Non-equilibrium processes for Current Reservoirs

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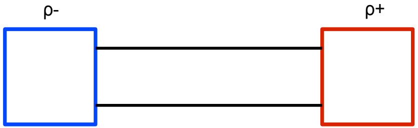
1. Introduction

Hydrodynamic limits and macroscopic theory for systems out of equilibrium.
Typical example: Fourier/Fick's law



Let Q_T be the number of particles which flows through the system during time T.
Fourier's / Fick's law:

$$\frac{\langle Q_T \rangle}{T} \sim \frac{1}{L}(\rho_+ - \rho_-)$$

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- Boundary conditions? Periodic, Dirichlet or other?

3. Periodic vs Dirichlet boundary conditions

Ω is the unit circle $(-1, 1]$ and $F(\rho) = \int_\Omega f(\rho(r))dr$ is the free energy.

Equation:

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The **free energy is monotone:**

$$\frac{dF(\rho(\cdot,t))}{dt} = \int_\Omega \frac{\delta F(\rho)}{\delta \rho(r)}\frac{\partial \rho}{\partial t}\,dr = -\int_{-1}^{1}\kappa(\rho)\Big(\frac{\partial f'(\rho(r,t))}{\partial r}\Big)^2\,dr$$

(integrating by parts and using periodicity)

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Density reservoirs: complement the equation with Dirichlet b. c.:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial r}, \qquad J = -\kappa(\rho)\frac{\partial f'(\rho)}{\partial r}, \quad r \in (-1, 1)$$

$$\rho(-1, t) = \rho_{-1}, \qquad \rho(1, t) = \rho_1, \quad \rho(r, 0) \text{ given}$$

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The **free energy is not monotone:**

$$\frac{dF(\rho(\cdot,t))}{dt} = -\int_{-1}^{1} \kappa(\rho)\left(\frac{\partial f'(\rho(r,t))}{\partial r}\right)^2 dr + J(-1,t)f'(\rho_{-1}) - J(1,t)f'(\rho_1)$$

Reservoirs

But the “total” mass is conserved: let

$$X_-(t) = \int_0^t J(-1,s)ds, \qquad X_+(t) = \int_0^t J(1,s)ds$$

be the mass that the system exchanges with the reservoirs (enters at 1 and exits at -1) within time t.

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Similarly for the free energy.

Define the “total” free energy:

$$F^{\text{total}} = F(\rho(\cdot, t)) + F_{\Lambda_-,t} + F_{\Lambda_+,t}$$

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Assume that $|\Lambda_-|$ and $|\Lambda_+|$ are so large that the reservoirs "instantaneously" homogenize any change of mass, then

$$F_{\Lambda_-,t} = |\Lambda_-| f(\rho_{-1} - \frac{X_-(t)}{|\Lambda_-|}) \approx F_{\Lambda_-,0} - f'(\rho_{-1}) X_-(t)$$

$$F_{\Lambda_+,t} = |\Lambda_+| f\big(\rho_1 + \frac{X_+(t)}{|\Lambda_+|}) \approx F_{\Lambda_+,0} + f'(\rho_1) X_+(t)$$

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Hence,

$$F^{\text{total}} \approx F(\rho(\cdot,t)) + F_{\Lambda_-,0} - f'(\rho_{-1}) X_-(t) + F_{\Lambda_+,0} + f'(\rho_1) X_+(t)$$

and it is monotone non increasing:

$$\frac{dF^{total}}{dt} = \frac{dF(\rho(\cdot,t))}{dt} - f'(\rho_{-1}) J(-1,t) + f'(\rho_1) J(1,t) = -\int_{-1}^{1} \kappa(\rho) \Big(\frac{\partial f'(\rho(r,t))}{\partial r} \Big)^2 \, dr$$

Fluctuations

Given a fluctuation $(\rho, X_{\pm})$ there is a unique smooth function $E(x,t)$ s.t.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial r}, \qquad J = -\kappa(\rho)(E + \frac{\partial f'(\rho)}{\partial r}), \quad r \in (-1,1)$$

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Suffices to choose:

$$J(x,t) := \frac{d}{dt}X_{-}(t) - \int_{-1}^{x} \frac{d}{dt}\rho(y,t)\, dy, \qquad E := \kappa^{-1}(J + \kappa \frac{d}{dx} f'(\rho))$$

Note that E has a non-local dependence on ρ.

