Finite temperature free fermions and the Kardar-Parisi-Zhang equation at finite time

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In collaboration with

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- Pierre Le Doussal (LPT ENS, Paris)
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Acknowledgements to Christophe Salomon (LKB, ENS Paris)

Ultra-cold atoms in confining potentials

- Recent progress in the experimental manipulation of cold atoms

→ to investigate the interplay between quantum and thermal behaviors in many-body systems at low temperature

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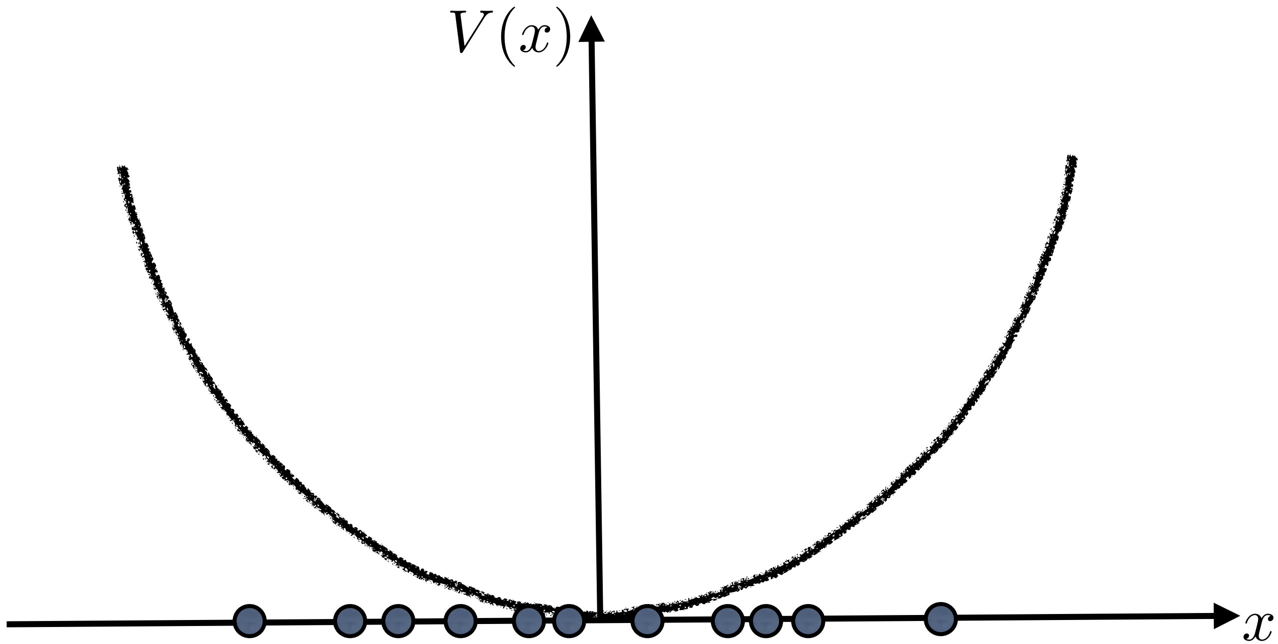
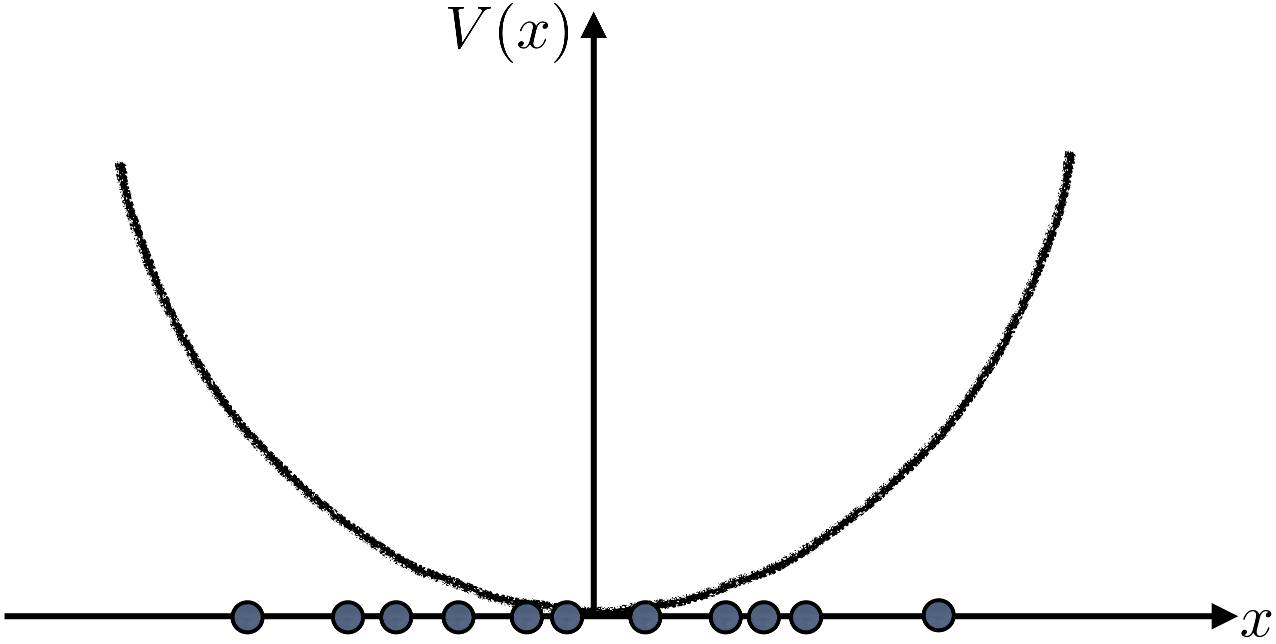
- Recent progress in the experimental manipulation of cold atoms

→ to investigate the interplay between quantum and thermal behaviors in many-body systems at low temperature

- A common feature of these experiments: presence of a confining potential that traps the atoms within a limited spatial region



Spinless free fermions in a 1d harmonic potential



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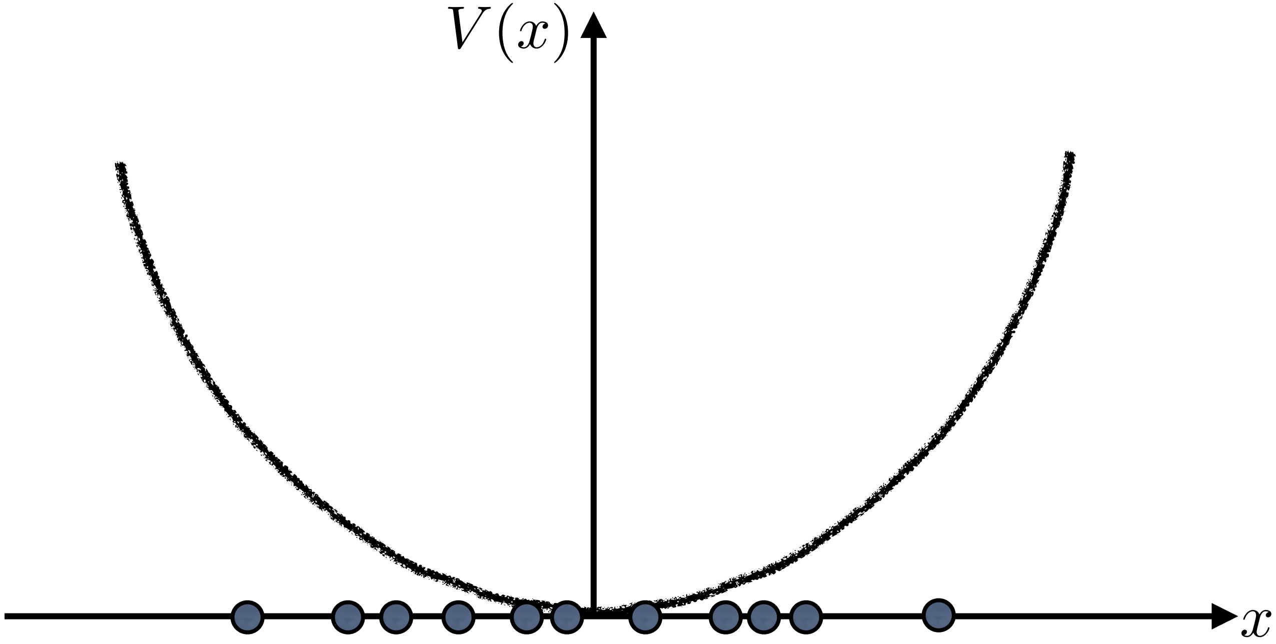

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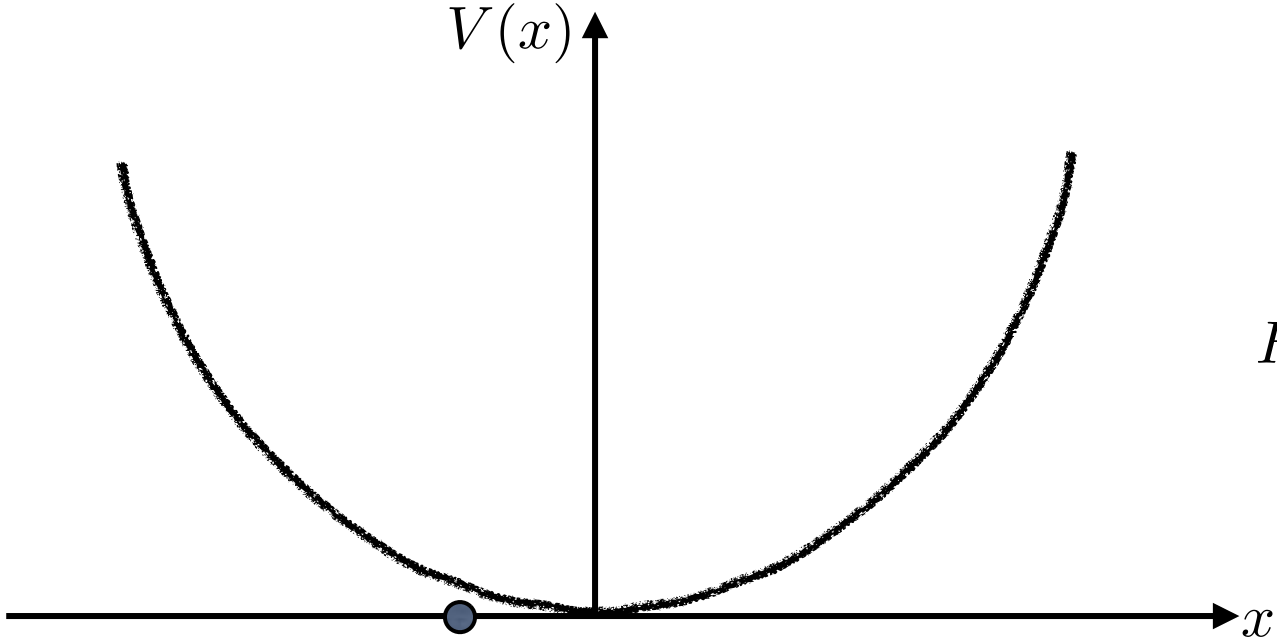
At zero temperature: connection between spinless free fermions in a harmonic trap and Random Matrix Theory (GUE)

Outline

- Free fermions in d=1 & T=0 and Random Matrix Theory (RMT)
- Free fermions in d=1 & T>0 and KPZ equation: main results
- Sketch of the derivation of our results
- Extension to higher dimensions, d>1
- Conclusion

Connection between free fermions at T=0 and RMT

- A single quantum particle in a harmonic potential



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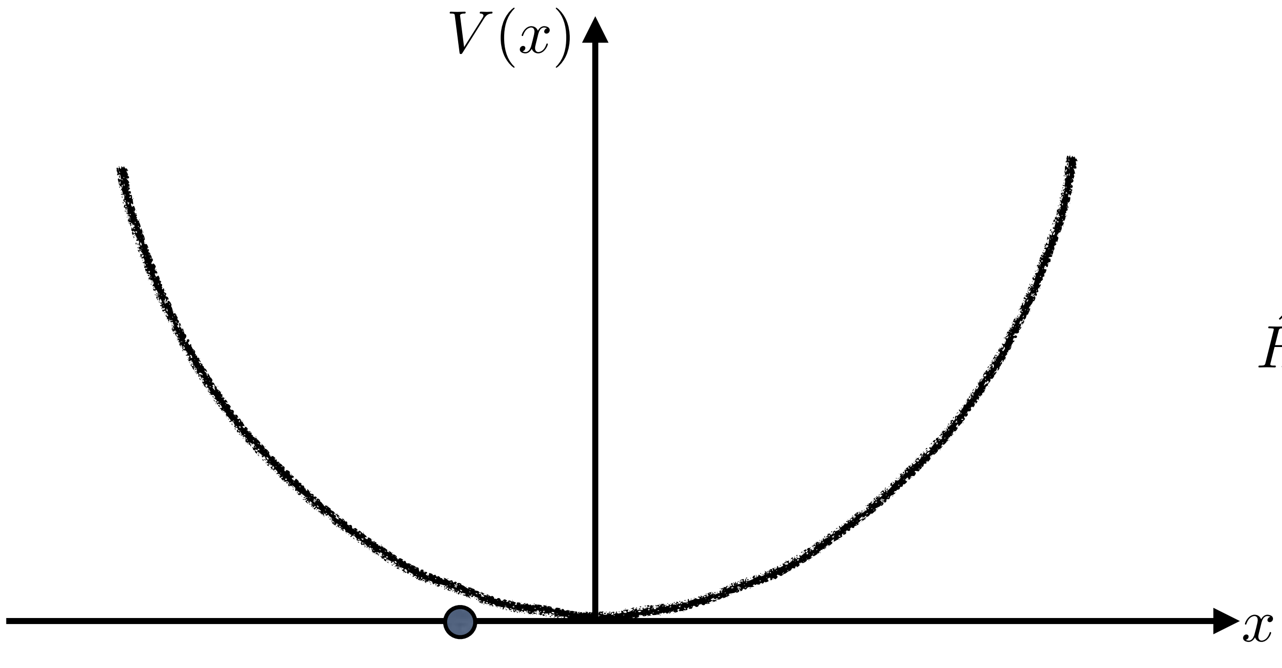

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- Single particle eigenfunctions

$$\hat{H}\,\varphi_E(x) = E\,\varphi_E(x)$$

with $\varphi_E(x \to \pm\infty) = 0$

$$\varphi_k(x) = \left[\frac{\alpha}{\sqrt{\pi}2^k k!}\right]^{1/2} e^{-\frac{\alpha^2 x^2}{2}} H_k(\alpha x)$$

$$\epsilon_k = \hbar\omega(k + 1/2) \quad , \quad \alpha = \sqrt{m\omega/\hbar}$$

$$k \in \mathbb{N}$$

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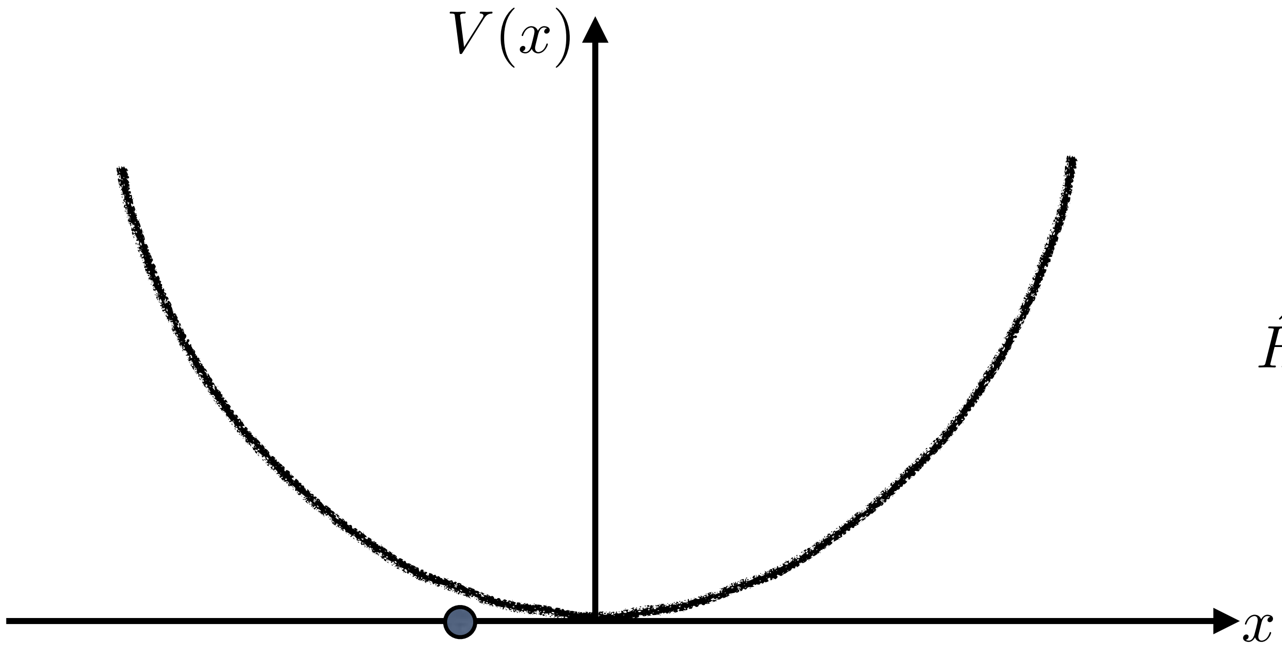

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N spinless free fermions in a 1d harmonic trap at T=0

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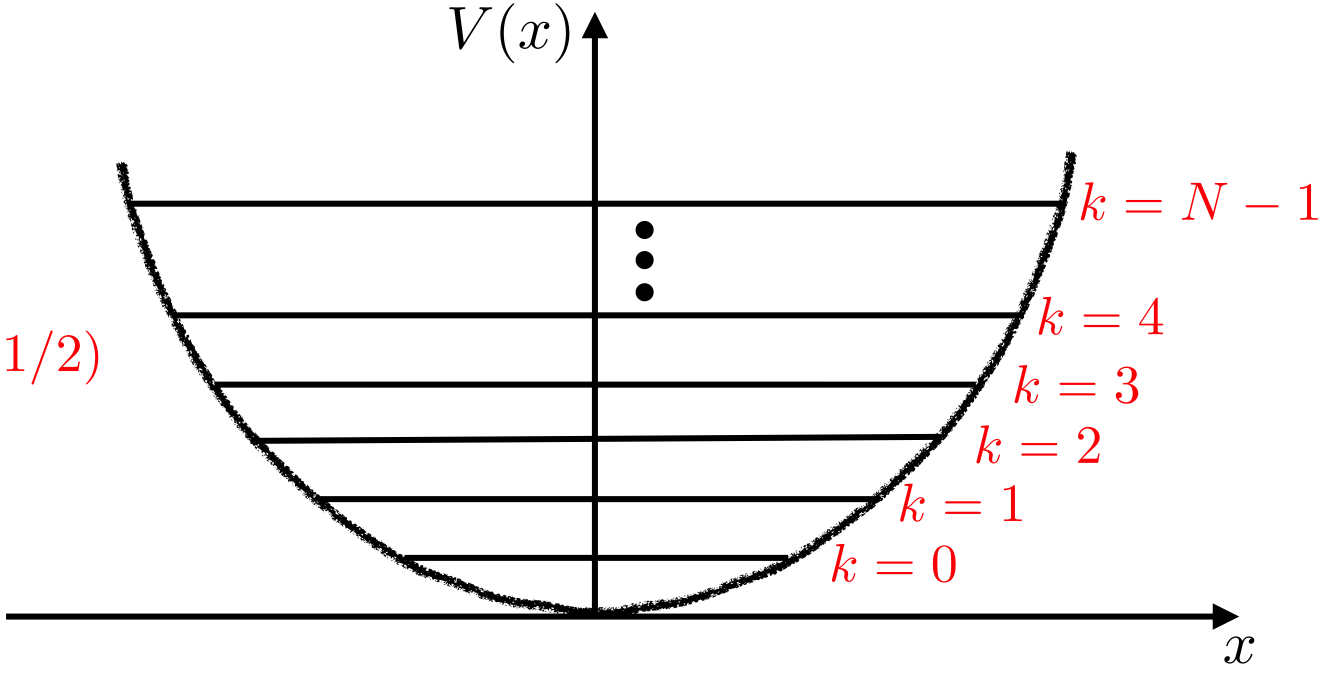

