The Nonlinear Schrodinger Equation, Nonlinear Fluctuating Hydrodynamics and the Kardar-Parisi-Zhang Universality Class

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PRA, 2015 (**Kulkarni**, Huse, Spohn)
PRA Rapid, 2013 (**Kulkarni** & Lamacraft)
Kulkarni, 2015 (unpublished)

Collaboration

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Contents

- Why NLS/GPE ?
- Brute Force discrete-NLS Numerics
- Nonlinear Fluctuating Hydrodynamics and KPZ family

Why NLS/GPE ?

$$H = \int dx \left[\frac{|\partial_x \psi|^2}{2m} + \frac{g}{2} |\psi|^4 \right]$$

$$i \partial_t \psi = -\frac{1}{2m} \partial_x^2 \psi + g |\psi|^2 \psi$$

Discrete-NLS (DNLS), non-integrable
Discrete-NLS (Ablowitz-Ladik), integrable

Vector-NLS, 2D NLS realizations

Experimental realizations in cold atoms and non-linear optics (continuum and discrete)

Gain-Loss NLS (non-Hamiltonian) realizations

Rich regimes (emergent-conserved quantities)

Non-local NLS (lattice-non-integrable, lattice-integrable, continuum)

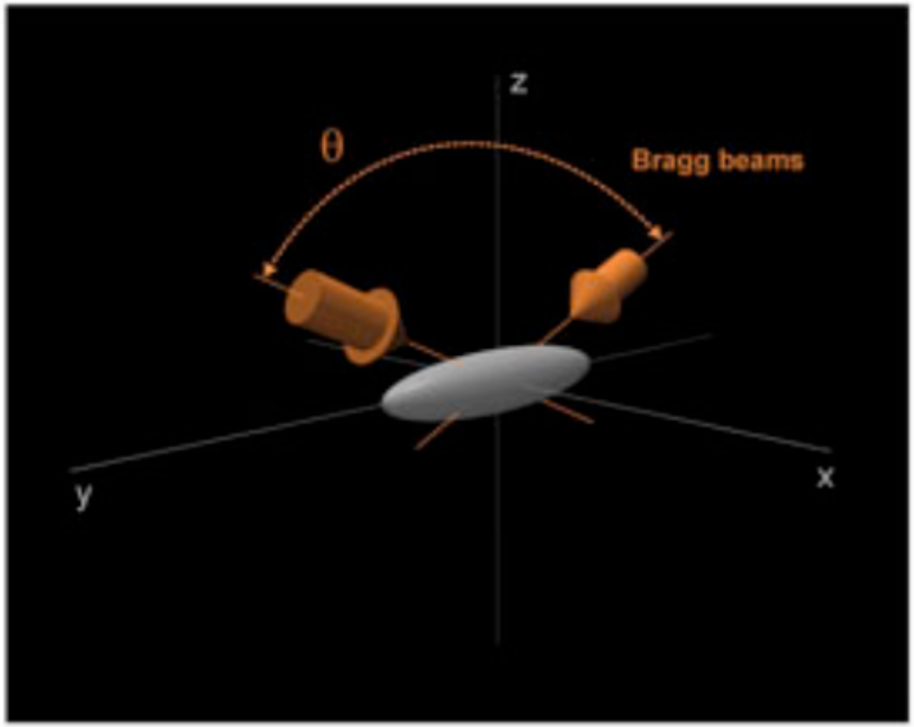
Dynamical Structure Factor

S(k,ω) measures dynamical density fluctuations in a system

$$S(k,\omega) = \int dx dt e^{i(\omega t - kx)} \langle \rho(x,t)\rho(0,0)\rangle$$

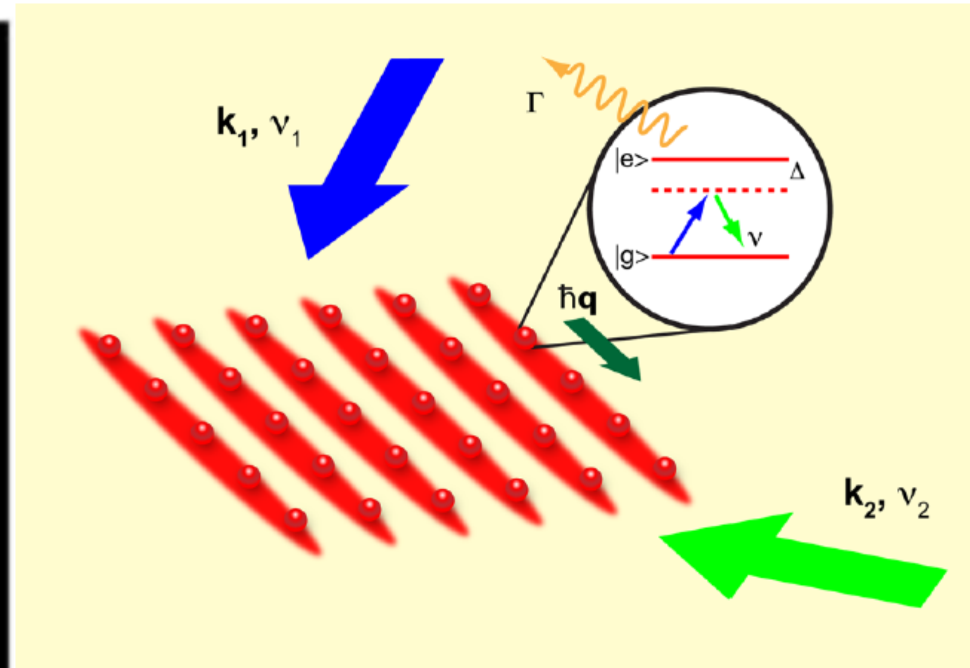
Linearized hydrodynamics fails at long wavelengths

