Additivity property and mass fluctuation in conserved-mass transport processes

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References

- A. Das, S. Chatterjee, P. Pradhan, and P. K. Mohanty (accepted in Phys. Rev. E).
- S. Chatterjee, P. Pradhan, and P. K. Mohanty, Phys. Rev. Lett. **112**, 030601 (2014).

Outline of the talk

Introduction

- Additivity and equilibrium thermodynamics

Question: Why do certain distributions frequently occur in systems irrespective of whether the systems are in or out of equilibrium?

- Gamma/gammalike distributions
- Simple power-law scaling

Additivity in conserved-mass transport processes

Calculation of mass distributions using additivity

Summary and outlooks

Additivity and equilibrium thermodynamics

Equilibrium: No current

- Detailed balance: $P(C)w(C \to C') = P(C')w(C' \to C)$

Additivity and thermodynamics

- Additivity property (short-ranged interaction):

$$\mathcal{P}(N_1, N_2) \sim \text{const.} e^{-\beta F(N_1)} e^{-\beta F(N_2)},$$

  where $\exp[-\beta F(N)] = \sum_{C_k} \exp[-\beta H(C_k)]$ obtained from a priori known probability weights $P(C) \propto e^{-\beta H(C)}$.
- Additivity leads to a remarkable thermodynamic structure: Unified characterization of fluctuations in terms of thermodynamic variables.

Nonequilibrium steady state (NESS)

- Nonzero current (no detailed balance)
- Time-stationary fluctuations;

Steady state probability in most cases are not known.

Can we extend equilibrium thermodynamics to the domain of NESS?

Additivity in NESS

Provided conservation law and additivity

(i) Total mass $\sum_i m_i = M \equiv \rho V$ is conserved (ρ density).
(ii) Joint probability distribution of subsystem masses has a factorized form

$$\mathcal{P}(m_1, m_2, \cdots, m_\nu) = \frac{\prod_{k=1}^{\nu} w_v(m_k)}{Z(M,V)} \delta\left(\sum_{k=1}^{\nu} m_k - M\right), \quad (1)$$

not too restrictive (expected to hold for $\xi \ll v$); $w_v(m_k)$ still unknown
[Lebowitz *et. al.*, J. Stat. Phys. (1996); Bertin *et. al.*, Phys. Rev. Lett. (2006)]

Consequence: Subsystem mass distribution in the steady state follows

$\mathrm{Prob}(m_k = m) \equiv P_v(m)$ determined solely from $\sigma^2(\rho) = (\langle m^2 \rangle - \langle m \rangle^2)/v$.

$P_v(m)$ in a broad class of NESS can actually be calculated using additivity.

Fluctuation-response relation ⇒ chemical potential and free energy

$$\frac{d\rho}{d\mu} = \sigma^2(\rho) \Rightarrow \mu(\rho) = \frac{df}{d\rho} \tag{2}$$

$$\lambda_v(s) = v\{\inf_\rho[f(\rho) + s\rho]\}$$

$$\tilde{w}_v(s) = \int_0^\infty w_v(m) e^{-sm} dm = e^{\lambda_v(s)}$$

Mass distribution function

$$P_v(m) \propto w_v(m) e^{\mu m} \tag{3}$$

Illustration

We substantiate our claims by calculating $P_v(m)$ from the variance in the following two cases:

$$\sigma^2(\rho) = \frac{\rho^2}{\eta}$$

$$\sigma^2(\rho) = \frac{g(\rho)}{(\rho_c - \rho)^n} \; ; \; \rho \leq \rho_c, n = 1$$

Mass transport processes: masses on a lattice (ring)

Dynamics

- Chipping, diffusion and coalescence masses (continuous and discrete)
- Symmetric and asymmetric mass transfer
- Random sequential (RSU), parallel (PU) or mixed updates

Features

- Total “mass” conserved
- Homogeneous systems (periodic boundary)

Contd.