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Power dissipated by E:

$$P_T(\rho, X_{\pm}) = \int_0^T dt \int_{-1}^{1} dx\ \kappa(\rho)E(x,t)^2$$

4. Microscopic stochastic model

Symmetric simple exclusion process on $\Lambda_\varepsilon = [0, \varepsilon^{-1}] \cap \mathbb{Z} = \{0, 1, ..., N\}$, $N = [\varepsilon^{-1}]$.
$\{\eta_t(x) \in \{0,1\},\ x \in \Lambda_\varepsilon,\ t \geq 0\}$ is the process with generator:

$$L_0 f(\eta) = \frac{1}{2} \sum_{x \in \Lambda_\varepsilon} \sum_{y:|y-x|=1} \Big(f(\eta^{(x,y)}) - f(\eta) \Big)$$

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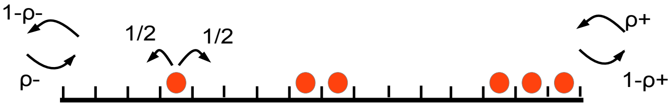
Put two independent **Poisson clocks of intensity** $\frac{1}{2}$ at the pairs $(-N, -N+1)$ and $(N, N+1)$.



When it rings at $(N, N+1)$, we put a particle at N with prob. ρ_1 and remove with probability $1 - \rho_1$. Similarly at $(-N, -N+1)$.

Hydrodynamic limit exists:

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2}, \qquad r \in (-1, 1)$$

with Dirichlet b.c. $\rho(-1, t) = \rho_{-1}$, $\rho(1, t) = \rho_1$.
The unique invariant measure μ_ε is such that for any $x \in \Lambda_\varepsilon$

$$\lim_{\varepsilon \to 0, \varepsilon x \to r} \mu_\varepsilon\big(\eta(x)\big) = (\rho_1 - \rho_{-1})r + \rho_{-1}$$

Microscopic current and Fick's law

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}\ \mu_\varepsilon\big(\eta(x) - \eta(x+1)\big) = \rho_1 - \rho_{-1}$$

Large deviations for the density and the current

Some of the references:

T. Bodineau, B. Derrida, J. Lebowitz (2010)
T. Bodineau, B. Derrida (2006)
L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim (2001,...)
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G. Eyink, J.L. Lebowitz, H. Spohn (1991)
H. Spohn (1983)
A. Galves, C. Kipnis, C. Marchioro, E. Presutti (1981)
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A flux of mass $J(1, t)$ enters into the system at the point 1 and
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(producing a change of density $\rho(\pm 1, t) \pm J(\pm 1, t)\, dt$).

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How about the case $\lambda \equiv 1$?

6. Microscopic theory, current reservoirs

In the interior simple exclusion process for $\eta(x) \in \{0,1\}$. Let f be a test function

$$L_0 f(\eta) := \frac{1}{2} \sum_{x=-N}^{N-1} [f(\eta^{(x,x+1)}) - f(\eta)],$$

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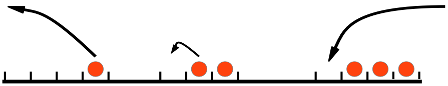
On the boundary ($|I_\pm| = K$, finite!) we impose a (microscopic) current $\frac{\varepsilon j}{2}$ with $\varepsilon = 1/N$

$$L_{b,\pm} f(\eta) := \frac{\varepsilon j}{2} \sum_{x \in I_\pm} D_\pm \eta(x) [f(\eta^{(x)}) - f(\eta)],$$