Experiments: S(k,ω) measurements in Cold Atoms



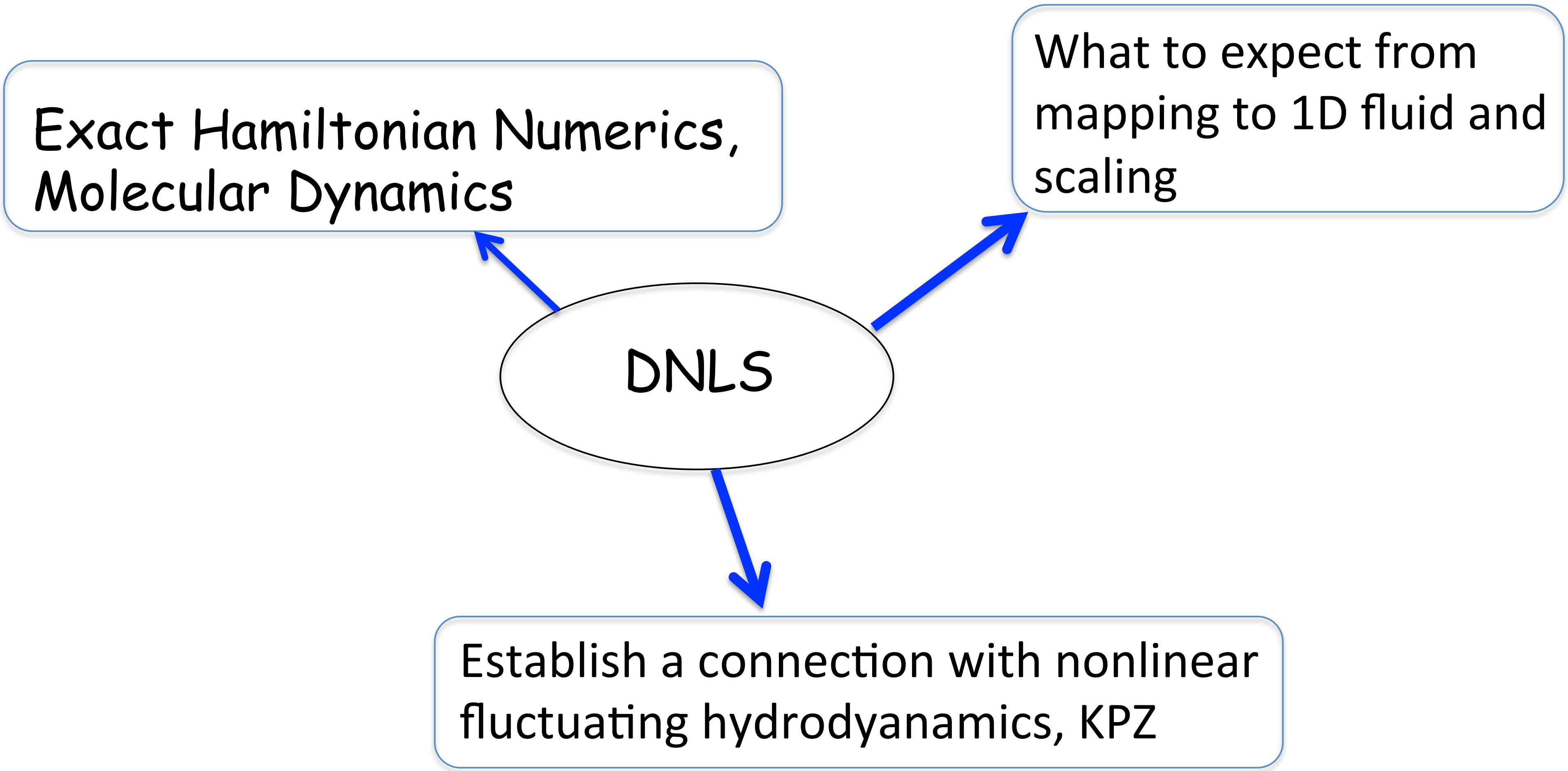
Bragg spectroscopy

Florence, MIT.....



Fabbri et al, PRA 79, 043623
Fabbri et al, PRA 83, 031604

$$E(\nu,q) \propto \nu S(\nu,q)$$


Exact Hamiltonian Numerics, Molecular Dynamics
What to expect from mapping to 1D fluid and scaling
DNLS
Establish a connection with nonlinear fluctuating hydrodynamics, KPZ

1D Discrete-NLS (generic, non-integrable)

$$H = \sum_{j=0}^{N-1} \left(\frac{1}{2m} \, |\psi_{j+1} - \psi_j|^2 + \frac{1}{2} \, g \, |\psi_j|^4 \right)$$

DNLS

$\downarrow$

$$\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}t} \psi_j = -\frac{1}{2m} \Delta \psi_j + g \, |\psi_j|^2 \, \psi_j$$

g>0, finite lattice spacing “a”

$$\mathcal{T}_\tau : \qquad \tilde{\psi}(k_q, t) \to e^{-ik_q^2 \tau/2} \tilde{\psi}(k_q, t)$$

$$\mathcal{V}_\tau : \qquad \psi(j, t) \to e^{-i\tau |\psi(j,t)|^2} \psi(j, t)$$

$$\mathcal{V}_{\frac{\tau}{2}} \cdot \mathcal{T}_\tau \cdot \mathcal{V}_{\frac{\tau}{2}} \qquad \text{numerical protocol}$$