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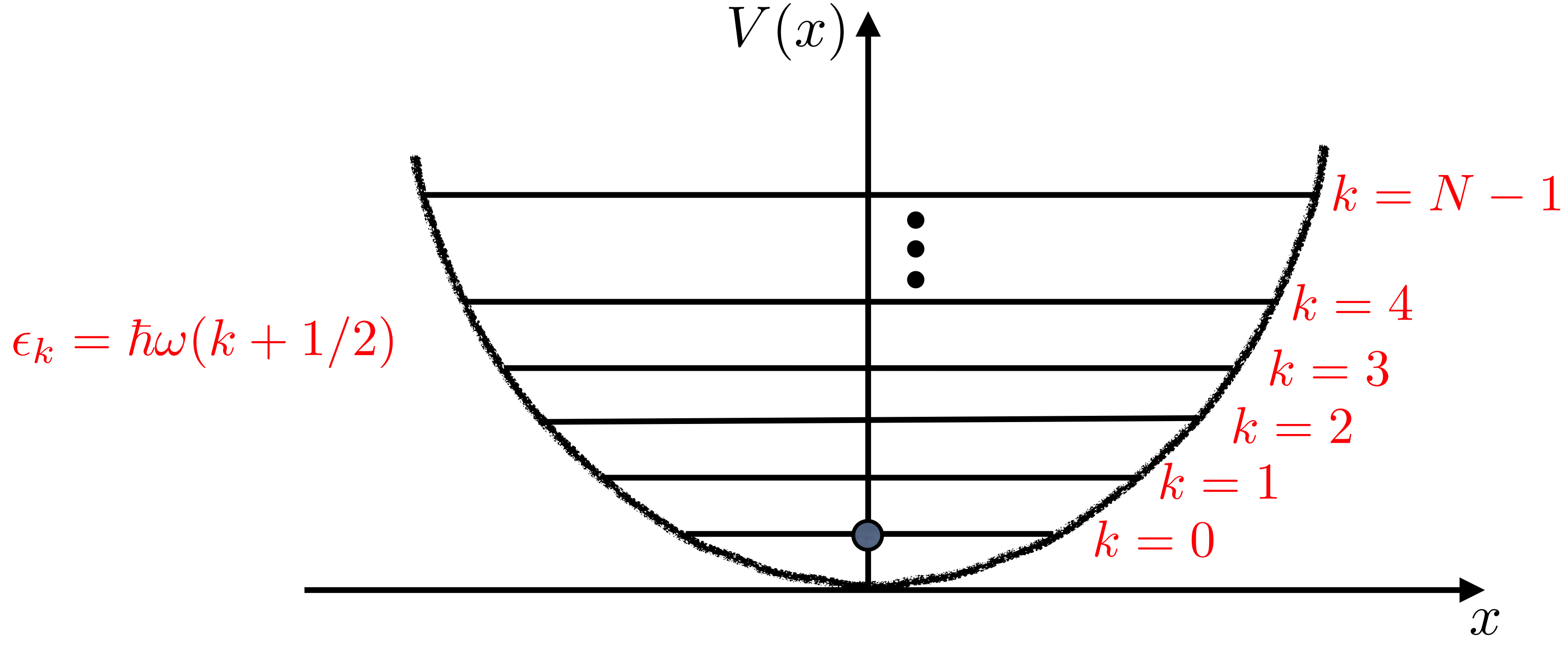

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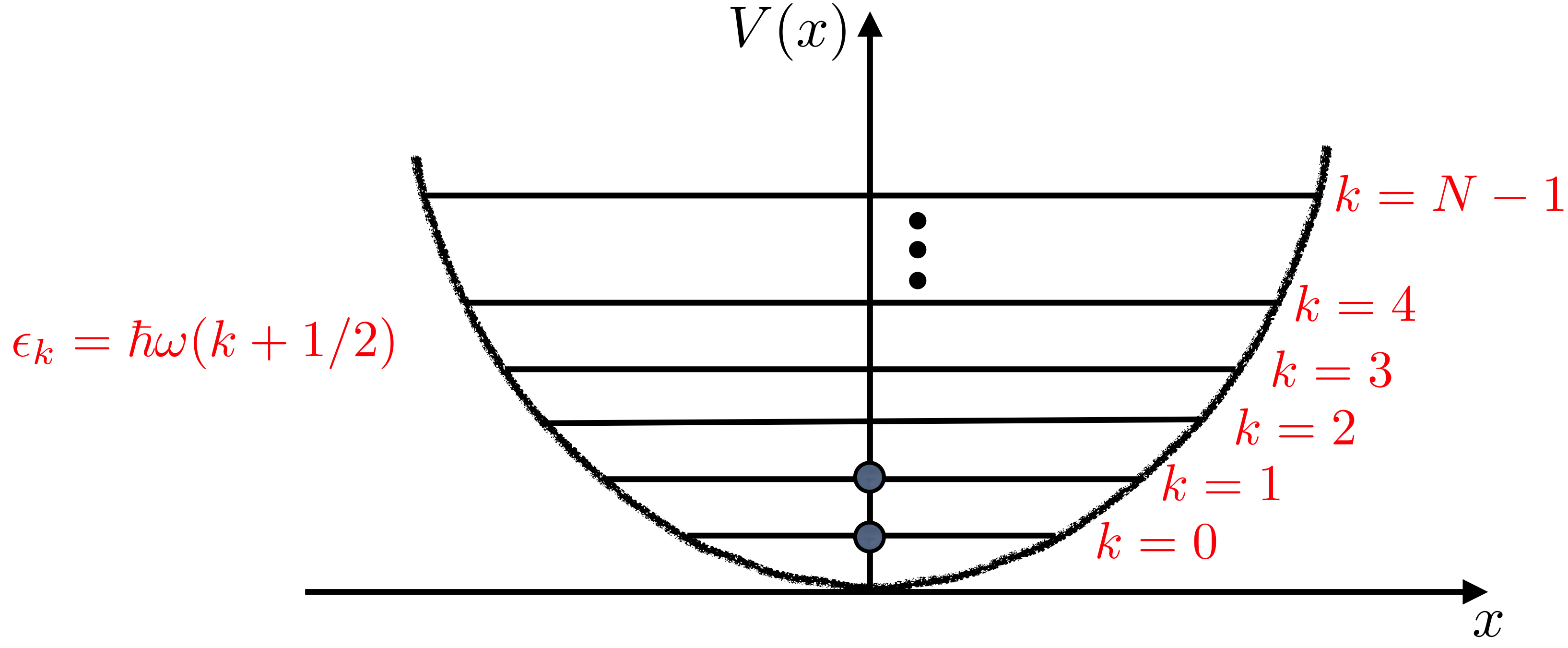

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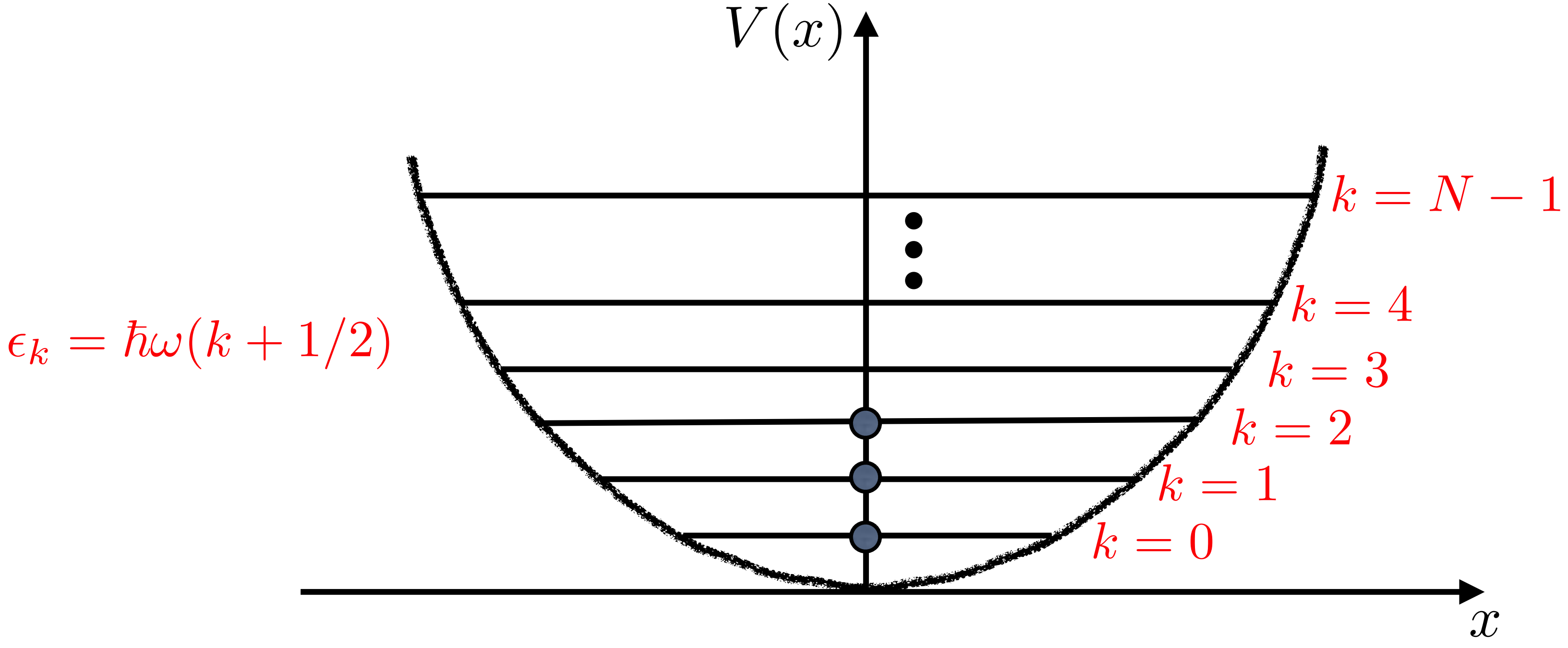

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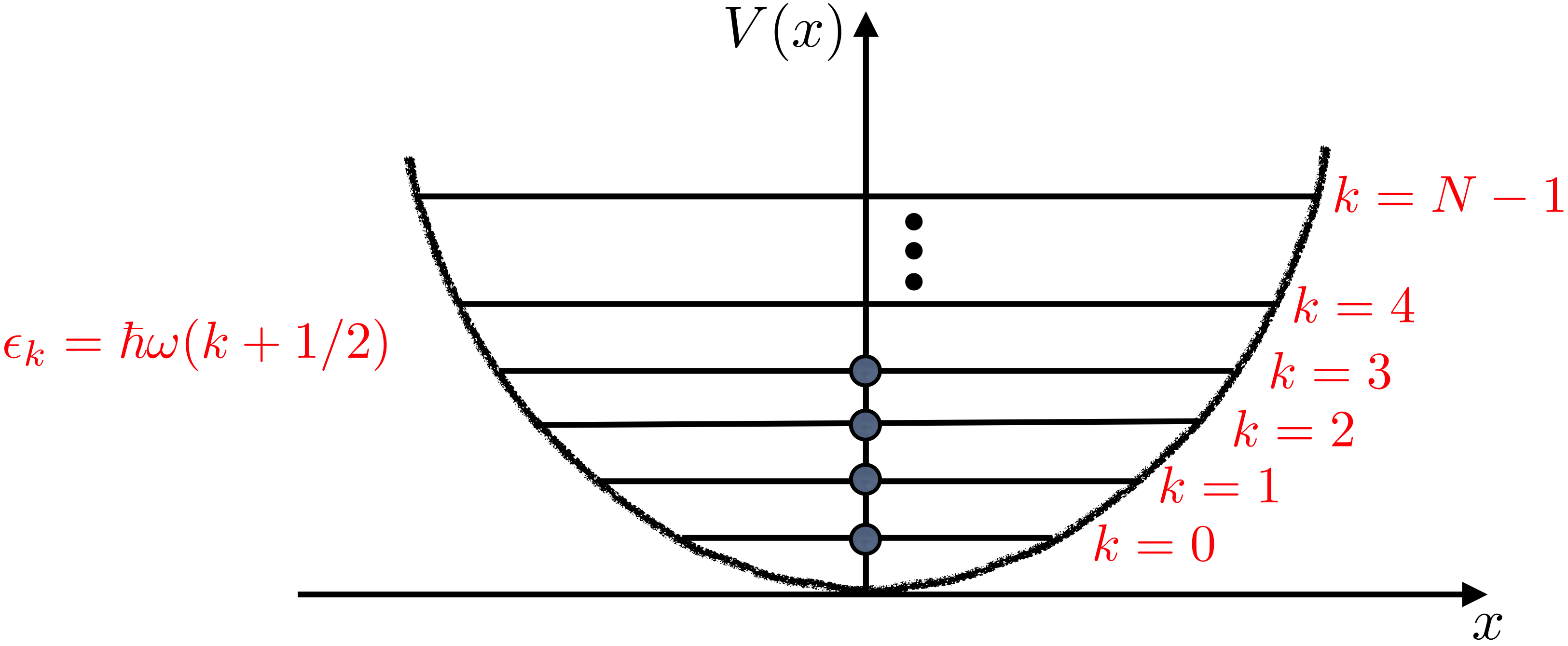

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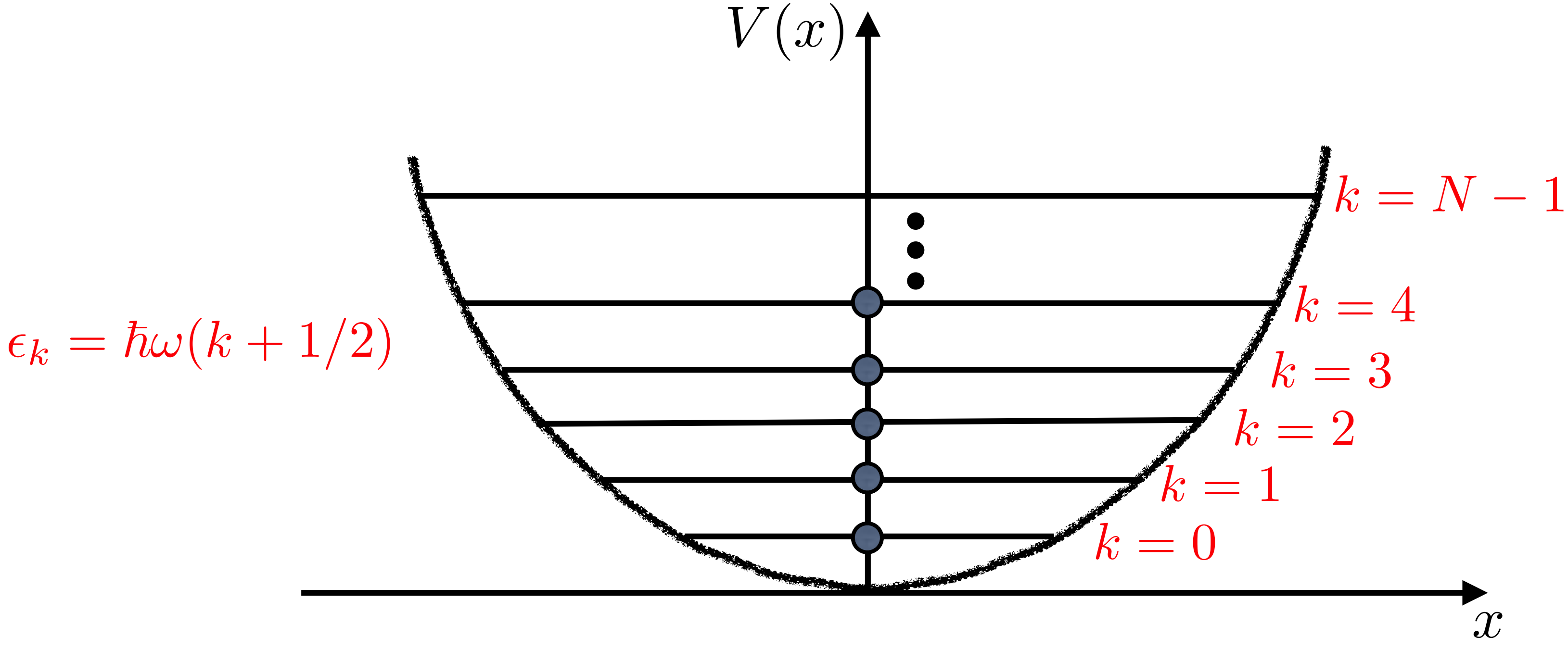

