Matrix and Bethe Ansatz for the Exclusion Process

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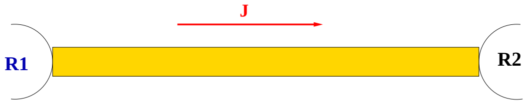
Introduction

A stationary driven system in contact with reservoirs is out of equilibrium:



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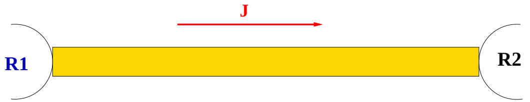
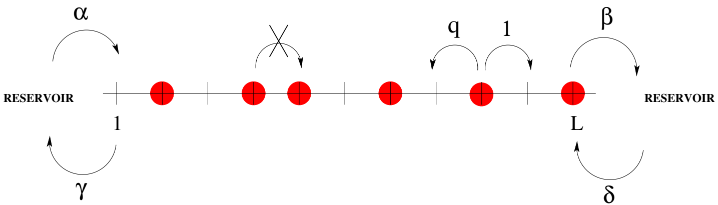
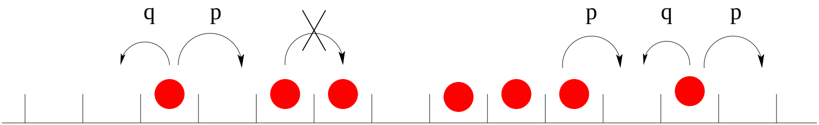


The asymmetric exclusion model (ASEP)



Thousands of articles devoted to this model in the last 20 years: A paradigm for non-equilibrium statistical physics.



Asymmetric Exclusion Process. A paradigm for non-equilibrium Statistical Mechanics.

- EXCLUSION: Hard core-interaction; at most 1 particle per site.
- ASYMMETRIC: External driving; breaks detailed-balance
- PROCESS: Stochastic Markovian dynamics; no Hamiltonian

An Elementary Model for Protein Synthesis

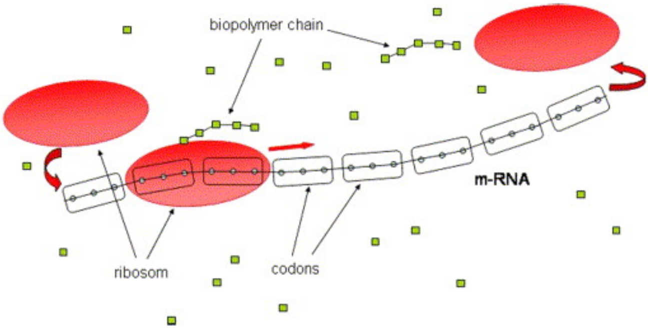


Figure: courtesy of Andreas Schadscheider

C. T. MacDonald, J. H. Gibbs and A.C. Pipkin, Kinetics of biopolymerization on nucleic acid templates, *Biopolymers* (1968).

An ubiquitous minimal model

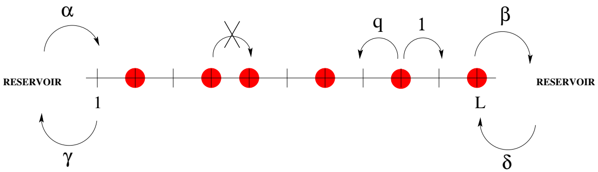
ORIGINS

- Interacting Brownian Processes (Spitzer, Harris, Liggett).
- Driven diffusive systems (Katz, Lebowitz and Spohn).
- Transport of Macromolecules through thin vessels.
  Motion of RNA templates.
- Hopping conductivity in solid electrolytes.
- Directed Polymers in random media. Reptation models.
- Interface dynamics. KPZ equation

APPLICATIONS

- Traffic flow.
- Sequence matching.
- Brownian motors.

Matrix Ansatz for ASEP (DEHP, 1993)



The key to the solution of the ASEP is the Matrix Product Representation of the stationary probabilities. The weight of a configuration $\mathcal{C}$ is given by:

$$P(\mathcal{C}) = \frac{1}{Z_L} \langle W | \prod_{i=1}^{L} \left(\tau_i D + (1 - \tau_i) E \right) | V \rangle$$

where $\tau_i = 1$ (or 0) if the site i is occupied (or empty) and the normalization constant is $Z_L = \langle W | (D + E)^L | V \rangle$

The weights of the system satisfy exact recursion relations between size L and size $L - 1$. This combinatorial structure will be encoded in the algebra generated by D, E, $\langle W|$ and $|V\rangle$.

Quadratic Algebra

The Matrix Ansatz will represent the steady state weights if the operators D and E, the vectors $\langle W|$ and $|V\rangle$ satisfy

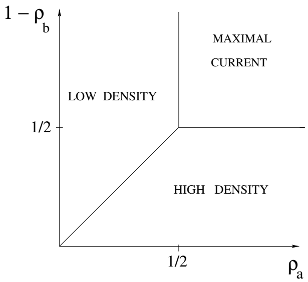
$$\begin{aligned} D\,E - qE\,D &= (1-q)(D+E) \\ (\beta\,D - \delta\,E)\,|V\rangle &= |V\rangle \\ \langle W|(\alpha\,E - \gamma\,D) &= \langle W| \end{aligned}$$

(B. Derrida, M. R. Evans, V. Hakim and V. Pasquier, 1993)

The Matrix Ansatz allows one to derive the Phase Diagram in the infinite size limit and to calculate many Stationary State Properties such as currents, correlations, fluctuations, finite size corrections, large deviations of the density profile... (see the review of R. Blythe and M. R. Evans).