Leap-frog, additional terms generated by discretizing time are higher order in spacial gradients and not expected to change low k behavior

$\downarrow$

$S(k,\omega)$, $S(j,t)$

PRA Rapid, 2013 (**Kulkarni** & Lamacraft)

1D Discrete-NLS (generic, non-integrable)

Initial conditions (near thermal):

$$\varrho(x) = \sqrt{\frac{\rho_0}{2L}} \sum_{k \neq 0} e^{-\kappa_k} \left(b_k e^{ikx} + \text{c.c} \right)$$

$$\theta(x) = \frac{i}{\sqrt{2\rho_0 L}} \sum_{k \neq 0} e^{\kappa_k} \left(b_k e^{ikx} - \text{c.c} \right)$$

$$\langle |b_k|^2 \rangle = \frac{T}{\mathcal{E}_k}$$

temperature

Claim: DNLS equilibrates fairly rapidly, should not depend on choice of initial conditions

Kulkarni & Lamacraft (PRA-Rapid, 2013)

Kulkarni, Huse, Spohn (PRA, 2015)

Confirmed: Mendl & Spohn (J. Stat. Mech, 2015),

Simulations using true $e^{-\beta H}$

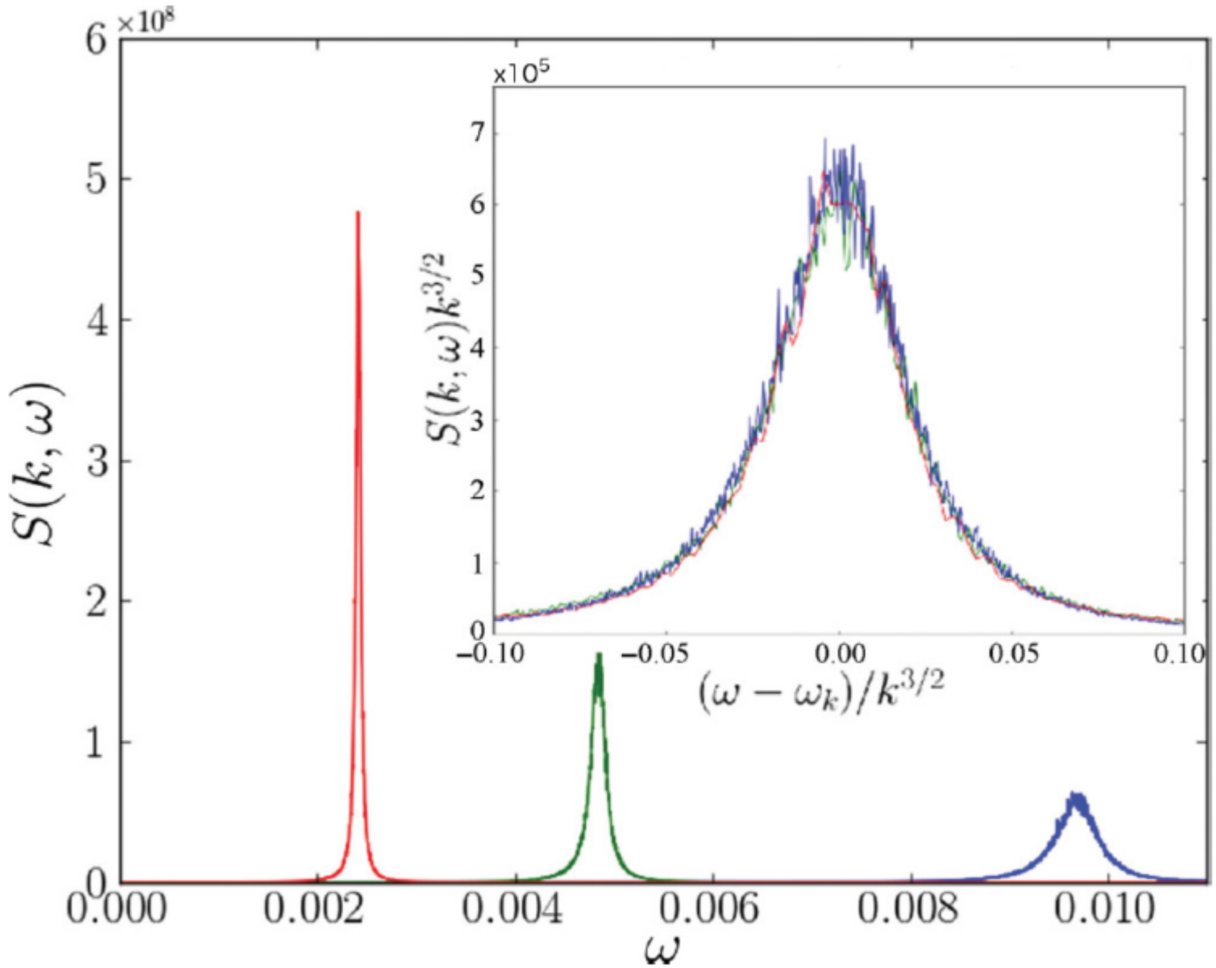
What to expect ?

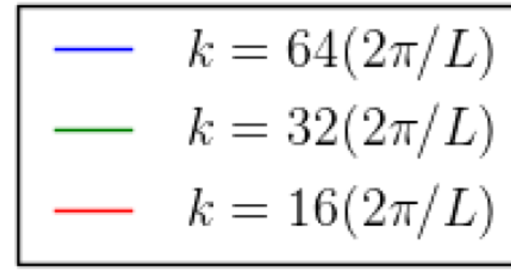
Henk van Beijeren (PRL, 2012) : Generic non-integrable 1D classical Hamiltonians with short range interactions in general fall into KPZ.