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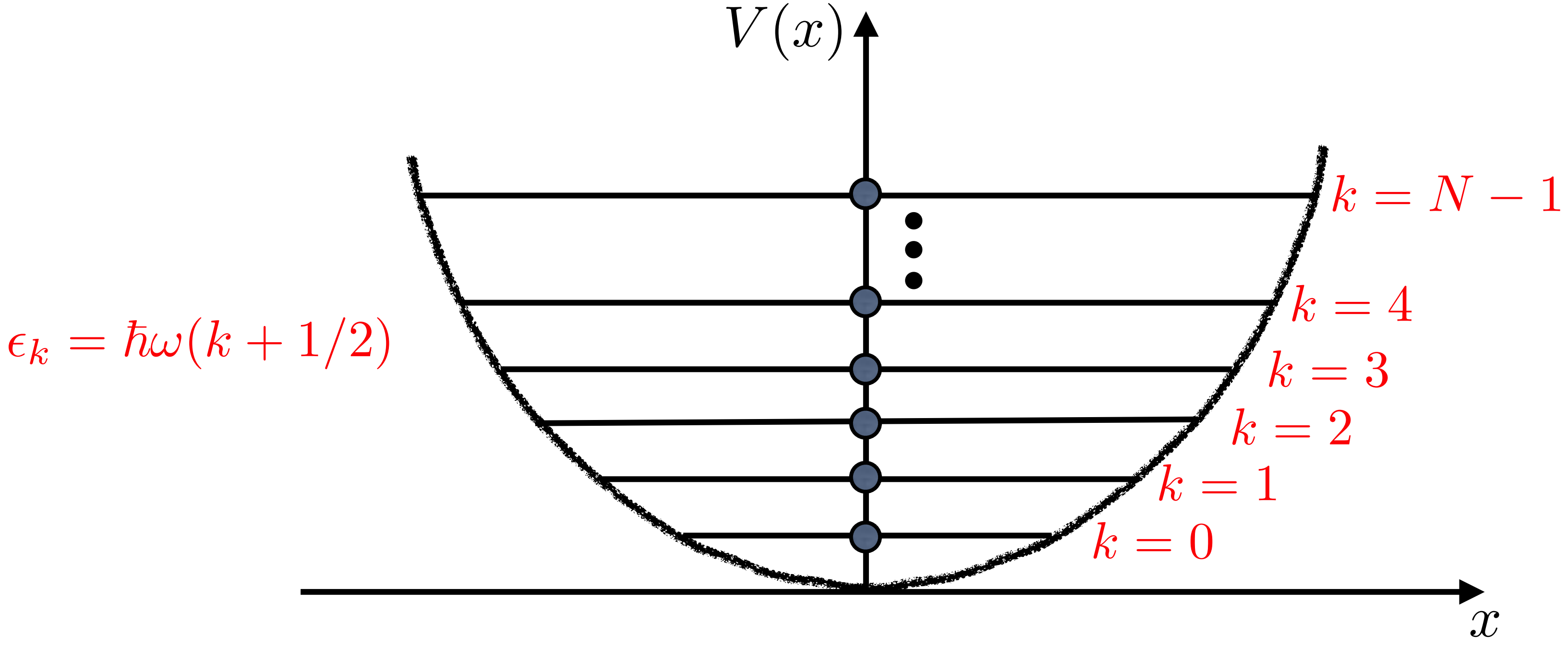

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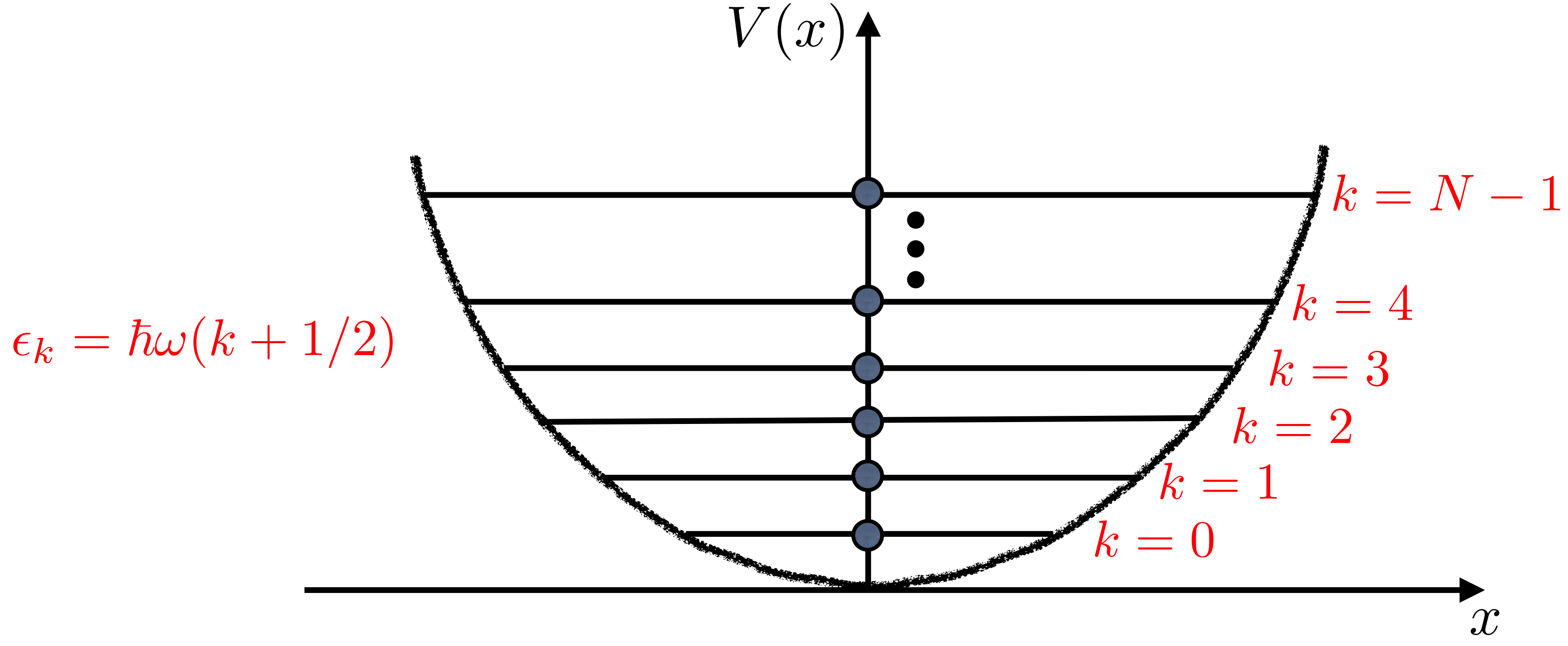

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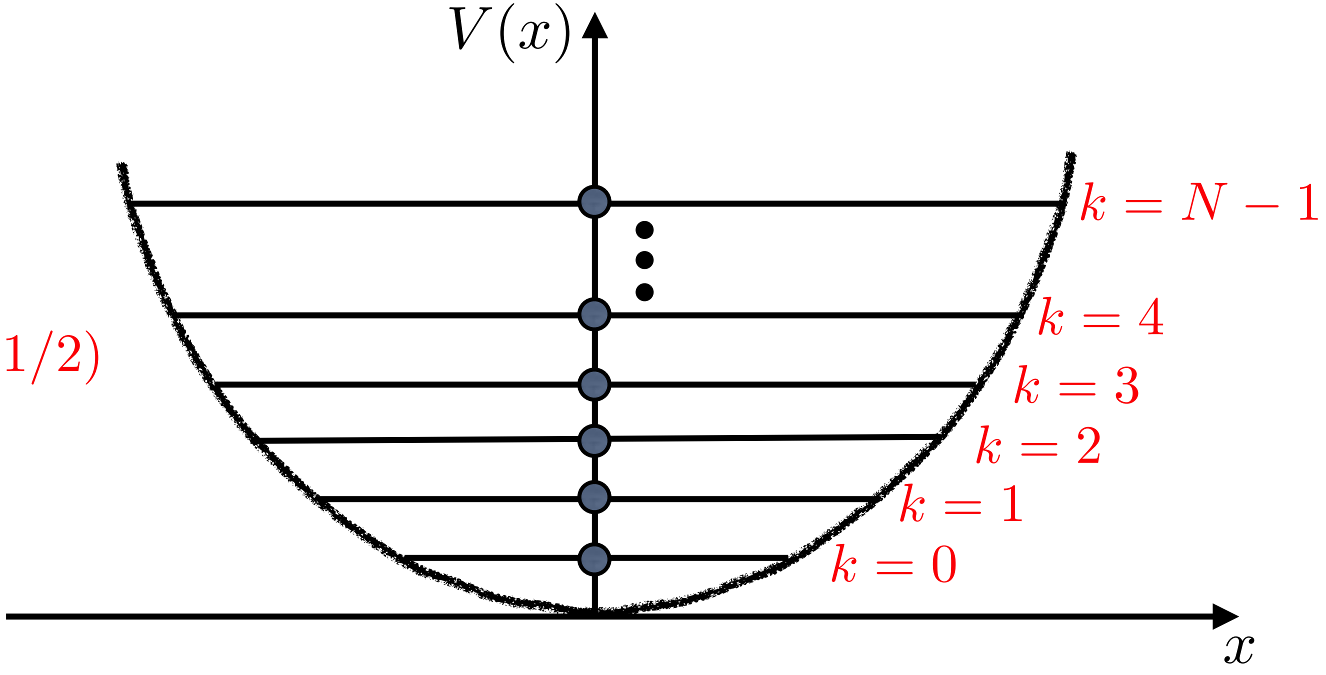

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Ground state energy $E_0 = \sum_{k=0}^{N-1} \epsilon_k = \frac{N^2}{2}$

Connection between free fermions at T=0 and RMT

- Ground-state wave function

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- Squared many-body wave function (T= 0 quantum probability)

$$|\Psi_0(x_1, \cdots, x_N)|^2 = \frac{1}{z_N(\alpha)} \prod_{i<j} (x_i - x_j)^2 e^{-\alpha^2 \sum_{i=1}^N x_i^2}$$

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- Let M be a $N \times N$ random Hermitian matrix with Gaussian (complex) entries. The PDF of the (real) eigenvalues $\lambda_i' s$ is given by

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⇒ The positions of the free fermions behave statistically like the eigenvalues of GUE random matrices

$$(\alpha\, x_1, \alpha\, x_2, \cdots \alpha\, x_N) \stackrel{d}{=} (\lambda_1,\lambda_2,\cdots,\lambda_N)$$

Properties of fermions in a 1d harmonic trap at T=0

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⇨ The spatial properties of free fermions in a harmonic trap at T=0 can directly be obtained from the known results in RMT

Eisler '13/Marino, S. N. M., Schehr, Vivo '14/Calabrese, Le Doussal, S. N. M. '15

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- Average density of free fermions: Wigner semi-circle law

$$\rho_N(x,T=0) = \frac{1}{N}\sum_{i=1}^N \langle\delta(x-x_i)\rangle$$

for $N \gg 1$ $\quad \rho_N(x,T=0) \approx \frac{\alpha}{\sqrt{N}} f_W\left(\frac{\alpha\, x}{\sqrt{N}}\right)$, $\boxed{f_W(z) = \frac{1}{\pi}\sqrt{2-z^2}}$

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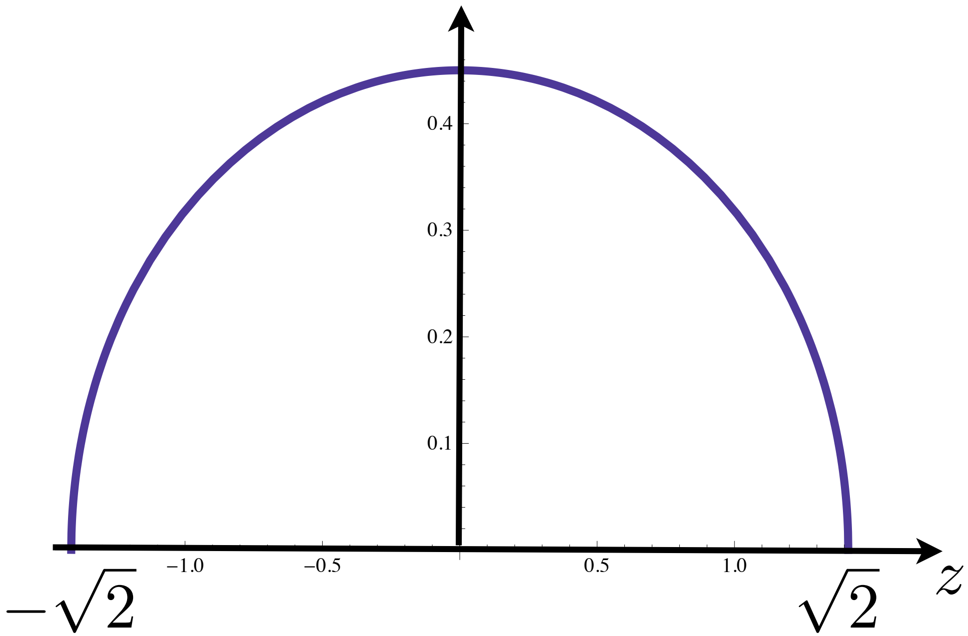
See also Local Density (or Thomas-Fermi) Approx. in the literature on fermions



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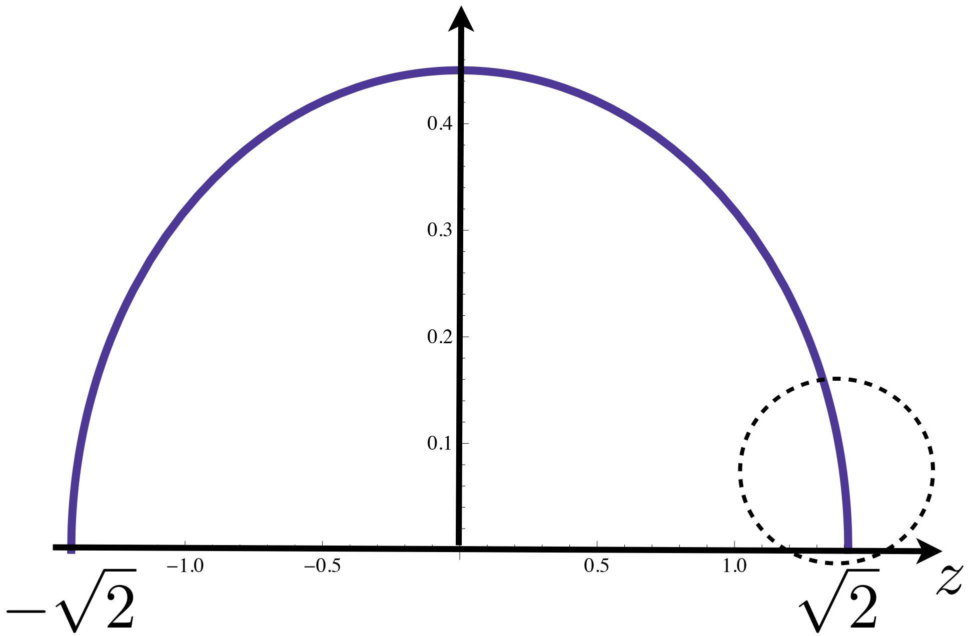

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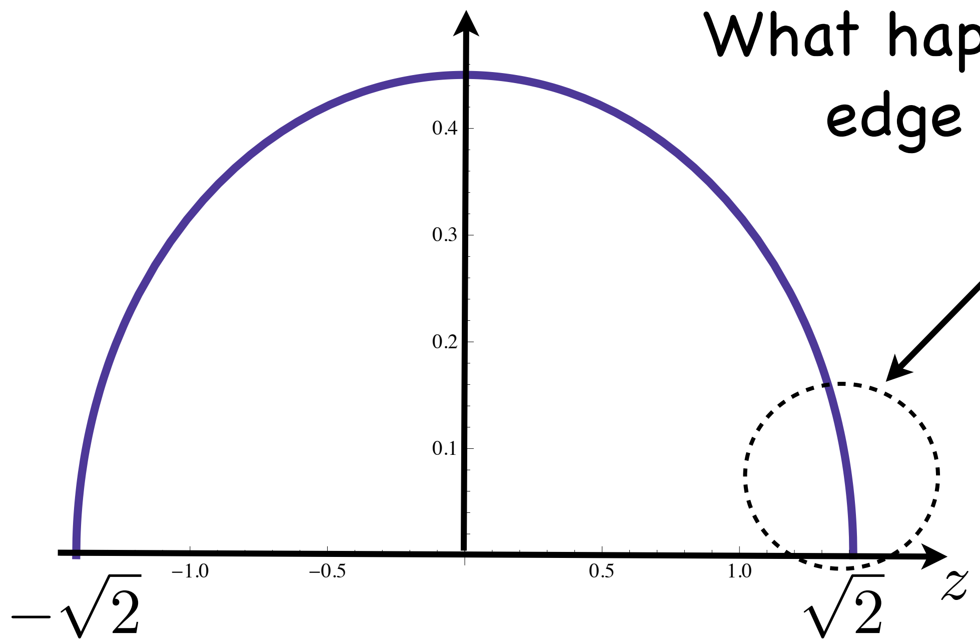

What happens close to the edge for finite N ?

Properties of fermions in a 1d harmonic trap at T=0

- Edge density of free fermions

Bowick, Brézin '91/Forrester '93

$$\rho_N(x) \approx \frac{1}{N w_N} F_1\left(\frac{x - \sqrt{2N}/\alpha}{w_N}\right)$$

with $w_N = \frac{N^{-1/6}}{\sqrt{2}\,\alpha}$ and $F_1(z) = [Ai'(z)]^2 - z[Ai(z)]^2$

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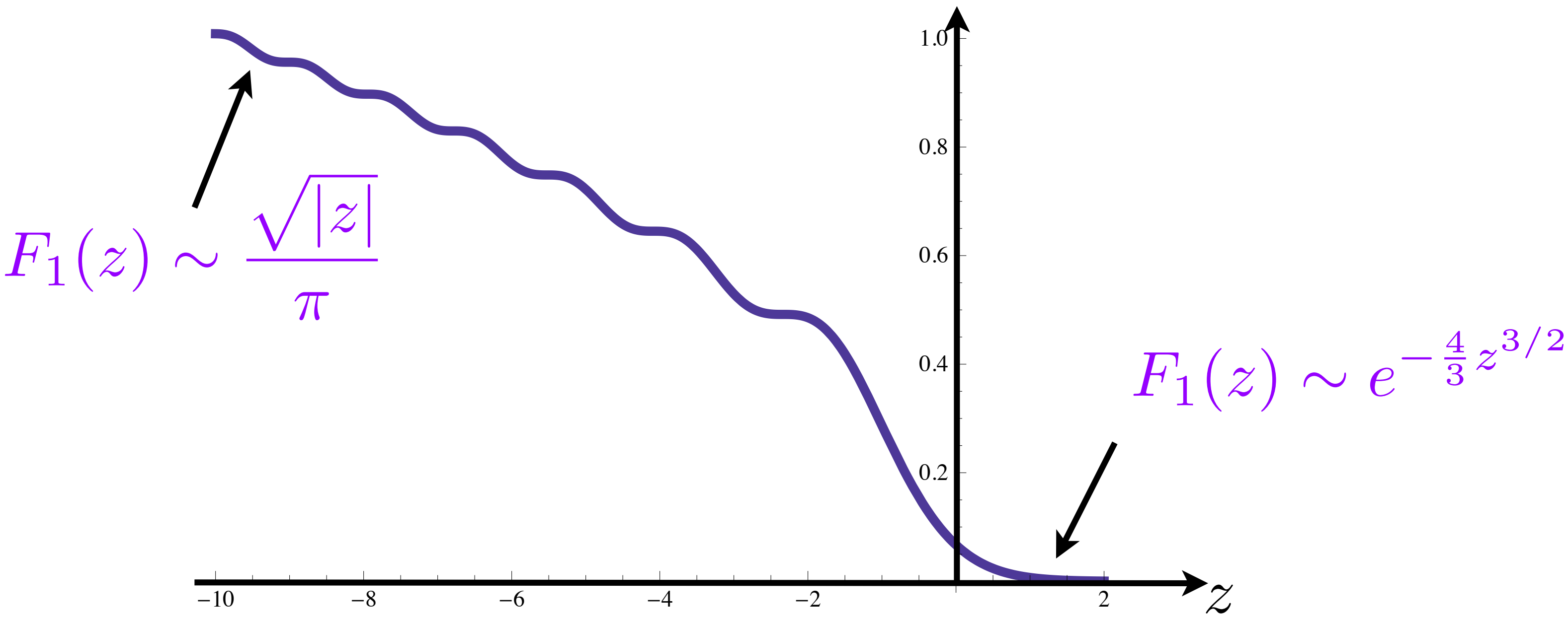