Note that the recursions can also be encoded through generating functions (Derrida, Domany and Mukamel, 1992: $q=0, \alpha=\beta=1$; Schütz and Domany, 1993: $q=0$ arbitrary α, β).

The Phase Diagram



$\rho_a = \frac{1}{a_+ + 1}$: effective left reservoir density.
$\rho_b = \frac{b_+}{b_+ + 1}$: effective right reservoir density.

$$a_\pm = \frac{(1-q-\alpha+\gamma) \pm \sqrt{(1-q-\alpha+\gamma)^2 + 4\alpha\gamma}}{2\alpha}$$

$$b_\pm = \frac{(1-q-\beta+\delta) \pm \sqrt{(1-q-\beta+\delta)^2 + 4\beta\delta}}{2\beta}$$

Representations of the quadratic algebra

$D = 1 + d$ where d is a q-destruction operator.

$E = 1 + e$ where e is a q-creation operator.

$$d = \begin{pmatrix} 0 & \sqrt{1-q} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{1-q^2} & 0 & \dots \\ 0 & 0 & 0 & \sqrt{1-q^3} & \dots \\ & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad e = d^\dagger$$

Generalized exclusion processes

1. Multispecies Exclusion Processes on a ring

(C. Arita, A. Ayyer, P. Ferrari, M. R. Evans and S. Prolhac)

2. 2-ASEP with open boundaries

(N. Crampé, E. Ragoucy, M. Vanicat)

3. Current fluctuations in the open ASEP

(A. Lazarescu)

Multispecies Models on a periodic lattice

The dynamical rules

We consider the N-TASEP model on a periodic RING. There are N classes of particles and holes.
During an infinitesimal time step dt, the following processes take place on each bond with probability dt:

$$I\ 0 \to 0\ I \qquad \text{for} \quad I \neq 0$$
$$I\ J \to J\ I \qquad \text{for} \quad 1 \leq I < J \leq N$$

Hierarchical priority rules: First-class particles have highest priority and overtake all the others; Second-class particles overtake all the other ones except first class particles etc... Note that particles can always overtake holes (= 0-th class particles).

There are P_I particles of class I. Total number of configurations:

$$\Omega = \frac{L!}{P_0!P_1!P_2!\ldots P_N!}$$

What is the Stationary Measure ?

The Two Species case

If there is single species the stationary measure is uniform.

Matrix Product for the Stationary Measure *(Derrida, Janowski, Lebowitz and Speer, 1993)*:

A Configuration is represented by a string e.g. 01220211. The corresponding Stationary Weight is given by

$$p(01220211) = \frac{1}{Z}\mathrm{Tr}(EDAAEADD)$$

where E, D and A, operators belong to a *quadratic algebra*

$$\begin{aligned} DE &= D + E \\ DA &= A \\ AE &= A \end{aligned}$$

This Matrix Ansatz leads to steady state properties. This invariant measure *is not a Boltzmann-Gibbs measure* (E. Speer).

Infinite dimensional Representations

$D = 1 + \delta$ where $\delta =$ is the right-shift

$E = 1 + \epsilon$ where ϵ is the left-shift.

$A = |1\rangle\langle 1| = [\delta, \epsilon]$ (projector on the first coordinate).

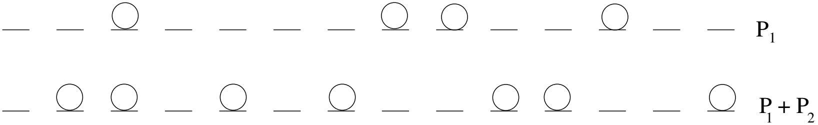
$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \; E = D^\dagger, \; A = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ . & . & . & . \end{pmatrix}$$

This algebra is the same as the one for the open TASEP (with a single species of particles).

The $N \geq 3$ case remained unsolved for more than a decade.

Geometric Construction of the 2-TASEP stationary measure (P. Ferrari, J. Martin)

A procedure to construct a configuration of the 2-TASEP with P_1 First Class Particles and P_2 Second Class Particles starting from two independent configurations of the 1 species TASEP.