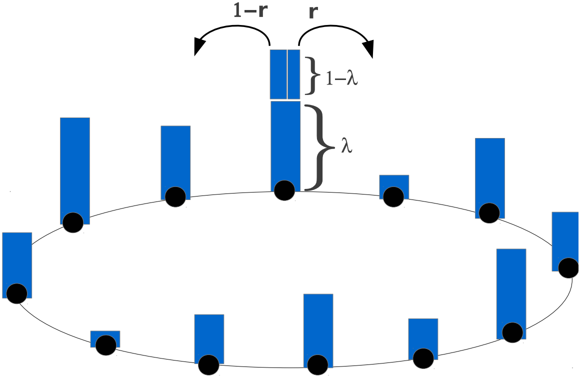
Mass chipping models (MCM)

Mass exchange models (MEM)

Mass aggregation models (MAM)

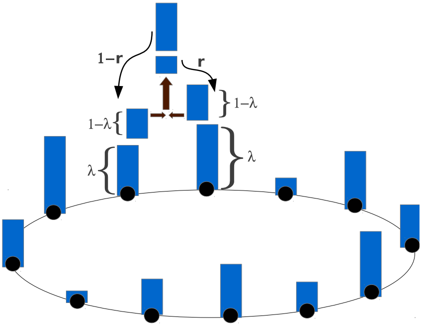
Mass chipping models (MCM)

$(1 - \lambda)$ fraction chipped off, r and $(1 - r)$ fractions of which diffuse and coalesce with one of the nearest neighbours; random $r \in [0, 1]$;



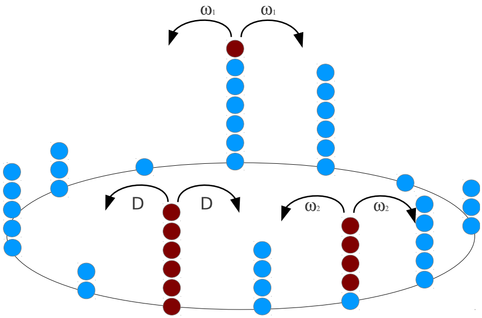
Mass exchange models (MEM)

$(1-\lambda)$ fraction chipped off from two nearest neighbours; r and $(1-r)$ fractions of total chipped-off masses exchanged.



Mass aggregation model (MAM)

A finite mass of $\Delta = 1, 2, \ldots, \Delta_0$ chipped off and diffuse to one of the neighbours; whole mass diffuse and aggregate with one of the neighbours.



Previous studies

MCM: $\lambda = 0$ and PU $\Rightarrow$ product measure; $\lambda \neq 0$, spatial structure unknown

- "q" model of force fluctuation (Liu *et. al.*, Science 1995; Coppersmith *et. al.*, PRE 1996)
- Asymmetric random average process (Krug and Garcia, J. Stat. Phys. 2000; Rajesh and Majumdar, J. Stat. Phys. 2000; Zielen and Schadsneider, J. Stat. Phys. 2002 and J. Phys. A 2003)
- Mass chipping models (Bondyopadhyay and Mohanty, J. Stat. Mech. 2012)

MEM: $\lambda = 0$ and RSU (equilibrium KMP) $\Rightarrow$ product measure; $\lambda \neq 0$, spatial structure unknown

- Energy distribution (Kipnis, Marchioro, and Presutti, J. Stat. Phys. 1982)
- Wealth distribution (Chakraborti and Chakrabarti, Eur. Phys. J. B 2000; Patriarca, Chakraborti, and Kaski, Phys. Rev. E 2004; Mohanty, Phys. Rev. E 2006; Matthes and Toscani, J. Stat. Phys. 2008; Yakovenko and Rosser, Rev. Mod. Phys. 2009; Patriarca, Heinsalu and Chakraborti, Eur. Phys. J. B 2010)

Contd.