Fermions in a 1d harmonic trap at T=0: kernel

- Higher order correlations

e.g., 2-point correlation function: $R_2(y,z) = \sum_{i \neq j} \langle \delta(y - x_i)\delta(z - x_j) \rangle$

$$R_n(x_1, \cdots, x_n) = \frac{N!}{(N-n)!} \int_{-\infty}^{\infty} dx_{n+1} \cdots \int_{-\infty}^{\infty} dx_N |\Psi_0(x_1, \cdots, x_n, x_{n+1}, \cdots, x_N)|^2$$

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$$K_N(x,y) = \sum_{k=0}^{N-1} \varphi_k(x)\varphi_k(y)$$

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kernel

in particular, the average density is given by $\rho_N(x) = \frac{1}{N} K_N(x,x)$

Limiting form of the kernel for trapped fermions at T=0

- Bulk limit: when $x \; \& \; y$ are far from the edge and

  and $|x-y| \sim \dfrac{1}{N\rho_N(x)} \equiv$ inter-particle distance

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$$K_N(x,y) \equiv \frac{1}{\ell} \mathcal{K}_{\rm bulk}\left(\frac{|x-y|}{\ell}\right) \; , \; \ell = \frac{2}{\pi N \rho_N(x)}$$

$$\mathcal{K}_{\rm bulk}(z) = \frac{\sin(2\,z)}{\pi\,z}$$

Sine-kernel

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$$\mathcal{K}_{\rm bulk}(z) = \frac{\sin(2\,z)}{\pi\, z}$$

Sine-kernel

- Edge scaling limit: for x & y close to the edge $r_{\rm edge} = \sqrt{2N}/\alpha$

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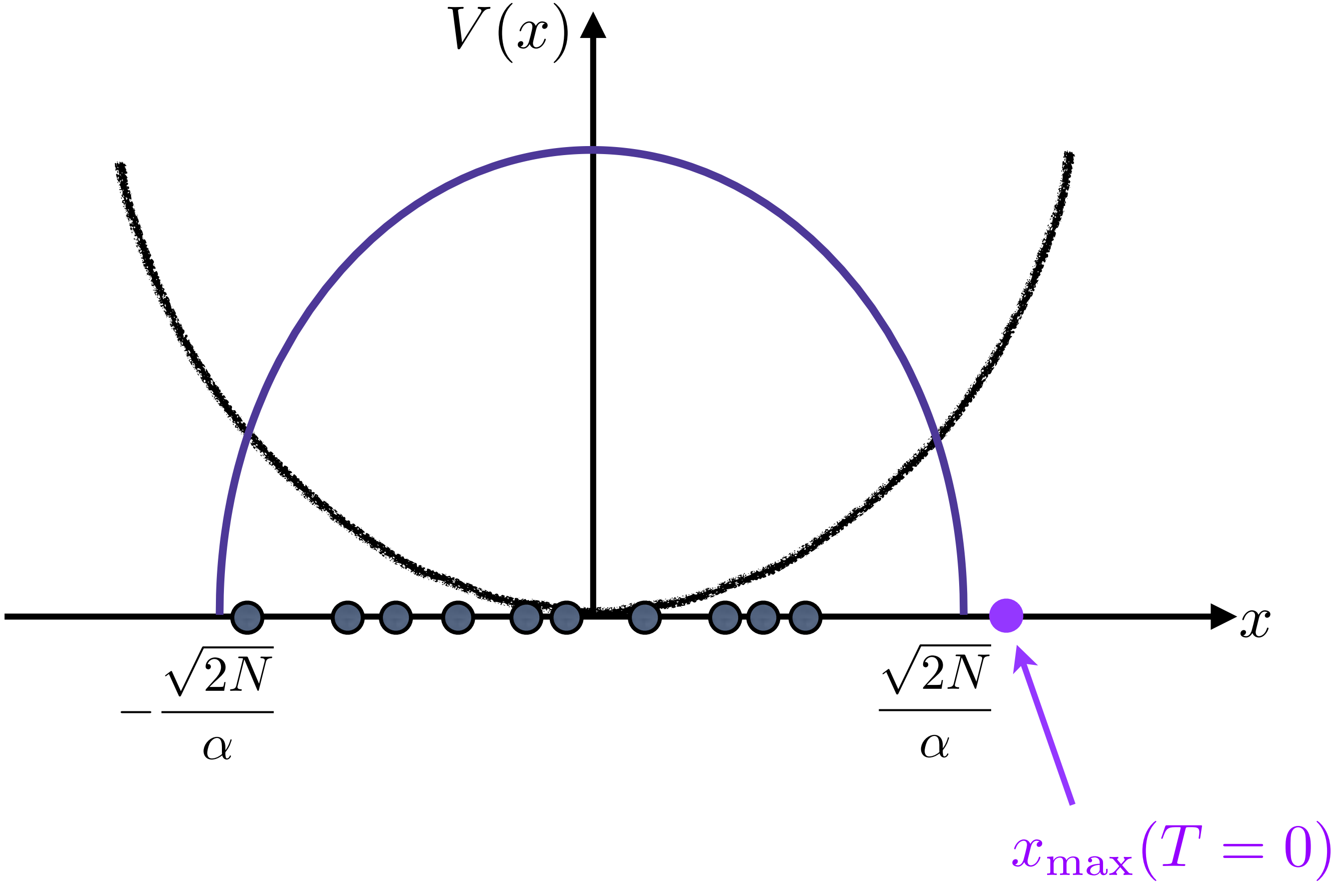
Airy-kernel

Position of the rightmost fermion at T=0



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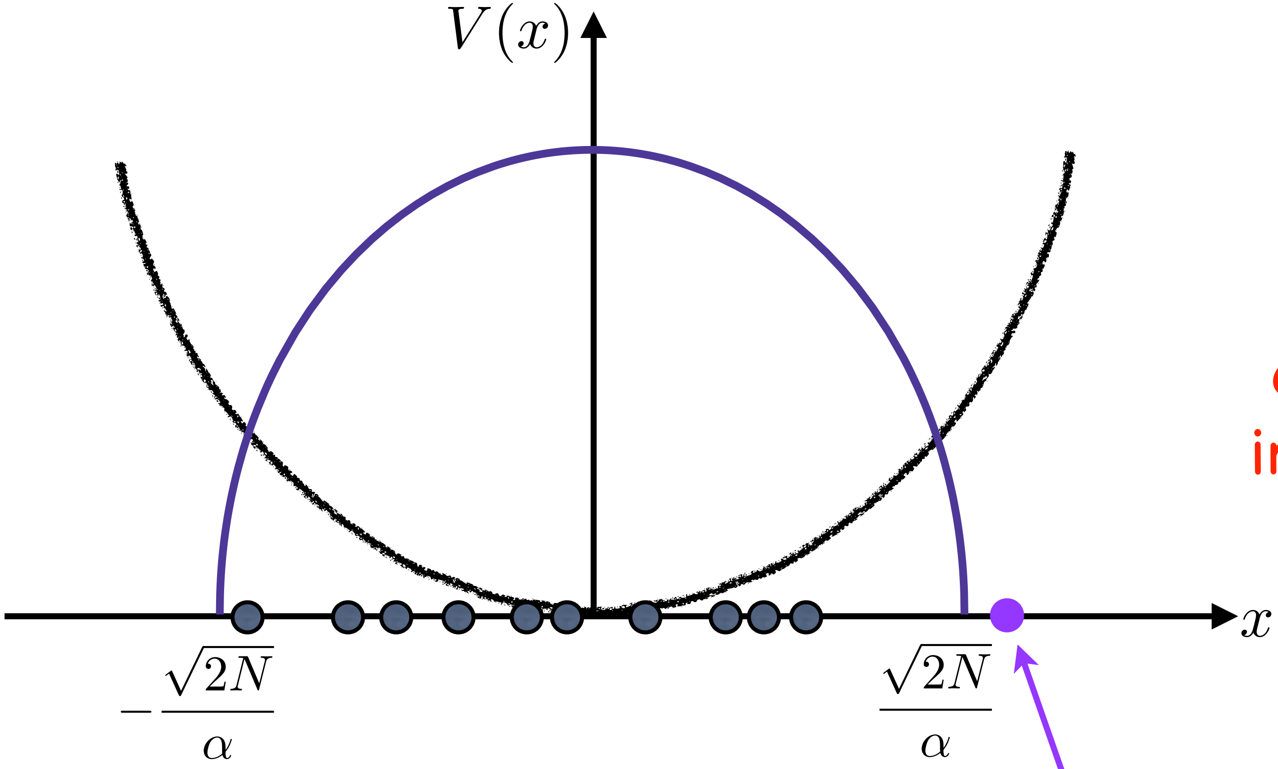

largest eigenvalue
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from the connection with RMT: $x_{\max}(T=0) \overset{d}{=} \frac{\lambda_{\max}}{\alpha}$

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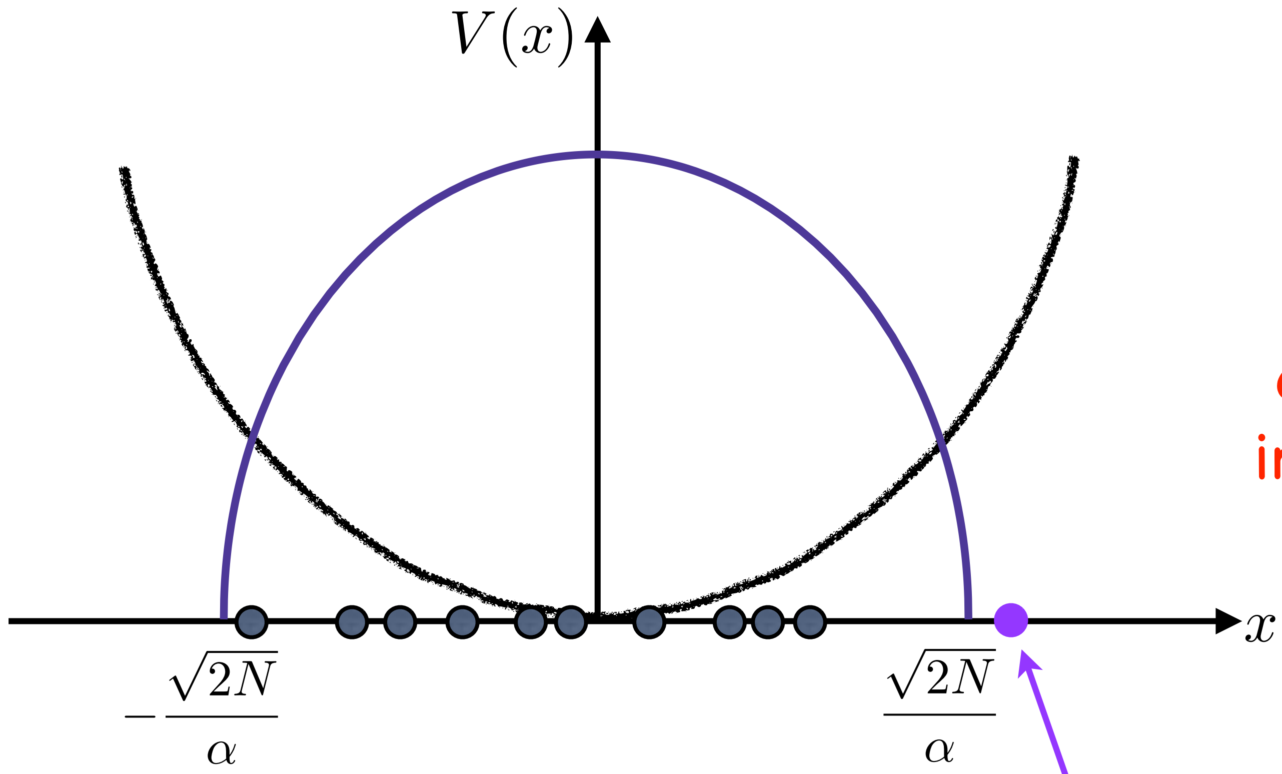

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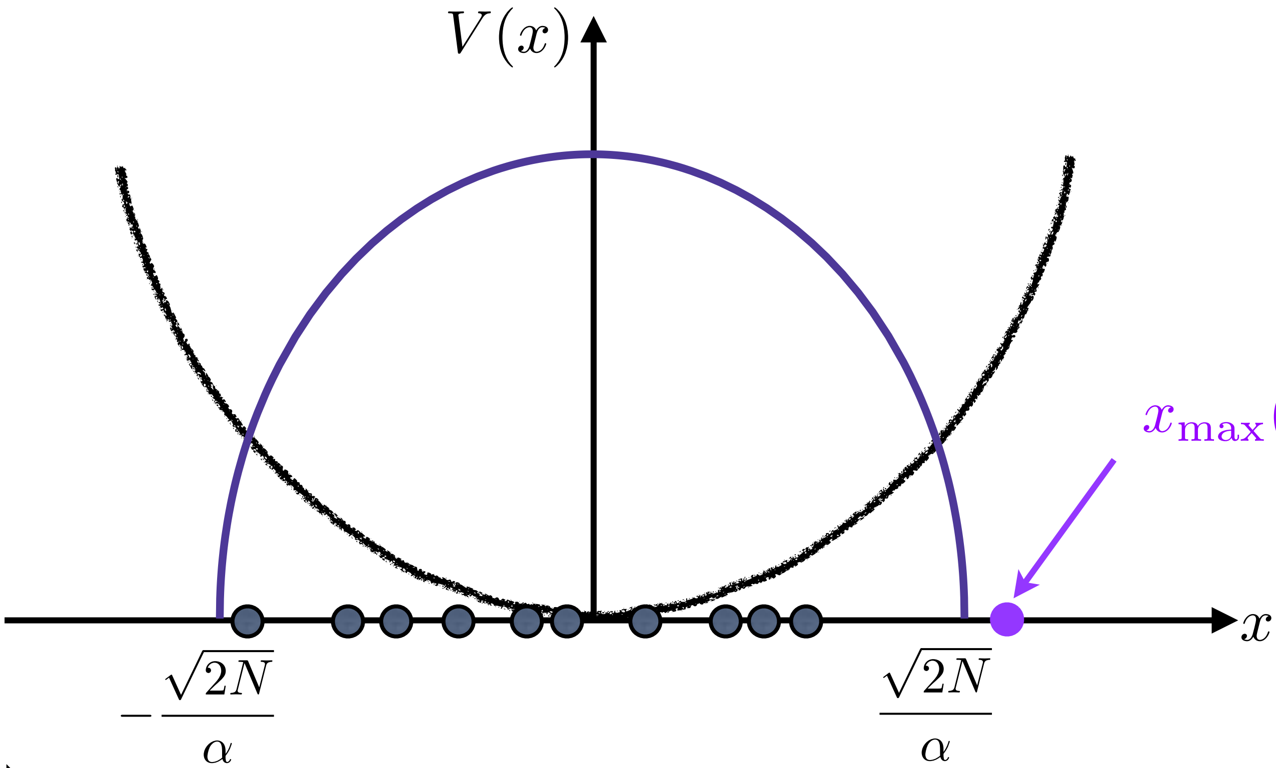

$$r_{\text{edge}} = \sqrt{2N}/\alpha$$

$$w_N = \frac{N^{-1/6}}{\sqrt{2}\,\alpha}$$

⇨ fluctuations of $x_{\max}(T=0)$ are governed by the Tracy-Widom distribution for GUE

$$\Pr.(x_{\max}(T=0) \leq M) \approx \mathcal{F}_2\left(\frac{M - r_{\text{edge}}}{w_N}\right)$$

$$\mathcal{F}_2(\xi) = \det(I - P_\xi K_{\text{edge}} P_\xi)\ ,\ K_{\text{edge}}(a,b) = \frac{Ai(a)Ai'(b) - Ai'(a)Ai(b)}{a-b}$$

Fredholm determinant Airy-kernel

Fermions in a 1d confining trap at T=0: summary

- Form a determinantal process (c.f. GUE for a harmonic well)

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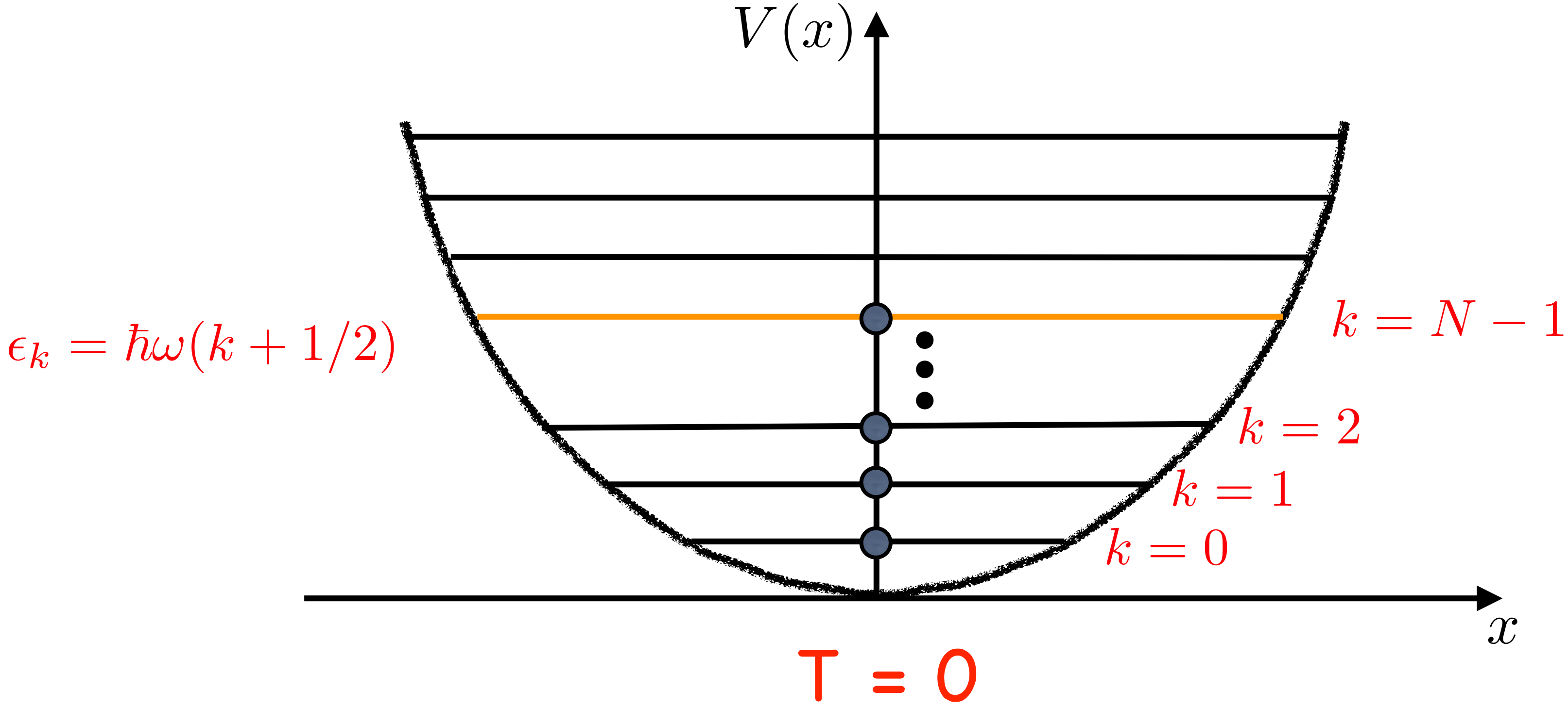
$V(x) \sim |x|^p$ with a single minimum Eisler '13

What happens at finite temperature

$T > 0$?