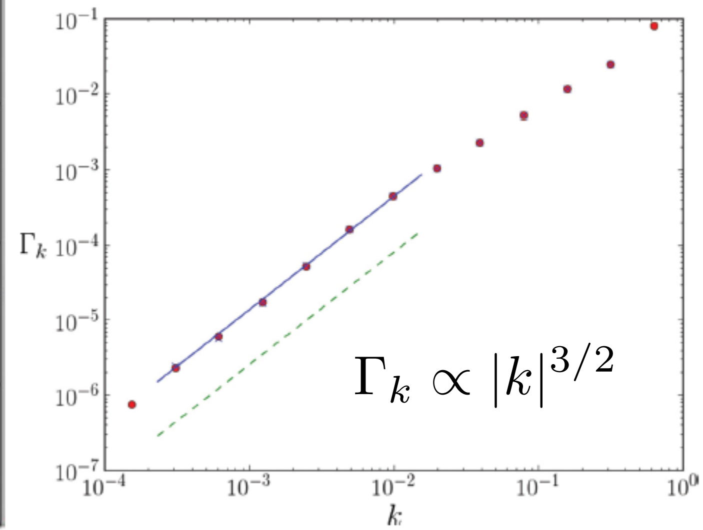
Several other reasons to expect that DNLS shows anomalous behaviour.

Exact Hamiltonian Numerics (Discrete-NLS)

Kulkarni & Lamacraft, PRA, Rapid (2013)







$$z = 1.510 \pm 0.018$$

$$S^{(\pm)}_{\text{phonon}}(k,\omega) \propto \frac{1}{\Gamma_k} f_{\text{PS}}\left(\frac{\omega \pm c|k|}{\Gamma_k}\right)$$

Prahofer-Spohn Scaling function ?

Similar works in other models:
Dhar, Spohn, Mendl,
Saito ......

NLS to classical 1D fluids

$$i\partial_t \psi = -\frac{1}{2m}\partial_x^2 \psi + g|\psi|^2\psi$$

$$\psi = \sqrt{\rho} e^{im \int_0^x v(x')dx'}$$

$$\begin{aligned} \partial_t \rho &= -\partial_x (\rho v) \\ \partial_t v &= -\partial_x \left(\frac{v^2}{2} + \frac{g\rho}{m} + \frac{1}{2}\frac{\partial_x^2 \sqrt{\rho}}{\sqrt{\rho}} \right) \end{aligned}$$

Sometimes relevant,
Sometimes irrelevant

Defining, $v_\pm = v \pm \sqrt{g\rho/m}$

Above can be written as,

$$\partial_t v_\pm + v_\pm \partial_x v_\pm = \frac{1}{3}(\partial_t + v_\pm \partial_x) v_\mp$$

One sector acts as a noise and dissipative agent to the other sector (noisy-burgers-like equation, KPZ)

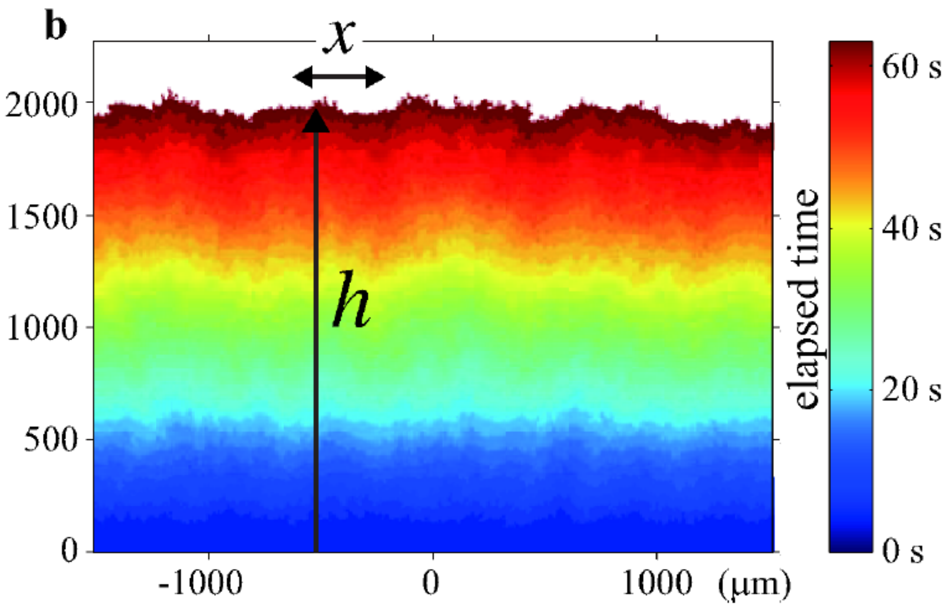
KPZ / Noisy-Burgers

$$\partial_t h = \frac{\lambda}{2}(\partial_x h)^2 + \nu \partial_x^2 h + \sqrt{D}\eta$$