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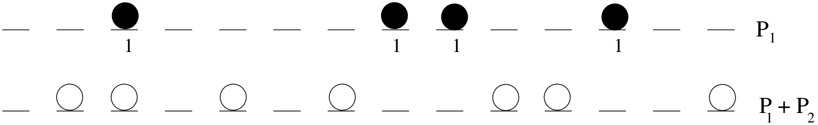

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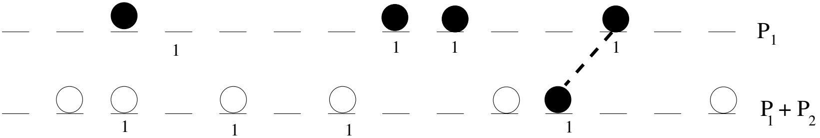

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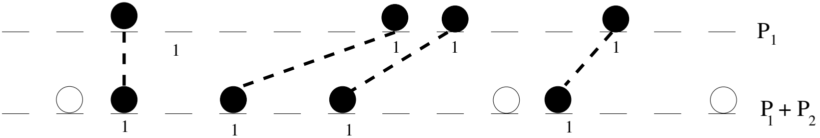

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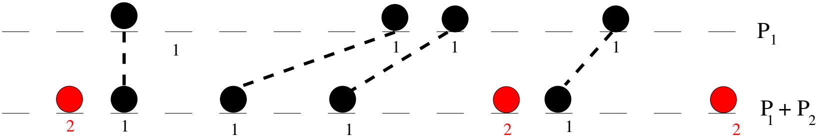

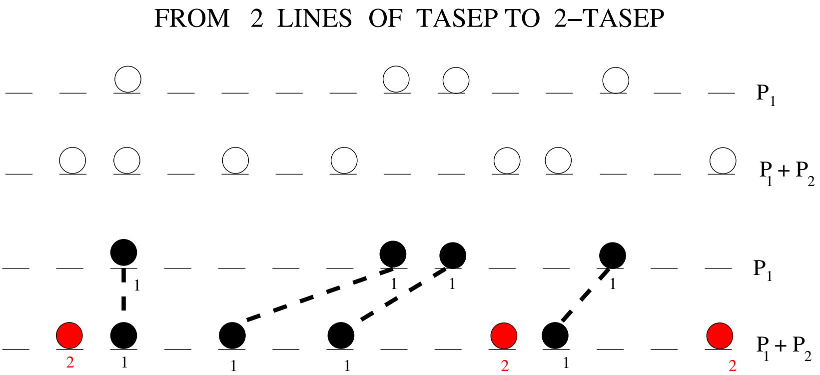

This construction is NOT one-to one: different configurations on the 1st line can produce the same configuration on the second line.

The weight of a 2-TASEP configuration is proportional to the total number of ways you can generate it by this construction.

Relation to the Matrix Ansatz

Characterization of the stationary weights:

- A 1 (on the 1st line) can not be located above a 2 (on the 2nd line).
- **Factorisation Property:** All the 1's (on the 2nd line) situated between two 2's MUST be linked to 1's (on the 1st line) that are located between the positions of the two 2's *(No Crossing Condition)*.
- **'Pushing' Procedure:** The *'ancestors'* of a string of the type 210102 are the strings obtained by pushing the 1's to the right i.e., $210102, 210012, 201102, 201012, 200112$.

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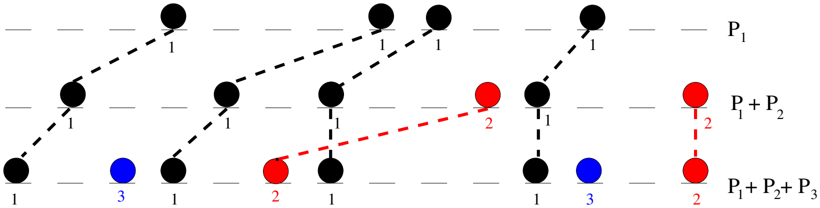
This Geometric Construction is encoded by the Matrix Ansatz:

- **Factorisation Property:** A is a **PROJECTOR.**
- **Pushing Procedure:** D and E are **SHIFT OPERATORS** (right-shift and left-shift, respectively).

From 3 lines of TASEP to the 3-TASEP



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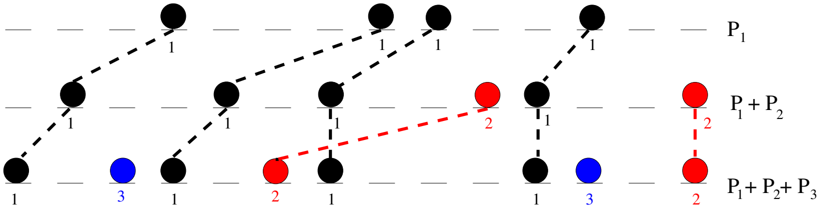
From 3 lines of TASEP to the 3-TASEP



The weight of a 3-TASEP configuration is proportional to the total number of ways you can generate it by this construction.

- FIND an ALGORITHM for constructing all ancestors of a given N-TASEP configuration.
- ENCODE this algorithm into an ALGEBRA (Matrix Product Representation).
- CALCULATE the stationary weights $\rightarrow$ TRACES over this algebra.

Nested Matrix Ansatz for the 3-TASEP

Tensor Products of Quadratic Algebras:
Hierarchical construction of representations of 'nested algebras' using the D, A and E matrices and the shift operators $\delta = D - 1$ and $\epsilon = E - 1$.

