chains, we prove usual diffusive behavior with heat equation.

Phonon modes: diffusive scale

$$\mathfrak{f}_i^{\pm}(t) = p_i(t) \pm \alpha^{1/2}\left(r_i(t) \pm \frac{3\gamma - 1}{2}(r_{i+1} - r_i)\right)$$

- Ballistic movement on the hyperbolic scale:

$$\frac{1}{N}\sum_i G(i/N)\, \mathfrak{f}_i^{\pm}(N^1 t) \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} G(x)\, \bar{\mathfrak{f}}^{\pm}(0, x \pm \alpha^{1/2} t) dx$$

- Then diffusive after re-centering

$$\frac{1}{N}\sum_i G(N^{-1} i \mp \alpha^{1/2} N^1 t)\, \mathfrak{f}_i^{\pm}(N^2 t) \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} G(x)\, \bar{\mathfrak{f}}^{\pm,d}(t, x) dx$$

$$\partial_t \bar{\mathfrak{f}}^{\pm,d} = \frac{3\gamma}{2}\, \partial_x^2 \bar{\mathfrak{f}}^{\pm,d}$$

Conjectured result for FPUT dynamics

This behavior at large space-time scale (diffusive for phonon and superdiffusive for heat mode) is also conjectured for the *equilibrium fluctuations* dynamics for the Fermi-Pasta-Ulam-Tsingou model

$$V(r) = r^2 + \alpha r^3 + \beta r^4$$

Recent *fluctuation hydrodynamics and mode coupling* calculation by Herbert Spohn (JSP 2014) for the deterministic FPU dynamics:

- If V symmetric ($\alpha = 0$ or β-FPU), and $\tau = 0$: then diffusion for phonons and $|\Delta|^{3/4}$-superdiffusion for heat (temperature),
- If V asymmetric (α-FPU), or symmetric but tension $\tau \neq 0$, then phonon KPZ-superdiffuse, and $|\Delta|^{5/6}$-superdiffusion for heat.

A simpler model: Bernardin's dynamics

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$$\eta_{2j} = r_j, \qquad \eta_{2j+1} = p_j$$

$$\frac{d}{dt}\eta_j = \eta_{j+1} - \eta_{j-1}$$

and random exchanges $\eta_j \leftrightarrow \eta_{j+1}$.
$\sum_j \eta_j, \ \sum_j \eta_i^2$ are the conserved quantities.
Bernardin-Goncalvez-Jara, ARMA 2015: fractional equation for the energy fluctuations in equilibrium.

Non-acoustic chains

These are *tensionless* chains, obtained by considering also non-nearest neighbor interactions, for example:

$$\mathcal{H} = \sum_j \left[\frac{p_j^2}{2} + V(\Delta q_j) \right], \qquad \Delta q_j = q_{j+1} + q_{j-1} - 2q_j$$

Here the relevant balanced quantity is not the volume strain $r_j = q_j - q_{j-1}$, but the *curvature* $\kappa_j = \Delta q_j$ (*Beam problem*). Equilibrium measures are parametrized by temperature β^{-1}, momentum p and bending stress τ_2:

$$d\mu_{\beta,\tau,p} = \prod_j e^{-\beta(\mathcal{E}_j - pp_j - \tau_2 \kappa_j) - \mathcal{G}(\beta,\tau,p)} dp_j d\kappa_j$$

$$\mathcal{E}_j = \frac{p_j^2}{2} + V(\kappa_j)$$

Non-acoustic chains: heat equation

For the harmonic dynamics with random moment exchange we obtain:

- Thermal conductivity is finite! (This provides a first rigorous counterexample for a one dimensional system with momentum conservation).

Starting with initial conditions with 0-curvature and a non-constant temperature profile we have (T. Komorowski, S.O., 2015):

$$\frac{1}{N}\sum_i G(i/N)\left\langle \mathcal{E}_i(N^2 t)\right\rangle \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} G(x)\ T(t,x)dx$$

$$\boxed{\partial_t T(x,t) = D\Delta_x T(x,t)}$$

Non-acoustic chains: mechanical and thermal non-equilibrium

With a non constant initial profiles of curvature and momentum and temperature, everything evolves on the diffusive scale

$$\left(\kappa_{[Nx]}(N^2 t), p_{[Nx]}(N^2 t), e_{[Nx]}(N^2 t)\right) \rightharpoonup (\kappa(x,t), p(x,t), e(x,t))$$