MAM: Spatial structure unknown

- Models for mass aggregation and polymerization (Majumdar, Krishnamurthy and Barma, Phys. Rev. Lett. 1998; Majumdar, Krishnamurthy and Barma, J. Stat. Phys. 2000; Rajesh and Krishnamurthy, Phys. Rev. E 2002; Rajesh, Das, Chakraborty and Barma, Phys. Rev. E 2002; Shrivastav, Banerjee and Puri, Phase Transit. 2010)

- **Exact steady-state weights in most cases are not known.**

- **Calculation of $P_v(m)$, even within MFT, can be difficult!**

- **Additivity can provide an analytic (exact or approximate) expression of $P_v(m)$.**

Gamma distribution

- The scaled variance:

$$\sigma^2(\rho) = \frac{\rho^2}{\eta}$$

- Chemical potential:

$$\mu(\rho) = \int \frac{1}{\sigma^2(\rho)} d\rho = -\frac{\eta}{\rho} + \alpha$$

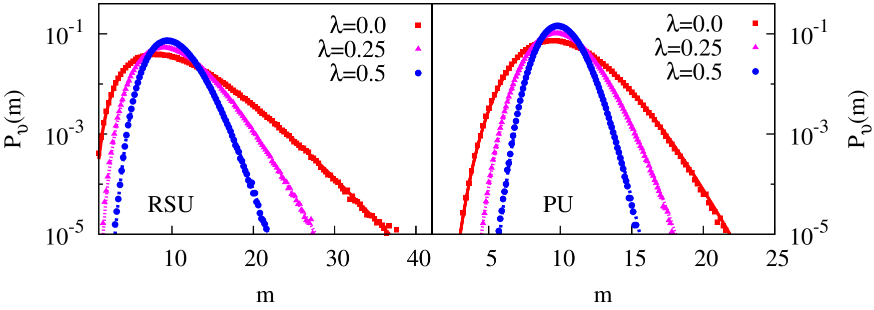
- Mass distribution:

$$\tilde{w}_v(s) = \frac{\text{const.}}{(s-\alpha)^{\eta v}} \Rightarrow P_v(m) = \text{const.} \times m^{\eta v - 1} e^{\mu m}$$

(Chatterjee, Pradhan and Mohanty, PRL 2014)

Asymmetric MCM (gamma distribution)

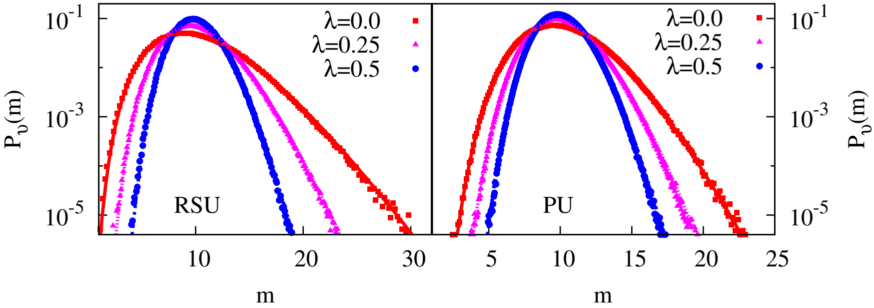
Correlation $c(r) = 0$ for $r \neq 0$; $P_v(m)$, $v \gg 1$, exact within additivity.
(Chatterjee, Pradhan and Mohanty, PRL 2014; Das, Chatterjee, and Pradhan)



Simulations: $\rho = 1$, $v = 10$ and $L = 5000$; random sequential update (RSU) and parallel update (PU).

Symmetric MCM (gamma distribution)

Correlation $c(r) \neq 0$; $P_v(m)$, $v \gg 1$, exact within additivity.
(Das, Chatterjee, and Pradhan)

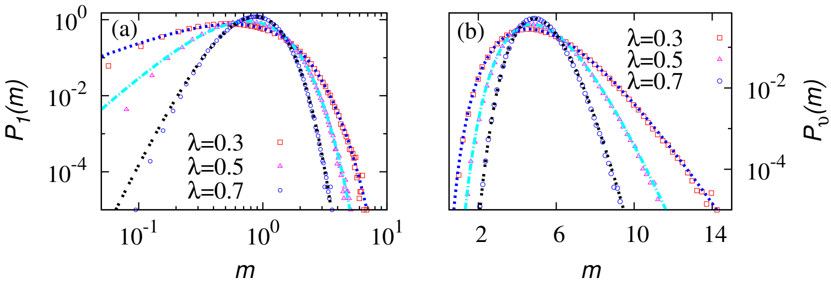


Simulations: $\rho = 1$, $v = 10$ and $L = 5000$; random sequential update (RSU) and parallel update (PU).

