Large Deviations Functionals for 2D Strongly Asymmetric Systems

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- Recent interest in higher dimensions: [Hurtado, Pérez-Espigares, del Pozo, P.L. Garrido], [Akkermans, Bodineau, Derrida, Shpielberg], [Pérez-Espigares, Redig, Giardinà]... .

Weakly Asymmetric Driven Diffusive Systems

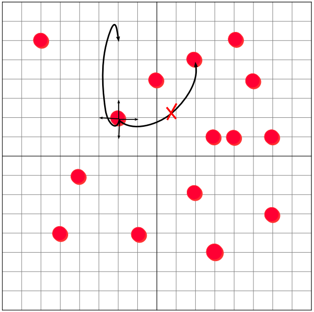
Consider a (closed) $2D$ driven diffusive conservative (in the density $\rho(t, \mathbf{x})$) system whose hydrodynamics (in the diffusive time scale) are given by the PDE

$$\partial_t \rho + \mathrm{div}\, j(\rho) = 0, \quad j(\rho) = -D(\rho)\nabla\rho + \nu\, f(\rho)$$

- ρ is the density,
- j is the density current,
- $D := D(\rho)$ is a square diffusion matrix,
- $f := f(\rho)$ a two dimensional vector,
- $\nu > 0$ a parameter regulating the strength of the drift.
- $t > 0$ is the time and $\mathbf{x} \in \mathbb{T}^2 := [0,1]^2$.

Examples

- Weakly asymmetric lattice gas
- Active particles



To take into account fluctuations around this typical behavior we replace the previous PDE by the conservative SPDE (fluctuating hydrodynamics)

$$\partial_t \rho + \mathrm{div}\, j(\rho) = 0, \quad j(\rho) = -D(\rho)\nabla\rho + \nu\, f(\rho) + \sqrt{\varepsilon\sigma(\rho)}\,\eta$$

where

- η is a space-time white noise,
- $\sigma := \sigma(\rho)$ is a symmetric matrix.

The SPDE describes the behavior of the empirical density ρ^ε (resp. current j^ε) of an extended system of size ε^{-1} ($\varepsilon \ll 1$), with a *weak* drift term of order $\varepsilon\nu$, in the time scale $\varepsilon^{-2}t$, where t is the macroscopic time:

$$\rho^\varepsilon(\varepsilon^{-2}t, \varepsilon^{-1}\mathbf{x}) \approx \rho(t, \mathbf{x}), \quad t > 0,\ \mathbf{x} \in \mathbb{T}^2,$$
$$\partial_t \rho^\varepsilon + \mathrm{div}\, j^\varepsilon = 0.$$

Current-density LDF

We have also

$$\mathbb{P}\left[(j^{\varepsilon},\rho^{\varepsilon}) \approx (j,\rho), \text{on the time window } [0,T]\right] \approx e^{-\varepsilon^{-2}\mathcal{I}_{[0,T]}(j,\rho)}.$$

where

$$\mathcal{I}^{\nu}_{[0,T]}(j,\rho) = \frac{1}{2}\iint_{\Omega} \left\langle \left[j + D\nabla\rho - \nu f\right], \sigma^{-1}\left[j + D\nabla\rho - \nu f\right]\right\rangle ds d\mathbf{x}$$

if the constraint

$$\partial_s \rho = -\mathrm{div}\, j$$

is satisfied and equal to infinity otherwise.
Here $\Omega = [0,T] \times \mathbb{T}^2$.

- This is the starting point of the Macroscopic Fluctuation Theory.
- The form of the LDF $\mathcal{I}$ has been proved to be valid for a large class of (weakly) asymmetric interacting particles systems.

Density LDF

The LDF $H^{\nu}_{[0,T]}$ is given by

$$H^{\nu}_{[0,T]}(\rho) = \inf_j \mathcal{I}^{\nu}_{[0,T]}(j, \rho)$$

describes the cost to observe a density profile ρ during the time window $[0, T]$.

Strongly Asymmetric Driven Systems

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- The typical behavior of the system, in the *hyperpolic time scale* $\varepsilon^{-1}t$, is now given by the scalar conservation law

$$\partial_t \rho + \mathrm{div}\ f(\rho) = 0.$$

- The (heuristic) derivation of this equation is simply based on the assumption of propagation of local equilibrium in the hyperbolic time scale.

