NESP - ICTS, Bangalore, Oct. 2015

Mon. Oct. 26

Thermalization and many-body Anderson localization (MBL) in closed quantum many-body systems.

– Vadim Oganesyan (CUNY/CSI) + many others

review paper: Nandkishore + Huse arXiv 1404, ARCMP 2015
+ many other collaborators.

Some Fundamentals of (quantum) stat mech.:

Closed system: $\boxed{H(t),\ \rho(0)}$ fully isolated from any external degrees of freedom.

strongly-interacting internally. (e.g. spins, cold atoms or molecules, q-bits, etc.)

Unitary time evolution: $\rho(t) = U(t)\rho(0)U^\dagger(t)$ — state $\rho(t)$ gives probability distribution of any observable.

$$U(\tau) = T\, e^{-i\int_0^\tau dt\, H(t)}$$

Special cases of interest: $H(t) = H$: conserved energy $i\dot\rho = [H,\rho]$

e.g. spin chain $H = H_z + H_x = \sum_i J_i\sigma_i^z\sigma_{i+1}^z + h_i\sigma_i^z + \sum_i g_i\sigma_i^x$

or $H(t) = H(t+\tau)$: Floquet system:

e.g. $U(\tau) = e^{-iH_x}e^{-iH_z}$

$$\rho(n\tau) = U^n(\tau)\,\rho(0)\left(U^\dagger(\tau)\right)^n$$

have eigenstates but no extensive conserved energy when $H(t)$ <u>not</u> time-independent.

Eigenstates: $U(\tau)|n\rangle = \lambda_n|n\rangle$ or $H|n\rangle = E_n|n\rangle$

- "Sensible" initial state $\rho(0)$: Uncertainties in any extensive conserved quantities are <u>sub</u>-extensive. (No "Schrödinger cats")
  States that if they thermalize have nonzero entropy density. $(T \neq 0)$
  ~~subextensive~~ definite densities in TDL

- Interested in: dynamics of <u>observables</u> (definition?) and of <u>entanglement</u>

First:

- Long-time states: $t \to \pm\infty$

  Two <u>generic</u> possibilities (two dynamic "phases"):

  - thermalization (system goes to thermal equilibrium)
  - many-body localization. (does not)

  also special cases:
  - other integrable systems (probably not stable to small local changes to U)
  - critical points
  - any other "intermediate" cases?

Standard assumption in stat. mech.: system goes to thermal equilibrium at long time.

This does <u>not</u> mean for the full system (in Hamiltonian case)

$$\rho(t) \xrightarrow[t\to\infty]{} \frac{e^{-\beta H}}{Z}$$

since $e^{-\beta H}$ is a stationary state that does not "attract" any other states.

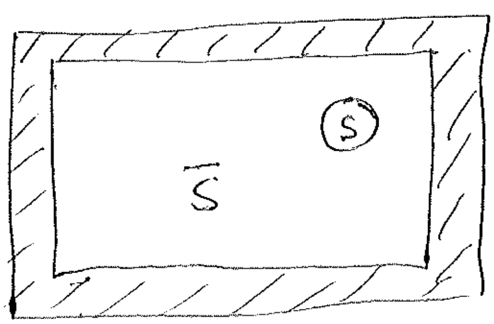
Time evolution of the "probability distribution" $\rho(t)$ is "deterministic", so all initial information in $\rho(0)$ remains in the full system. (it never "forgets", it only "hides" information about $\rho(0)$: "decoherence")

Then what is thermalization?

- System successfully acts as a reservoir and brings its <u>sub</u>systems to thermal equilibrium.

  Some generic systems do this: "the thermal phase", while others fail to do so: "many-body localized".

For systems that do thermalize, what subsystems do thermalize?



S = degrees of freedom in subsystem

$\bar{S}$ = all other degrees of freedom

Full system $F = S \oplus \bar{S}$ is <u>all</u> degrees of freedom.