N free fermions in 1d-harmonic trap at T > 0



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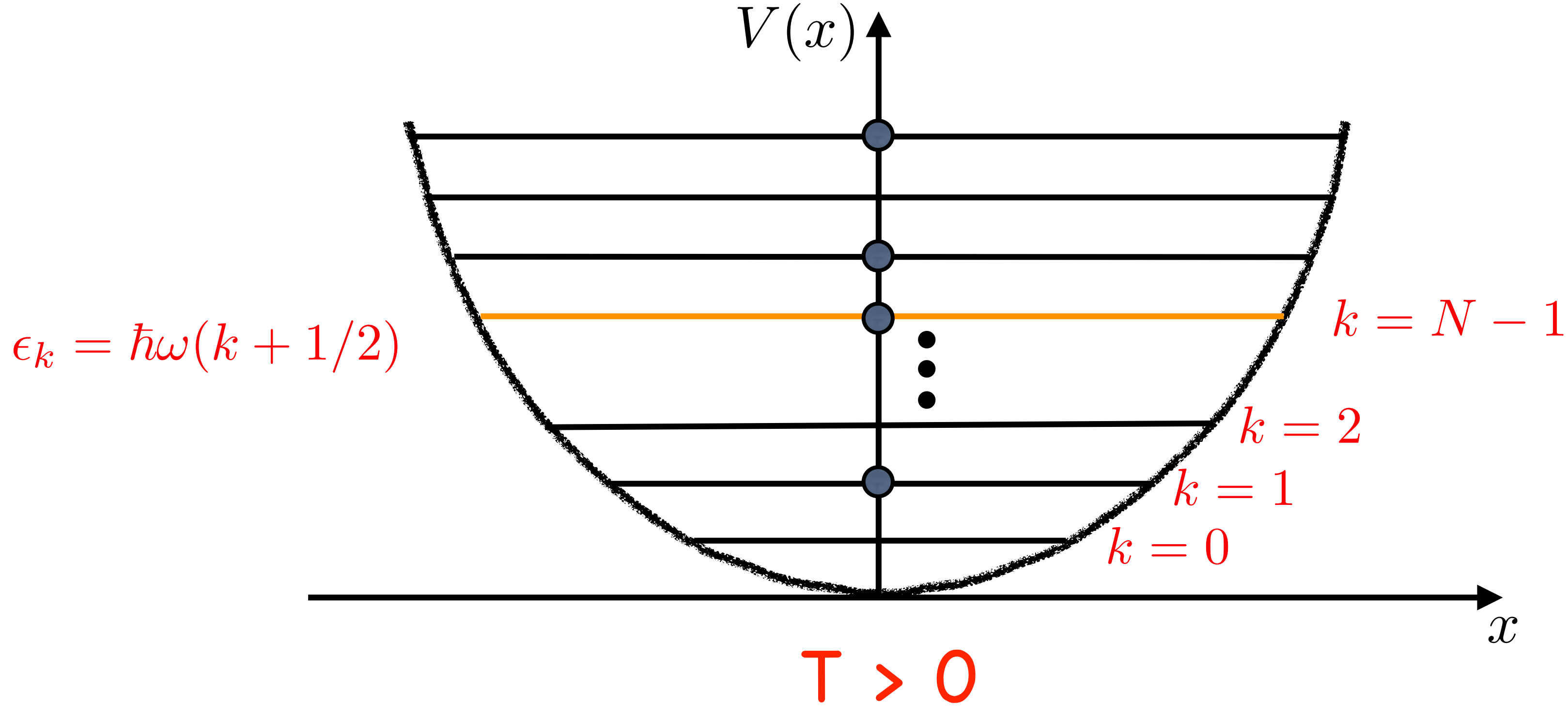

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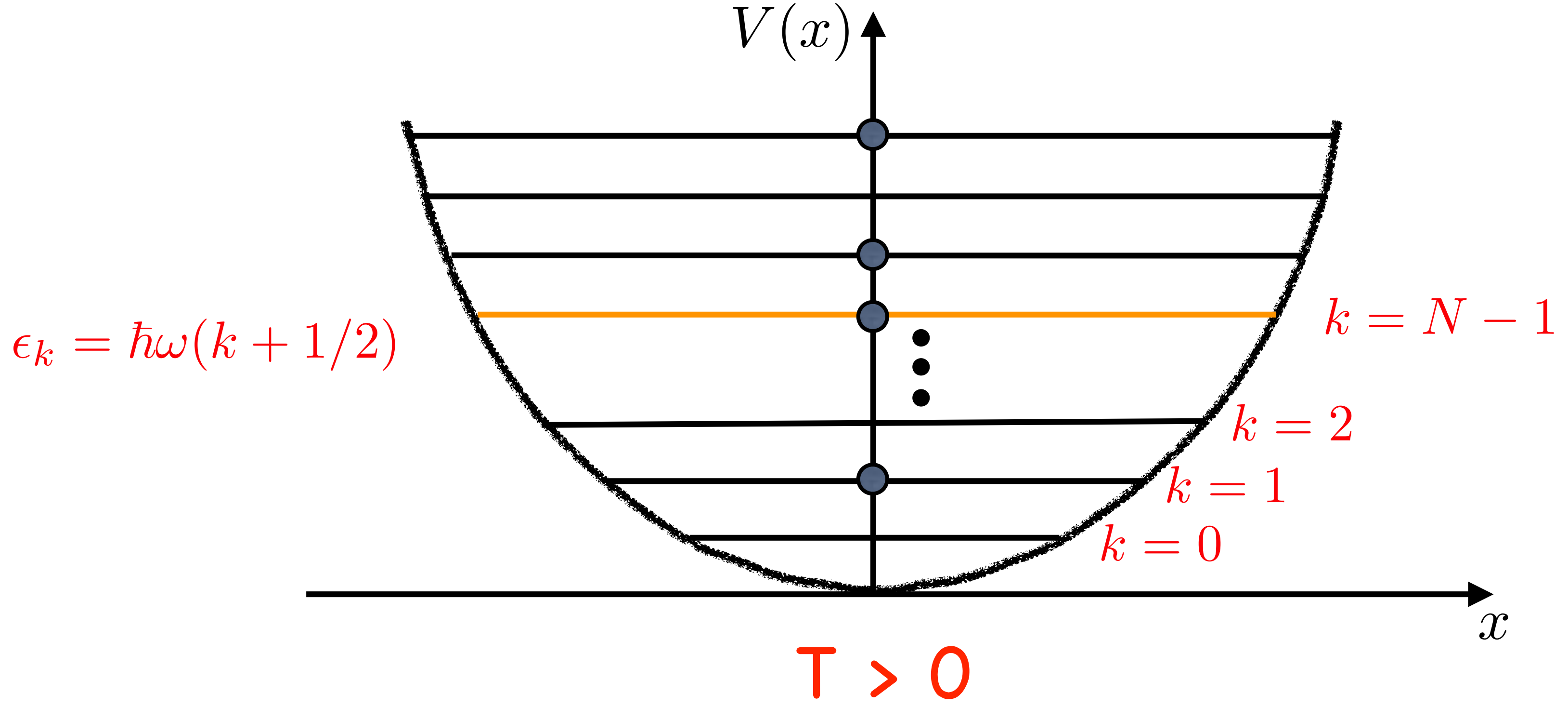

- Probability density function (PDF) of the positions $x_i's$

$$P_{\text{joint}}(x_1, \cdots x_N) = \frac{1}{N! Z_N(\beta)} \sum_{k_1 < \cdots < k_N} \left[\det_{1 \le i,j \le N} (\varphi_{k_i}(x_j)) \right]^2 e^{-\beta(\epsilon_{k_1} + \cdots + \epsilon_{k_N})}$$

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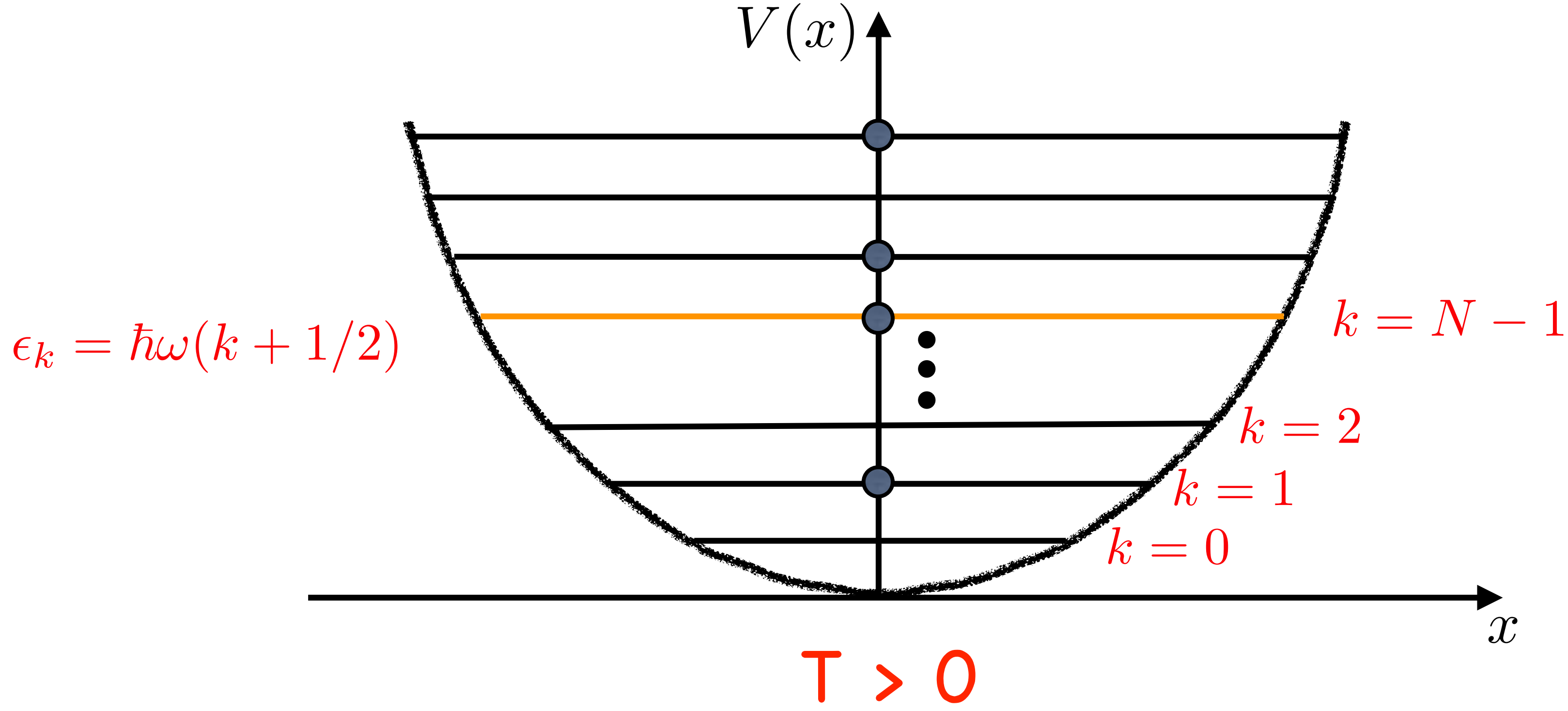

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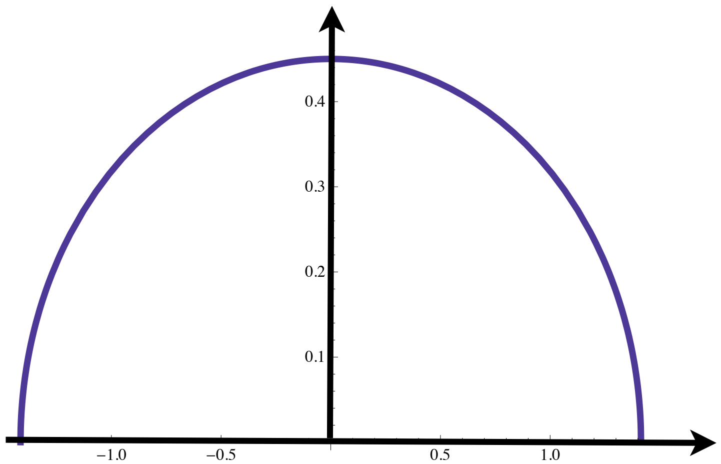
$T = 0$



$$\rho_N(x, T\to 0) \approx \frac{\alpha}{\sqrt{N}} f_W\left(\frac{\alpha\, x}{\sqrt{N}}\right) ,$$

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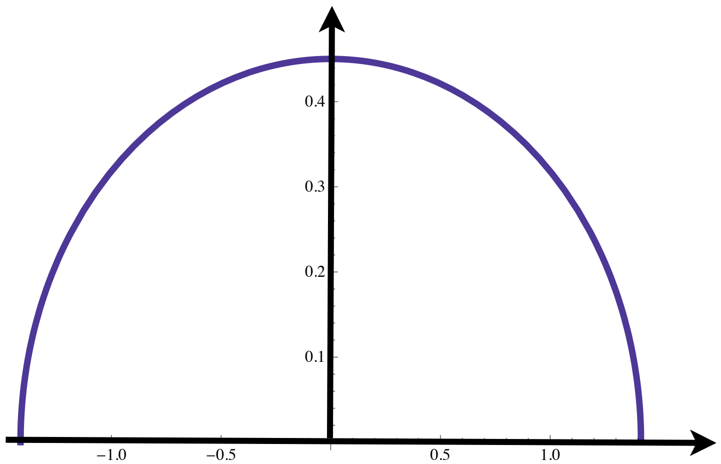

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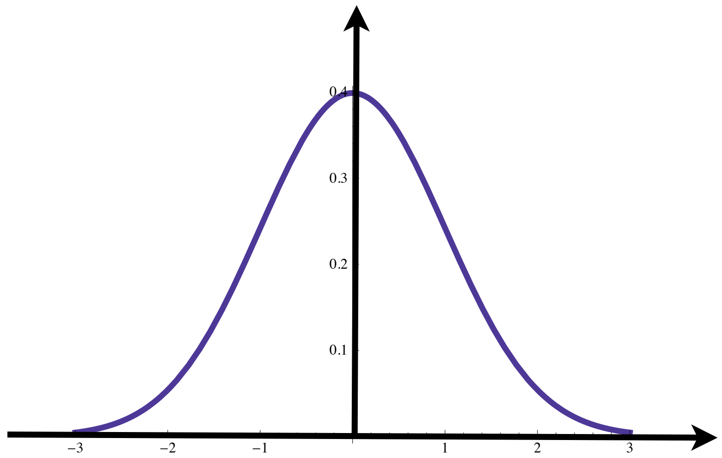

$$\rho_N(x, T\to\infty) \sim \sqrt{\frac{\beta\, m\, \omega^2}{2\pi}} \exp\left[-\frac{\beta}{2}\, m\, \omega^2\, x^2\right]$$

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Boltzmann-Gibbs

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Interpolation at finite T ?

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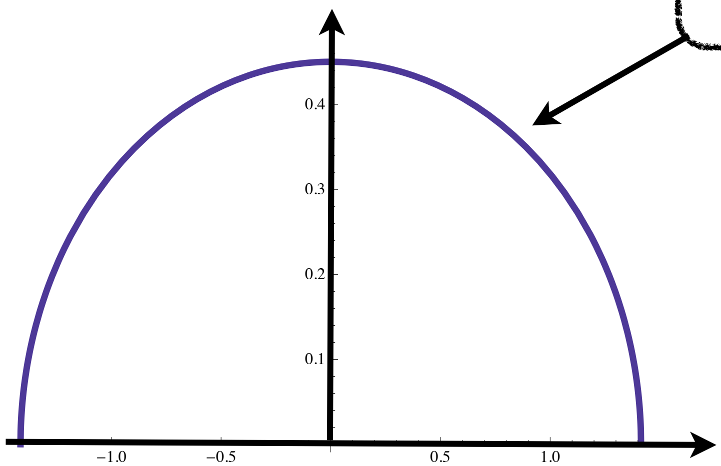

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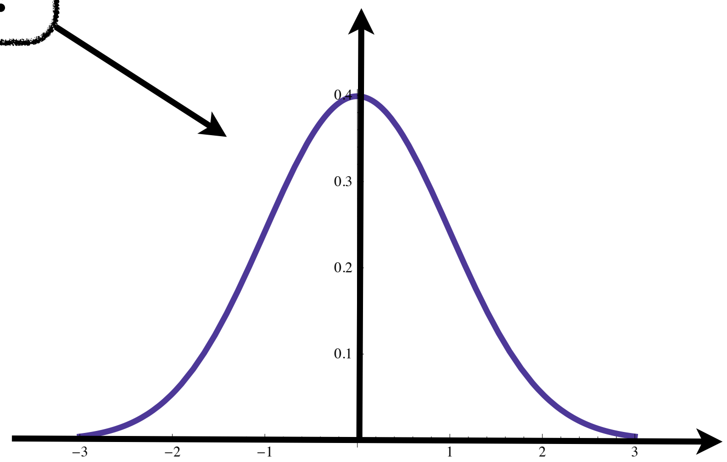

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Dean, Le Doussal, S. N. M., Schehr '14

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$$R(y,z) = -\frac{1}{\sqrt{2\pi\, y}}\,\mathrm{Li}_{1/2}\left(-\left(e^y - 1\right)\, e^{-z^2}\right) \qquad \mathrm{Li}_n(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^n}$$

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See also Local Density (or Thomas-Fermi) Approx. in the literature on fermions