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- We say that a function $\rho : [0,T] \times \mathbb{T}^2 \to \mathbb{R}$ is a weak solution if

$$\iint_{\Omega} \{\rho\,\partial_t \varphi + \langle f(\rho), \nabla \varphi \rangle\}\; dt\, d\mathbf{x} = 0.$$

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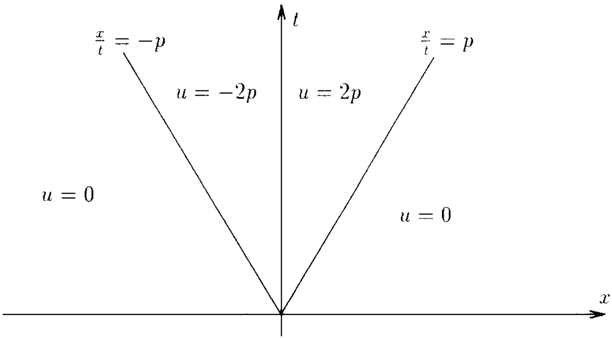
  for any smooth test function $\varphi : [0, T] \times \mathbb{T}^2 \to \mathbb{R}$.
- Weak solutions are not unique! What is the weak solution which describes correctly the typical behavior of the microscopic system?



For any $p > 0$, there is a weak solution to the $1D$ Burgers equation $\partial_t u + \partial_x(u^2) = 0$ with initial condition $u(0, x) = 0$.

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- A weak solution is called entropic if for each entropy-entropy flux pair (η, q) with η convex, and $\varphi \geq 0$ a test function, the **entropic inequality** (second principle) holds:

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- Existence and uniqueness of an entropic solution has been proved under generic conditions. The entropic solution is the only weak solution dissipating entropy.

Entropic solution: vanishing viscosity limit

- The entropic solution ρ can also be obtained as $\rho = \lim_{\varepsilon \to 0} \rho^\varepsilon$ of

$$\partial_t \rho^\varepsilon + \mathrm{div} f(\rho^\varepsilon) = \varepsilon \Delta \rho^\varepsilon.$$

- This also explains the entropic inequality, which holds for ρ^ε and will thus persist in the limit $\varepsilon \to 0$.

Large deviations of the density

- Entropic solution to the conservation law

$$\partial_t \rho + \mathrm{div}\; f(\rho) = 0.$$

  describes the typical macroscopic behavior of the strongly asymmetric microscopic system (size ε^{-1}) in the hyperbolic time scale $\varepsilon^{-1} t$.
- We are interested in the cost to produce a fluctuation of the density in this hyperbolic time scale.

- Let ε^{-1} the size of the microscopic system. The LDF $H^{\nu}_{[0,T]}(\rho)$ gives the cost to produce a given density fluctuation equal to ρ for the weakly asymmetric system (drift strength is $\nu\varepsilon$) in the diffusive time scale $t\varepsilon^{-2}$.

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- We thus expect that the cost $H^{\infty}_{[0,T]}(\rho)$ is given by

$$H^{\infty}_{[0,T]}(\rho) = \lim_{\nu\to\infty} H^{\nu}_{[0,T/\nu]}(\rho).$$

The one dimensional case

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- *One dimensional TASEP*: the LDF (**JV functional**) of the empirical density of the TASEP in the hyperbolic time scale has been rigorously derived by [Jensen'00-Varadhan'04] starting directly from the microscopic system [TASEP] ($f(\rho) = \sigma(\rho) = \rho(1-\rho)$, $D(\rho) = 1$).

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- [Bellettini-Bertini-Mariani-Novaga'10] proposed a generalization of the JV functional for general f, D, σ in $1D$ by studying the limit (for the Γ-convergence)

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- These results show that $H^{\infty}_{[0,T]}(u) < \infty$ only if u is a weak solution of the scalar conservation law. **This gives a physical meaning of weak solutions: there are exactly the profiles appearing in a large fluctuation**.