For the 3-species TASEP case:

$$\begin{aligned}
\hat{\mathbf{P}}_0 &= \mathbf{1} \otimes \mathbf{1} \otimes E + \mathbf{1} \otimes \epsilon \otimes A + \epsilon \otimes \mathbf{1} \otimes D\,. \\
\hat{\mathbf{P}}_1 &= \mathbf{1} \otimes \mathbf{1} \otimes D + \delta \otimes \epsilon \otimes A + \delta \otimes \mathbf{1} \otimes E \\
\hat{\mathbf{P}}_2 &= A \otimes \mathbf{1} \otimes A + A \otimes \delta \otimes E \\
\hat{\mathbf{P}}_3 &= A \otimes A \otimes E
\end{aligned}$$

Matrix Ansatz for the N-Species TASEP: M.R. Evans, P. Ferrari, K.M., *J.Stat.Phys, 2009.*

The algebraic proof bypasses the combinatorial pictures: Generalization for the N-Species ASEP, for which no geometric construction exists.

Generalization to the N-ASEP

If backward jumps are allowed (rate $q \neq 0$)

$$\begin{aligned} DE - qED &= (1-q)(D+E) \\ DA - qAD &= (1-q)A \\ AE - qEA &= (1-q)A \end{aligned}$$

$\rightarrow$ Replace the previous shift-operators by **deformed shift-operators:**

$$\boxed{\delta\epsilon = 1 \rightarrow \delta\epsilon - q\epsilon\delta = 1}$$

Recursive Matrix Ansatz:

$$X_J^{(N)} = \sum_{M=0}^{N-1} a_{JM}^{(N)} \otimes X_M^{(N-1)} \quad \text{for } 0 \leq J \leq N \text{ with } X_0^{(1)} = X_1^{(1)} = 1$$

Matrix Ansatz for the N-Species TASEP: M.R. Evans, K.M., S. Prolhac *J.Phys.A, 2009.*

Transfer Matrix

The operators $a_{JM}^{(N)}$ define a Transfer Matrix between the (N-1)-species ASEP and the N-ASEP:

$$|\Omega_N\rangle = T_{N-1\to N}\ |\Omega_{N-1}\rangle$$

$T_{N-1\to N}$ lifts the (N-1)-ASEP into the N-ASEP, allowing to construct whole sectors of the spectrum and it 'inverts' the identification operator.

$$\begin{tikzcd}
\Omega_{N-1} \arrow[r, "M_{N-1}"] \arrow[d, "T_{N-1,N}"'] & \Omega_{N-1} \arrow[d, "T_{N-1,N}"] \\
\Omega_N \arrow[r, "M_N"'] & \Omega_N
\end{tikzcd}$$

Yang-Baxter type equations

The operators $a_{JM}^{(N)}$ generate a quadratic algebra, that can be expressed in a Yang-Baxter form, using the local update Markov matrix:

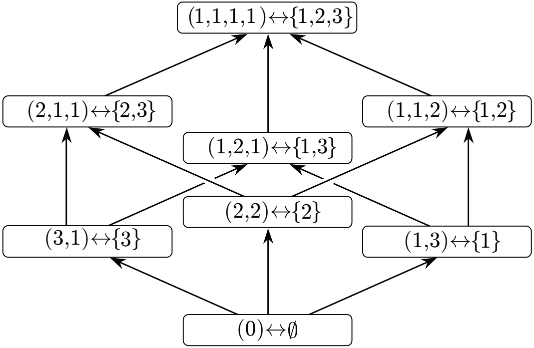
$$M_{\mathrm{Loc}}^{(N)}(\mathbf{a}\otimes\mathbf{a}) - (\mathbf{a}\otimes\mathbf{a})M_{\mathrm{Loc}}^{(N-1)} = \mathbf{a}\otimes\widehat{\mathbf{a}} - \widehat{\mathbf{a}}\otimes\mathbf{a}$$

The $a_{JM}^{(N)}$ can be written as a rectangular tableau

$$a^{(2)} = \begin{pmatrix} \mathbb{1} & \epsilon \\ \delta & \mathbb{1} \\ A & 0 \end{pmatrix} \quad \text{and} \quad a^{(3)} = \begin{pmatrix} \mathbb{1}\otimes\mathbb{1} & \epsilon\otimes\mathbb{1} & \mathbb{1}\otimes\epsilon \\ \delta\otimes\mathbb{1} & \mathbb{1}\otimes\mathbb{1} & \delta\otimes\epsilon \\ A\otimes\delta & 0 & A\otimes\mathbb{1} \\ A\otimes A & 0 & 0 \end{pmatrix}$$

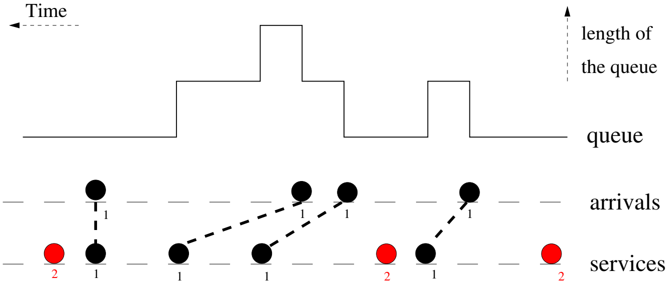
A. Ayyer, KM: *J. Phys. A, 2010.*
C. Arita, A. Ayyer, KM, S. Prolhac: *J. Phys. A, 2011; 2012.*
C. Arita, KM: *J. Phys. A 2013.*

Hasse Diagram of the N-ASEP



To each arrow corresponds a different quadratic algebra that leads to different lifting operators and to generalized Ferrari-Martin constructions. These algebras have been studied in C. Arita et al. (2011).