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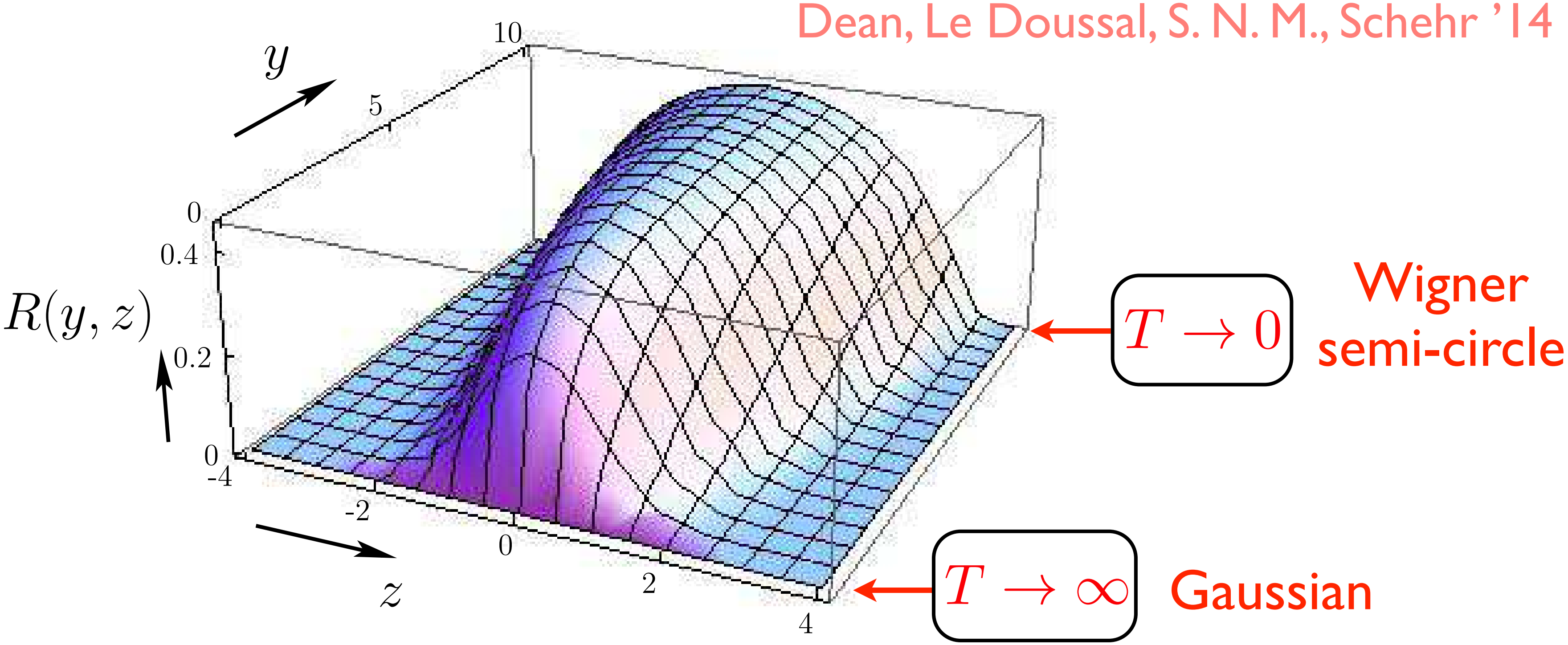

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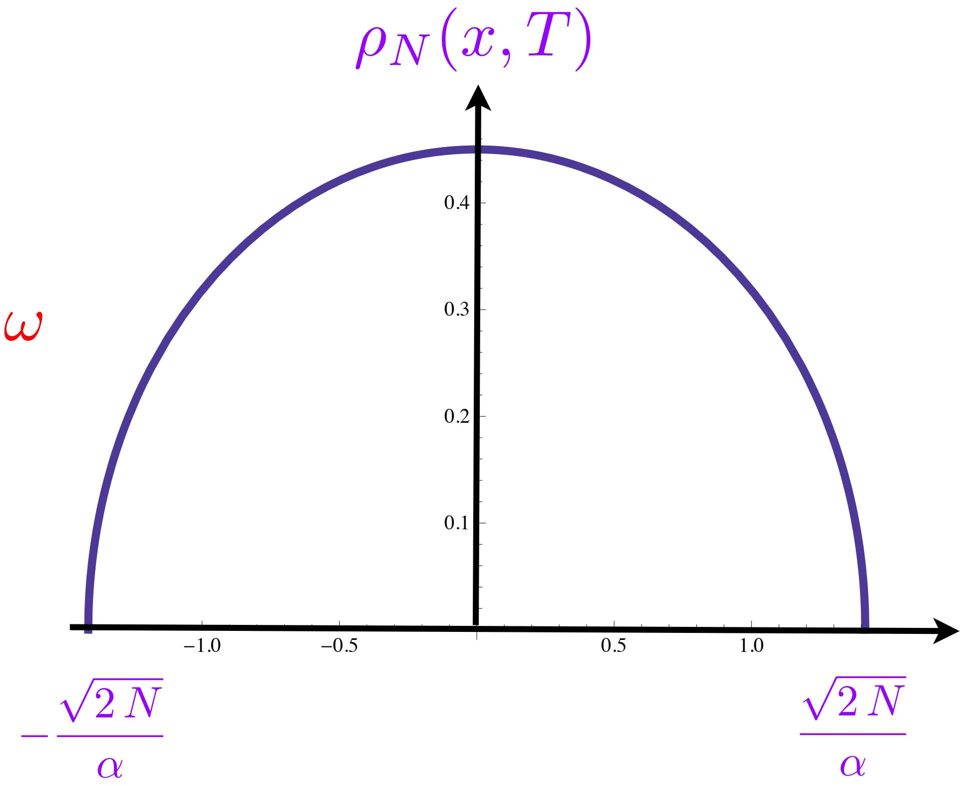

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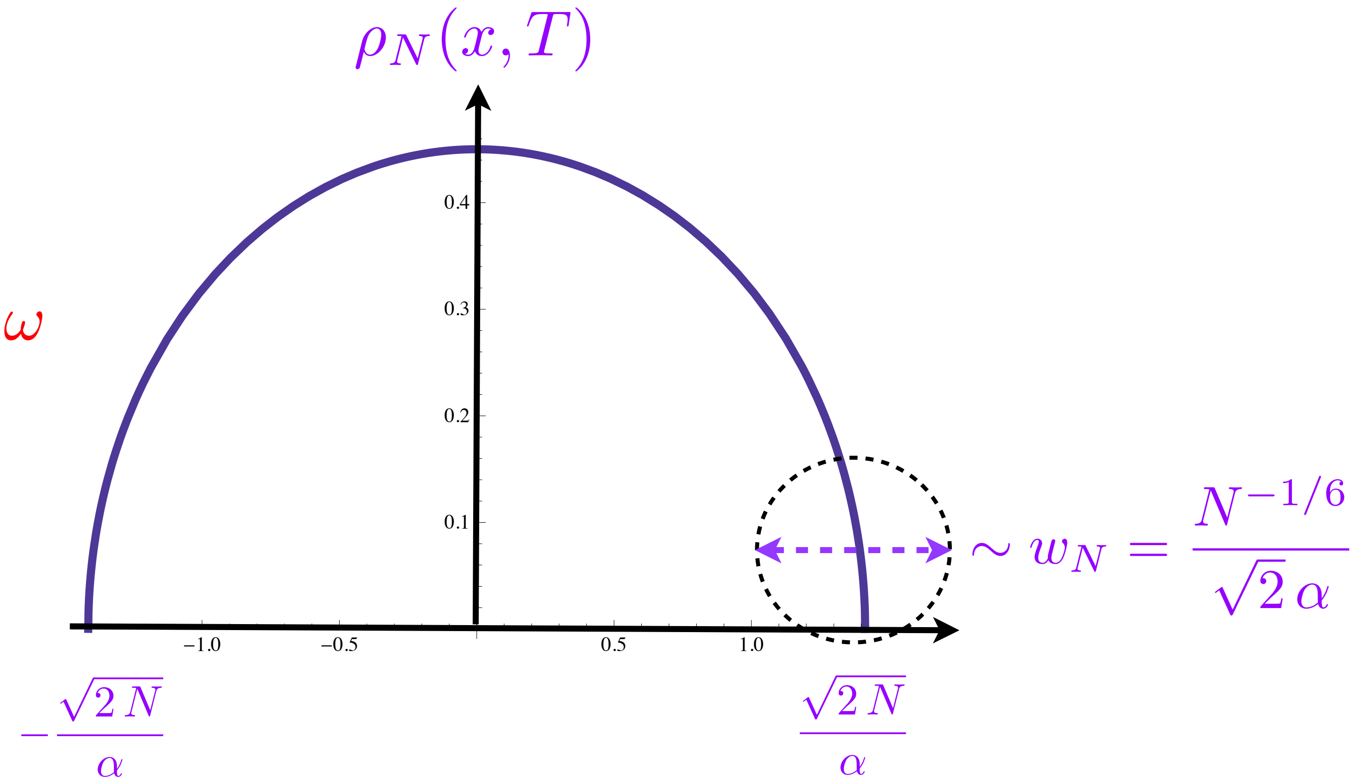

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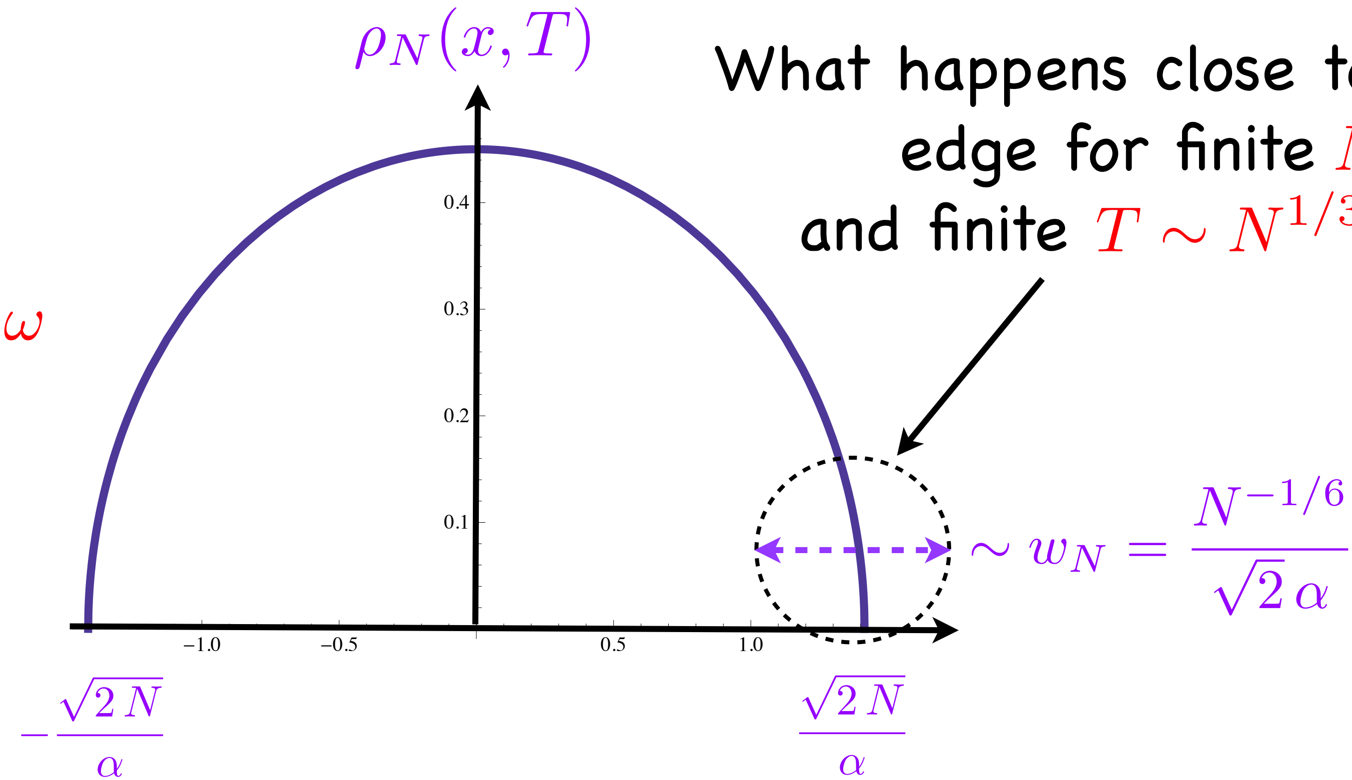

What happens close to the edge for finite N and finite $T \sim N^{1/3}$?

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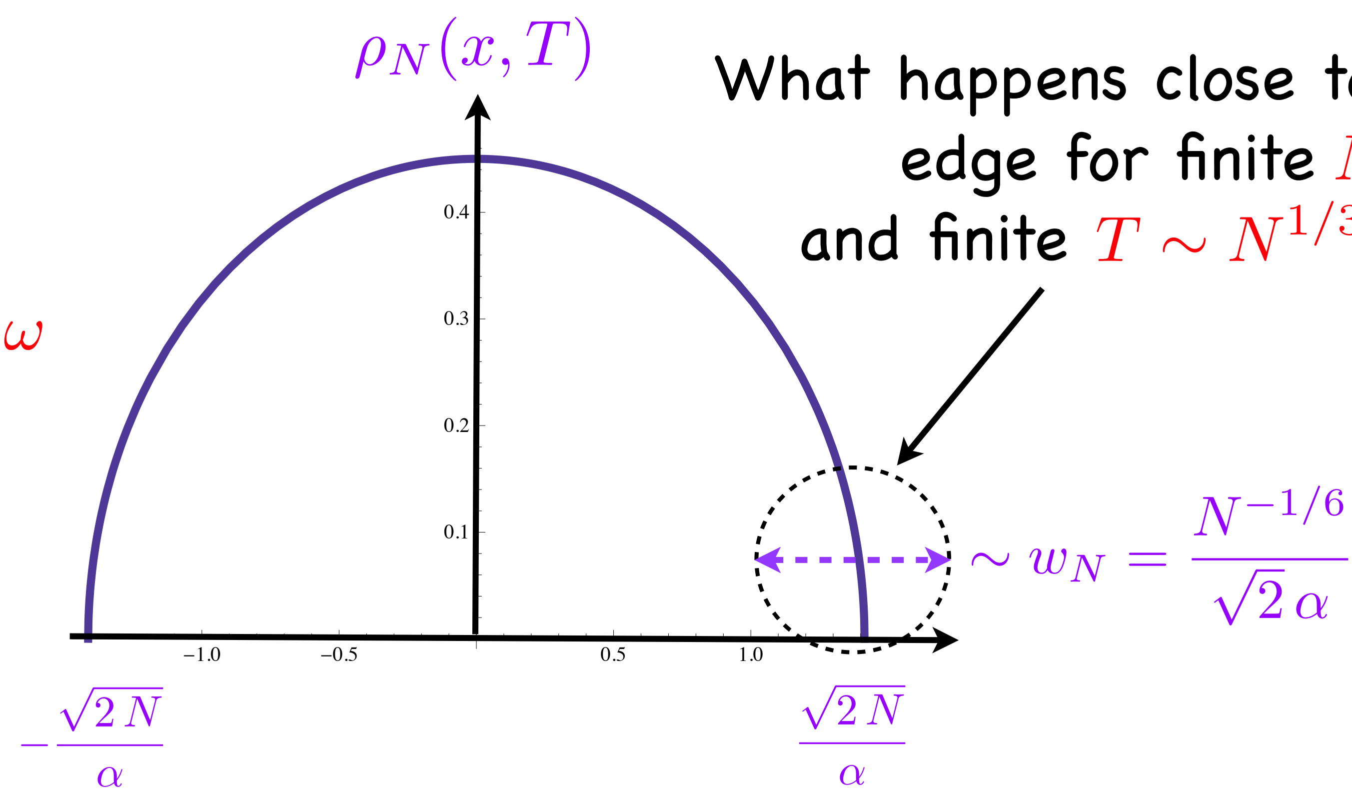

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$$\rho_N(x,T) \approx \frac{1}{N w_N} F_1\left(\frac{x - \sqrt{2N}/\alpha}{w_N}\right)$$

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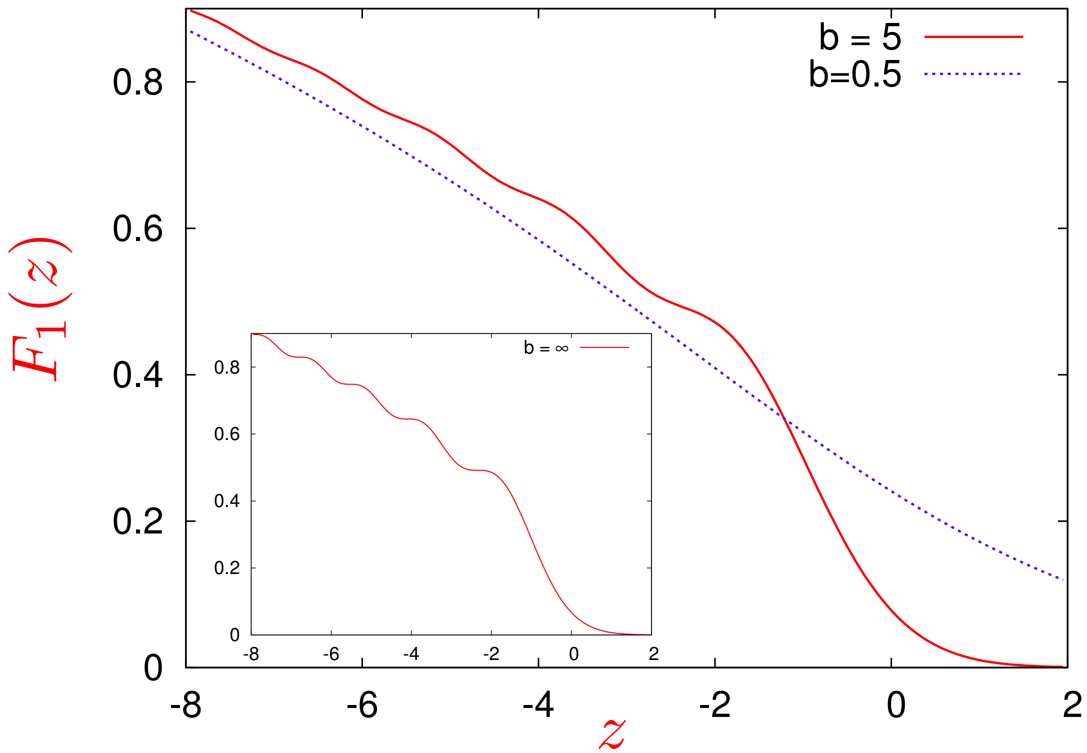