Nonlinearity

Relaxation

Noise



Growing Interface

Burning Paper, Bacterial Growth, Liquid Crystal interfaces

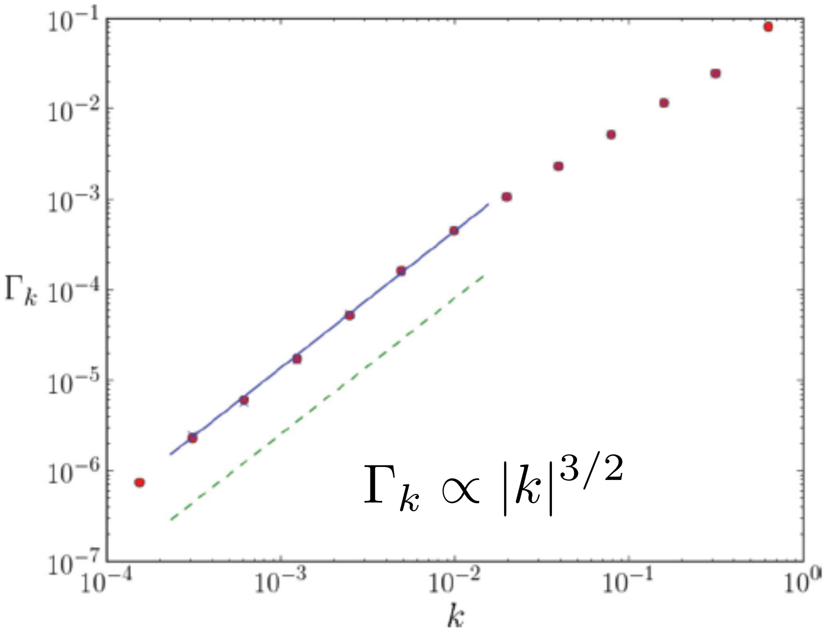
Hope: Add NLS to the experimental list

$$\langle \partial_x h(0,0)\, \partial_x h(x,t)\rangle \longleftrightarrow S(k,\omega) \qquad \Gamma_k \propto |k|^{3/2}$$

$$S^{(\pm)}_{\text{phonon}}(k,\omega) \propto \frac{1}{\Gamma_k} f_{\text{PS}}\left(\frac{\omega \pm c|k|}{\Gamma_k}\right)$$ (Prahofer-Spohn function)

$$\lambda = 1,\; v = -\partial_x h \Longrightarrow \partial_t v + v\partial_x v = \nu \partial_x^2 v + \sqrt{D}\partial_x \eta$$

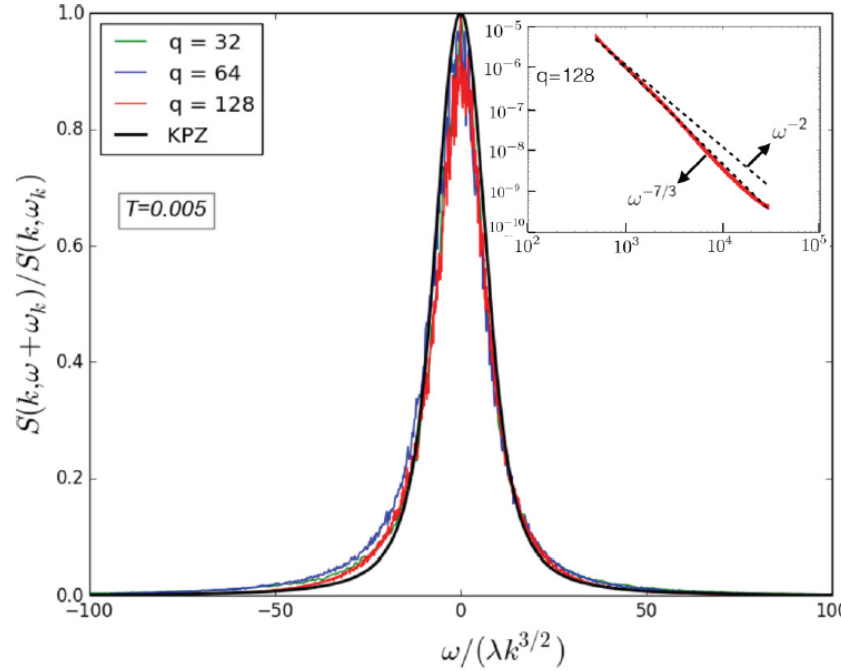
Agreement with 3/2 and the Prahofer-Spohn Scaling function



Kulkarni & Lamacraft, PRA, Rapid (2013)

$$z = 1.510 \pm 0.018$$

(DNLS)



Kulkarni, Huse, Spohn (PRA, 2015)

Next, adapting Nonlinear fluctuating hydrodynamics formalism to DNLS

DNLS and Nonlinear fluctuating hydrodynamics

$$\partial_t \rho + \partial_x(\rho v) = 0\,, \quad \partial_t v + \partial_x\left(\frac{v^2}{2} + \frac{g}{m}\rho\right) = 0$$

$\rho \to \rho_0 + \varrho$ and $v \to 0 + v$ Linearization

$$\partial_t \vec{u} + \partial_x A\vec{u} \quad \text{with} \quad \vec{u} = \begin{pmatrix} \varrho \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \rho_0 \\ \frac{g}{m} & 0 \end{pmatrix}$$

Adding, noise + dissipation, computing Hessians (non-linearity)

$$\partial_t \vec{u} + \partial_x \left[A\vec{u} + \frac{1}{2}\sum_{\alpha,\beta=1}^{2} \vec{H}_{\alpha,\beta} u_\alpha u_\beta - \partial_x(D\vec{u}) + B\vec{\xi} \right] = 0$$