Queueing Theory Interpretation



The matrices D and E act on $|n\rangle$ the length of the queue:

Service Time: $D|n\rangle = |n\rangle + |n-1\rangle$

Non-Service Time: $E|n\rangle = |n\rangle + |n+1\rangle$

This queueing process can be generalized to the N-TASEP. *The matrices act on the queue at each time step:* they are constructed by inspection of the different possible arrivals at a given time.

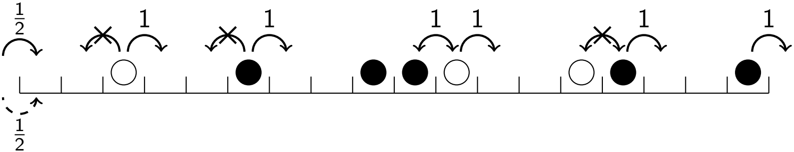
The 2-ASEP with open boundaries

An open problem

Consider a 2 species ASEP bulk rules:

$10 \xrightarrow{1} 01$	$01 \xrightarrow{q} 10$
$20 \xrightarrow{1} 02$	$02 \xrightarrow{q} 20$
$12 \xrightarrow{1} 21$	$21 \xrightarrow{q} 12$

Is it possible to find a set of boundary rules ensuring that the model remains exactly solvable?



An example of integrable 2-TASEP (full circles represent species 1, empty circles are species 2). On the left boundary, the continuous line means injection of 1 whereas the dashed line means injection of 2.

Integrability

Consider the process on a periodic ring: its Markov matrix can be written as a sum of local update 9×9 operators

$$M = \sum_i M_{i,i+1}$$

This process in integrable: this Markov matrix can be embedded in a family of commuting transfer matrices $t(x)$ that depend on a spectral parameter x:

$$[t(x), t(x')] = 0$$

and

$$M = t'(1)$$

This commutation is crucial as it implies the existence of a sufficient number of independent conserved quantities and warrants integrability.

The Yang-Baxter Equation

The transfer matrix $t(x)$ is built by making tensor products of elementary local operators that act on two lattices sites $R_{ij}(x)$. The crucial identity satisfied by the R-matrix and that implies in turn the commutation property is the Yang-Baxter Equation:

$$R_{12}\left(\frac{x_1}{x_2}\right) R_{13}\left(\frac{x_1}{x_3}\right) R_{23}\left(\frac{x_2}{x_3}\right) = R_{23}\left(\frac{x_2}{x_3}\right) R_{13}\left(\frac{x_1}{x_3}\right) R_{12}\left(\frac{x_1}{x_2}\right)$$

This defines a system of cubic algebraic relations satisfied by the entries of the R-matrix.

It is a standard procedure to start from solutions of the YBE, define a family of commuting transfer matrices and take derivatives at special points to obtain integrable Hamiltonians.

The R-Matrix for 2-ASEP

$$R(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{(x-1)q}{qx-1} & 0 & \frac{(q-1)x}{qx-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(x-1)q}{qx-1} & 0 & 0 & 0 & \frac{(q-1)x}{qx-1} & 0 & 0 \\ 0 & \frac{q-1}{qx-1} & 0 & \frac{x-1}{qx-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(x-1)q}{qx-1} & 0 & \frac{(q-1)x}{qx-1} & 0 \\ 0 & 0 & \frac{q-1}{qx-1} & 0 & 0 & 0 & \frac{x-1}{qx-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{q-1}{qx-1} & 0 & \frac{x-1}{qx-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This matrix acts on a bound and it satisfies the YBE.

One can show that $R(1)$ is the permutation operator P and that the local update operator is given by $M_{i,i+1} = (q-1)PR'(1)$.

System with boundaries: K-matrices

The update rules on the right and the left boundaries can be encoded by boundary operators $K(x)$ and $\tilde{K}(x)$.

Integrability will be preserved only of the boundary conditions satisfy some compatibility conditions with the bulk rules. Technically, the K-matrices are to satisfy the Sklyanin reflection equations:

$$R_{12}\left(\frac{x_1}{x_2}\right) K_1(x_1)\, R_{21}\left(x_1 x_2\right) K_2(x_2) = K_2(x_2)\, R_{12}\left(x_1 x_2\right) K_1(x_1)\, R_{21}\left(\frac{x_1}{x_2}\right)$$

(and $\tilde{K}$ satisfies a similar equation.)

Finding all integrable 2-ASEP models with open boundaries is thus equivalent to solving the Sklyanin reflection equations, with the known R-Matrix for 2-ASEP.

Classification of boundary K-matrices

The different steps to derive the classification are: take a generic K-matrix with 9 unknown functions as entries; use the reflection equation to get 81 relations; solve these functional equations and among the solutions retain only the Markovian ones.