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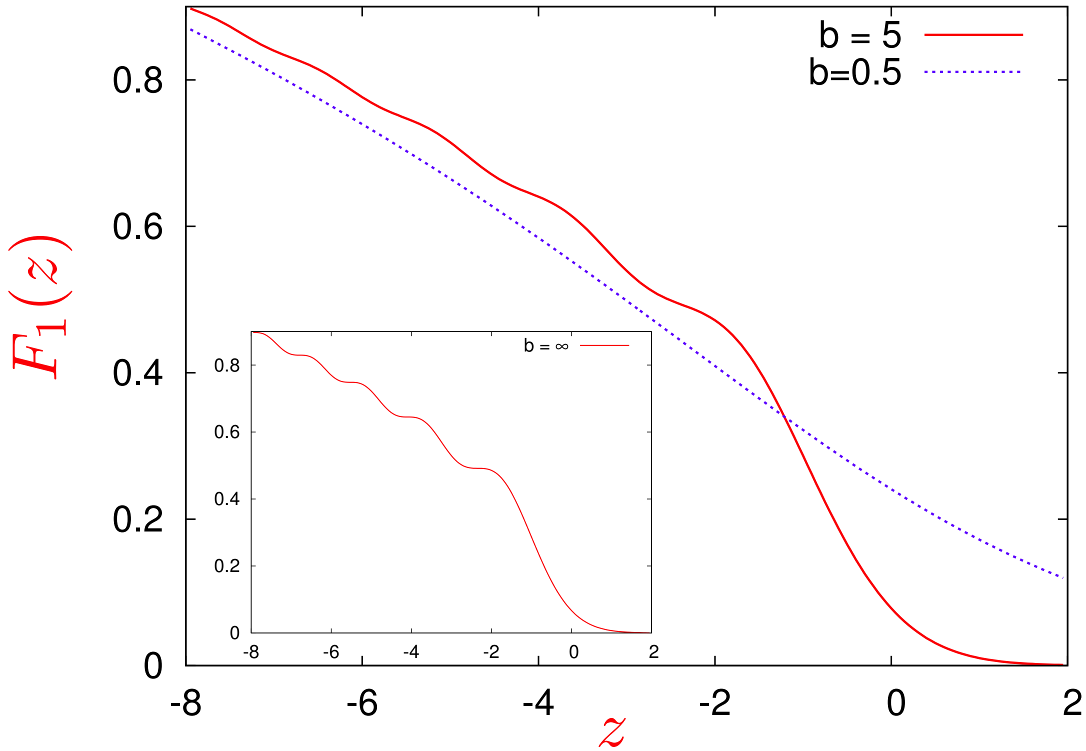

Asymptotic behaviors

$$F_1(z) \sim \begin{cases} \dfrac{\sqrt{|z|}}{\pi} \, , \; z \to -\infty \\ \exp(-b\,z) \, , \; z \to +\infty \end{cases}$$

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Correlation kernel for N free fermions at T > 0

- For $N \gg 1$ the canonical and grand-canonical ensembles coincide

canonical: number of particles N is fixed

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where $\hat{H}\,\varphi_k = \epsilon_k\,\varphi_k$ ← single-particle eigenfunction

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see also Moshe, Neuberger, Shapiro '94/Johansson '07

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Dean, Le Doussal, S. N. M., Schehr '14

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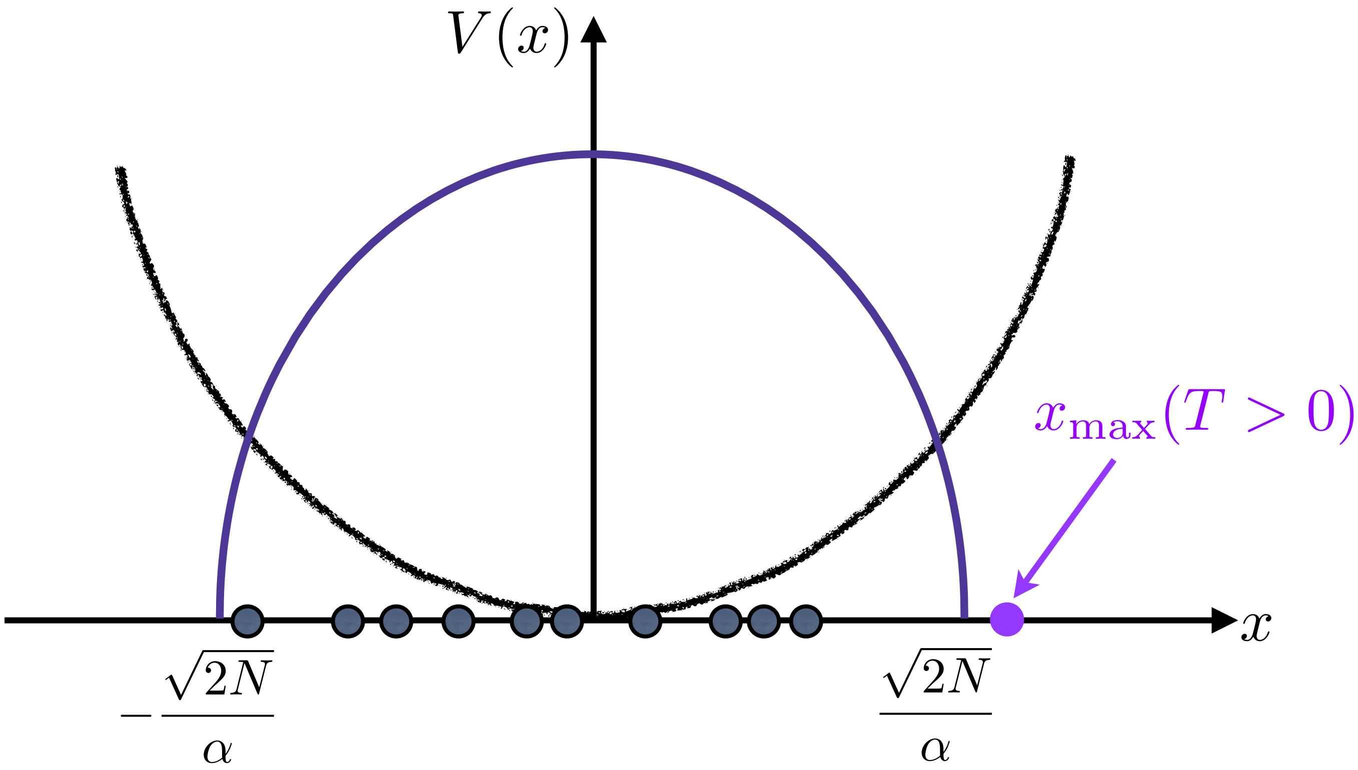
Position of the rightmost fermion at finite but low T



$T \sim N^{1/3}$

$r_{\text{edge}} = \sqrt{2N}/\alpha$

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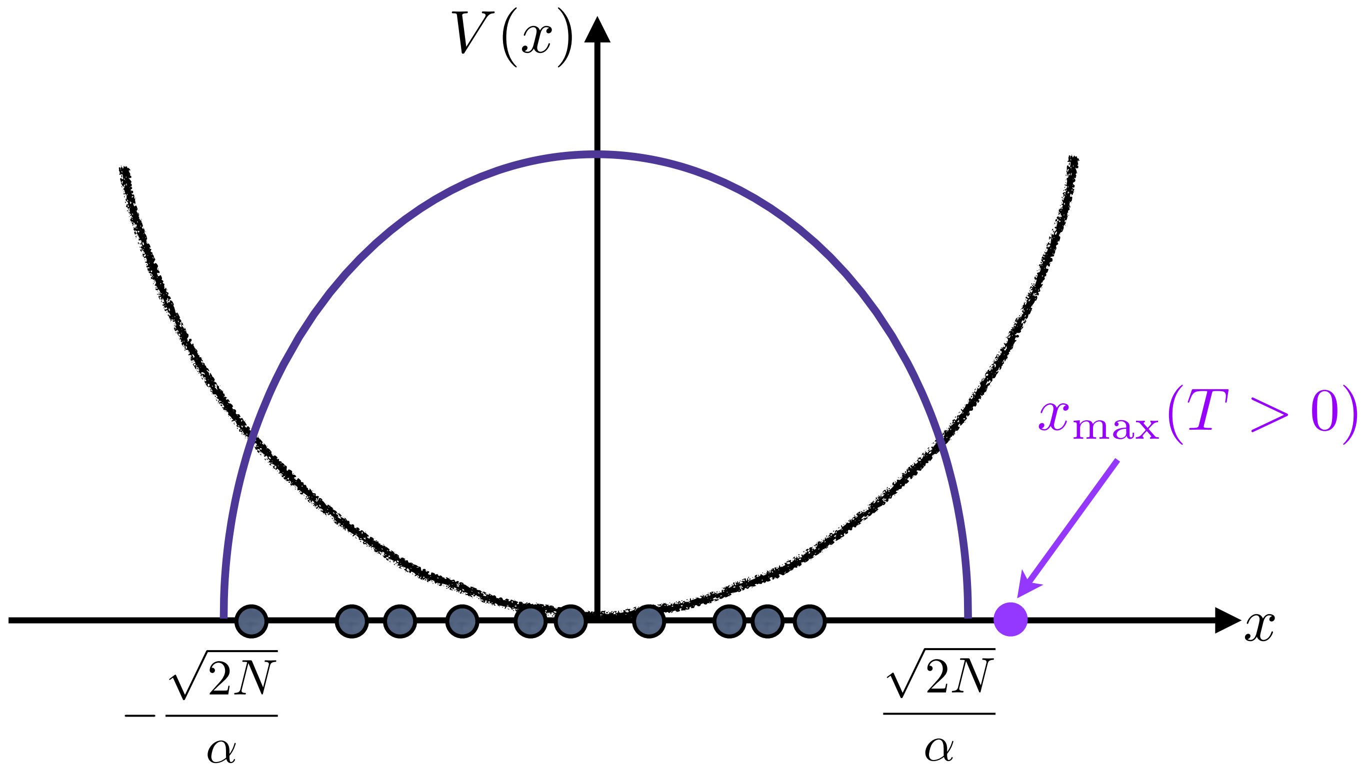

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- Fluctuations of $x_{\max}(T>0)$ — Dean, Le Doussal, S. N. M., Schehr '14

$$\Pr.(x_{\max}(T>0) \le M) \approx \mathcal{F}\left(\frac{M - r_{\rm edge}}{w_N}\right)$$

$$\mathcal{F}(\xi) = \det(I - P_\xi K_{\rm edge} P_\xi)\;,\; K_{\rm edge}(a,b) = \int_{-\infty}^{\infty} \frac{Ai(a+u)Ai(b+u)}{e^{-b\,u}+1}\,du$$

Kardar-Parisi-Zhang (KPZ) equation at finite time

- KPZ equation in 1+1 dimensions and curved geometry

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2} (\partial_x h)^2 + \sqrt{D}\eta(x,t)$$

$$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$$

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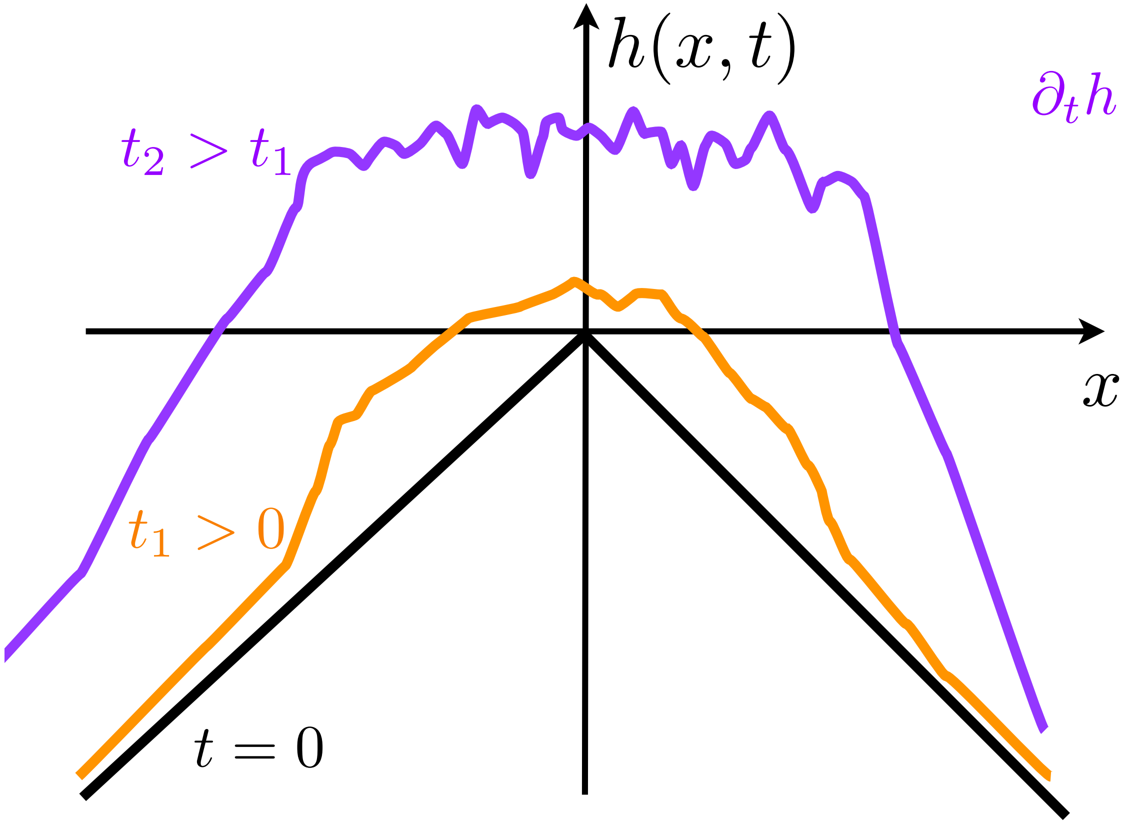

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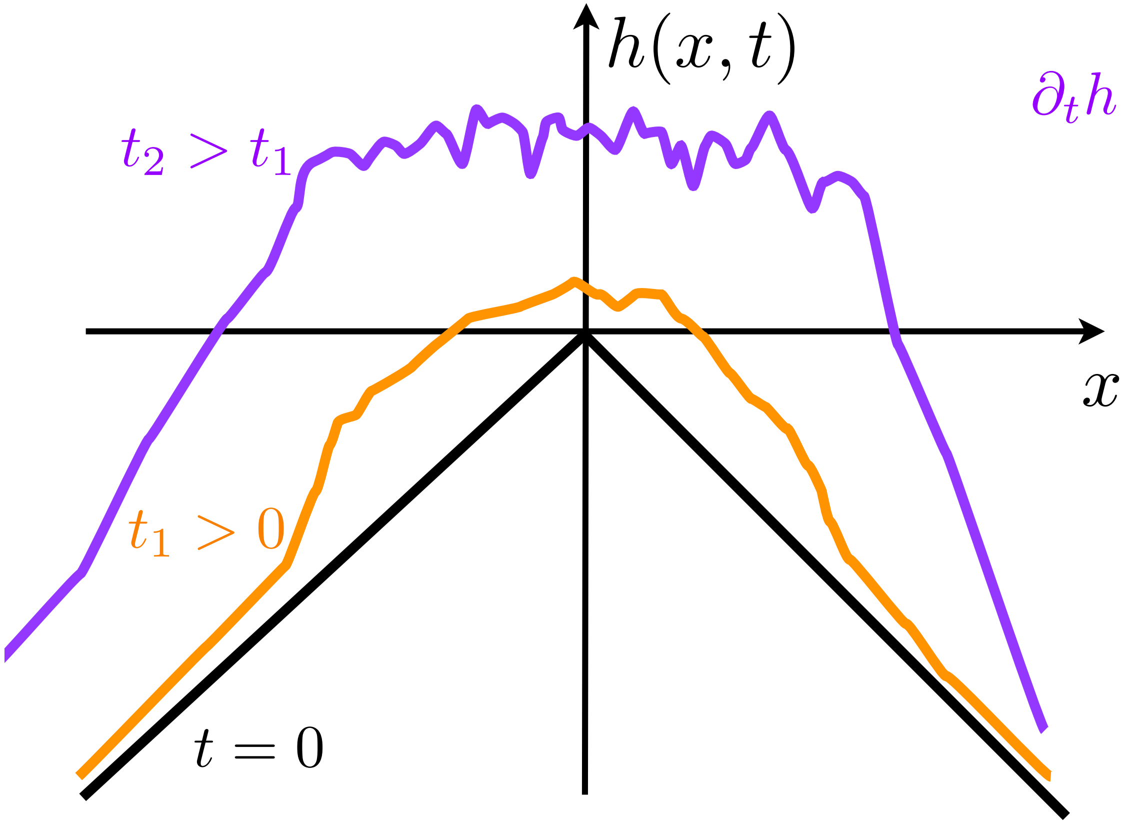

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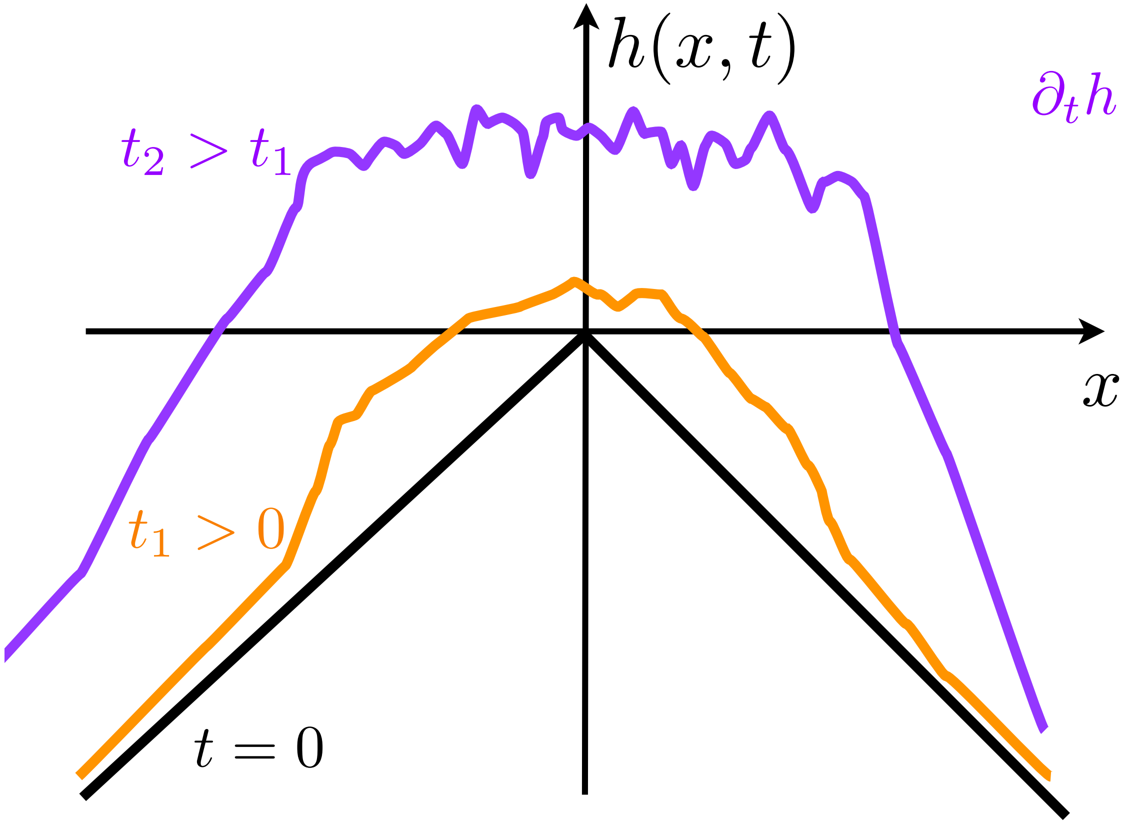

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Sasamoto, Spohn '10/Calabrese, Le Doussal, Rosso '10/Dotsenko '10/ Amir, Corwin, Quastel '11
Imamura, Sasamoto, Spohn '13