DNLS and Nonlinear fluctuating hydrodynamics

Eigen-modes / Chiral Fields

$$\begin{pmatrix}\phi_- \\ \phi_+\end{pmatrix} = R \begin{pmatrix}\varrho \\ v\end{pmatrix} \quad \text{with} \quad R = \frac{1}{c\sqrt{2c_1}} \begin{pmatrix}-c & \rho_0 \\ c & \rho_0\end{pmatrix}$$

$$RAR^{-1} = \mathrm{diag}(-c, c)$$

Speed of sound

After rotating non-linear "u" equation

$$\partial_t \phi_\sigma + \partial_x[\sigma c \phi_\sigma + \langle \vec{\phi}, G^\sigma \vec{\phi} \rangle - \partial_x (D_{\mathrm{rot}} \phi)_\sigma + (B_{\mathrm{rot}} \xi)_\sigma] = 0$$

with $G^- = \frac{c}{2\rho_0}\sqrt{\frac{c_1}{2}}\begin{pmatrix}3 & 1 \\ 1 & -1\end{pmatrix}, \quad G^+ = \frac{c}{2\rho_0}\sqrt{\frac{c_1}{2}}\begin{pmatrix}-1 & 1 \\ 1 & 3\end{pmatrix}$

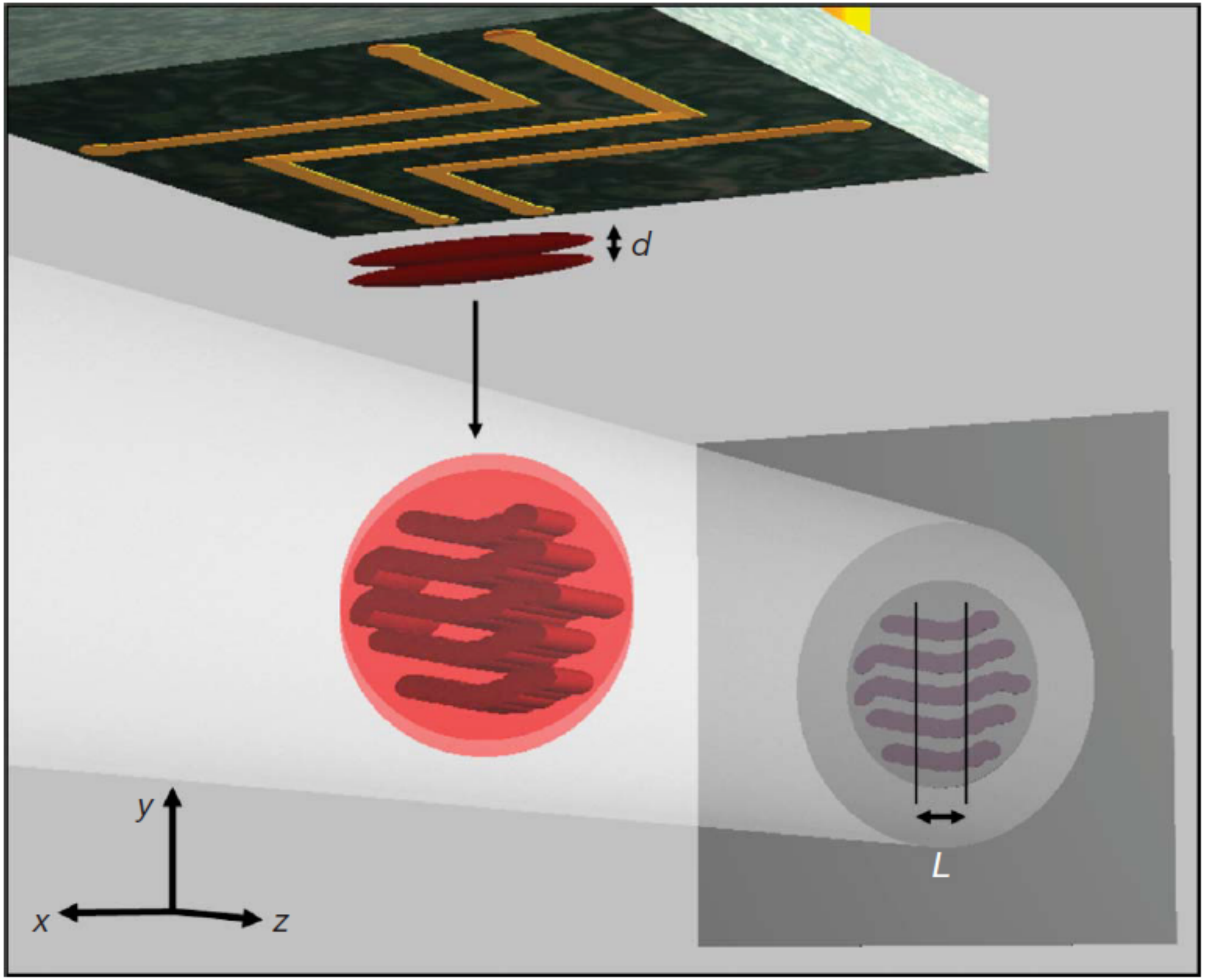
Keeping relavant terms gives us scalar KPZ

$$\langle \phi_\sigma(x,t)\phi_\sigma(0,0)\rangle = (\lambda t)^{-2/3} f_{\mathrm{KPZ}}((\lambda t)^{-2/3}(x - \sigma c t))$$

$$\lambda = 2\sqrt{2}|G^\sigma_{\sigma\sigma}| = \frac{3c}{\rho_0}\sqrt{c_1}$$

Application of KPZ scaling: condensate dephasing

Hofferberth *et al.* (2007)