$$T(x,t) = e(x,t) - \frac{\kappa(x,t)^2 + p(x,t)^2}{2}$$

and the macroscopic equations are the *damped Euler-Bernoulli beam equations*:

$$\partial_t \kappa(x,t) = -\partial_x^2 p(x,t), \qquad \partial_t p(x,t) = \frac{1}{4}\partial_x^2 \kappa(x,t) + \gamma \partial_x^2 p(x,t)$$

$$\partial_t T(x,t) = D_\gamma \partial_x^2 T(x,t) + \frac{\gamma}{2}\left[(\partial_x p)^2 - p\partial_x^2 p\right]$$

Damped Euler-Bernoulli beam equation

$$
\begin{aligned}
\partial_t^2 \kappa(x,t) &= -\frac{1}{4}\partial_x^4 \kappa(x,t) - \gamma \partial_x^2 p(x,t), \\
\partial_t T(x,t) &= D_\gamma \partial_x^2 T(x,t) + \frac{\gamma}{2}\left[(\partial_x p)^2 - p\partial_x^2 p\right] \\
p(x,t) &= \partial_t \kappa(x,t)
\end{aligned}
$$

Further results and work in progress:

- acoustic chains in two dimensions: logarithmic (super)-diffusion, normal heat equation.
- Finite system in contact with Langevin heat bath at different temperature: proper definition of the fractional laplacian in finite volume with boundary conditions.

Wave function and Wigner distribution

Proofs of these results do not use relative entropy but L^2 norms in Fourier coordinates.

Wave function and Wigner distribution

$r_y = q_y - q_{y-1}$, $y \in \mathbb{Z}$:

$$\begin{aligned} \dot{q}_x(t) &= p_x(t) \\ dp_x(t) &= -(\alpha * q(t))_x dt \; + \gamma^{1/2} \sum_{\ell=-1,0,1} Y_{x+\ell} p_x(t) \circ dw_{x+k}(t), \end{aligned}$$

α_x symmetric, compact support or $|\alpha_x| \le Ce^{-|x|/C}$, $\hat{\alpha}(k) > 0$ and

- $\hat{\alpha}(0) = 0$ unpinned chain
- $\hat{\alpha}''(0) > 0$ acustic chain

Dispersion function:

$$\omega(k) = \hat{\alpha}(k)^{1/2} \qquad (= 2|\sin(\pi k)|), \qquad k \in \mathbb{T}.$$

This implies

$$\int_{|k|\le\delta} \frac{\omega'(k)^2}{k^2} dk = +\infty$$

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Wave function

$$\hat{f}(k) = \sum_{y \in \mathbb{Z}} f(y) e^{-i2\pi yk}$$

Complex wave function in Fourier space:

$$\boxed{\hat{\psi}(t,k) := \omega(k)\hat{q}(t,k) + i\hat{p}(t,k)}$$

Wave function

$$\hat{f}(k) = \sum_{y \in \mathbb{Z}} f(y) e^{-i2\pi yk}$$

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$$\begin{aligned} d\hat{\psi}(t,k) :=& - i\omega(k)\hat{\psi}(t,k)dt - \gamma\frac{\hat{\beta}(k)}{4}[\hat{\psi}(t,k) - \hat{\psi}(t,-k)^*]dt \\ &+ \gamma^{1/2} i \int_{\mathbb{T}} r(k,k')[\hat{\psi}(k-k') - \hat{\psi}(-k+k')^*]dw(t,k') \end{aligned}$$

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$$< dw^*(t,k) dw(s,k') >= \delta(k-k')\delta(t-s) dt\, dk, \quad < dw(t,k) dw(s,k') >= 0$$

$$r(k,k') = 2r_1^2(k) r_1(2(k-k')) + 2r_1(2k) r_1^2(k-k'), \qquad r_1(k) = \sin(\pi k)$$

$$\hat{\beta}(k) = 8r_1^2(k) + 4r_1^2(2k) \sim k^2$$

Wigner distribution

$$\widehat{W}_\epsilon(\eta, k, t) = \langle \psi^*(k - \epsilon\eta, t)\psi(k + \epsilon\eta, t)\rangle$$

$$W_\epsilon(y, k, t) = \epsilon \int_{(\mathbb{T}/\epsilon)} e^{i2\pi y\eta}\widehat{W}_\epsilon(\eta, k, t)\, d\eta$$