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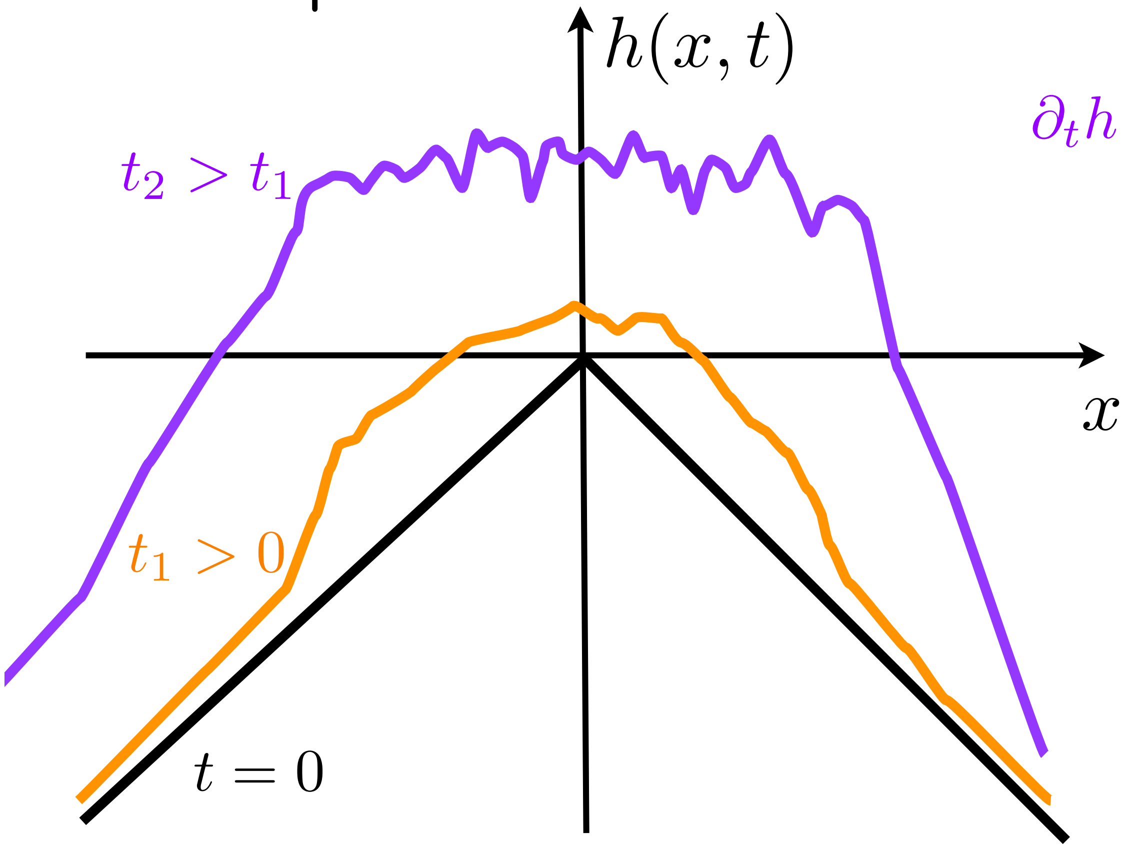

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- Time-dependent generating function of the height field

$$g_t(\zeta) = \langle \exp(-e^{\gamma_t(\tilde{h}(0,t)-\zeta)})\rangle$$

$$g_t(\zeta) = \det(I - P_\zeta K_{\text{bulk}} P_\zeta)$$

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Connection between fermions at finite temperature and KPZ at finite time

Connection between fermions at finite temperature and KPZ at finite time

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→ formal connection between the two problems with $1/T \Longleftrightarrow t^{1/3}$

Dean, Le Doussal, S. N. M., Schehr '14

Outline

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Average density of free fermions at T > 0

$$P_{\rm joint}(x_1, \cdots x_N) = \frac{1}{N! Z_N(\beta)} \sum_{k_1 < \cdots < k_N} \left[\det_{1 \leq i,j \leq N} (\varphi_{k_i}(x_j)) \right]^2 e^{-\beta(\epsilon_{k_1} + \cdots + \epsilon_{k_N})}$$

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- Exact expression for the av. density $\rho_N(x,T) = \frac{1}{N} \sum_{i=1}^{N} \langle \delta(x - x_i) \rangle$

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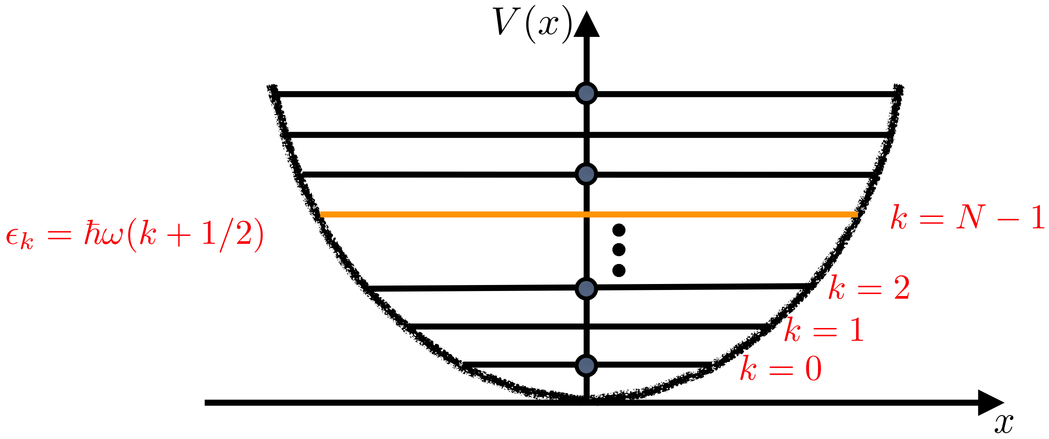
introduce the occupation numbers



$$m_k = \begin{cases} 0 \,, & \text{if state } k \text{ is empty} \\ 1 \,, & \text{if state } k \text{ is occupied} \end{cases}$$

Average density of free fermions at T > 0

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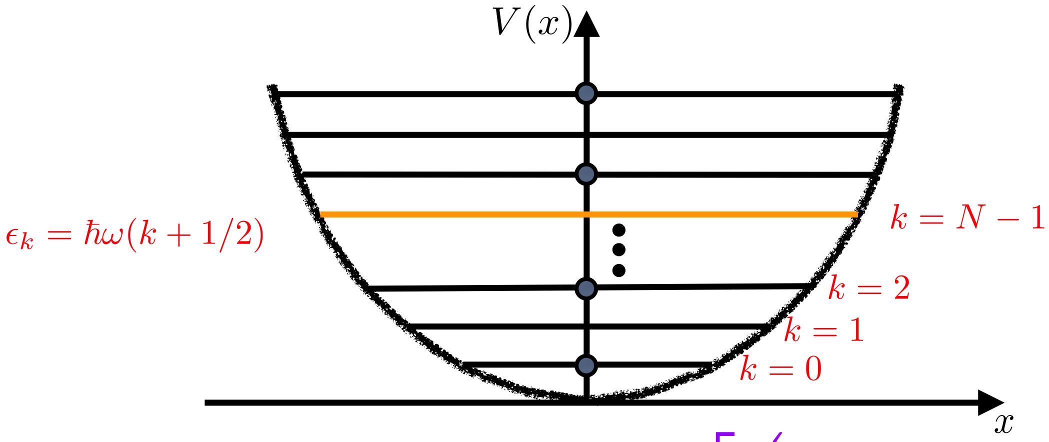

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$$\rho_N(x) = \frac{\mathcal{N}_N(x)}{N Z_N(\beta)}$$ **with** $$\mathcal{N}_N(x) = \sum_{\{m_k\}} \left[\left(\sum_{k\geq 0} m_k (\varphi_k(x))^2 \right) e^{-\beta \sum_{k\geq 0} m_k \epsilon_k} \delta\left(\sum_{k\geq 0} m_k , N \right) \right]$$

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similarly $\displaystyle \sum_{N\geq 0} z^N Z_N(\beta) = \prod_{j=0}^{\infty} (1 + z\, e^{-\beta\epsilon_j})$ grand-canonical partition function

Average density of free fermions at T > 0

- An exact formula (using Cauchy formula)

$$\rho_N(x) = \frac{1}{N} \frac{\oint \frac{dz}{2i\pi} z^{-(N+1)} \prod_{j=0}^{\infty} (1 + z\, e^{-\beta \epsilon_j}) \sum_{k \geq 0} (\varphi_k(x))^2 \frac{z\, e^{-\beta \epsilon_k}}{1 + z\, e^{-\beta \epsilon_k}}}{\oint \frac{dz}{2i\pi} z^{-(N+1)} \prod_{j=0}^{\infty} (1 + z\, e^{-\beta \epsilon_j})}$$

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Fermi factor

Correlation kernel for free fermions at T > 0

- Final result for the density for large N

$$\rho_N(x) \approx \frac{1}{N} \sum_{k \geq 0} (\varphi_k(x))^2 \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

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- n-point correlation functions for large N (by similar computations)

$$R_n(x_1, \cdots, x_n) \approx \det_{1\leq i,j \leq n} K_N(x_i, x_j)$$

where the correlation kernel is given by

$$K_N(x, x') = \sum_{k=0}^{\infty} \frac{\varphi_k(x)\varphi_k(x')}{e^{\beta(\epsilon_k - \mu)} + 1} \quad \text{and} \quad N = \sum_{k=0}^{\infty} \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

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Free fermions in a d-dimensional harmonic trap (T=0)

- Single particle Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}\right) + \frac{1}{2}m\omega^2\,\underbrace{\left(x_1^2 + \cdots x_d^2\right)}_{r^2}$$

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- Global density (at T=0)

$$\rho_N(\mathbf{x}) \approx \frac{1}{N}\left(\frac{m}{2\pi\hbar^2}\right)^{d/2} \frac{[\mu - \frac{1}{2}m\omega^2 r^2]^{d/2}}{\Gamma(d/2+1)}$$

with $\mu \approx \hbar\omega[\Gamma(d+1)\, N]^{1/d}$

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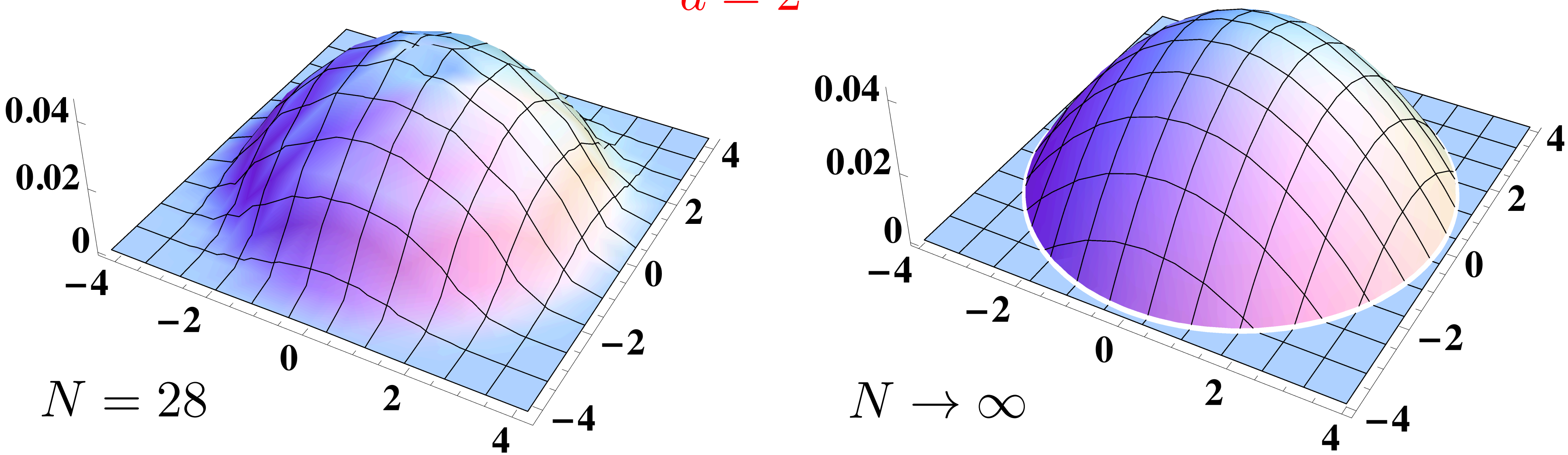
Dean, Le Doussal, S. N. M., Schehr, arXiv: 1505.01543

$d = 2$



Free fermions in a d-dimensional harmonic trap (T=0)

Dean, Le Doussal, S. N. M., Schehr, arXiv: 1505.01543

- Edge density of free fermions

$$\rho_{\mathrm{edge}}(\mathbf{x}) \approx \frac{1}{N}\frac{1}{w_N^d} F_d\left(\frac{r - r_{\mathrm{edge}}}{w_N}\right)$$

with $w_N = b_d\, N^{-\frac{1}{6d}}$ and $F_d(z) = \frac{1}{\Gamma(\frac{d}{2}+1)2^{\frac{4d}{3}}\pi^{\frac{d}{2}}}\int_0^\infty du\; u^{\frac{d}{2}} Ai(u + 2^{2/3}\, z)$

Free fermions in a d-dimensional harmonic trap (T=0)

Dean, Le Doussal, S. N. M., Schehr, arXiv: 1505.01543

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recall that $F_1(z) = [Ai'(z)]^2 - z[Ai(z)]^2$

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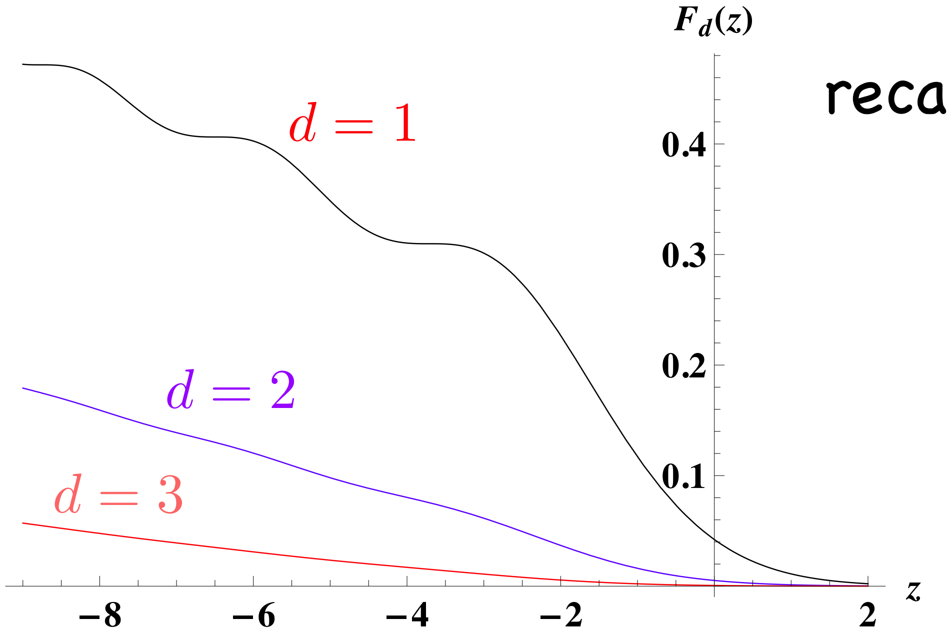

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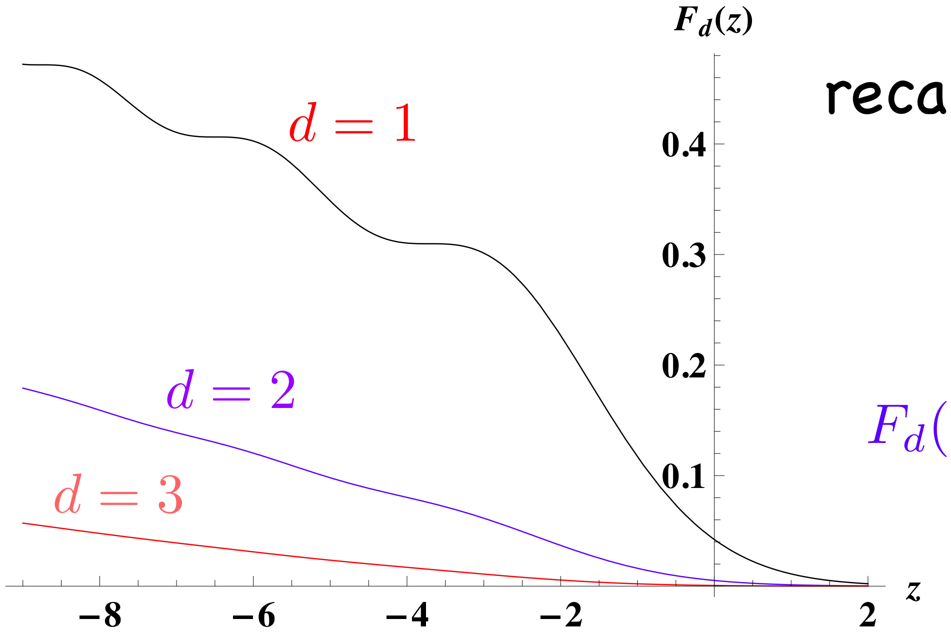

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Free fermions in a d-dimensional harmonic trap (T=0): limiting correlation kernels

$$K_N(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k}} \theta(E_F - \epsilon_{\mathbf{k}}) \psi_{\mathbf{k}}(\mathbf{x}) \psi_{\mathbf{k}}(\mathbf{y})$$

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$$K_N(\mathbf{x},\mathbf{y}) \approx \frac{1}{w_N^d}\mathcal{K}_{\rm edge}\left(\frac{\mathbf{x}-\mathbf{r}_{\rm edge}}{w_N}, \frac{\mathbf{y}-\mathbf{r}_{\rm edge}}{w_N}\right)$$

with

$$\mathcal{K}_{\rm edge}(\mathbf{a},\mathbf{b}) = \int \frac{d^d q}{(2\pi)^d} e^{-i\mathbf{q}\cdot(\mathbf{a}-\mathbf{b})} Ai_1\left(2^{\frac{2}{3}}q^2 + \frac{a_n+b_n}{2^{1/3}}\right)$$

$a_n = \mathbf{a}\cdot\mathbf{r}_{\rm edge}/r_{\rm edge}$ and $b_n = \mathbf{b}\cdot\mathbf{r}_{\rm edge}/r_{\rm edge}$

$$Ai_1(z) = \int_z^\infty Ai(u)du$$

Outline

- Free fermions in d=1 and T=0 and Random Matrix Theory (RMT)
- Free fermions in d=1 and T>0 and KPZ equation: main results
- Sketch of the derivation of our results
- Extension to higher dimensions, d>1
- Conclusion

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- Exact results for Free fermions in d=1 at finite temperature T>0

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- Universality of these new kernels: they do not depend on the trapping potential (for spherically symmetric trap with a single minimum)

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- Can one observe these kernels in cold atoms experiments ?