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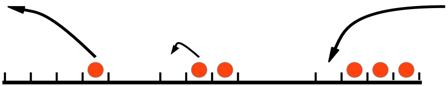

where

$$D_+\eta(x) = [1 - \eta(x)]\eta(x+1)\eta(x+2)\ldots\eta(N), \quad x \in I_+$$
$$D_-\eta(x) = \eta(x)[1 - \eta(x-1)][1 - \eta(x-2)]\ldots[1 - \eta(-N)], \quad x \in I_-.$$

7. Results

$$\begin{aligned}\frac{d}{dt}\mathbb{E}_\varepsilon[\eta(x,t)] &= \mathbb{E}_\varepsilon[L_0(\eta)+L_b(\eta)] \\ &= \frac{1}{2}\Delta_\varepsilon\mathbb{E}_\varepsilon[\eta(x,t)] + \mathbb{E}_\varepsilon\frac{\varepsilon j}{2}\sum_{x\in I_\pm} D_\pm\eta(x)[f(\eta^{(x)})-f(\eta)]\end{aligned}$$

Can we close it with respect to $\rho_\varepsilon(x,t) := \mathbb{E}_\varepsilon[\eta(x,t)]$?

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- Propagation of chaos. Considering the correlation functions:

$$v^\varepsilon(\underline{x},t|\mu^\varepsilon) := \mathbb{E}_\varepsilon\Big[\prod_{i=1}^{n}\{\eta(x_i,t)-\rho_\varepsilon(x_i,t)\}\Big], \quad \underline{x}\in\Lambda_N^{n,\neq},\ n\geq 1$$

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Theorem (propagation of chaos, EJP '12)

$\exists\tau>0, c^*>0$, s.t. $\forall\beta^*>0, n\in\mathbb{Z}_+$, $\exists c_n$ s.t. $\forall\varepsilon>0$

$$\sup_{\underline{x}\in\Lambda_N^{n,\neq}}|v^\varepsilon(\underline{x},t|\mu^\varepsilon)| \leq \begin{cases} c_n(\varepsilon^{-2}t)^{-c^* n}, & t\leq\varepsilon^{\beta^*} \\ c_n\varepsilon^{(2-\beta^*)c^* n} & \varepsilon^{\beta^*}\leq t\leq\tau\log\varepsilon^{-1} \end{cases}$$

- In the limit $\varepsilon \to 0$: heat equation with special boundary conditions: (JSP '11):

$$\frac{\partial}{\partial t}\rho(r,t) = \frac{1}{2}\frac{\partial^2}{\partial r^2}\rho(r,t), \qquad r \in (-1,1),$$

$$\frac{\partial \rho(r,t)}{\partial r}\Big|_{r=1} = j(1-\rho(1,t)^K), \quad \frac{\partial \rho(r,t)}{\partial r}\Big|_{r=-1} = j(1-(1-\rho(-1,t))^K)$$

In the sense that for any $t_1 > t_0 > 0$:

$$\lim_{\varepsilon\to 0} \sup_{x\in\Lambda_N} \sup_{t_0\le t\le t_1} |\rho_\varepsilon(x,t) - \rho(\varepsilon x,t)| = 0$$

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Hence, we found in our model the reservoir mobilities:

$$\lambda\big(\rho(1,t)\big) = 1-\rho(1,t)^K \quad \text{and} \quad \lambda(\rho(-1,t)) = 1-(1-\rho(-1,t))^K$$

- In the limit $\varepsilon \to 0$: heat equation with special boundary conditions: (JSP '11):

$$\frac{\partial}{\partial t}\rho(r,t) = \frac{1}{2}\frac{\partial^2}{\partial r^2}\rho(r,t), \qquad r \in (-1,1),$$

$$\frac{\partial \rho(r,t)}{\partial r}\Big|_{r=1} = j(1-\rho(1,t)^K), \quad \frac{\partial \rho(r,t)}{\partial r}\Big|_{r=-1} = j(1-(1-\rho(-1,t))^K)$$

In the sense that for any $t_1 > t_0 > 0$:

$$\lim_{\varepsilon\to 0}\sup_{x\in\Lambda_N}\sup_{t_0\le t\le t_1}|\rho_\varepsilon(x,t) - \rho(\varepsilon x,t)| = 0$$