MEM (gamma distribution)

Correlation $c(r) = 0$ for $r \neq 0$; $P_v(m)$, for $v \gg 1$, exact within additivity;
$P_1(m)$ still approximate!

(Chatterjee, Pradhan and Mohanty, PRL 2014; Das, Chatterjee, and Pradhan)



Simulations: $\rho = 1$ and $L = 1000$.

Mean-field moments in MEM

[Repetowicz, Hutzler, and Richmond, Physica (2005)]

$$\langle m \rangle = \rho$$

$$\langle m^2 \rangle = \frac{(\lambda+2)}{(2\lambda+1)}\rho^2$$

$$\langle m^3 \rangle = \frac{3(\lambda+2)}{(2\lambda+1)^2}\rho^3 = \frac{(6+3\lambda)}{(4\lambda^2+4\lambda+1)}\rho^3$$

$$\langle m^4 \rangle = \frac{(72+12\lambda-2\lambda^2+9\lambda^3-\lambda^5)}{(2\lambda+1)^2(2\lambda^3-\lambda^2+6\lambda+3)}\rho^4$$

$$= \frac{(72+12\lambda-2\lambda^2+9\lambda^3-\lambda^5)}{(3+18\lambda+35\lambda^2+22\lambda^3+4\lambda^4+8\lambda^5)}\rho^4$$

Contd.

$$\langle m^5 \rangle = \frac{5(288 + 456\lambda - 208\lambda^2 + 278\lambda^3 + 56\lambda^4 - 83\lambda^5 + 28\lambda^6 + \lambda^7 - 6\lambda^8)}{(1 + 2\lambda)^3(12 + 48\lambda + 35\lambda^2 + 55\lambda^4 - 12\lambda^5 + 12\lambda^6)}\rho^5$$

$$= \frac{5(288 + 456\lambda - 208\lambda^2 + 278\lambda^3 + 56\lambda^4 - 83\lambda^5 + 28\lambda^6 + \lambda^7 - 6\lambda^8)}{(12 + 120\lambda + 467\lambda^2 + 882\lambda^3 + 859\lambda^4 + 598\lambda^5 + 600\lambda^6 + 368\lambda^7 + 48\lambda^8 + 96\lambda^9)}$$

$$\langle m^6 \rangle = \frac{\rho^6}{(1 + 2\lambda)^4(3 + 6\lambda - \lambda^2 + 2\lambda^3)(4 + 8\lambda - 3\lambda^2 + 6\lambda^3)(5 + 10\lambda - 6\lambda^2 + 13\lambda^3 - 3\lambda^4 + 2\lambda^5)}$$
$$\times\ (43200 + 120960\lambda - 2160\lambda^2 + 936\lambda^3 + 192456\lambda^4 - 168906\lambda^5 +$$
$$87650\lambda^6 + 2170\lambda^7 - 35652\lambda^8 + 19643\lambda^9 - 3758\lambda^{10} - 2265\lambda^{11} +$$
$$1070\lambda^{12} - 194\lambda^{13} - 24\lambda^{14} + 24\lambda^{15})$$

$$= \rho^6(43200 + 120960\lambda - 2160\lambda^2 + 936\lambda^3 + 192456\lambda^4 - 168906\lambda^5 +$$
$$87650\lambda^6 + 2170\lambda^7 - 35652\lambda^8 + 19643\lambda^9 - 3758\lambda^{10} - 2265\lambda^{11} +$$
$$1070\lambda^{12} - 194\lambda^{13} - 24\lambda^{14} + 24\lambda^{15})/(60 + 840\lambda + 4903\lambda^2 + 15442\lambda^3 + 28869\lambda^4$$
$$+35697\lambda^5 + 38177\lambda^6 + 41969\lambda^7 + 34695\lambda^8 + 21038\lambda^9 + 17804\lambda^{10} + 9848\lambda^{11} + 2128\lambda^{12}$$
$$+3488\lambda^{13} - 192\lambda^{14} + 384\lambda^{15})$$