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- By following Bellettini-Bertini-Mariani-Novaga approach in 2D, we propose a generalization of the JV formula.
- The derivation is based on three simple principles:
  1. Locally around a (space-time) discontinuity point, a weak solution looks like a moving step function between u^- and u^+ propagating at some velocity v in a direction $\mathbf{k}$.
  2. For a weak solution $u(t,\mathbf{x}) = g(\langle \mathbf{k}, \mathbf{x} \rangle - vt)$ in the form of a moving step function, $H^{\infty}_{[0,T]}(u)$ can be evaluated by computing explicitly $\lim_{\nu \to \infty} H^{\nu}_{[0,T/\nu]}(u^\nu)$ for a good smooth approximation u^ν of u.
  3. The cost of a weak solution u is obtained by summing the individual costs of each discontinuity of u (space-time additive principle).

Local structure of a weak solution

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- For $(t, s_t(\alpha)) = (t, \mathbf{x}) \in J_u$ let

$$\mathbf{n} := (\mathbf{n}^t, \mathbf{n}^{\mathbf{x}}) = \frac{1}{\mathcal{N}} \left(- \left\langle \frac{ds_t}{dt}, \left[\frac{ds_t}{d\alpha}\right]^{\perp} \right\rangle, \left[\frac{ds_t}{d\alpha}\right]^{\perp} \right) \in \mathbb{R}^3,$$

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- $\Omega \backslash J_u = \Omega^+ \cup \Omega^-$ and for $(t, \mathbf{x}) = (t, s_t(\alpha)) \in J_u$

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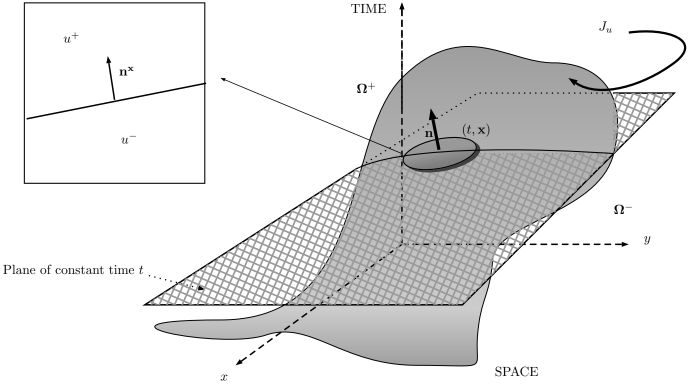
- Since u is a weak solution (Rankine-Hugoniot condition)

$$(u^+ - u^-)\mathbf{n}^t + \langle (f(u^+) - f(u^-)), \mathbf{n}^{\mathbf{x}} \rangle = 0$$


u^+
$\mathbf{n^x}$
u^-
TIME
J_u
Ω^+
$(t, \mathbf{x})$
$\mathbf{n}$
Ω^-
y
Plane of constant time t
x
SPACE

Cost of a moving step function

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- Take an **entropic** moving step function in the form

$$\rho(t,\mathbf{x}) = g(-\langle \mathbf{k}, \mathbf{x}\rangle + vt)$$

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- The space-time reversed function

$$u(t, \mathbf{x}) = \rho(-t, -\mathbf{x}) = g(\langle \mathbf{k}, \mathbf{x} \rangle - vt)$$

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- The entropic solution ρ can be approximated by a traveling wave $\rho^\nu(t, \mathbf{x})$ (vanishing viscosity approximation of order ν^{-1}).

- We use the space-time reversed approximation $u^{\nu}(t,\mathbf{x}) = \rho^{\nu}(-t,-\mathbf{x})$ to show that

$$\lim_{\nu\to\infty} H^{\nu}_{[0,T/\nu]}(u^{\nu}) = 2|J_u \cap \Omega| \int \frac{\langle \mathbf{k}, D(g)\,\mathbf{k}\rangle}{\langle \mathbf{k}, \sigma(g)\,\mathbf{k}\rangle}\, \Gamma^{+}_{\mathbf{k}}(u^-, u^+, g)\, dg$$

with

$$\Gamma_{\mathbf{k}}(g, u^-, u^+) = \frac{\left\langle f(u^-)(u^+ - g) + f(u^+)(g - u^-) - f(g)(u^+ - u^-), \mathbf{k}\right\rangle}{|u^+ - u^-|}$$