Decoherence Dynamics in Low-Dimensional Cold Atom Interferometers

Measure *coherence*

$$\mathcal{C}(t) \equiv \frac{1}{L}\mathrm{Re}\int dx\,\langle e^{i[\theta_1(x,t)-\theta_2(x,t)]}\rangle$$

Phase analogous to height $h(x,t)$ in KPZ problem

$$\theta_1(x,t) - \theta_2(x,t) \sim t^{1/3}\chi_x$$

χ_x random variable

$$\mathcal{C}(t) \sim \exp\left[-(t/t_0)^{2/3}\right]$$

Burkov, Lukin, Demler (2007)

Multi component systems

$$\partial_t \phi_i + \partial_x \left[D\partial_x \phi_i + \sqrt{2D}\xi_i + G^i_{11}\phi_1^2 + (G^i_{12} + G^i_{21})\phi_1\phi_2 + G^i_{22}\phi_2^2 \right] = 0$$

Two-component noisy-Burgers

$$G^{i=1} = \begin{bmatrix} G^1_{11} & G^1_{12} \\ G^1_{21} & G^1_{22} \end{bmatrix} \qquad G^{i=2} = \begin{bmatrix} G^2_{11} & G^2_{12} \\ G^2_{21} & G^2_{22} \end{bmatrix}$$

Spohn & Stolz (Jstat. Phys, 2015)

Peak 1 & 2 can be KPZ, 3/2 Levy, 5/3 Levy, Diffusive, gold-Levy …

In NLS physical realizations, can we try to forbid some terms due to symmetry ?

This way, may be we will realize atleast some of above cases ?

Lets us start with coupled NLS :

$$H = \int dx \left[\frac{|\partial_x \psi_1|^2}{2m} + \frac{g_1}{2}|\psi_1|^4 + \frac{|\partial_x \psi_2|^2}{2m} + \frac{g_2}{2}|\psi_2|^4 + h|\psi_1|^2|\psi_2|^2 \right]$$

Multi component systems

Kulkarni, Huse (unpublished)

$$H = \int dx \Big[\frac{\rho_1 v_1^2}{2} + \rho_1 \epsilon(\rho_1) + \frac{(\partial_x \rho_1)^2}{8m^2 \rho_1} + \frac{\rho_2 v_2^2}{2} + \rho_2 \epsilon(\rho_2) + \frac{(\partial_x \rho_2)^2}{8m^2 \rho_2} + h\rho_1\rho_2 \Big]$$

$$\rho_{c,s} = \rho_1 \pm \rho_2$$
$$v_{c,s} = \frac{v_1 \pm v_2}{2}$$

Define spin & charge variables

Normal Modes in symmetric NLS case is given by

$$g_1 = g_2$$
$$\rho_{01} = \rho_{02} = \rho_0$$

$$\phi_s^{\mp} = \pm \frac{1}{4}\sqrt{\frac{g-h}{\rho_0}}\rho_s - \frac{v_s}{2}$$
$$\phi_c^{\mp} = \mp \frac{1}{4}\sqrt{\frac{g+h}{\rho_0}}\rho_c + \frac{v_c}{2}$$

$$c_{\text{spin}}^{\mp} = \mp \sqrt{g-h\rho_0}$$
$$c_{\text{charge}}^{\mp} = \mp \sqrt{g+h\rho_0}$$

Multi component systems

Kulkarni, Huse (unpublished)

Let us focus only on the right moving sectors of both spin and charge :

$$\phi_i = \phi_c^+, \phi_s^+$$

$$\phi_c^+ + \partial_x \left[\phi_c^{+2} + \phi_s^{+2} + \ldots\right] = 0$$

$$\phi_s^+ + \partial_x \left[\phi_c^+ \phi_s^+ + \ldots\right] = 0$$

Due to $1 \leftrightarrow 2$ symmetry

$$\phi_c \rightarrow \phi_c$$

$$\phi_s \rightarrow -\phi_s$$

Charge mode is “KPZ-like” and Spin is non-KPZ (conjectured to be diffusive).