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- Validity of Fourier law: the expected current through $x+\frac{1}{2}$ is

$$j^{(\varepsilon)}(x,t) = \frac{\varepsilon^{-2}}{2}\,\mathbb{E}_\varepsilon\big[\varepsilon\{\eta(x,t)-\eta(x+1,t)\}\big] = -\frac{1}{2}\mathbb{E}_\varepsilon\Big[\frac{\eta(x+1,t)-\eta(x,t)}{\varepsilon}\Big].$$

and we prove that for $r \in (-1,1)$

$$\lim_{\varepsilon\to 0} j^{(\varepsilon)}([\varepsilon^{-1}r],t) = -\frac{1}{2}\,\frac{d\rho(r,t)}{dr}.$$

- As $\varepsilon \to 0$, the profile corresponding to the (unique) invariant measure = stationary solution of the equation. (JSP '12).

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$$\rho''(x) = 0, \quad \rho'(1) = j(1 - \rho_+^K), \quad \rho'(-1) = j(1 - (1 - \rho_-)^K)$$

  We obtain $\rho^*(x) = j_{\text{eff}} x + \frac{1}{2}$ where $j_{\text{eff}} = j(1 - \alpha^K)$
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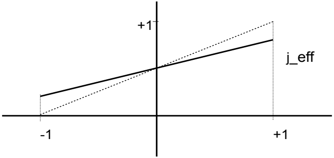
  E.g. for $K = 1$:

$$j_{\text{eff}} = \frac{1}{2} \frac{j}{1+j}$$



Stationary measure, (JSP '12)

Let μ_N be the *unique* invariant measure.

$$\lim_{N\to\infty} \max_{(x_1,\ldots,x_k)\in\Lambda_N^{k,\neq}} \left|\mu_N\big(\eta(x_1)\cdots\eta(x_k)\big) - \rho^*(x_1/N)\cdots\rho^*(x_k/N)\right| = 0$$

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Main idea: both process and equation preserve order.

- Let η_0 and ξ_0 be two particle configurations such that $\eta_0 \le \xi_0$, and let $\mathbb{P}_{\eta_0}$, respectively $\mathbb{P}_{\xi_0}$, be the law of the process starting from η_0, respectively ξ_0.
  Then there is a coupling $\mathbb{Q}$ of $\mathbb{P}_{\eta_0}$ and $\mathbb{P}_{\xi_0}$ such that

$$\mathbb{Q}\{(\eta,\xi) : \eta_t \le \xi_t, \forall t\} = 1$$

- The analogous monotonicity property holds for the macroscopic equation.
  Hence, if $\bar{\rho}(r,t)$ denotes the solution with initial datum $\rho \equiv 1$, and $\underline{\rho}(r,t)$ the one corresponding to initial datum $\rho \equiv 0$, then for any ρ_0:

$$\underline{\rho}(r,t) \le \rho(r,t|\rho_0) \le \bar{\rho}(r,t).$$

Hence, after times of order N^2, the measure μ_N shrinks concentrating on a L^1-neighborhood of the limit profile ρ^*.

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Spectral gap, (Bernoulli '15)

In our case we have that for any initial measure

$$\|\mu_N^{(t)} - \mu_N^{\text{st}}\| \leq cNe^{-bN^{-2}t}$$

where for any signed measure λ, $\|\lambda\| = \sum_\eta |\lambda(\eta)|$.

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In some respect surprising!

- With $j = 0$, $L = L_0$ (stirring process) restricted to any of the invariant subspaces $\{\eta\colon \sum \eta(x) = M\}$ has a spectral gap that scales as N^{-2} (Lu-Yau, CMP'93).

- The full process with $L = L_0 + \frac{j}{N} L_b$ in a time of the same order N^2 manage to equilibrate among all the above subspaces according to μ_N^{st}.

- Density reservoirs: $L = L_0 + L'$ same spectral gap: $\|\mu_N^{(t)} - \mu_N^{\text{st}}\| \le cNe^{-bN^{-2}t}$. (Here the birth-death events are not scaled down with N.)

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Thank you!