This is different from the energy $< \mathcal{E}_{[\epsilon^{-1}y]} >$, but can be proven that

$$W_\epsilon(y, k, t) - < \mathcal{E}_{[\epsilon^{-1}y]}(t) > \underset{\epsilon \to 0}{\longrightarrow} 0.$$

Start with initial distribution such that $\sup_\epsilon \epsilon < \|\phi(0)\|^2 > \leq C$, and

$$\bar{W}_\epsilon(\eta, 0) = \int \widehat{W}_\epsilon(\eta, k, 0)\, dk \longrightarrow W_0(\eta)$$

Theorem

$$\widehat{W}_\epsilon(\eta, k, \epsilon^{-3/2} t) \underset{\epsilon \to 0}{\longrightarrow} e^{-\hat{c} t |\eta|^{3/2}} W_0(\eta)$$

with $\hat{c} = \frac{\hat{\alpha}''(0)^{3/4}}{2^{9/4}(3\gamma)^{1/2}}$.

previous results

- *Basile, O. Spohn, ARMA 2009:* kinetic limit/weak noise: $\gamma = \epsilon\gamma'$, $\delta = 1$, convergence to a linear Boltzmann equation

$$\partial_t \widehat{W}(\eta,k,t) + i\eta\omega'(k)W(\eta,k,t) = \gamma\mathcal{L}\widehat{W}(\eta,k,t)$$

with the scattering operator

$$\mathcal{L}\widehat{W}(\eta,k,t) = \int R(k,k')(\widehat{W}(\eta,k',t) - \widehat{W}(\eta,k,t))$$

$$R(k,k') = R(k',k) = r^2(k,k-k') + r^2(k,k+k')$$

$$= \frac{3}{4}\sum_{i=\pm 1} R_i(k)R_{-i}(k') \quad \text{for the continous noise}$$

$$= \frac{9}{4}R(k)R(k) \quad \text{for the velocity exchange noise}$$

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$$\lambda X(\lambda^{-3/2} t) \;=\; \lambda \int_0^{\lambda^{-3/2} t} \omega'(K(s))\, ds \underset{\lambda \to 0}{\longrightarrow} \frac{3}{2} - \textit{stable Levy}$$

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- *Cedric Bernardin, Milton Jara, Patricia Goncalves, 2014*: 2 conserved quantity model, equilibrium fluctuations.

Time evolution of the Wigner distribution

Anti-Wigner distribution:

$$\widehat{Z}_\epsilon(\eta, k, t) = \langle \psi(k - \epsilon\eta, t)\psi(k + \epsilon\eta, t)\rangle$$

for small ϵ, $\delta = 2/3$,

$$\begin{aligned}
\partial_t \widehat{W}_\epsilon(\eta, k, \epsilon^{-\delta} t) &= -i\epsilon^{-\delta+1}\omega'(k)\eta\widehat{W}_\epsilon + \gamma\epsilon^{-\delta}\mathcal{L}\widehat{W}_\epsilon \\
&\quad - \gamma\epsilon^{-\delta}\mathcal{L}\left[\widehat{Z}_\epsilon(\eta, k, t) + \widehat{Z}_\epsilon(-\eta, k, t)\right] \\
&\quad - \gamma\epsilon^{2-\delta}\eta^2\widetilde{\mathcal{L}}\,\widehat{W}_\epsilon + o(\epsilon) \\
\partial_t \widehat{Z}_\epsilon(\eta, k, \epsilon^{-\delta} t) &= \dots
\end{aligned}$$

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\partial_t \widehat{Z}_\epsilon(\eta, k, \epsilon^{-\delta} t) &= \dots
\end{aligned}$$

- Because of fast time fluctuations $\widehat{Z}_\epsilon(\eta, k, \epsilon^{-\delta} t) \longrightarrow 0$.
- By averaging for many collisions $\widehat{W}_\epsilon(\eta, k, \epsilon^{-\delta} t) \longrightarrow \bar{W}(\eta, t)$, constant in k.