Examples of subsystems that do <u>not</u> thermalize:

Any Projections on to eigenstates of full system $|n\rangle\langle n|$ is Hermitian but is not an observable for $\bar{S}\to\infty$

or operators involving projections such as $|n\rangle\langle m|$

Any conserved quantity e.g. $U(\tau) = \sum_n |n\rangle \lambda_n \langle n|$

or $H = \sum_n |n\rangle E_n \langle n|$ is an observable.

is made of such projections.

Take subsystem S to be finite (e.g., a finite number of points in real-space or in momentum-space)

and not ~~[illegible]~~ to contain a conserved quantity.

~~Thermalization~~ State of subsystem is given by the reduced state

$$\rho_S(t) = \underset{\bar{S}}{\mathrm{Trace}}\, \rho(t)$$

Thermalization: (as a statement about dynamics)

$$\lim_{\substack{t\to\infty \\ \bar{S}\to\infty}} \rho_S(t) = \rho_S^{(eq.)}(T,\mu,\dots) = \underset{\bar{S}}{\mathrm{Trace}} \frac{e^{-\beta(H-\mu N \dots)}}{Z}$$

↑ in general need to take limits together appropriately.

~~Need~~ Can not take $t\to\infty$ before $\bar{S}\to\infty$ because for finite $\bar{S}$ ~~there are~~

$\rho_S(t)$ is quasiperiodic and has no limit. "~~Poincaré recurrences~~" so no ~~limit $t\to\infty$ is [illegible]~~

If there are any extensive conserved quantities (energy, particles, spin, ...)

can not take $\vec{S}\to\infty$ before $t\to\infty$ since the transport times $\to\infty$.

For Floquet system with no extensive conserved quantities, presumably can do

$$\lim_{t\to\infty} \lim_{\vec{S}\to\infty} \rho_S(t)$$

and even allow S to be anything less than 1/2 of system.

Strong statement of thermalization:

Systems that thermalize, do so for all subsystems that do not contain conserved quantities and for all initial states with subextensive uncertainties (no "cats")
(sometimes have thermalization only within some energy range)

It is a strong statement, hard to test, and stronger than is needed for Eq'm Stat Mech to apply. We test it as best as can be done by exact diagonalizations.

Quantum thermalization is stronger than classical, due to quantum's intrinsic randomness.

Eigenstate Thermalization (Hypothesis (ETH))

If we have thermalization from all initial states, then exact eigenstates of H or of $U(\tau)$ must all be thermal.

$$\text{ETH:} \quad \lim_{\bar{S}\to\infty} \operatorname{Trace}_{\bar{S}} |n\rangle\langle n| = \rho_S^{(eq.)}(T, \mu, \cdots)$$

eigenstate.

in all eigenstates, all subsystems have thermal eq'm distributions.

This is a little easier to test: Do exact diagonalization and find the eigenstate that is farthest from thermal. ~~Find~~ Increase L, see deviation decrease $\sim e^{-cL}$, as expected Kim, Ikeda, Huse P.R.E. 2014

→ These eigenstates have thermal, "volume-law", entanglement, since subsystems have thermal entropy.

Also, thermalization requires (off-diagonal matrix elements of observables)

$$\langle n | A | m \rangle \xrightarrow[\bar{S}\to\infty]{} 0$$

any observable

to avoid the possibility of observable ~~[illegible]~~ oscillations in $\langle A \rangle$ in e.g. state $\frac{1}{\sqrt{2}}(|n\rangle + |m\rangle)$

in fact this → 0 exponentially in size of system, in practice, for systems that do thermalize.

For a system that thermalizes, the nonlocal Hermitian operators made out of exact MB eigenstates that do not thermalize, such as $|n\rangle\langle n|$ and

$|n\rangle\langle m| + |m\rangle\langle n|$ are not "observables" in the limit $\vec{S} \to \infty$. "Observables" are of finite-order in the bare degrees of freedom.