We have found four different classes of solutions, K_1, K_2, K_3, K_4, (each depending on α and γ, two free parameters)

$$K_1(x) = \begin{pmatrix} \frac{x((\alpha^2-\gamma^2)(x-1)+(\gamma x+\alpha)(q-1))}{(\alpha x+\gamma)((\alpha+\gamma)(x-1)+q-1)} & \frac{(x^2-1)(\alpha+\gamma)\alpha}{(\alpha x+\gamma)((\alpha+\gamma)(x-1)+q-1)} & \frac{(x^2-1)(\alpha+\gamma)\alpha}{(\alpha x+\gamma)((\alpha+\gamma)(x-1)+q-1)} \\ \frac{(x^2-1)(\alpha+\gamma+1-q)\gamma}{(\alpha x+\gamma)((\alpha+\gamma)(x-1)+q-1)} & -\frac{(\alpha^2-\gamma^2)(x-1)+(\alpha x+\gamma)(1-q)}{(\alpha x+\gamma)((\alpha+\gamma)(x-1)+q-1)} & \frac{(x^2-1)(\alpha+\gamma)\gamma}{x(\alpha x+\gamma)((\alpha+\gamma)(x-1)+q-1)} \\ 0 & 0 & -\frac{(\alpha+\gamma)(x-1)+x(1-q)}{x((\alpha+\gamma)(x-1)+q-1)} \end{pmatrix}$$

And similarly for the right boundary $\tilde{K}$.

Classification of boundary conditions

Four sets of integrable Left-Rules depending on two parameters α, γ:

L_1	L_2	L_3	L_4
$0 \xrightarrow{f(\alpha,\gamma)} 2$	$0 \xrightarrow{\gamma} 2$	$0 \xrightarrow{\alpha} 1$	$0 \xrightarrow{\alpha} 1$
$2 \xrightarrow{\alpha} 0$	$0 \xrightarrow{\alpha} 1$	$2 \xrightarrow{\gamma} 0$	$1 \xrightarrow{\gamma} 0$
$1 \xrightarrow{\alpha} 0$	$2 \xrightarrow{\alpha} 1$	$2 \xrightarrow{\alpha} 1$	
$1 \xrightarrow{\gamma} 2$	$1 \xrightarrow{g(\alpha,\gamma)} 2$	$1 \xrightarrow{\gamma} 0$	

Four sets of integrable Right-Rules depending on two parameters β, δ:

R_1	R_2	R_3	R_4
$0 \xrightarrow{\delta} 2$	$0 \xrightarrow{g(\beta,\delta)} 2$	$0 \xrightarrow{\delta} 1$	$0 \xrightarrow{\delta} 1$
$0 \xrightarrow{\beta} 1$	$2 \xrightarrow{\beta} 0$	$2 \xrightarrow{\beta} 0$	$1 \xrightarrow{\beta} 0$
$2 \xrightarrow{\beta} 1$	$1 \xrightarrow{\beta} 0$	$2 \xrightarrow{\delta} 1$	
$1 \xrightarrow{f(\beta,\delta)} 2$	$1 \xrightarrow{\delta} 2$	$1 \xrightarrow{\beta} 0$	

$$f(x,y) = \frac{y(x+y+1-q)}{x+y} \quad \text{and} \quad g(x,y) = \frac{y(x+y+q-1)}{x+y}$$

Some remarks

These boundary conditions are very restrictive: this was not the case for the standard 1-species exclusion process. Our classification contains previously studied models (Ayyer, Lebowitz and Speer; Duchi and Schaeffer) but allows the exchange of all types of particles.

For the 2-TASEP case, the integrable boundaries can be obtained by taking the limit $q = 0$. The simplest specific example is given by:

Left	Bulk	Right
$0 \xrightarrow{1/2} 1$	$10 \xrightarrow{1} 01$	$1 \xrightarrow{1} 0$
$0 \xrightarrow{1/2} 2$	$12 \xrightarrow{1} 21$	$2 \xrightarrow{1} 0$
$2 \xrightarrow{1/2} 1$	$20 \xrightarrow{1} 02$	

Note that the boundaries are permeable to all the species.

With these rules, Integrability becomes 'visible'.

Matrix Ansatz for the steady state

We look for a Matrix Ansatz for the stationary weights:

$$P(\tau_1, \dots, \tau_L) = \frac{\langle W_2 | X_{\tau_1} X_{\tau_2} \cdots X_{\tau_L} | V_2 \rangle}{\langle W_2 | (X_1 + X_2 + X_3)^L | V_2 \rangle}$$

Here X_0, X_1, X_2 are constructed from the Zamolodchikov, Faddeev and Ghoshal relations, related to YBE and Sklyanin equations. Writing