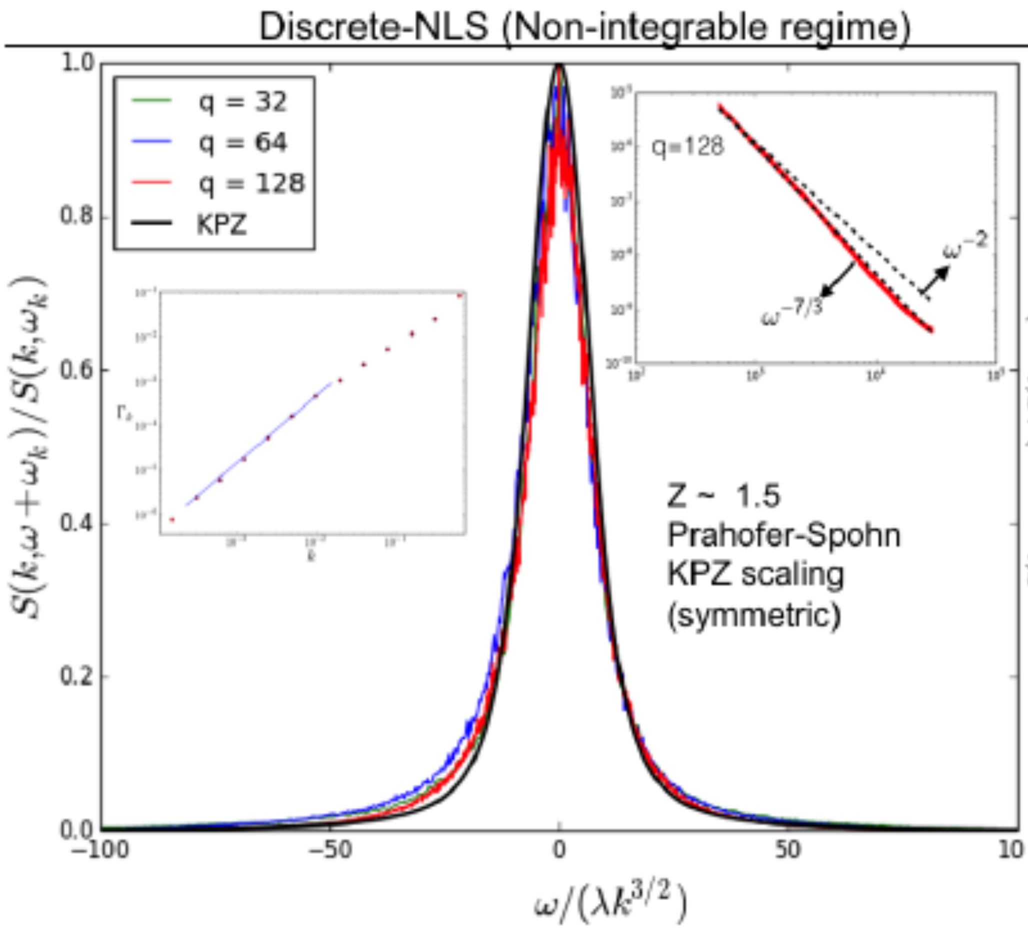
Broken Symmetry case

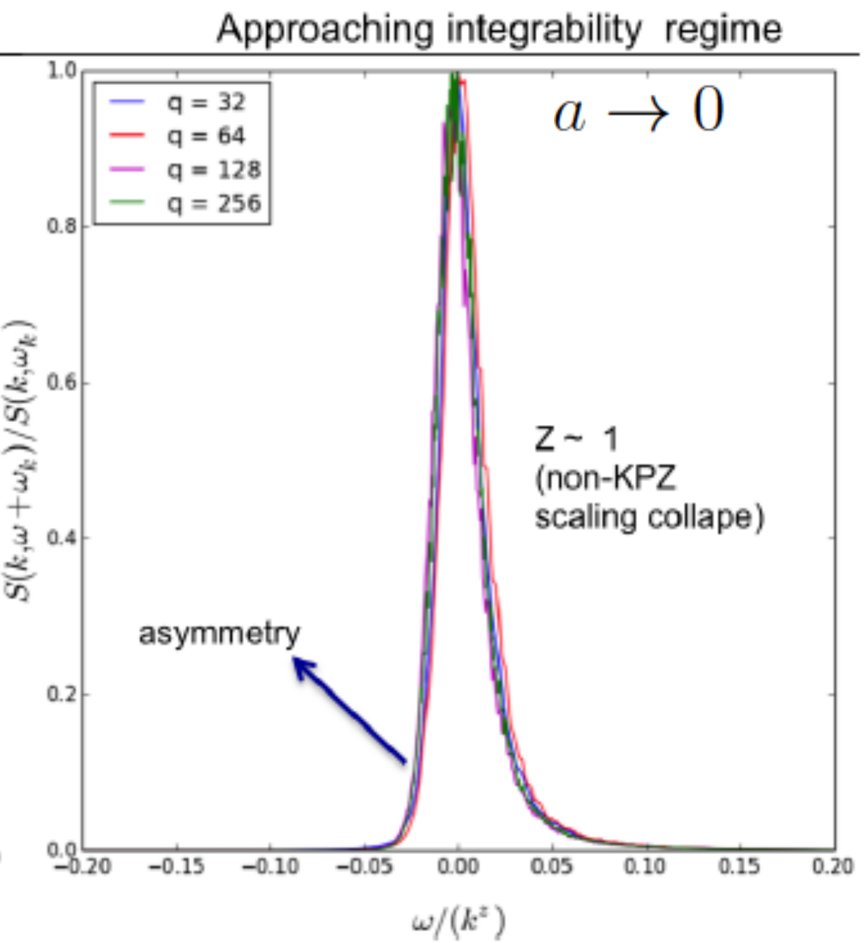
Both modes become KPZ-like

Symmetry breaking leads to crossover from non-KPZ to KPZ behavior

Note on Approaching Integrability

(preliminary data)





Is this line-shape universal for integrable models? Probably not.

Exponent, Line-shapes, Lyapunov Exponents …

Conclusions & Future

Discrete-NLS (DNLS), non-integrable

- NFH, KPZ
- universal line shape, exponents,
- Nonlinearities crucial
- Ideal Platform to probe dynamical critical phenomenon
- Possible future Experimental test bed for KPZ physics

Vector-NLS

- Multi-component noisy Burgers
- More universality classes
- Crossovers from exploiting symmetry

$$H = \int dx \left[\frac{|\partial_x \psi|^2}{2m} + \frac{g}{2}|\psi|^4 \right]$$

$$i\partial_t \psi = -\frac{1}{2m}\partial_x^2 \psi + g|\psi|^2\psi$$

Rich regimes

- High temp (Diffusive)
- Low temp (KPZ & Levy)
- Ultra-Low temp (pseudo-intgrable)

Gain-Loss NLS

- What happens to Universality ?
- Do we still have KPZ ?

Non-local NLS (lattice-non-integrable, lattice-integrable, Continuum)

2D NLS realizations

- 2D is still anamolous
- What happens to exact numerics?
- Can we write NFH and 2D versions of KPZ ?