Thus in the thermodynamic limit, <u>observables</u> are only an ~~small~~ infinitesimal subset of all Hermitian operators.

– If all eigenstates are at thermal equilibrium, how do we make a nonequilibrium (thus low entropy) state? Need <u>very</u> special <u>coherences</u> $\langle n|\rho|m\rangle$ <u>off</u> the diagonal in the energy eigenbasis. With ~~th~~ time these

$$\langle n|\rho(t)|m\rangle = \langle n|\rho(0)|m\rangle \, e^{i(E_n - E_m)t}$$

↑ pick up phases that become essentially random phases for $t \to \infty$:

– "Equilibration is "just" dephasing" –

What is a "reservoir"? Even for a Floquet system with no conservation laws, the system can be or fail to be a reservoir.

(when it succeeds $\rho_S^{(eq.)} = \dfrac{\mathbb{1}}{Z_S}$ = maximum entropy)

→ A reservoir is something ~~to~~ that can "accept" an unlimited amount of quantum entanglement. This is the <u>essential</u> feature of a reservoir. In some cases it also exchanges energy and/or particles, etc. with subsystem.

Single-eigenstate microcanonical ensembles

for systems that do thermalize and obey ETH, these are fine ensembles.

But unlike the usual ensembles, they can also see the breakdown of ETH, e.g. MBL

(Later, we may return to consider $0<t<\infty$ transport.)

But first:

for ICTS Oct 2015

Main generic exception to thermalization: Tues., Oct 27

Many-body Anderson localization.

Anderson 1958: immobile, randomly placed spins in a semiconductor, H_{spin}

ignore coupling of spins to phonons or other "external" degrees of freedom.

He asked: Is this many-spin system a reservoir that can thermalize its subsystems?

When no: this is Anderson localization: reservoir failure

broadly speaking:

3 cases of localization in literature:

- Noninteracting particles: most of the theory literature before ~2003
- Ground states with interactions, metal-insulator or superconductor-insulator transitions: most of the expt'l literature.

→ Highly excited states with interactions: Now called many-body localization (MBL)

this is today's topic, a strong recent theory literature ≥ 2006

MBL, fortunately, can occur at "high T" (even "$T\to\infty$") in 1D, which makes it somewhat accessible to numerics.

(Oganesyan + Huse PRB 2007)

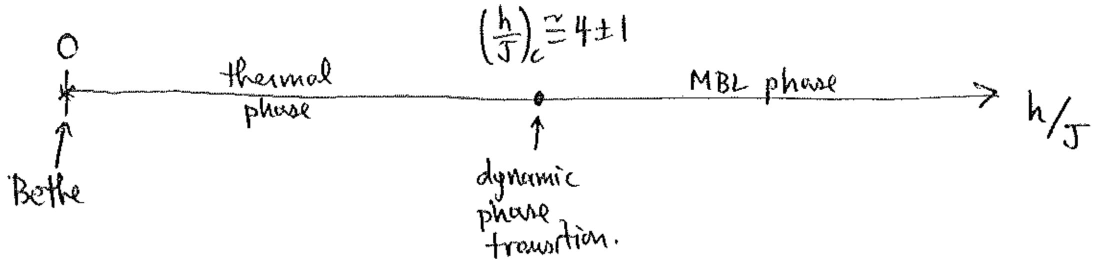
(Znidaric, Prosen, Prelovcek 2008)
Pal + Huse 2010

The "Standard Model" of MBL: spin-$\frac{1}{2}$ chain.

$$H = \sum_i \Big(\underbrace{h_i}_{\text{static random fields}} S_{iz} + J \vec{S}_i \cdot \vec{S}_{i+1} \Big) \qquad h_i \text{ in } [-h, h]$$

has extensive conserved energy and $\left(\sum_i S_{iz}\right)$

Phase diagram at <u>max</u> of MBDOS $(\langle H \rangle = E = 0)$ ("$T=\infty$")



— Trivially localized limit $J=0, h>0$, $h/J \to \infty$

Dynamics is independent Larmor precession of each spin: no transport of energy or S_z. System does not thermalize, no reservoir.