$X_1 = 1 + G_1 + G_2 + G_3, \; X_2 = G_4 + G_5 + G_6$ and $X_0 = 1 + G_7 + G_8 + G_9$

a quadratic algebra is obtained:

$$\begin{array}{llll}
[G_1, G_2] = 0, & & & \\
[G_1, G_3] = 0, & [G_2, G_3] = 0, & & \\
G_1 G_4 = G_5, & G_2 G_4 = G_6, & G_3 G_4 = 0, & \\
[G_1, G_5] = G_6 - G_4 G_2, & G_2 G_5 = G_1 G_6, & G_3 G_5 = 0, & [G_4, G_5] = 0, \\
[G_1, G_6] = -G_4 G_3, & [G_2, G_6] = -G_5 G_3, & G_3 G_6 = 0, & [G_4, G_6] = 0, \\
G_1 G_7 = G_8, & G_2 G_7 = G_9, & G_3 G_7 = 1, & G_4 G_7 = 0, \\
[G_1, G_8] = G_9 - G_7 G_2, & G_2 G_8 = G_1 G_9, & G_3 G_8 = G_1, & [G_4, G_8] = -G_7 G_5, \\
[G_1, G_9] = 1 - G_7 G_3, & [G_2, G_9] = G_1 - G_8 G_3, & G_3 G_9 = G_2, & [G_4, G_9] = -G_7 G_6, \\
[G_5, G_6] = 0, & & & \\
G_5 G_7 = 0, & G_6 G_7 = 0, & & \\
G_5 G_8 = G_4 G_9, & G_6 G_8 = G_4, & [G_7, G_8] = 0, & \\
[G_5, G_9] = G_4 - G_8 G_6, & G_6 G_9 = G_5, & [G_7, G_9] = 0, & [G_8, G_9] = 0.
\end{array}$$

Some results

- One can show, using the previous algebra, that all the weights of all configurations can be calculated (PBW Thm).

- Some explicit weights of some special types of configurations can be obtained.

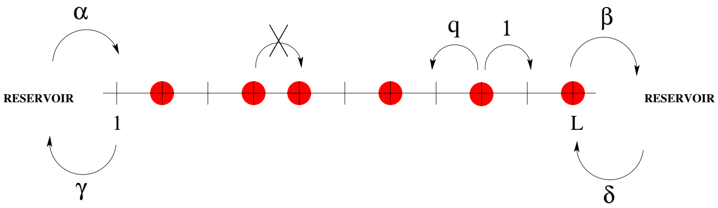
- Exact formulas for the currents can be derived:

$$j_1 = \frac{L+1}{2(2L+1)}\,, \quad j_2 = \frac{1}{2(2L+1)}\,, \quad j_0 = -\frac{L+2}{2(2L+1)}$$

- To obtain an explicit Matrix Ansatz, the key observation is that one should look for a representation of the quadratic ZF algebra $(G_1, \ldots, G_9)$ that contains 9 elements (and not try to find directly the matrices X_0, X_1, X_2).

Current Fluctuations in the open ASEP

Total Current in the ASEP with Open Boundaries



The observable Y_t counts the total number of particles exchanged between the system and the left reservoir between times 0 and t. Hence, $Y_{t+dt} = Y_t + y$ with

- $y = +1$ if a particle enters at site 1 (at rate α),
- $y = -1$ if a particle exits from 1 (at rate γ)
- $y = 0$ if no particle exchange with the left reservoir has occurred during dt.

These three mutually exclusive types of transitions lead to a three parts decomposition of the Markov Matrix: $M = M_+ + M_- + M_0$.

Current Statistics as an eigenvalue

The statistics of Y_T can be probed by the cumulant-generating function $E(\mu)$ when $t \to \infty$:

$$\langle \mathrm{e}^{\mu Y_t} \rangle \simeq \mathrm{e}^{E(\mu) t}$$

$E(\mu)$ is shown to be the dominant eigenvalue of the deformed matrix

$$\boxed{M(\mu) = M_0 + \mathrm{e}^{\mu} M_+ + \mathrm{e}^{-\mu} M_-}$$

Expanding, one has: $E(\mu) = 0 + J\mu + \Delta \frac{\mu^2}{2} + C_3 \frac{\mu^3}{3!} \ldots$

- Average current J: obtained by the DEHP Matrix Ansatz (1993).
- Variance Δ: calculated by a tensor product of three DEHP algebras (B. Derrida, M. R. Evans, KM, 1995).

For the k-th term in the expansion of $E(\gamma)$, we built a Matrix Ansatz at order k, by making $(2k-1)$ Tensor Products of Quadratic Algebras.

Mimics the construction used for the multispecies exclusion process.

Generalized Matrix Ansatz

We have proved that the dominant eigenvector of the deformed matrix $M(\mu)$ is given by the following matrix product representation:

$$F_\mu(\mathcal{C}) = \frac{1}{Z_L^{(k)}} \langle W_k | \prod_{i=1}^{L} (\tau_i D_k + (1-\tau_i) E_k) | V_k \rangle + \mathcal{O}\left(\mu^{k+1}\right)$$

The matrices D_k and E_k are the same as above

$$\begin{aligned} D_{k+1} &= (1 \otimes 1 + d \otimes e) \otimes D_k + (1 \otimes d + d \otimes 1) \otimes E_k \\ E_{k+1} &= (1 \otimes 1 + e \otimes d) \otimes E_k + (e \otimes 1 + 1 \otimes e) \otimes D_k \end{aligned}$$

The boundary vectors $\langle W_k |$ and $| V_k \rangle$ are constructed recursively:

$|V_k\rangle = |\beta\rangle|\tilde{V}\rangle|V_{k-1}\rangle$ and $\langle W_k| = \langle W^\mu|\langle \tilde{W}^\mu|\langle W_{k-1}|$

$$[\beta(1-d) - \delta(1-e)]\,|\tilde{V}\rangle = 0$$

$$\langle W^\mu|[\alpha(1 + \mathrm{e}^\mu\, e) - \gamma(1 + \mathrm{e}^{-\mu}\, d)] = (1-q)\langle W^\mu|$$

$$\langle \tilde{W}^\mu|[\alpha(1 - \mathrm{e}^\mu\, e) - \gamma(1 - \mathrm{e}^{-\mu}\, d)] = 0$$

Asymptotic behaviour in the Phase Diagram

- Maximal Current Phase:

$$\mu = -\frac{L^{-1/2}}{2\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(2k)!}{k!k^{(k+3/2)}} B^k$$

$$\mathcal{E} - \frac{1-q}{4}\mu = -\frac{(1-q)L^{-3/2}}{16\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(2k)!}{k!k^{(k+5/2)}} B^k$$

- Low Density (and High Density) Phases:
  Dominant singularity at a_+: $\phi_k(z) \sim F^k(z)$. By Lagrange Inversion:

$$E(\mu) = (1-q)(1-\rho_a)\frac{\mathrm{e}^{\mu}-1}{\mathrm{e}^{\mu}+(1-\rho_a)/\rho_a}$$

(cf de Gier and Essler, 2011).

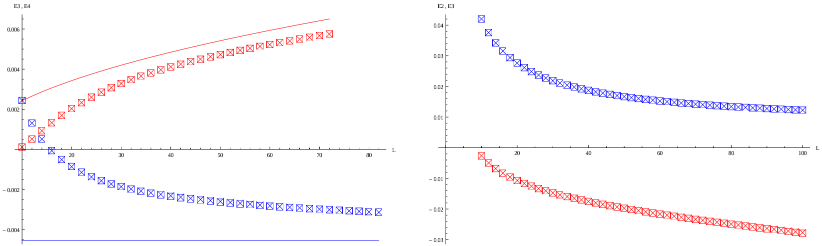
Current Large Deviation Function:

$$\boxed{\Phi(j) = (1-q)\left\{\rho_a - r + r(1-r)\ln\left(\frac{1-\rho_a}{\rho_a}\frac{r}{1-r}\right)\right\}}$$

where the current j is parametrized as $j = (1-q)r(1-r)$.

Matches the predictions of Macroscopic Fluctuation Theory in the Weak Asymmetry Limit, as observed by T. Bodineau and B. Derrida.

Numerical results (DMRG)



Left: **Max. Current** ($q = 0.5$, $a_+ = b_+ = 0.65$, $a_- = b_- = 0.6$), Third and Fourth cumulant.

Right: **High Density** ($q = 0.5$, $a_+ = 0.28$, $b_+ = 1.15$, $a_- = -0.48$ and $b_- = -0.27$), Second and Third cumulant.

A. Lazarescu and K. Mallick, J. Phys. A 44, 315001 2011).

M. Gorissen, A. Lazarescu, K.M., C. Vanderzande, PRL **109** 170601 (2012).

A special TASEP case

In the case $\alpha = \beta = 1$, a parametric representation of the cumulant generating function $E(\mu)$:

$$\mu = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)]!}{[k(L+1)]!\,[k(L+2)]!} \frac{B^k}{2k},$$

$$E = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)-2]!}{[k(L+1)-1]!\,[k(L+2)-1]!} \frac{B^k}{2k}.$$

First cumulants of the current

- Mean Value : $J = \frac{L+2}{2(2L+1)}$
- Variance : $\Delta = \frac{3}{2} \frac{(4L+1)![L!(L+2)!]^2}{[(2L+1)!]^3(2L+3)!}$
- Skewness :
  $E_3 = 12 \frac{[(L+1)!]^2[(L+2)!]^4}{(2L+1)[(2L+2)!]^3} \left\{ 9 \frac{(L+1)!(L+2)!(4L+2)!(4L+4)!}{(2L+1)![(2L+2)!]^2[(2L+4)!]^2} - 20 \frac{(6L+4)!}{(3L+2)!(3L+6)!} \right\}$
  For large systems: $E_3 \to \frac{2187-1280\sqrt{3}}{10368}\pi \sim -0.0090978\ldots$

Conclusion

Systems out of equilibrium are ubiquitous in nature. They break time reversal invariance. Often, they are characterized by non-vanishing stationary currents. In general, the steady-state measures are not given by the Boltzmann-Gibbs Law.

Exact solutions have been obtained for one-dimensional processes, thanks to various techniques: Bethe Ansatz, Determinantal Processes and Matrix Product Representations.

Tensor Products of Matrix Product States have allowed us to study multispecies generalizations of the exclusion process as well as current fluctuations in the open ASEP.

Many partially solved/open questions: multispecies processes with open boundaries? Systematic construction of the Matrix Ansatz and relation to integrability. Applications to other single-file models and queueing processes.