Any analytic expression of $P_1(m)$ could not be obtained earlier.

Additivity provides a closed form, though approximate, analytic expression of $P_1(m)$.

Comparison between MFT and additivity

The following table is for $\lambda = 1/2$, $\rho = 1$.

Table : Numerical values of moments

Moments	Additivity	Mean-field
$\langle m \rangle$	1	1
$\langle m^2 \rangle$	1.25	1.25
$\langle m^3 \rangle$	1.875	1.875
$\langle m^4 \rangle$	3.281	3.274
$\langle m^5 \rangle$	6.562	6.511
$\langle m^6 \rangle$	14.765	14.490

CMAM: Variance and chemical potential

- $D = 1, w_1 = 1, w_2 = 0$:

$$\sigma^2(\rho) \simeq \frac{\rho(1+\rho)(1+\rho^2)}{(\sqrt{2}+1+\rho)(\rho_c-\rho)}$$

$$\mu(\rho) \simeq -2\tan^{-1}\rho + \ln\left(\frac{\rho}{1+\rho}\right) + \alpha$$

- $D = 1, w_1 = 0, w_2 = 1$:

$$\sigma^2(\rho) \simeq \frac{\rho(1-\rho)(2\rho^2-2\rho+1)}{2\rho^2-4\rho+1}$$

$$\mu(\rho) \simeq 2\tan^{-1}(1-2\rho) - \ln\left[\frac{1}{2\rho(1-\rho)} - 1\right] + \alpha$$

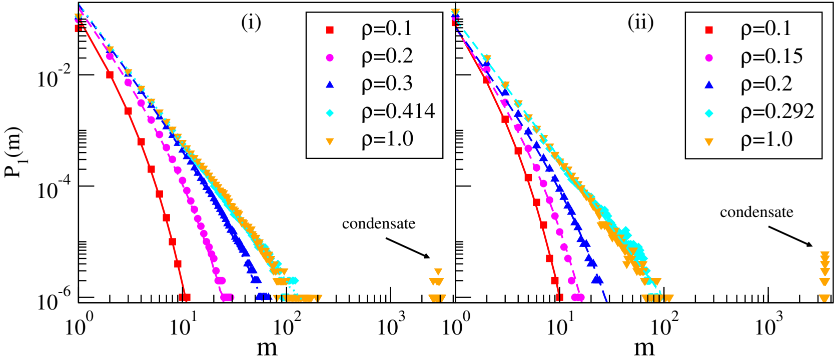
Single-site mass distribution

- $\tilde{w}(s) = e^{-\lambda_v(s)} \simeq e^{va_0}[1 + va_1(s - s_c) + va_2(s - s_c)^{3/2}]$
- Mass distribution for $m \gg 1$:

$$P_1(m) \propto \frac{1}{m^{5/2}} e^{(\mu(\rho)-\mu(\rho_c))m}$$

(Das, Chatterjee, Pradhan and Mohanty, PRE 2015)

Single-site mass distribution



(Das, Chatterjee, Pradhan and Mohanty, PRE 2015)

Summary and outlooks

- Characterization of mass fluctuation in out-of-equilibrium systems with a mass conserving dynamics and short-ranged spatial correlations.
- Explanation of why conserved-mass transport processes often exhibit gammalike mass distribution and simple power-law scaling.
- Formulation of a steady-state thermodynamics.
- What happens in the absence of conservation law, in continuum, etc?

THANK YOU