Exact eigenstates are: $\ldots\uparrow\downarrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\cdots$ clearly do not obey ETH
zero entanglement.

Is an integrable system; $[H, S_{iz}] = 0$ for all i.

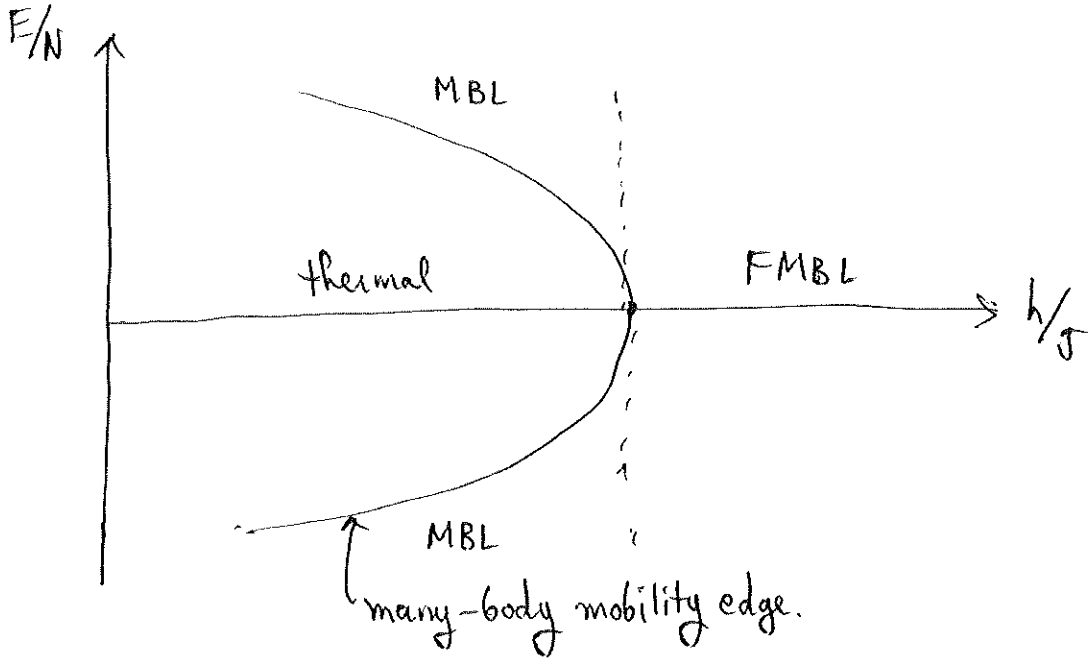
Now turn on small nonzero J. System remains localized. (until J gets too large)
Basko, Aleiner, Altshuler 2006

Eigenstates get "dressed" with a little entanglement ("quantum fluctuations"), but entanglement remains "boundary-law" (with "tails" to large entanglement at rare locations)

Conserved spins S_{iz} get "dressed" to make localized conserved pseudospins τ_{iz}, "l-bits" Serbyn, Papic, Abanin 2013
Huse, Nandkishore, Oganesyan 2014

Eigenstates remain non-entangled products of the eigenstates of the $\{\tau_{iz}\}$ Imbrie 2014

Phase transition is sharp contrast in: dynamics, eigenstate properties, but not in ~~...~~ thermal equilibrium properties.



Fully MBL: All eigenstates localized. Complete set of localized pseudospins τ_{iz}. ("Emergent integrability")

MBL but not FMBL regime: (might not exist (?))