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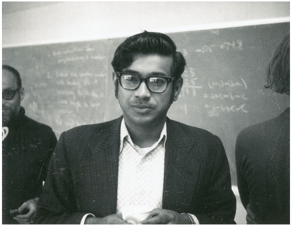
- Usually, no. We have to use a density argument. Mathematical proof is missing.
- "One does not see at the moment how to produce a general non-entropic solution, partly because one does not know what it is." *S.R.S. Varadhan*



Additive principle

Assuming space-time additive principle holds we get

$$H^{\infty}_{[0,T]}(u) = 2\int_0^T dt \int_{\mathbf{x}\in s_t} ds_t \left\{ \int \frac{\langle \mathbf{n}^{\mathbf{x}}, D(g)\,\mathbf{n}^{\mathbf{x}}\rangle}{\langle \mathbf{n}^{\mathbf{x}}, \sigma(g)\,\mathbf{n}^{\mathbf{x}}\rangle} \Gamma^{+}_{\frac{\mathbf{n}^{\mathbf{x}}}{\|\mathbf{n}^{\mathbf{x}}\|}}(u^-, u^+, g)\, dg \right\}$$

with

$$\Gamma_{\mathbf{k}}(g, u^-, u^+) = \frac{\left\langle f(u^-)(u^+ - g) + f(u^+)(g - u^-) - f(g)(u^+ - u^-), \mathbf{k}\right\rangle}{|u^+ - u^-|}$$

- With respect to the $1D$ case, the difference in the $2D$ case for the JV functional is the replacement of the second derivative of the Einstein entropy

$$S''(g) = \frac{2D(g)}{\sigma(g)}$$

by the scalar

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- One may also obtain a variational form of the generalized JV functional, based on a kinetic interpretation of the scalar conservation law.

Kinetic formulation (Lions, Perthame, Tadmor'94)

Let $\chi(g,u) = \mathbf{1}_{0<g\leq u} - \mathbf{1}_{u\leq g<0}$. Then u is a weak solution of

$$\partial_t u + \mathrm{div}\ f(u) = 0$$

iff $h(t,\mathbf{x},g) = \chi(g,u(t,\mathbf{x}))$ is solution of

$$\partial_t h + \langle f'(g), \nabla_{\mathbf{x}} h \rangle = -\partial_g \mu$$

for some locally finite space-time measure $\mu := \mu_u(g, dt, d\mathbf{x})dg$. Observe that

$$\int h(t,\mathbf{x},g)dg = u(t,\mathbf{x}).$$

If u is the entropic solution then μ_u is a negative measure.

- We say that $\mathcal{V}(g, t, \mathbf{x})$ is an entropy sampler if for any $(t, \mathbf{x})$ the function $g \longrightarrow \mathcal{V}(g, t, \mathbf{x})$ is convex.

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- The entropy production of a weak solution u w.r.t. the entropy sampler $\mathcal{V}$ is defined by the number

$$\mathcal{P}_{\mathcal{V}}(u) := \int dg \iint_{\Omega} \mathcal{V}''(g, t, \mathbf{x})\, \mu_u(g, dt, d\mathbf{x})$$

Variational form of the 2D JV functional

We have that

$$H^{\infty}_{[0,T]}(\rho) = \sup_{\mathcal{V} \in \hat{\mathbf{V}}} \mathcal{P}_{\mathcal{V}}(\rho).$$

where $\hat{\mathbf{V}}$ is the set of entropy samplers which are such that (in matrix sense)

$$2D(g) - \sigma(g)\mathcal{V}''(g, t, \mathbf{x}) \geq 0$$

Perspectives

- A complete rigorous proof is missing.
- Generalization to 2D strongly asymmetric systems in contact with baths at different densities ([Bahadoran, Bodineau-Derrida]).
- Obtain the corresponding quasi-potential, i.e. the non-equilibrium free energy of the NESS (hyperbolic MFT for 2D Asymmetric Systems).
- Case of multi-component systems (hard).
- Applications to deduce properties of the steady state of active particles.