Let's drop $\widehat{Z}_\epsilon = 0$ and smaller terms in ϵ, we are left

$$\partial_t \widehat{W}_\epsilon(\eta, k, \epsilon^{-\delta} t) = -i\epsilon^{-\delta+1}\omega'(k)\eta\widehat{W}_\epsilon + \gamma\epsilon^{-\delta}\mathcal{L}\widehat{W}_\epsilon$$

$$\mathcal{L}\widehat{W}_\epsilon(\eta, k, t) = \int R(k, k')(\widehat{W}(\eta, k', t) - \widehat{W}_\epsilon(\eta, k, t))dk'$$

$$R(k, k') = \frac{3}{4}\sum_{i=\pm 1} R_i(k)R_{-i}(k')$$

To simplify the argument assume the simpler scattering rate

$$R(k, k') = R(k)R(k'), \qquad \int R(k)dk = 1, \qquad R(k) \sim k^2$$

$$\mathcal{L}\widehat{W}_\epsilon(\eta, k, t) = R(k)\int R(k')\widehat{W}_\epsilon(\eta, k', t)dk - R(k)\widehat{W}_\epsilon(\eta, k, t)$$

$$\equiv R(k) < R, \widehat{W}_\epsilon > -R(k)\widehat{W}_\epsilon(\eta, k, t)$$

Taking Laplace Transform $\hat{w}_\epsilon(\eta, k, \lambda) = \int_0^\infty e^{-\lambda t} \widehat{W}_\epsilon(\eta, k, \epsilon^{-\delta} t) dt$, and denoting $< R, \hat{w}_\epsilon >= \int R(k') \hat{w}_\epsilon(\eta, k', \lambda) dk'$

$$\epsilon^\delta \lambda \hat{w}_\epsilon + i\epsilon\omega'(k)\eta \hat{w}_\epsilon - R(k) < R, \hat{w}_\epsilon > + R(k)\hat{w}_\epsilon = \epsilon^\delta W_0(\eta, k)$$

Taking Laplace Transform $\hat{w}_\epsilon(\eta,k,\lambda) = \int_0^\infty e^{-\lambda t}\widehat{W}_\epsilon(\eta,k,\epsilon^{-\delta}t)dt$, and denoting $< R, \hat{w}_\epsilon >= \int R(k')\hat{w}_\epsilon(\eta,k',\lambda)dk'$

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rearranging

$$w_\epsilon(\eta,k,\lambda) = \frac{\epsilon^\delta W_0(\eta,k) + R(k) < R, \hat{w}_\epsilon >}{\epsilon^\delta\lambda + R(k) + i\epsilon\omega'(k)\eta}$$

Multiplying by $R(k)$ and integrating in k, after rearrangement:

$$< R, \hat{w}_\epsilon > \epsilon^{-\delta}\left(1 - \int \frac{R(k)^2}{\epsilon^\delta\lambda + R(k) + i\epsilon\omega'(k)\eta}dk\right) = \int \frac{R(k)W_0(\eta,k)}{\epsilon^\delta\lambda + R(k) + i\epsilon\omega'(k)\eta}dk$$

$$< R, \hat{w}_\epsilon > \epsilon^{-\delta} \left(1 - \int \frac{R(k)^2}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk \right)$$

$$= \int \frac{R(k)\widehat{W}_0(\eta, k)}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk$$

$$< R, \hat{w}_\epsilon > \epsilon^{-\delta}\left(1 - \int \frac{R(k)^2}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk\right)$$

$$= \int \frac{R(k)\widehat{W_0}(\eta, k)}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk$$

$$\int \frac{R(k)W_0(\eta, k)}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk \longrightarrow \int \widehat{W_0}(\eta, k) dk = \bar{W}_0(\eta)$$

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$$\epsilon^{-\delta}\left(1 - \int \frac{R(k)^2}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk\right) \longrightarrow \lambda + \hat{c}\eta^{3/2}$$

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$$< R, \hat{w}_\epsilon >= \int R(k)\hat{w}_\epsilon(\eta, k, \lambda) dk \longrightarrow w(\eta, \lambda).$$

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$$\epsilon^{-\delta}\left(1 - \int \frac{R(k)^2}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk\right) \longrightarrow \lambda + \hat{c}\eta^{3/2}$$

$$< R, \hat{w}_\epsilon >= \int R(k)\hat{w}_\epsilon(\eta, k, \lambda) dk \longrightarrow w(\eta, \lambda).$$

$$\Longrightarrow \quad \left(\lambda + \hat{c}\eta^{3/2}\right) w(\eta, \lambda) = \bar{W}_0(\eta) \qquad \square$$