For eigenstates in this energy range there exists a local unitary V that disentangles: $V|n\rangle$ = product state

contrasts between "phases"

MBL	thermal.
not	system is a reservoir that thermalizes its subsystems. (reservoir of entanglement)
zero DC thermal or other conductivity	DC transport happens (may be subdiffusive in 1D)
system is a local memory	no local memory. local info about initial state ends up spread through entire system and thus inaccessible.
boundary ~~area~~-law eigenstate entanglement (thus like ground states) Can have stable LRO or TO even in 1D at high T.	volume-law and thermal eigenstate entanglement (entanglement entropy of eigenstates is maximized)
from product initial state, entanglement grows as $S \sim \log t$ distance $l \sim \log t$	faster spread of entanglement ~~(presumably linear in time except in $d=1$)~~ sometimes sublinear $S \sim t^{1/z}$ often $z=1$, but in 1D can have $1<z<\infty$ due to quantum Griffiths effects.

sub regime: **F**MBL: A complete set of localized operators that mutually commute and commute with U determine the eigenstates.

↑

Novel and poorly understood dynamic quantum phase transition

scaling?

universality classes?

Initial approximations with ~~strong~~ versions of strong-randomness R.G.

The "l-bit" picture of the FMBL phase. (Huse, ~~Oganesyan~~, Nandkishore, Oganesyan ~~2014~~ PRB 2014)

localized pseudospins operators $\vec{\tau}_i$ exist with $[\vec{\tau}_i, \vec{\tau}_j] = 0$ for $i \neq j$

and Pauli algebra

$$[H, \tau_{iz}] = 0 \quad \forall\, i$$

or $[U, \tau_{iz}] = 0$ for Floquet system

τ_{iz} are the localized conserved quantities $\rightarrow$ 2^N eigenstates each one mapped to eigenstates of $\{\tau_{iz}\}$

N "bare" spins $\{\vec{\sigma}_i\}$ $\Longleftrightarrow$ N "l-bits" $\{\vec{\tau}_i\}$

When H is written in terms of $\vec{\tau}_i$'s, it only contains τ_{iz}'s

it is thus an Ising Hamiltonian of $\vec{\tau}_i$'s. Eigenstates of U are product states of eigenstates of each $\{\tau_{iz}\}$.

Challenge is given $H(\{\vec{\sigma}\})$, find the $\vec{\tau}$'s.

Has multi-l-bit interactions $\sim \tau_{iz}\tau_{jz}\tau_{kz}\cdots\cdots\tau_{wz}$ whose strength typically falls off exponentially with distance and order.

$$H = \sum_i h_i \tau_{iz} + \sum_{ij} J_{ij}\tau_{iz}\tau_{jz} + \sum_{ijk} K_{ijk}\tau_{iz}\tau_{jz}\tau_{kz} + \cdots$$

When ~~l-bits~~ $\vec{\tau}_i$ are written in terms of bare spins $\{\vec{\sigma}_j\}$ again are multispin terms whose weight typically falls off exponentially with distance and order. So product states of $\vec{\tau}_i$'s are (eigenstates) area-law entangled in terms of $\vec{\sigma}_i$'s.

long-distance Entanglement spread in MBL phase comes from interactions between $\vec{\tau}_i$ and $\vec{\tau}_j$.

If ~~neither~~ state is <u>not</u> an eigenstate of either τ_{iz} or τ_{jz}, then interaction $\tau_{iz}\tau_{jz}\cdot$(other l-bits) means $\vec{\tau}_i$ precesses about its z-axis at a rate that depends on value of τ_{jz} (and vice versa) $\Rightarrow$ entanglement between i and j

Interaction $\sim e^{-r_{ij}/\xi}$ ← decay length

so time to entangle $\sim e^{+r_{ij}/\xi}$

entanglement "spreads" distance $r \sim \xi \log t$

Wed., Oct 28, 2015

- Area-law entanglement means MPS /DMRG methods should work well. Can we use MPS to learn more about MBL than we can learn from ED? (as happened long ago for ground states, e.g., spin-1 chain)

Why is this difficult?

One reason: Many-body resonances:

They are always present in the MBL phase.

It is the "proliferation" of these resonances that destabilizes MBL → thermalization (phase transition)

Also, such resonances dominate the low-frequency dynamics in the MBL phase (Gopalakrishnan, et al. PRB 2015)

Many-body resonances:

Consider first one eigenstate $|n\rangle$ of an infinite spin-1/2 chain in the MBL phase.

$|\tilde{n}\rangle$

Now also consider the $\sim 2^L$ other eigenstates $|\tilde{n}\rangle$ that differ on a stretch of length L spins, and are the same ~~(essentially)~~ elsewhere ("$T=\infty$")

$\longleftarrow \; L \; \longrightarrow$

$|\tilde{n}'\rangle$

$|\tilde{n}\rangle$ and $|\tilde{n}'\rangle$ typically differ by flipping $\sim L/2$ spins.

typical level spacing to the nearest (in λ, E) eigenstate $\delta \sim 2^{-L}$

typical smallest level spacing of all pairs $\sim 4^{-L} = e^{-L(\ln 4)}$ (assuming Poisson level statistics)

(smallest gap is distributed all the way $\to 0$)

When we diagonalize U, there is an effective matrix element $\langle n|U|n'\rangle$ due to $\sim L$th order perturbation theory that $\sim e^{-L/\xi}$

$$U \sim \begin{pmatrix} +\delta/2 & e^{-L/\xi} \\ e^{-L/\xi} & -\delta/2 \end{pmatrix}$$

for a ~~resonant~~ a near-resonant pair of states.

A many-body resonance is present here when there is a pair of eigenstates with level spacing $\lesssim e^{-L/\xi}$, which occurs with probability

$\sim e^{L(\ln 4 - \frac{1}{\xi})}$, ~~so~~ For any finite L they occur with some nonzero density in all eigenstates of an ∞ MBL system.

Stability of the MBL phase:

A typical eigenstate is not involved in a resonance at a typical location:

$$2^{-L} > e^{-L/\xi} \quad \text{in MBL phase,}$$

But this seems to allow ~~s~~ a regime of the MBL phase where

$$2^{-L} > e^{-L/\xi} > 4^{-L}$$

so there are resonances present in some eigenstates at all locations.

For $e^{-L/\xi} < 4^{-L}$ resonances ~~need to~~ are present at some nonzero density, but are dilute for large L.

Can we systematically understand these resonances, enough to accurately catch their physics with MPB, and maybe even to see ~~and~~ how they proliferate as we approach the MBL-thermal transition?

So far, my discussion did not need ~~use~~ a random U (or H). It should be the same for quasiperiodic U (or H). (which has MBL, Iyer, et al. PRB 2013)

since they also have Poisson level stats in the MBL phase

And no strong qualitative differences between random and quasiperiodic have yet been seen in the numerical work (?)

This needs to looked for more systematically, which we are doing.

But eventually (on approaching the transition), rare-regions Griffiths effects and the Harris-Chayes inequality should constrain the random systems, but not the quasiperiodic systems. (Anushya)

Scaling studies of the transition from numerics (ED and the like) find an effective exponent $\nu \simeq 1 << \frac{2}{d} = 2$ strongly violating the Harris-Chayes inequality.

MBL is a problem with a real-space and Fock-space structure:

System of length L: random potentials at L sites:
L random numbers.
or L "pseudorandom" numbers for Q.P. problem.

Fock space of 2^L "sites": both cases: 2^L pseudorandom "local" (in Fock space) potentials. Effectively $d \to \infty$ Fock space, even if $d=1$ real space

Scenario: System does not "know" its real space is 1D and/or it is random (or quasiperiodic) until rather low energy (long length scale). At higher energy (small L), the physics is localization in Fock space (or perhaps 1D, but not random) (Altshuler + collaborators), which crosses over to the appropriate real-space-determined universality class at low energy.

~~Most ED~~
Perhaps ED studies are seeing mostly the Fock-space physics. (?)